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Crack paths at multiple-crack systems in anisotropic structures: simulation and experiment

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Abstract

This paper is targeted on numerical methods for accurate crack tip loading analysis and crack path prediction. Those are based on finite element calculations of the boundary value problem. Applying path-independent integrals to curved cracks in order to accurately calculate the J-integral, energy release rate (ERR) or stress intensity factors (SIF) is still not state of the art. Contours which are not confined to the crack tip require special analytical preparation and numerical treatment to supply results which are sufficiently precise for reliable crack path prediction. Methods to improve the calculation of the J-integral and the interaction integral (I-integral) are presented. In particular, the latter has never been applied to strongly curved cracks. Also, efficient methods for the loading analysis and crack growth simulation of multiple interacting cracks based on path-independent integrals are presented. The anisotropy of fracture toughness is taken into account being a crucial part of the numerical model. Experiments are carried out with specimens made of aluminum alloy Al-7075, comparing subcritically grown cracks with simulations.

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1. Introduction

Predicting the correct crack path in engineering structures is still a cumbersome procedure as many influencing parameters have to be accounted for e.g. crack tip loading, crack deflection criteria, anisotropic and inelastic material behavior. Many researchers of the recent years have focussed on the crack path predictions in plane structures applying different methods and theories and comparing their results to experimental findings (Miranda et al., 2003; Meyer et al., 2006; Španiel et al., 2009). Path-independent integrals are widely applied to calculate loading quantities such as the ERR (Griffith, 1921), SIF (Irwin, 1957), or the J-integral (Rice, 1968). Budiansky and Rice (1973) extended Rices’ approach of $J$, which was limited to straight cracks, by a formulation of the two-dimensional $J_2$-integral vector which is composed of the coordinate $J_1 = J$ and $J_2$. It is well-known that the calculation of the $J_2$-integral is challenging since the numerical treatment of the singular stresses at the crack tip is going along with problems finally leading to inaccurate results.

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The interaction integral is a conservation integral based on the superposition of two loading scenarios (Stern et al., 1976), i.e. the physical (a) and an auxiliary loading (b). In general, the near tip solution is employed to obtain auxiliary fields originally limiting this method to straight cracks in homogeneous materials without interfaces. After the introduction of path-independent integrals, two new methods for calculating accurate values of $J_2$ valid for straight and curved cracks are presented. Further, difficulties and solutions are pointed out for the calculation of the $I_k$-interaction integral considering arbitrary curved crack faces. A second focus is directed at the crack loading analysis of multiple cracks systems. Here, a new procedure is introduced, based on a global $I_k$-integral calculating accurate loading quantities related to the i-th crack tip by fading out all other crack tips. Anisotropy in fracture toughness has a strong influence on crack paths and is thus included in the model. A corresponding crack deflection criterion is suggested based on the ERR. Crack paths are calculated and compared to those resulting from experimental findings in rolled Al-7075 plates.

2. Path-independent contour integrals

Within the theory of Linear Elastic Fracture Mechanics (LEFM) the $J_k$-integral vector is a path-independent energy conservation integral. Applying an integration contour $\Gamma_\varepsilon$ in the vicinity of the crack tip at a distance $\varepsilon$, the $J_k$-integral is defined as

$$J_k = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} Q_{kj} n_j d\varepsilon, \quad Q_{kj} = \frac{1}{2} \sigma_{mn} \varepsilon_{mn} \delta_{kj} - \sigma_{ij} u_{ik},$$

with Eshelby’s tensor $Q_{kj}$, including the stress tensor $\sigma_{mn}$, the strain tensor $\varepsilon_{mn}$ and the displacement derivatives $u_{ik}$. The Kronecker identity tensor is denoted as $\delta_{kj}$. In LEFM the coordinates of Eq. (1) are related directly to the SIF:

$$J_1 = \frac{K_1^2 + K_\Pi^2}{E'}, \quad J_2 = -2 \frac{K_1 K_\Pi}{E'}.$$

For plane stress $E' = E$ and for plane strain $E' = E/(1 - \nu^2)$. If the $J_k$-integral is calculated, assuming two different superimposed loading scenarios (a) and (b) for an arbitrary crack configuration, one obtains the following expression:

$$J_k^{(a)+(b)} = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} Q_{kj}^{(a)+(b)} n_j d\varepsilon = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} (Q_{kj}^{(a)} + Q_{kj}^{(b)} + Q_{kj}^{(a)(b)}) n_j d\varepsilon = J_k^{(a)} + J_k^{(b)} + J_k^{(a)(b)}.$$

The third term of Eq. (3) is the interaction integral vector $J_k^{(a)(b)}$ and will be denoted from now on as $I_k$,

$$I_k = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} Q_{kj}^{(a)(b)} n_j d\varepsilon, \quad Q_{kj}^{(a)(b)} = \frac{1}{2} (\sigma_{mn}^{(a)} \varepsilon_{mn}^{(b)} + \sigma_{mn}^{(b)} \varepsilon_{mn}^{(a)}) \delta_{kj} - (\sigma_{ij}^{(a)} u_{ik}^{(b)} + \sigma_{ij}^{(b)} u_{ik}^{(a)}).$$

with $Q_{kj}^{(a)(b)}$ being Eshelby’s tensor related to the interaction integral. For straight crack faces, the near-tip solution yields valid fields $\varepsilon_{mn}$, $\sigma_{mn}$, $u_{ik}$ associated to an auxiliary loading configuration and is therefore usually applied as auxiliary field. The relation between the coordinates of Eq. (4) and SIF is as follows:

$$I_1 = 2 \frac{K_1^{(a)} K_1^{(b)} + K_\Pi^{(a)} K_\Pi^{(b)}}{E'}, \quad I_2 = -2 \frac{K_1^{(a)} K_\Pi^{(b)} + K_\Pi^{(a)} K_1^{(b)}}{E'}.$$

Auxiliary fields are generally chosen according to an unit mode-I ($K_1^{(b)} = 1$, $K_\Pi^{(b)} = 0$) or unit mode-II ($K_1^{(b)} = 0$, $K_\Pi^{(b)} = 1$) loading. If finite integration contours $\Gamma_0$ are considered, see Fig. 1, the coordinates of $J_k$ and $I_k$ in general become path-dependent (Judt and Ricoeur, 2013b). In case of straight crack faces subjected to mixed-mode loading the path-dependence is restricted to the second coordinates $J_2$ and $I_2$ if crack surface loads are neglected. If curved crack faces are considered, both coordinates of $J_k$ and $I_k$ are depending on the chosen integration contour $\Gamma_0$. To hold path-independence, crack face integrals have to be introduced, describing the jump of Eshelby’s tensor across the physical crack faces $d\Gamma_c^+ = -d\Gamma_c^- = d\Gamma_c$ and the fictitious crack faces $d\Gamma_f^+ = -d\Gamma_f^- = d\Gamma_f$, see Fig. 1(a):
The asymptotic crack tip solutions are selected to be the corresponding auxiliary fields and thus, the fictitious crack faces $\Gamma_I$ and the physical ones $\Gamma_c$ always coincide at the crack tip. The integration in the vicinity of the crack tip based on numerical values provided from the FE-calculation is challenging. As the numerical representation of the singularity in stresses and strains deviates strongly from analytic solutions, the calculation of crack face integrals needs a special treatment (Judt and Ricoeur, 2013a) which is usually circumvented applying small contours at the crack tip.

### 3. Approaches for the accurate calculation of $J_2$ and $I_2$

The analytic expression for crack face integrals along a small segment $d\Gamma_c = \delta$ in the vicinity of the crack tip, where crack faces are approximately straight $\varphi = \pm \pi$ and $n_j = -\mathbf{e}_2$, is

$$J_k = \int_{\Gamma_c} Q_{kj} n_j ds + \int_{\Gamma_c} \left[ Q_{kj} \right]^+ n_j ds$$

$$I_k = \int_{\Gamma_c} Q_{kj}^{(a/b)} n_j ds + \int_{\Gamma_c} \left[ Q_{kj}^{(a/b)} \right]^+ n_j ds + \int_{\Gamma_c} \left[ Q_{kj}^{(a/b)} \right]^- n_j ds.$$  

The constant T-stress $T_{11}$ at a small distance to the crack tip follows from the representation of tangential normal stress on the crack faces:

$$T_{11} = \frac{1}{2} \left( \sigma_{11} (\delta, +\pi) + \sigma_{11} (\delta, -\pi) \right).$$  

Those parts of $J_k$ and $I_k$ which are based on reliable numerical data are calculated from Eqs. (6) where a small part $\delta$ at the crack faces is excluded. The remaining part of the integrals is expressed analytically by Eqs. (7). Here, the analytic part $I_k$ is directly calculated applying $T_{11}$ according to Eq. (8). In contrast, $J_k$ is evaluated by an iterative procedure, as the analytic integral depends on the values of $J_k$. This becomes obvious, as the unknown value $K_II$ is calculated from the coordinates $J_k$ by rearranging Eqs. (2).

A second approach for the accurate calculation of $J_k$ is the extrapolation of tangential normal stresses and strains on the crack faces. The crack face integral incorporates the jump of the strain energy density across the positive and
negative crack faces $[\mu]^\times$, with $\mu = (\sigma_{11} e_{11}) / 2$. Investigations show that values of tangential normal stresses $\sigma_{11}$ and strains $e_{11}$ on the crack faces related to a mode-I loading should reach a constant value at the crack tip. In contrast, values related to a mode-II loading are singular there. Numerically calculated values of $\sigma_{11}$ and $e_{11}$, respectively, become highly inaccurate approaching the crack tip. The values related to the single mode cases are separated as follows:

\[
\sigma_{11}^I = \frac{1}{2} (\sigma_{11}^T + \sigma_{11}^C), \quad \sigma_{11}^{II} = \sigma_{11}^T - \sigma_{11}^C, \quad e_{11}^I = \frac{1}{2} (e_{11}^T + e_{11}^C), \quad e_{11}^{II} = e_{11}^T - e_{11}^C. \tag{9}
\]

Mode-I values $\sigma_{11}^I$ and $e_{11}^I$ within the region $[0, \delta]$ are replaced by those calculated from a linear regression based on values at $r > \delta$. The integral is calculated classically according to Eq. (6a) considering the extrapolated values.

4. Global approach for crack tip loading analyses in multiple cracks systems

In LEFM the ERR $G^{(n)}$ of a crack $n$ equals the projection of the $J_k$-integral vector onto the unit vector of crack propagation $z_k$:

\[
\frac{1}{B} \frac{d\Pi}{da^{(n)}} = G^{(n)} = j_k^{(n)} z_k. \tag{10}
\]

The width $B$ of the specimen will be dropped from now on whereupon the ERR has the unit J/m and must be divided by $B$ to obtain the physical quantity. The virtual change of the total potential energy equals the product of the ERR and the virtual crack extension $\delta a$:

\[
\delta \Pi = \frac{d\Pi}{da} \delta a = -G \delta a. \tag{11}
\]

Taking into account that $\delta a^{(n)} = \delta a$ for all crack tips, the change of potential energy of a system with $N$ cracks reads

\[
\delta \Pi = - \sum_{n=1}^{N} G^{(n)} \delta a^{(n)} = - \sum_{n=1}^{N} G^{(n)} \delta a. \tag{12}
\]

It becomes clear from Eq. (12), that the total energy release of a system with $N$ cracks is the sum of individual ERRs for each single crack $n$, according to

\[
\tilde{G} = \frac{d\Pi}{da} = \sum_{n=1}^{N} G^{(n)} = \sum_{n=1}^{N} j_k^{(n)} z_k. \tag{13}
\]

where $\tilde{G}$ is a global ERR. Within a local crack tip coordinate system $\tilde{e}_k^{(n)}$ related to the $n$-th crack, see Figs. 1(b) and 1(c), the first coordinate $j_k^{(n)}$ is related to the ERR $G^{(n)}$ in case of a self-similar crack propagation $z_k^{(n)} = \tilde{e}_k^{(n)}$. Consistent with Eq. (13) and considering Eq. (5) the interaction integral of multiple cracks systems equals the sum of the interaction integrals related to each crack tip:

\[
\tilde{G}^{(a/b)} = \tilde{I}_1 = \sum_{n=1}^{N} I_k^{(n)} = \frac{2}{E'} \sum_{n=1}^{N} [(a) K_1^{(n)} (b) K_1 + (a) K_2^{(n)} (b) K_2]. \tag{14}
\]

A global interaction integral approach according to Eq. (14) basically has to deal with interacting auxiliary fields and thus additional terms in Eq. (4). The integration path $\Gamma_0$ is including all crack tips, thus the approach is denoted as "global". In Eq. (14) $2N$ SIF of the physical problem are unknown. To determine these values, the choice of SIF related to the auxiliary fields is similar as explained in Sec. 2 for the single-crack problem. Now, all SIF of the auxiliary fields $K^{(m)}_{II}$ are chosen to be zero except for the $m$-th crack. In this case, the global interaction integral according to Eq. (14) simplifies and for auxiliary unit mode-I (superscript I) and unit mode-II loadings (superscript II) reads:

\[
\tilde{I}_1^{(m)} = \frac{2}{E'} (a) K_1^{(m)}, \quad \tilde{I}_1^{II} = \frac{2}{E'} (a) K_{II}^{(m)}. \tag{15}
\]
This choice equals the procedure of defining only one auxiliary field related to the $m$-th crack tip. Thus, the interaction of multiple auxiliary fields and therefore additional terms in the $I_k$-integral calculation according to Eq. (4) is prevented as these terms are canceled out. Now that all cracks except one are faded out, the integration contour may be chosen on a global level, i.e. all crack tips are included. The integration contours chosen for this global approach, are exemplarily shown in Figs. 1(b) and 1(c) for a system with $N = 2$ cracks. The general formulation of the path invariant $I_k$-integral thus considers integrations along the external contour $\Gamma_0$ and along all $N$ physical and fictitious crack faces, $\Gamma^n_m$ and $\Gamma^{in}_m$. As the auxiliary fields are inserted according to the above discussed choice of the auxiliary SIF, according to which all are zero except for the $m$-th crack, the $I_k$-integral simplifies as follows:

$$ I^{(m)}_k = \int_{\Gamma_0} Q_{kj}^{(a/b)} n_j \, ds + \sum_{n=1}^{N} \int_{\Gamma^n} \left[ Q_{kj}^{(a/b)} \right]^+ n_j \, ds + \int_{\Gamma^{in}} \left[ Q_{kj}^{(a/b)} \right]^+ n_j \, ds. \quad (16) $$

5. Comparison of experimental and numerical crack paths

The concepts presented in the previous sections are applied to accurate crack path predictions. The $J_{Ik}$- and $I_k$-integrals are implemented as post processors into the commercial FE code ABAQUS. The simulation of stable and unstable crack growth is realized by an incremental extensions of the crack faces. In the past, many crack deflection criteria were presented e.g. by Erdogan and Sih (1963) or Hussain et al. (1974). The deviation of crack paths derived from different deflection criteria is negligible for small mixed-mode ratios $|K_{II}/K_1| < 0.1$. The $J$-integral criterion assumes that the crack always grows in the direction of the $J_k$-vector, as the ERR is maximized in that case. According to Eq. (10) the ERR can be expressed as a quantity depending on the crack deflection angle $\alpha$

$$ G (\alpha) = J_1 \cos \alpha + J_2 \sin \alpha. \quad (17) $$

Most fracture criteria assume isotropic critical parameters $K_{lc}$ or $G_c$ which are independent of the crack growth direction. $G_c$ is related to the fracture toughness $K_{lc}$ by

$$ G_c = \frac{K_{lc}^2}{E}. \quad (18) $$

Due to rolling process during production, plates of Al-7075 show anisotropic effects in fracture toughness which must be considered in the modeling and prediction of crack growth. In general, the fracture toughness in rolling direction (RD) $K_{lc}^{RD}$ is smaller than in transversal direction (TD) $K_{lc}^{TD}$. Kfouri (1996) presented an elliptical interpolation function describing the fracture toughness as a function of the orientation angle $\alpha$ with respect to the RD:

$$ \left( \frac{1}{K_c (\alpha)} \right)^2 = \left( \frac{\cos \alpha}{K_{lc}^{RD}} \right)^2 + \left( \frac{\sin \alpha}{K_{lc}^{TD}} \right)^2, \quad \frac{1}{G_c (\alpha)} = \frac{\cos^2 \alpha}{G_{c}^{RD}} + \frac{\sin^2 \alpha}{G_{c}^{TD}}. \quad (19) $$

Besides $G_c (\alpha)$ the loading quantity $G (\alpha)$ depends on the crack deflection angle. On the one hand, the system attempts to minimize the total potential energy and therefore maximize the energy release rate during the crack advance $G (\alpha) \rightarrow \max$. On the other hand, the crack tends to grow into the direction of the minimum material resistance $G_c (\alpha) \rightarrow \min$. The ratio of ERR and crack resistance according to Eqs. (17) and (19) is defined as

$$ G_R (\alpha) = \frac{G (\alpha)}{G_c (\alpha)} = \frac{J_1 \cos^3 \alpha + J_2 \sin \alpha \cos^2 \alpha}{G_{c}^{RD}} + \frac{J_2 \cos \alpha \sin^2 \alpha + J_2 \sin^3 \alpha}{G_{c}^{TD}}. \quad (20) $$

It is assumed, that the crack grows in the direction of maximum $G_R$, thus $\partial G_R / \partial \alpha = 0$ and $\partial^2 G_R / \partial \alpha^2 < 0$. Numerically predicted crack paths for different ratios of anisotropy $\chi = K_{lc}^{TD}/K_{lc}^{RD}$ are presented in Fig. 2 and compared with experiments. Experiments with CT specimens were carried out for rolled Al-7075 providing a ratio of $\chi = 1.14$. Comparing crack paths in Fig. 2(a) it is obvious that the anisotropy of fracture toughness is crucial for crack path prediction although the ratio $\chi = 1.14$ slightly overestimates the crack deflection by the hole.
6. Closure

New approaches have been presented for the accurate calculation of crack face contribution to \( J_k \)-and \( I_k \)-integrals enabling the application of remote integration contours. This leads to efficient simulations of crack problems considering internal boundaries or multiple crack faces. A new method for the separation of loading quantities at multiple cracks systems based on global \( I_k \)-integrals is presented. It is shown that the fracture toughness anisotropy has a strong impact on crack paths and must be considered in numerical models. Experiment and prediction are in good agreement.

References