Analytical and experimental study of a circular membrane in Hertzian contact with a rigid substrate

Dewei Xu, Kenneth M. Liechti *

Research Center, Mechanics of Solids, Structures and Materials, Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, TX 78712, USA

Abstract

The problem that is addressed here is that of a pressurized circular membrane in contact with a rigid substrate. A closed-form membrane analysis with Hertz-type contact is developed to describe the relationship between pressure, contact radius and contact force. Both the variation in the slope of the deflection profile of the portion of the membrane outside the contact zone and the contact radius itself are measured by an apparatus based on moiré deflectometry. Contact experiments with a 3 μm PET film and a glass substrate show that this analysis predicts both the slope field and contact radius well.

© 2010 Elsevier Ltd. All rights reserved.

1. Introduction

The adhesion, contact and deformation of thin membranes have played important roles in many fields. For instance, in biological science, cell membrane and substratum adhesion is vital in normal cell functioning and locomotion (Fisher, 1993) and vesicle membrane fusion is of practical importance for targeted drug delivery (Bakowsky et al., 2008). In micro- or micro-opto-electro-mechanical systems (MEMS or MOEMS), electrostatically driven bridges or diaphragm membranes operate over trillions of cycles in their life span and the study of reliability and durability of such MEMS/MOEMS devices relies on a quantitative understanding and determination of change in adhesion and contact over time (Rebeiz, 2003). Furthermore, an accurate determination of contact size is necessary to evaluate contact resistance, heat dissipation and contact temperature in DC-contact-switch MEMS (Hyman and Mehregany, 1999; Rebeiz, 2003).

The contact mechanics of two elastic solids has been well established by the advent of the Hertz (1881), JKR (Johnson et al., 1971), DMT (Derjaguin et al., 1975; Maugis, 1992) and Maugis (Maugis, 1992) theories. The application conditions and selection guidelines of these theories are summarized in the Johnson-Greenwood map (Johnson and Greenwood, 1997), which is based on the dimensionless parameter initiated by Tabor (1977). However, these theories for elastic solids are not applicable to thin membranes in contact. This is due to the fact that the elastic strain energy is determined by the membrane stresses which result from the large out-of-plane deflections of the thin membrane. Consequently, geometrical non-linearity has to be considered and exact closed-form solutions are not possible. The first configuration studied by membrane contact mechanics was a cellular membrane compressed between two parallel plates (Cole, 1932; Harvey, 1938; Hiramoto, 1963), which was used to characterize the mechanical properties of cellular membranes. The other extensively studied configuration is a spherical capsule adhered to a substrate (Shanahan, 1997; Wan and Liu, 2001). This configuration is widely used to explore cell/vesicle/liposome/microcapsule–substrate contact and adhesion which play critical roles in biological and biomedical science. The third class of problems includes one-dimensional strips or axisymmetric membranes contacting a rigid substrate or punch under adhesive surface forces (Plaut et al., 1999, 2001; Wan and Julien, 2009; Wong et al., 2007; Yang, 2004). These configurations have been used to study the contact and adhesion between thin membranes and substrates and stiction and adhesion in MEMS structures.

In this paper, a pressurized circular membrane clamped peripherally and contacting a rigid substrate is studied, which can be regarded as a contact configuration in the third category mentioned above. This geometry is also reminiscent of the constrained blister...
test (Chang et al., 1989; Napolitano et al., 1988) but instead the edge of the blister is clamped and the contact and adhesion between the membrane and the constrained plate is of interest. Plaut et al. (2003) obtained extensive numerical solutions for the contact mechanics of this configuration under linear plate, nonlinear plate and membrane assumptions, with or without adhesion. The same author has also studied the mechanical response of axisymmetric membranes with or without contact under various loading conditions and theories (Plaut, 2009b). However, residual stresses were not considered in these analyses and so far, no experimental verification of these analyses has been reported. Furthermore, the use of uniform pressure over the contact region makes this configuration attractive for the study of contact and adhesion between thin membranes and substrates and in MEMS structures. Therefore, it is desirable to develop closed-form analytical solutions to predict the relationships between contact radius, contact force and pressure for simplicity, as the Hertz, DMT, JKR and Maugis contact theories for elastic bodies did.

In the following, a combined analytical and experimental study is pursued. An approximate closed-form analysis is used to predict the relationship between the pressure, the contact radius and the contact force without considering adhesion, which is reminiscent of the Hertz contact theory for two elastic bodies. Furthermore, residual stress is included in the analysis and this analysis is verified by contact experiments. An apparatus modified from a previous bulge tester (Xu and Liechti, 2009) based on moiré deflectometry (Kafri, 1980) is used to simultaneously measure both the contact radius and the slope of the deflection profile of the membrane outside the contact zone. The direct measurement of this slope, which has not been reported in the literature, is critical for thin membrane contact because the membrane strain and the corresponding stress are best determined from the slope of the deflection rather than the deflection itself.

This paper is organized as follows: the Hertz-type contact theory is developed in Section 2; this is followed by an introduction of the principles and formulation for measuring the contact area and the slope of the deflection profile using moiré deflectometry in Section 3; sample preparation, a description of the apparatus and resulting moiré patterns are presented in Section 4; finally, experimental and analytical results are presented and discussed in Section 5.

2. Hertz theory

Fig. 1 shows a schematic of the contact configuration for this study. A smooth thin film with a thickness $h$, Young’s modulus $E$ and Poisson’s ratio $v$ is clamped peripherally to a substrate with a circular opening with a diameter of $2a$. As a pressure $p$ is applied on one side, the thin film bulges with a profile $w(r)$ and contacts a smooth, parallel and rigid surface which is placed above the substrate with a prescribed gap $g$ when the pressure approaches a certain value. The goal of the present analysis is to find the relationship between the pressure $p$, the contact radius $c$ and the contact force $F$. The assumptions made in the analysis are as follows: (i) the thin film has negligible flexural rigidity and only membrane stresses are considered, i.e., $a.g > h$; (ii) the gap $g < a$ and $\sin \theta \approx dw/dr$, where $\theta$ is defined in Fig. 1b, which, as will be seen later, places a restriction on the extent of the contact radius; (iii) a constant radial stress, i.e., $\sigma_r = \sigma$, is assumed; (iv) the contact between the film and the rigid surface is frictionless.

The free body diagram of the thin membrane and the frame of reference are shown in Fig. 1b. Equilibrium in the $r$-direction is given by (Timoshenko and Woinowsky-Krieger, 1959)

$$T_r - T_t + \frac{dr}{dr} = 0,$$

where $T_r$ and $T_t$ are the radial and tangential forces per unit length. According to assumption (iii) ($T_r = \sigma_r h = \text{const}$) and an equi-biaxial stress state, $T_r = T_t = \sigma h$, is obtained. Outside the contact region ($c < r \leq a$), the equilibrium equation in the $w$-direction is

$$\frac{dw}{dr} = 2 \pi r (r^2 - c^2) \cdot p = 0. \quad (2)$$

Considering the boundary conditions $w(c) = g$, $w'(c) = 0$ and $w(a) = 0$, $\sigma$ and $w$ are, respectively, obtained as

$$\sigma = \frac{pa^2 \left( 1 - (c/a)^2 + (c/a)^4 \ln(c/a)^2 \right)}{4gh}, \quad (3)$$

and

$$w = \begin{cases} g, & 0 < r \leq c, \\ \frac{g(1 - (r/a)^2 - (c/a)^2 \ln(c/a)^2)}{1 - (c/a)^2 + (c/a)^4 \ln(c/a)^2}, & c < r \leq a. \end{cases} \quad (4)$$

Accordingly, the slope $dw/dr$ is given by

$$\frac{dw}{dr} = \begin{cases} 0, & 0 < r \leq c, \\ \frac{-2cg(r/a - (c/a)^2 \ln(c/a))}{a(1 - (c/a)^2 + (c/a)^4 \ln(c/a)^2)}, & c < r \leq a. \end{cases} \quad (5)$$

The resulting equi-biaxial stress state from assumption (iii) cannot be satisfied over the entire domain. For example, the tangential strain $\varepsilon_t$ varies from the equi-biaxial state over the domain $0 < r \leq c$ to $\varepsilon_t = 0$ at $r = a$. Considering that $\varepsilon_t = du/dr + (dw/dr)^2/2$ and $\varepsilon_r = u/r$, where $u$ is the radial displacement, a mean strain (Williams, 1997) is defined as

$$\varepsilon = \frac{1}{2} \left( \varepsilon_t + \varepsilon_r \right) = \frac{1}{2} \left( \frac{du}{dr} + u + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \right) = \frac{\sigma(1 - v)}{E}. \quad (6)$$

Consequently, the constitutive relationship in integral form under the resulting equi-biaxial stress state from assumption (iii) is represented as

$$\int_0^a \frac{d}{d\theta} (ur) + \int_0^a \frac{1}{2} \left( \frac{dw}{dr} \right)^2 rdr = 2\frac{\sigma(1 - v)}{E} \int_0^a rdr. \quad (7)$$

Noting that $u(0) = u(a) = 0$ and $dw/dr = 0$ for $0 < r \leq c$ and $dw/dr = -p(r^2 - c^2)/2ahr$ for $c < r \leq a$, Eq. (7) becomes

$$\int_0^a \frac{d}{d\theta} (ur) + \int_0^a \frac{1}{2} \left( \frac{dw}{dr} \right)^2 rdr = 2\frac{\sigma(1 - v)}{E} \int_0^a rdr. \quad \text{(7)}$$
\[ \sigma = \frac{E \pi^2 a^2}{32(1 - \nu)^2} \left( 1 - 4 \left( \frac{C_0}{C_1} \right)^2 + 3 \left( \frac{C_0}{C_1} \right)^4 - \left( \frac{C_0}{C_1} \right)^6 \ln \left( \frac{C_0}{C_1} \right)^6 \right). \]  

Comparing Eqs. (3) and (8), the relationship between the prescribed pressure \( p \) and the corresponding contact radius \( c \) is

\[ p = \frac{2Eg^2h(1 - 4(c/a)^2 + 3(c/a)^4 - (c/a)^4 \ln(c/a)^4)}{(1 - \nu)a^2(1 - (c/a)^2 + (c/a)^2 \ln(c/a)^2)^2}. \]

If an equi-biaxial residual stress \( \sigma_0 \) exists before pressurization and noting that the strain in Eq. (6) is referred to the deformed state, the quantity \( \sigma \) in Eq. (7) should be substituted with \( \sigma - \sigma_0 \). As a result, the relationship between the prescribed pressure \( p \) and the corresponding contact radius \( c \) with an equi-biaxial residual stress \( \sigma_0 \) is obtained as

\[ p = \frac{2Eg^2h(1 - 4(c/a)^2 + 3(c/a)^4 - (c/a)^4 \ln(c/a)^4) + 4(1 - \nu)\sigma_0 gha^2C_0^2}{(1 - \nu)a^2C_0^2}, \]

where \( C_0 = 1 - (c/a)^2 + (c/a)^2 \ln(c/a)^2 \). Note that from Eq. (10)

\[ \lim_{c \to 0} p = \frac{2Eg^2h + 4(1 - \nu)\sigma_0 gha^2}{(1 - \nu)a^2}, \]

which is consistent with the governing equation that was used for circular bulge tests (Xu and Liechti, 2009).

The contact force exerted on the rigid plate can be represented as

\[ F = \pi c^2 p, \]

where \( c \) can be obtained by solving Eq. (9) or (10) for a known pressure \( p \). Alternatively, the direct relationship between the contact force \( F \) and the pressure \( p \) with or without residual stress \( \sigma_0 \) can be obtained by combining Eq. (12) with Eq. (10): \n
\[ p = \frac{2Eg^2h(1 - 4\frac{f}{\pi f} + 3\left( \frac{f}{\pi f} \right)^2 - \left( \frac{f}{\pi f} \right)^2 \ln \left( \frac{f}{\pi f} \right)^2 \right)}{(1 - \nu)a^2C_1^2}. \]

Alternatively, combining Eq. (12) with Eq. (9) yields

\[ p = \frac{2Eg^2h(1 - 4\frac{f}{\pi f} + 3\left( \frac{f}{\pi f} \right)^2 - \left( \frac{f}{\pi f} \right)^2 \ln \left( \frac{f}{\pi f} \right)^2) + 4(1 - \nu)\sigma_0 gha^2C_1^2}{(1 - \nu)a^2C_1^2}, \]

where \( C_1 = 1 - \frac{f}{\pi f} + \left( \frac{f}{\pi f} \right)^2 \ln \left( \frac{f}{\pi f} \right). \)

### 3. Moiré deflectometry for contact radius and slope measurements

The deflection profile of the membrane outside the contact region, its corresponding slope and the relationship between the pressure and the contact radius were derived above in Section 2. In this study, the contact radius and the slope are also measured simultaneously by moiré deflectometry, which was introduced by Kafri (1980) as an incoherent light technique that measures ray deflection of a collimated beam instead of measuring differences in optical path length. Therefore, compared to interferometry, its setup is much simpler and is superior with respect to mechanical stability. Furthermore, the fringes generated from moiré deflectometry map the slope field of the deflected surface and this unique technique has found numerous applications in optical mapping, fluid density analysis, studies of transient phenomena, etc. (Kafri and Glatt, 1990).

A schematic of the moiré deflectometry setup for measuring the contact size and the slope of the deflection profile of the thin membrane is shown in Fig. 2a. This arrangement is very similar to that used in the earlier bulge tester to determine the mechanical properties of transparent thin films by measuring the focal length of the lens structure formed by the bulged film and the pressurizing medium (Xu and Liechti, 2009). A collimated beam passes through the lens structure \( L \) formed by the bulged film and the pressurization medium with a refractive index \( n \) and two identical Ronchi gratings \( G_1 \) and \( G_2 \) with a pitch \( p \), which are placed behind the sample with a separation \( D \) and with a small rotation \( \theta \) (<10°) between them. The grating \( G_1 \) is placed at a distance \( D_0 \) from the sample. The \( y \)-axis is defined such that the angle between it and \( G_1 \) or \( G_2 \) is \( \theta/2 \). A diffuse screen \( S \) is attached to the rear of the grating \( G_2 \). Before pressurization, the resulting moiré fringes on \( S \) (Fig. 2b) are shown in gray. The black fringes in Fig. 2b result when the pressurized thin membrane contacts the transparent rigid plate with a contact diameter \( 2c \). It can be seen that these black fringes inside the contact zone remain in the same direction as the gray fringes and they deviate from the gray ones outside the contact zone. The contact diameter is determined to be the distance between two turning points on the central black fringe from straight to slant. For example, \( c \) and \( C \) shown in Fig. 2b are the two turning points on the central black fringe and \( C/C = 2c \).

The slope at \( A \) on the sample can be determined as (Karny and Kafri, 1982; Xu and Liechti, 2009)

\[ \frac{dw}{dt} \big|_A \approx \frac{\sigma}{1 - \nu} \approx \frac{\theta y_A}{(n - 1)D_0}, \]

where \( \sigma \) is the angle between the original beam \( AA' \) and the deflected beam \( AA' \); \( y_A \) is the \( y \)-coordinate (Fig. 2c) of \( A \) which is the corresponding point on the moiré pattern to \( A \) on the sample; the refractive index of the pressurizing fluid is \( n \). The radial position of \( A \) can be represented in terms of coordinates \( r_A \) and \( y_A \) of \( A' \)

\[ r_A = r_{A'} + (D + D_0)\theta = r_{A'} + (D + D_0)\frac{\theta y_A}{(n - 1)D_0}. \]

where \( r_{A'} \) is the \( r \)-coordinate (Fig. 2c) of \( A' \). Because \( r_{A'} \) and \( y_{A'} \) can be directly measured from the moiré pattern recorded during contact experiments, the pointwise slope of the membrane can be obtained using Eqs. (16) and (15).

### 4. Experimental

In this section, we describe the specimen fabrication, the apparatus and the procedures for conducting the contact experiments. The resulting pressure history and moiré patterns are presented and examined.

#### 4.1. Sample preparation and apparatus

A polyethylene terephthalate (PET) film with a thickness of 3 \( \mu m \) was used in the contact experiments of a circular membrane on a rigid substrate. The PET films were first stretched and fixed to a 127 mm diameter ring. The films were then bonded to smaller aluminum frames with an aperture of 20.32 mm using NOA 68 optical adhesive (Norland Products Inc., Cranbury, New Jersey). After being cured under ultraviolet (UV) light for 20 min, samples were harvested from the 127 mm ring by cutting the films off around the corresponding aluminum frame. This frame was mounted onto the pressure manifold described in the following and the freestanding film over the central aperture was the sample area used for contact experiments.

A schematic view of the apparatus used in this study is shown in Fig. 3. It consists of a pressurization device along with video recording and data acquisition equipment and the components for moiré deflectometry. The collimated beams were produced by...
an 8mW He-Ne laser source and a beam expander. The aluminum frame with the PET sample was fixed to the manifold and sealed by a rubber o-ring. A transparent glass plate was fixed on steel shim strips (Precision Brand Products, Inc.) which were placed between the glass plate and the aluminum frame and acted as a spacer to prescribe the gap between them. Two different shim strips with thicknesses of 203 µm and 305 µm were used. The pressurizing medium was deionized water and a syringe pump (NE500, New Era Pump Systems Inc., Wantagh, New York), which can operate in both infusion and withdrawal, was used to pressurize the system. The pressure was measured with a pressure transducer (Sensotec Z/0761-092G, Columbus, Ohio) with a capacity of 103.4 kPa. It was connected to a data acquisition board (National Instruments PCI-MIO-16XE-50, Austin, Texas) which was installed in a PC with LabVIEW software. The pitch of the gratings used for moiré deflectometry was 0.254 mm. The separation $D$ between the two gratings was 12.7 mm and the rotation $\theta$ between them was about 7.3°. The value of $\theta/D$ was calibrated as 0.102 (Xu and Liechti, 2009).

4.2. Contact experiments, pressure history and moiré patterns

Before contact experiments, bulge tests with both rectangular and circular specimens were used to determine the mechanical properties of the PET sample (Xu and Liechti, 2009). A Young’s modulus of 4.65 GPa, Poisson’s ratio of 0.34 and an equi-biaxial residual stress of 7.50 MPa were obtained.

After the circular bulge test, the shim strips and the glass plate were fixed above the sample. The glass plate was sonicated in deionized water and dried in a dry nitrogen flow. Before pressurization, trapped air bubbles were carefully eliminated from the chamber in order to obtain satisfactory moiré patterns. The volume rate during both infusion and withdrawal was 3 ml/h. The acquisition of the changing pressure and moiré patterns during the experiment was synchronized and their history was then recorded.

Fig. 4 shows the pressure history during a contact experiment with a gap of 203 µm. The PET film first deflected under pressurization and behaved as a membrane until $A$ (200 s, 218 Pa) where...
the film started to contact the glass plate. Note that the pressure at A can be used to check the mechanical properties using Eq. (11). After A, the contact size increased as the syringe pump continued to pressurize to B (340 s, 1036 Pa). Then the syringe was reversed and consequently the pressure decreased. The pressure decreased as the syringe pump continued to withdraw until D (870 s, 210 Pa) when the PET film lost contact with the glass plate. The pressure was then released. The consistency of the pressures at A and D demonstrates that there was no adhesion and that the contact was fully reversible. Note that from B to C, the pressure varied much more slowly than from C to D in spite of the fact that the pumping rate was constant at 3 ml/h. This is due to mechanical hysteresis (e.g. backlash, viscous effects) in the pumping system.

The recorded moiré fringes were used to determine both the contact size and the slope of the membrane deflection outside the contact zone. Fig. 5 shows the moiré pattern that was obtained at 547 Pa. Note that the recorded fringes rotated in the opposite direction to that shown in Fig. 2b because they were recorded from behind the diffuser S. The central fringe was identified by LabVIEW software using an intensity threshold to establish the fringe width and then taking the middle of the fringe as its location. The diameter of the contact area was measured as the distance between the two turning points on the central fringe. The r- and y-coordinates of an arbitrary point on the middle of the central fringe with respect to the center were used to obtain the real position on the sample using Eq. (16) and the corresponding slope using Eq. (15), respectively.

The contact experiment and analysis were repeated with a gap of 305 µm and the results will be presented in next section.

5. Results and discussion

Fig. 6 shows the variation of the slope of the deflection profile with normalized radius obtained from the contact experiments with a 203 µm gap. Two sets of data that were obtained when the normalized contact radius was 0.247 and 0.428 are shown in the figure. These slope values were determined from the median line on the central fringe, as described in Section 4. The error bars represent the uncertainty of slopes which is equal to the slope value resulting from one quarter of the fringe width. Finer fringes and more sophisticated image processing would lead to more accurate measurements of the slope. The predictions shown in Fig. 6 are evaluated from Eq. (5), which predicts the slope of the membrane over \( c < r < a \) quite satisfactorily. Note that both the deflection profile (Eq. (4)) and the derivative (Eq. (5)) of the membrane are independent of mechanical properties of and residual stress in the film. This is due to the membrane assumption where the mechanical properties do not appear in the equilibrium equations (Eqs. (1) and (2)). In addition, we compare our data and approximate solution (Fig. 6) with the numerical solution (Lai and Dillard, 1996; Plaut, 2009a; Plaut et al., 2003). The results are very consistent.

Figs. 7 and 8 show the variation of contact radius with pressure during both loading and unloading for gaps of 203 µm and 305 µm, respectively. As shown in Fig. 7, contact in the experiments with the gap of 203 µm was made at 218 Pa, corresponding to A in Fig. 4. The minimum contact radius is limited by the slope and spatial resolution of the moiré deflectometry. The solid line represents the prediction obtained from Eq. (10). The data from both loading and unloading are consistent with the analysis and no hysteresis
was observed in either experiment. The numerical solutions (Lai and Dillard, 1996; Plaut, 2009a; Plaut et al., 2003) (dashed lines) are also added to Figs. 7 and 8 and are very consistent with our data and approximate solution.

The data indicates that the approximate analysis in Section 2 predicts both the slope of the deflection profile of the membrane and the contact area quite accurately. Assumption (i), which is related to classical membrane assumptions, was easily satisfied. For assumption (ii), the condition $g < a$ does not necessarily imply that $\sin \theta \approx dw/dr$, which is used to derive Eq. (2). For example, even if $g < a$, when $c/a$ is large, so is $dw/dr$. Consequently, an additional constraint on the contact radius should be considered as part of assumption (ii). According to Eq. (5), $dw/dr$ is a monotonically increasing function of $c/a$ and $r$ in the region $c < r < a$. Table 1 lists the maximum ratios $\theta_{\text{max}}$ that satisfy assumption (ii) for various values of $a/g$. This maximum ratio was obtained from Eq. (5) by assuming that $\theta_{\text{max}} = 10^\circ$ at $r = a$. The ratios of $a/g$ corresponding to the data shown in Figs. 7 and 8 were 33 and 50, respectively. Thus it can be seen that the corresponding ratios of $c/a$ in Figs. 7 and 8 are well within the limits shown in Table 1.

We focus on assumptions (iii) and (iv) in the following. The assumption (iii) of a constant radial stress and the introduction of an additional constraint on the contact radius should be considered as part of assumption (ii). According to Eq. (5), $dw/dr$ is a monotonically increasing function of $c/a$ and $r$ in the region $c < r < a$. Table 1 lists the maximum ratios $\theta_{\text{max}}$ that satisfy assumption (ii) for various values of $a/g$. This maximum ratio was obtained from Eq. (5) by assuming that $\theta_{\text{max}} = 10^\circ$ at $r = a$. The ratios of $a/g$ corresponding to the data shown in Figs. 7 and 8 were 33 and 50, respectively. Thus it can be seen that the corresponding ratios of $c/a$ in Figs. 7 and 8 are well within the limits shown in Table 1.

We focus on assumptions (iii) and (iv) in the following. The assumption (iii) of a constant radial stress and the introduction of an additional constraint on the contact radius should be considered as part of assumption (ii). According to Eq. (5), $dw/dr$ is a monotonically increasing function of $c/a$ and $r$ in the region $c < r < a$. Table 1 lists the maximum ratios $\theta_{\text{max}}$ that satisfy assumption (ii) for various values of $a/g$. This maximum ratio was obtained from Eq. (5) by assuming that $\theta_{\text{max}} = 10^\circ$ at $r = a$. The ratios of $a/g$ corresponding to the data shown in Figs. 7 and 8 were 33 and 50, respectively. Thus it can be seen that the corresponding ratios of $c/a$ in Figs. 7 and 8 are well within the limits shown in Table 1.

We focus on assumptions (iii) and (iv) in the following. The assumption (iii) of a constant radial stress and the introduction of an additional constraint on the contact radius should be considered as part of assumption (ii). According to Eq. (5), $dw/dr$ is a monotonically increasing function of $c/a$ and $r$ in the region $c < r < a$. Table 1 lists the maximum ratios $\theta_{\text{max}}$ that satisfy assumption (ii) for various values of $a/g$. This maximum ratio was obtained from Eq. (5) by assuming that $\theta_{\text{max}} = 10^\circ$ at $r = a$. The ratios of $a/g$ corresponding to the data shown in Figs. 7 and 8 were 33 and 50, respectively. Thus it can be seen that the corresponding ratios of $c/a$ in Figs. 7 and 8 are well within the limits shown in Table 1.

We focus on assumptions (iii) and (iv) in the following. The assumption (iii) of a constant radial stress and the introduction of an additional constraint on the contact radius should be considered as part of assumption (ii). According to Eq. (5), $dw/dr$ is a monotonically increasing function of $c/a$ and $r$ in the region $c < r < a$. Table 1 lists the maximum ratios $\theta_{\text{max}}$ that satisfy assumption (ii) for various values of $a/g$. This maximum ratio was obtained from Eq. (5) by assuming that $\theta_{\text{max}} = 10^\circ$ at $r = a$. The ratios of $a/g$ corresponding to the data shown in Figs. 7 and 8 were 33 and 50, respectively. Thus it can be seen that the corresponding ratios of $c/a$ in Figs. 7 and 8 are well within the limits shown in Table 1.
of a mean strain over the entire domain resulted in the integral form of the constitutive equation (Eq. (7)). When the contact radius \( c = 0 \) the resulting stress obtained from assumption (iii) is 11% lower than the true radial mean stress (Kelkar et al., 1985; Williams, 1997). However, when contact is made, the stresses are equi-biaxial inside the contact zone and therefore, the deviation between the resulting stress from assumption (iii) and the corresponding true radial mean stress is expected to be less than 11%. This can be seen by considering the state of stress when the Poisson’s ratio is greater than 0.3 the deviation decreases (Williams, 1997) in the noncontact case and lowers the upper bound when contact is made. Furthermore, the existence of residual stresses can significantly alleviate the deviation of the assumed average stress from true radial mean stress because it can be represented as \( \frac{1}{\alpha + 1} \), where \( \alpha \) is the ratio of the pre-existing residual stress to the true radial mean stress developed by the large deformation. In this study, the residual stress was 7.5 MPa and the corresponding ratio \( \alpha \) was approximately 6, making the maximum deviation approximately 2%. A quantitative error estimate based on contact radius was provided (Plaut, 2009a) via a direction comparison of the numerical solution to the problem (Lai and Dillard, 1996; Plaut et al., 2003) and the approximate solution presented here. The error increased with contact radius but was less than 2%. Therefore, assumption (iii) is quite reasonable here, where the residual stresses were relatively high, and explains why the agreement between the analysis (Eq. (10)) and the measurements was so good.

Finally, we consider the assumption (iv) of frictionless contact. In other words, it is assumed that there is no adhesive interaction between the PET film and the glass plate due to the close relationship between friction and adhesion (Frisbie et al., 1994; Park and Thiel, 2008; Xu et al., 2008). The very weak adhesion between the PET film and the glass plate can be seen from Fig. 4 where the pressures corresponding to \( A \) and \( D \) were almost the same and from Figs. 7 and 8 where no adhesive hysteresis was observed between experimental data obtained during loading and unloading, making the assumption of frictionless contact appear quite reasonable. The true surface energy of the PET is on the order of 100 mJ/m\(^2\) (Pandiyaraj et al., 2008) which would lead to a large contact hysteresis. The observed ultralow adhesion is due to the surface roughness of the PET film that was used in the contact experiments. Fig. 9 shows the surface topography of the PET film as imaged by an optical profiler (Veeco, NT1100\(^1\)). It can be seen that the surface of the PET film was quite rough and the corresponding RMS (root-mean-square) roughness was about 65 nm. The large surface roughness of the PET film was due to embedded particles with heights up to hundreds of nm. These small particles were added to the PET film surface during the manufacturing process to prevent self-adhesion of the film when stored as a roll. Furthermore, the elastic energy in thin membrane contact is stored by membrane forces, which are not sensitive to the surface roughness. This is quite different from the contact between two elastic bodies where the surface roughness plays a critical role (Greenwood and Williamson, 1966; Persson, 2001).

In the following, two issues related to the Hertzian contact analysis that was developed in Section 2 are discussed. First, this analysis was developed on the basis of displacement control, i.e., the gap between the substrate and the constrained plate was prescribed a priori. For contact under load control with a prescribed contact force \( F \), a combination of Eq. (12) and either Eq. (9) or (10) can be used to determine the contact radius \( c \) and the separation \( g \) without or with residual stresses, respectively. Second, it is also interesting to consider the use of Eq. (10) to curve-fit the contact radius-pressure responses as shown in Figs. 7 and 8 and extract the Young’s modulus and residual stress. A similar scheme is widely used in nanoindentation tests (Kiely and Houston, 1998; Wang et al., 2004) to obtain the mechanical properties of

<table>
<thead>
<tr>
<th>( \frac{\varepsilon}{\varepsilon_{\text{max}}^c} )</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.474</td>
<td>0.642</td>
<td>0.728</td>
<td>0.781</td>
<td>0.889</td>
<td></td>
</tr>
</tbody>
</table>

Table 1
The maximum ratio \( \varepsilon / \varepsilon_{\text{max}}^c \) in terms of \( a/g \).

1 The authors are grateful to Uni-Pixel for providing this data.
elastic bodies. The corresponding curve-fits of pressure–contact radius responses for both loading and unloading are shown in the insets of Figs. 7 and 8. Assuming a Poisson’s ratio of 0.34, from the data shown in Fig. 7 with the 203 µm gap, the resulting Young’s modulus and residual stress were 3.89 GPa and 7.83 MPa, respectively. Similarly, for the data shown in Fig. 8 with the 305 µm gap, the resulting Young’s modulus and residual stress were 4.42 GPa and 7.86 MPa, respectively. These values are quite consistent with the Young’s modulus 4.65 GPa and residual stress 7.50 MPa that were obtained from bulge testing and that were used in the analysis. This curve-fitting scheme for obtaining the mechanical properties can be applied to various situations, where two parameters among contact force \( F \), contact radius \( c \) and pressure \( p \) under a gap \( g \) may be used. This is very useful because the contact force \( F \) and pressure \( p \), which are relatively easy to measure, can be employed for property extraction. In a complementary manner, under gap control and measuring the contact force \( F \), the contact radius \( c \) and pressure \( p \) can be determined along with the material properties.

6. Conclusions

An approximate closed-form Hertz-type contact theory has been developed to describe the relationship between pressure, contact radius, contact force and gap. Both the slope of the deflection profile of the membrane outside the contact zone and contact radius itself were measured by an apparatus based on moiré deflectometry. Note that the slope field of the deflection profile is rather difficult to measure directly by other techniques. Contact experiments with a 3 µm PET film showed that this analysis predicts both the slope field and contact radius quite accurately. It turned out that the surface roughness of the PET film resulted in the contact radius \( c \) and pressure \( p \) and gap \( g \) can be used to obtain the Young’s modulus of and residual stress in the thin membrane.

Acknowledgements

The authors thank Uni-Pixel Inc. for financial and technical assistance in the course of this study. The authors also thank Professor Ray Plaut for his help on error estimation, via private communication. One of the reviewers was also extremely helpful with comments.

References


Fig. 9. An image of the surface topography of the PET film scanned by a Veeco optical profiler (courtesy of Uni-Pixel Inc.).
Plaut, R.H., 2009a. Private communication.