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Spectral localization in Banach Algebras

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ABSTRACT

The notions of the theory of spectral localization which are well-known for operators are generalized to an arbitrary Banach algebra. In this setting several improvements and corrections of the existing results for operators are obtained.

1. INTRODUCTION

Before giving an outline of this paper we define the notions of a local spectral theory in a general Banach algebra and describe their relation with the existing literature.

Throughout this paper $\mathbb{A}$ denotes a Banach algebra with identity element e.

1.1. DEFINITIONS. Let $a, b \in \mathbb{A}$. An analytic $\mathbb{A}$-valued function $u(\ , a; b)$ defined on an open subset Ω of $\mathbb{C}$ is a local resolvent of a at b on Ω if

$$(\lambda e - a)u(\lambda, a; b) = b,$$

for all $\lambda \in \Omega$.

The union of the domains of all local resolvents of a at b is called the local resolvent set of a at b; notation $\varrho(a; b)$. The complement $\mathbb{C}\backslash\varrho(a; b) =: \sigma(a; b)$ is called the local spectrum.

If $R(\ , a)$ denotes the ordinary resolvent of a, then $R(\ , a)b$ is a local resolvent of a at b on the ordinary resolvent set $\varrho(a)$.

It is obvious that $\varrho(a; b)$ is open in $\mathbb{C}$ and that $\varrho(a; b) \supset \varrho(a)$; $\sigma(a; b)$ is a closed subset of the spectrum $\sigma(a)$, but unlike the ordinary spectrum, $\sigma(a; b)$ may be empty (e.g. for $b=0$).

1.2. In the literature local resolvents and local spectra have only been studied for operators, bounded or closed, on Banach spaces and Fréchet spaces. Roughly spoken one looks at the action of the operator on a one-dimensional subspace only; that motivates the use of the adjective "local". We do not give a survey of the literature. The first author to study local spectra seems to have been Nelson Dunford as long ago as the fifties. In this paper we refer to Gray [2], Stampfli [3], Vasilescu [4], [5], [6] and Vrbová [7].

A typical way to define the concepts of local resolvent and local spectrum for operators is the following.

Let X be a Banach space, T a bounded operator on X and $x \in X$ a vector not equal to 0. An analytic X-valued functie $\tilde{u}(\ ,T;\ x)$ defined on an open subset Ω of $\mathbb{C}$ is a local resolvent of T at x on Ω if

$$(\lambda I - T)\tilde{u}(\ ,T;\ x) = x,$$

for all $\lambda \in \Omega$.

The union of the domains of all local resolvents of T at x is the local resolvent set $\tilde{\varrho}(T;\ x)$; the complement $\mathbb{C}\backslash\tilde{\varrho}(T;\ x)$ is the local spectrum $\tilde{\sigma}(T;\ x)$.

1.3. Many results for operators can be adapted to our situation by an easy translation; we freely use such results in the more general setting of a Banach algebra. That our approach is in fact a generalization is best seen from the following theorem. This theorem clarifies the relation between the usual spectral localization for operators on a Banach space X and our definition used in the algebra, $B(X)$, of the bounded linear operators on the same Banach space.

THEOREM. Let X be a Banach space, $T \in B(X)$, $x \in X\backslash\{0\}$. If $P \in B(X)$ is a projection on the one-dimensional subspace spanned by x then $\tilde{\sigma}(T;\ x) = \sigma(T;\ P)$.

PROOF. The mapping $f: X \to \mathbb{C}$ satisfying $Pz = f(z)x$ for all $z \in X$ is a bounded linear functional. If we define the mapping $\pi: X \to B(X)$ by

$$\pi(y)(z) := f(z)y \qquad y \in X,\ z \in X,$$

then π is linear and continuous, and $\pi(x) = P$. Let $\tilde{u}(\ ,T;\ x)$ be a local resolvent of T at x on a set Ω in the sense of 1.2. Then $\pi(\tilde{u}(\ ,T;\ x))$ is an analytic function on Ω with values in $B(X)$; moreover, for all $z \in X$ we have for λ in the set Ω:

$$(\lambda I - T)\pi(\tilde{u}(\lambda, T;\ x))(z) = (\lambda I - T)f(z)\tilde{u}(\lambda, T;\ x) = f(z)x = Pz.$$

So $\pi(\tilde{u}(\ ,T;\ x))$ is a local resolvent of T at P on Ω in the sense of 1.1 and $\tilde{\varrho}(T;\ x) \subset \varrho(T;\ P)$. On the other hand; if $U(\ ,T;\ P)$ is a local resolvent of T at P on Ω in the sense of 1.1 then $U(\ ,T;\ P)x$ is an analytic X-valued function on Ω

satisfying $(\lambda I - P)U(\lambda, T;\ P)x = Px = x$; hence it is a local resolvent in the sense of 1.2 and $\varrho(T;\ P) \subset \tilde{\varrho}(T;\ x)$. □

A detailed treatment of local resolvents and local spectra in a general Banach algebra is available in [1]. In the present paper we focus our attention on improvements of the existing theories. In $B(X)$ the advantage of our approach is that we are not restricted to localization at projections with one-dimensional range.

1.4. REMARKS. Every Banach algebra $\mathbb{A}$ can also be regarded as an algebra of operators working on itself e.g. by looking at the left regular representation which represents $a \in \mathbb{A}$ by the operator $\bar{a} \in B(\mathbb{A})$ where $\bar{a}: x \to ax$ $(x \in \mathbb{A})$. We can now distinguish four local spectra of a at b: $\sigma(a;\ b)$ in the algebra $\mathbb{A}$ in the sense of 1.1; $\tilde{\sigma}(\bar{a}, b)$ in the sense of 1.2. and $\sigma(\bar{a};\ \bar{b})$ in the sense of 1.1 in the algebra $B(\mathbb{A})$ but also in the subalgebra $\bar{\mathbb{A}} := \{\bar{a} \mid a \in \mathbb{A}\}$. Fortunately these four local spectra are equal.

It is merely a matter of choice that our localization is based on the functional equation $(\lambda e - a)u(\lambda) = b$ and not on $u(\lambda)(\lambda e - a) = b$. Requiring that local resolvents satisfy both equations is to severe a restriction, since common solutions only exist if $ab = ba$. In fact, if $(\lambda e - a)u = u(\lambda e - a) = b$ then

$$ab = \lambda b - (\lambda e - a)b = \lambda b - (\lambda e - a)u(\lambda e - a) = \lambda b - b(\lambda e - a) = ba.$$

It is very well possible that there are different local resolvents of a at b on a set Ω (see [3] or sec. 2 below). It is also possible that there is only one local resolvent, but that the functional equation has non analytic solutions on Ω as well. For a trivial example of the latter phenomenon, take $a = b = 0$ and $\Omega = \mathbb{C}$; then the function identical equal to 0 is the only local resolvent on $\mathbb{C}$ but every function of λ which is zero for $\lambda \neq 0$ satisfies the functional equation.

1.5. OUTLINE. Situations where the local resolvent is uniquely determined have drawn much attention in the literature. Section 2 of this paper is devoted to the question of uniqueness of local resolvents. In Section 3 we look at large local spectra (and also at small ones) and refute a conjecture of Gray's. The short Sections 4 and 5 contain some remarks on the radius of the local spectrum and on spectral mapping theorems, respectively.

2. THE SINGLE VALUED EXTENSION PROPERTY

2.1. DEFINITION. Let $a, b \in \mathbb{A}$. We say that a has the single valued extension property (abbreviated s.v.e.p.) at b if for all open sets $\Omega \subset \mathbb{C}$ there exists at most one local resolvent of a at b on Ω. Moreover, we say that a has s.v.e.p. if a has s.v.e.p. at b for every $b \in \mathbb{A}$.

2.2. If $u_1(\ , a;\ b)$ and $u_2(\ , a;\ b)$ are local resolvents of a at b on a set Ω, then the set of all complex λ for which $u_1(\lambda, a;\ b) \neq u_2(\lambda, a;\ b)$ is an open subset of $\varrho(a;\ b) \backslash \varrho(a)$. This is a consequence of the following lemma and the identity theorem for analytic functions.

LEMMA. Let $a, b \in \mathbb{A}$ and let $\Omega \subset \mathbb{C}$ be open and connected and $\Omega \cap \varrho(a) \neq \emptyset$. Let υ be an analytic $\mathbb{A}$-valued function on Ω. Then υ is a local resolvent of a at b on Ω iff $\upsilon(\lambda) = R(\lambda;\, a)b$ for all $\lambda \in \Omega \cap \varrho(a)$.

PROOF. [2] Theorem 1.2. □

Since $\varrho(a;\, b) \setminus \varrho(a) \subset \sigma(a)$, it is now obvious that a has s.v.e.p. if the interior of $\sigma(a)$ is empty. Important examples of elements for which the spectrum has empty interior are idempotents and quasinilpotents in a Banach algebra; all operators on finite dimensional spaces (matrices); all operators on Hilbert space which are compact or self-adjoint or unitary.

2.3. REMARK. The following example (which is a modification of Stampfli's example of absence of s.v.e.p., [3] p. 287) gives open sets Ω_1 and Ω_2 which are domains of local resolvents whereas $\Omega_1 \cup \Omega_2$ is not the domain of a local resolvent (of course $\Omega_1 \cap \Omega_2$ has to be non empty for this phenomenon). The restriction of a local resolvent to an open subset of its domain is also a local resolvent, so every open subset of $\varrho(a;\, b)$ is the domain of a local resolvent iff $\varrho(a;\, b)$ is itself the domain of a local resolvent. The fact that not every open subset of the local resolvent set is necessarily the domain of a local resolvent seems to have been overlooked in [2]; therefore, some results in [2], in fact [2] Lemma 2.2 and [2] Theorem 3.1. require a correction.

2.4. EXAMPLE. Let $\mathbb{A}$ be the algebra of the bounded linear operators on a separable Hilbert space H, with inner product $\langle\ ,\ \rangle$ and an orthonormal basis $\{f_n\}_{n=0}^{\infty}$. Let $W \in \mathbb{A}$ be the adjoint of the unilateral shift, so $Wf_0 = 0$, $Wf_n = f_{n-1}$ $(n = 1, 2, \ldots)$. Let $P_n \in \mathbb{A}$ be given by $P_n x := \langle x, f_0 \rangle f_n$ $(x \in H)$; then $\|P_n\| = 1$ $(n = 0, 1, \ldots)$. Let Ω_1 be the open unit disc: $\{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$. Then $\sum_{n=0}^{\infty} \lambda^n P_n$ is a local resolvent, different from O, of W at O on Ω_1; so W lacks s.v.e.p. at O.

The function $-\sum_{n=1}^{\infty} \lambda^{n-1} P_n$ is a local resolvent of W at P_0 on Ω_1; $\lambda^{-1} P_0$ is a local resolvent of W at P_0 on $\Omega_2 := \mathbb{C} \setminus \{0\}$. Hence $\varrho(W;\, P_0) = \mathbb{C} = \Omega_1 \cup \Omega_2$. There is no local resolvent of W at P_0 on $\mathbb{C}$, for, by 2.2, such a function would be equal to $R(\lambda, W)P_0 = \lambda^{-1} P_0$ for $|\lambda| > 1$ and by Liouville's theorem this is impossible for an entire function.

2.5. The phenomenon mentioned in 2.3 and illustrated in 2.4 can not occur for elements that have s.v.e.p.

LEMMA. If a has s.v.e.p. at b then there exists a unique local resolvent of a at b on $\varrho(a;\, b)$.

PROOF. By definition $\varrho(a;\, b) = \bigcup_{j \in J} \Omega_j$, where Ω_j is the domain of a local resolvent $u_j(\ , a;\, b)$ and J an index set. If for $\lambda \in \varrho(a;\, b)$ we set $u(\lambda, a;\, b) = u_j(\lambda, a;\, b)$ if $\lambda \in \Omega_j$ then by s.v.e.p. this defines an analytic function which is a local resolvent on $\varrho(a;\, b)$. □

2.6. The difference $v = u_1(\ ,a;\ b) - u_2(\ ,a;\ b)$ of two local resolvents with common domain Ω satisfies $(\lambda e - a)v(\lambda) = 0$ on Ω. This observation explains why we are interested in the following sets (see Vasilescu [4]).

DEFINITION. $S'(a)$ is the set of all complex λ for which there exists a local resolvent $u(\ ,a;\ 0)$ of a at 0 with $u(\lambda, a;\ 0) \neq 0$. $S(a)$ is the closure of $S'(a)$.

Note that $S'(a)$ is open, $S'(a) \subset \sigma(a)$ hence $S'(a) \subset$ interior $\sigma(a)$, and $S(a) \subset \sigma(a)$.

In the operator approach of spectral localization the analogous sets are in fact equal to $S'(T)$ and $S(T)$.

2.7. EXAMPLE. We continue 2.4. There we noted that $\sum_{n=0}^{\infty} \lambda^n P_n$ is a local resolvent, different from the O-function, of W at O on Ω_1. So $S'(W) \supset \Omega_1$. Since $\sigma(W) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$ we have $S(W) = \sigma(W)$.

2.8. Vasilescu ([6] Lemma 3.2) has proved that the set of all complex λ for which there is a nonzero vector $x \in X$ with $(\lambda I - T)x = 0$ and $\tilde{\sigma}(T;\ x) = \emptyset$ is a dense subset of $S(T)$. The next theorem implies that the set of all such λ is in fact equal to $S'(T)$.

(Note that Vasilescu ([4] Definition 2.3) calls $\tilde{\sigma}(T;\ x) \cup S(T)$ the local spectrum. See also 4.1. below).

THEOREM. Let $a \in \mathbb{A}$. Then $S'(a) = S''(a)$ where

$$S''(a) = \{\lambda \in \mathbb{C} \mid \exists_{c \in \mathbb{A}}[c \neq 0,\ (\lambda e - a)c = 0,\ \sigma(a;\ c) = \emptyset]\}.$$

PROOF. (i) Let $\lambda_0 \in S'(a)$ and let u be a local resolvent of a at 0 on a set Ω with $u(\lambda_0) \neq 0$. By a theorem of Vasilescu's, which is valid for our local spectrum also ([4] Proposition 2.2.) we have $\sigma(a;\ u(\lambda_0)) = \sigma(a;\ 0)$. Trivially $\sigma(a;\ 0) = \emptyset$. Taking $c = u(\lambda_0)$ we deduce $S'(a) \subset S''(a)$.

(ii) Let $\lambda_0 \in \mathbb{C}$ and $c \in \mathbb{A}$ be such that $c \neq 0$, $(\lambda_0 e - a)c = 0$, $\sigma(a;\ c) = \emptyset$. There is a local resolvent u of a at c on a neighbourhood Ω of λ_0. Since $(\lambda_0 e - a)u(\mu, a;\ c)$ is analytic in μ on Ω and satisfies

$$(\mu e - a)(\lambda_0 e - a)u(\mu, a;\ c) = (\lambda_0 e - a)c = 0$$

and

$$(\lambda_0 e - a)u(\lambda_0, a;\ c) = c \neq 0,$$

we have $\lambda_0 \in S'(a)$ and $S''(a) \subset S'(a)$. □

2.9. It is obvious that a has s.v.e.p. at b iff $S(a) \subset \sigma(a;\ b)$. Since $\sigma(a;\ 0) = \emptyset$ this implies that a has s.v.e.p. iff $S'(a) = \emptyset$. The next theorem gives another necessary and sufficient condition for s.v.e.p. A further result on s.v.e.p. will be given in sec. 3.5.

THEOREM. Let $a \in \mathbb{A}$. The following conditions are equivalent

(i) a has s.v.e.p.

(ii) $\forall_{b \in \mathbb{A}}[\sigma(a;\ b) = \emptyset$ iff $b = 0]$.

PROOF. We only show that $S(a)=\emptyset$ together with $\sigma(a;\,b)=\emptyset$ implies $b=0$. If $S(a)=\emptyset$ and $\varrho(a;\,b)=\mathbb{C}$ then a has s.v.e.p. at b and by Lemma 2.5 there exists a unique local resolvent $u(\ ,a;\,b)$ on $\mathbb{C}$. By Lemma 2.2 we have

$$\lim_{\lambda\to\infty} u(\lambda,a;\,b)=\lim_{\lambda\to\infty} R(\lambda,a)b=0.$$

Liouville's theorem now yields $u(\lambda,a;\,b)=0$ for all $\lambda\in\mathbb{C}$, hence

$$b=(\lambda e-a)u(\lambda,a;\,b)=0. \qquad \square$$

3. LARGE AND SMALL LOCAL SPECTRA; GRAY'S CONJECTURE

3.1. In [2] Gray states the following conjecture: For every operator $T\in B(X)$ there is a vector $x\in X$ such that $\tilde{\sigma}(T;\,x)=\sigma(T)$. Although Vrbová proves in [7] that the set

$$\{x\in X\,|\,\tilde{\sigma}(T;\,x)\cup S(T)=\sigma(T)\}$$

is of the second category in X, Gray's conjecture is not true. In the counterexample we give in 3.6. local spectra at the identity play a central rôle. Therefore we first study $\sigma(a;\,e)$ in some detail.

3.2. LEMMA. Let $a,b\in\mathbb{A}$. Then $\sigma(a;\,b)\subset\sigma(a;\,e)$. If b has a right inverse then $\sigma(a;\,b)=\sigma(a;\,e)$.

PROOF. If $u(\ ,a;\,e)$ is a local resolvent of a at e then $u(\ ,a;\,e)b$ is one at $eb=b$. So $\varrho(a;\,e)\subset\varrho(a;\,b)$. Let, moreover, c be a right inverse of b, i.e. $bc=e$. If $u(\ ,a;\,b)$ is a local resolvent of a at b then $u(\ ,a;\,b)c$ is one at $bc=e$, hence $\varrho(a;\,b)\subset\varrho(a;\,e)$. $\square$

3.3. For the ordinary resolvent $R(\ ,a)$ analyticity follows from the fact that it satisfies $(\lambda e-a)R(\lambda,a)=R(\lambda,a)(\lambda e-a)=e$. As we have seen in 1.4. the functional equation used in defining the local resolvent set may have non analytic solutions. Nevertheless $\varrho(a;\,e)$ can be characterized, just as $\varrho(a)$, by a purely algebraic condition.

THEOREM. Let $a\in\mathbb{A}$, then $\varrho(a;\,e)$ is the set of all $\lambda\in\mathbb{C}$ for which $\lambda e-a$ has a right inverse in $\mathbb{A}$. Let $\lambda_0\in\mathbb{C}$ and $u\in\mathbb{A}$ satisfy $(\lambda_0 e-a)u=e$. Then

$$\Omega_0:=\{\lambda\in\mathbb{C}\,\big|\,|\lambda-\lambda_0|<(\|u\|)^{-1}\}\subset\varrho(a;\,e).$$

PROOF. The function $\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n u^{n+1}$ is analytic for $|\lambda-\lambda_0|<(\|u\|)^{-1}$. An easy calculation shows that it is a local resolvent of a at e, hence $\Omega_0\subset\varrho(a;\,e)$. It is trivial that for $\lambda\in\varrho(a;\,e)$ the element $\lambda e-a$ has a right inverse in $\mathbb{A}$. $\square$

3.4. We denote the topological boundary of a set V by bdV and the closure of V by $\bar{V}$.

THEOREM. Let $a\in\mathbb{A}$. Then $bd\sigma(a)\subset bd\sigma(a;\,e)$.

PROOF. If $\lambda_0 \in bd\sigma(a)$ then $\lambda_0 e - a$ is a two-sided, hence right topological divisor of zero. If λ_0 were in $\varrho(a;\ e)$ then $e = (\lambda_0 e - a)u(\lambda_0, a;\ b)$ would also be a right topological divisor of zero, which is absurd. Hence $bd\sigma(a) \subset \sigma(a;\ e)$. Since $bd\sigma(a) \subset \overline{\varrho(a)} \subset \overline{\varrho(a;\ e)}$ we have $bd\sigma(a) \subset \sigma(a;\ e) \cap \overline{\varrho(a;\ e)} = bd\sigma(a;\ e)$. □

3.5. THEOREM. Let $a \in \mathbb{A}$. Then a has s.v.e.p. at e iff $\sigma(a;\ e) = \sigma(a)$.

PROOF. (i) If a has s.v.e.p. at e then there is a unique analytic function $u(\ , a;\ e)$ on $\varrho(a;\ e)$ satisfying $(\lambda e - a)u(\lambda, a;\ e) = e$. Moreover, we have

$$(\lambda e - a)[au(\lambda, a;\ e) - u(\lambda, a;\ e)a] = 0.$$

From the fact that a has s.v.e.p. at e it follows that $(\lambda e - a)v(\lambda) = 0$ on $\varrho(a;\ e)$ implies $v(\lambda) = 0$ on $\varrho(a;\ e)$. Hence $au(\lambda, a;\ e) = u(\lambda, a;\ e)a$; so $u(\lambda, a;\ e)$ equals the resolvent and $\varrho(a;\ e) \subset \varrho(a)$. This means $\sigma(a;\ e) = \sigma(a)$.

(ii) If $\sigma(a;\ e) = \sigma(a)$ then $S(a) \subset \sigma(a) = \sigma(a;\ e)$; by 2.9 a has s.v.e.p. at e. □

REMARK. By Lemma 3.2 we have: if a has s.v.e.p. at one right invertible element, then a has s.v.e.p. at every right invertible element.

3.6. EXAMPLE. Since $\sigma(a;\ e)$ is the largest local spectrum, we have a counterexample to Gray's conjecture if we find an operator T for which $\sigma(T;\ I) \neq \sigma(T)$ (see Theorem 1.3).

We continue 2.4 and 2.7. From 3.4 we deduce $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\} = bd\sigma(W) \subset \subset \sigma(W;\ I) \subset \sigma(W)$. We apply Theorem 3.3. From $WW^* = I$ it follows that $0 \in \varrho(W;\ I)$, and from $\|W^*\| = 1$ it follows then that $\Omega_1 \subset \varrho(W;\ I)$. The conclusion is $\sigma(W;\ I) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} \neq \sigma(W)$. As we have seen, this means that in Hilbert space Gray's conjecture is not true. In this example we also see that $\bigcup_{x \in H} \tilde{\sigma}(W;\ x) \neq \sigma(W)$; therefore the statements in [2] Theorem 2.5 and its corollary [2], 2.6 are not quite correct.

3.7. Looking in the opposite direction of Gray's conjecture we finish this section with a result on small local spectra. Trivially $\emptyset = \sigma(a;\ 0)$ is the smallest local spectrum. We study spectral sets, i.e. parts of the spectrum that are both open and closed in $\sigma(a)$. If σ_1 is such a spectral set, then the spectral projection e_1 with respect to σ_1 can be given by

$$e_1 = \frac{1}{2\pi i} \int_{(\sigma_1)} R(\xi, a)d\xi$$

where the integration path is in $\varrho(a)$ and once around σ_1, whereas $\sigma(a) \backslash \sigma_1$ is outside the path.

THEOREM. Let $a \in \mathbb{A}$; let $\sigma(a) = \sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_k$ where $\sigma_1, \ldots, \sigma_k$ are disjoint spectral sets; let $e_1, \ldots, e_k$ be the spectral projections with respect to $\sigma_1, \ldots, \sigma_k$.

Then

$$\sigma(a;\ e_i) \subset \sigma_i \qquad (i=1,\dots,k);$$

$$bd\sigma_i \subset bd\sigma(a;\ e_i) \qquad (i=1,\dots,k);$$

and

$$\sigma(a;\ e) = \sigma(a;\ e_1) \cup \sigma(a;\ e_2) \cup \dots \cup \sigma(a;\ e_k).$$

PROOF. If γ_i is an integration path in $\varrho(a)$ once around σ_i, with $\sigma(a) \setminus \sigma_i$ outside γ_i, then the function

$$w(\lambda) := \frac{1}{2\pi i} \int_{\gamma_i} (\lambda - \xi)^{-1} R(\xi, a) d\xi$$

is a local resolvent of a at e_i on the set of all λ outside γ_i. To see this, note that

$$(\lambda e - a) w(\lambda) = \frac{1}{2\pi i} \int_{\gamma_i} (\lambda - \xi)^{-1} (\lambda e - a) R(\xi, a) d\xi =$$

$$= \frac{1}{2\pi i} \int_{\gamma_i} (\lambda - \xi)^{-1} (\lambda - \xi) R(\xi, a) d\xi +$$

$$+ \frac{1}{2\pi i} \int_{\gamma_i} (\lambda - \xi)^{-1} (\xi e - a) R(\xi, a) d\xi = e_i + 0 = e_i$$

for all λ outside γ_i. Since every λ not in σ_i can be separated from σ_i by an integration path we have $\sigma(a;\ e_i) \subset \sigma_i$ $(i=1,\dots,k)$. By 3.2. one sees $\bigcup_{i=1}^{k} \sigma(a;\ e_i) \subset \subset \sigma(a;\ e)$. On the other hand $e = e_1 + \dots + e_k$ and, by [3] Proposition 3, this implies $\sigma(a;\ e) \subset \bigcup_{i=1}^{k} \sigma(a;\ e_i)$. Hence $\sigma(a;\ e) = \bigcup_{i=1}^{k} \sigma(a;\ e_i)$. Moreover, $\sigma(a;\ e) \cap \sigma_i = \sigma(a;\ e_i)$. The statement about the boundary of $\sigma(a;\ e_i)$ now follows from 3.4. □

4. THE RADIUS OF THE LOCAL SPECTRUM

4.1. DEFINITIONS. Let $a, b \in \mathbb{A}$. We define

$$r(a;\ b) := \begin{cases} \sup \{ |\lambda| \mid \lambda \in \sigma(a;\ b) \} & \text{if } \sigma(a;\ b) \neq \emptyset \\ 0 & \text{if } \sigma(a;\ b) = \emptyset. \end{cases}$$

We also define

$$r_V(a;\ b) := \sup \{ |\lambda| \mid \lambda \in \sigma(a;\ b) \cup S(a) \} \text{ if } S(a) \neq \emptyset \text{ or } b \neq 0$$

$$r_V(a;\ 0) := 0 \text{ if } S(a) = \emptyset.$$

The radii introduced for operators by Gray and Vasilescu are given as follows

$$\tilde{r}(T;\ x) := \begin{cases} \sup \{ |\lambda| \mid \lambda \in \tilde{\sigma}(T;\ x) \} & \text{if } \tilde{\sigma}(T;\ x) \neq \emptyset \\ 0 & \text{if } \tilde{\sigma}(T;\ x) = \emptyset. \end{cases}$$

([2] Definition 1.3) and

$$\tilde{r}_V(T;\ x) := \sup \{ |\lambda| \mid \lambda \in \tilde{\sigma}(T;\ x) \cup S(T) \}$$

(see [5]; the set of which $\tilde{r}_V(T;\ x)$ is the supremum is never empty since $x \neq 0$.)

4.2. THEOREM. 1. Let $a, b \in \mathbb{A}$. Then

$$r(a,b) \leq \limsup_{n\to\infty} \|a^n b\|^{1/n} \leq r_V(a;\, b).$$

2. Let X be a Banach space, $T \in B(X)$ and $x \in X$, $x \neq 0$. Then

$$\tilde{r}(T;\, x) \leq \limsup_{n\to\infty} \|T^n x\|^{1/n} \leq \tilde{r}_V(T;\, x).$$

PROOF. The proofs of 1 and 2 are completely analogous, so we only prove 1. On the set

$$\{\lambda \in \mathbb{C} \mid |\lambda| > \limsup_{n\to\infty} \|a^n b\|^{1/n}\}$$

the function $\sum_{n=0}^{\infty} \lambda^{-n-1} a^n b$ is a local resolvent of a at b, as one verifies easily. This proves the first inequality. For $|\lambda| > r_V(a;\, b)$ the local resolvent $u(\lambda, a;\, b)$ is unique and for $|\lambda| > r(a)$ (the ordinary spectral radius) Lemma 2.2 yields

$$u(\lambda, a;\, b) = R(\lambda, a)b = (\textstyle\sum_{n=0}^{\infty} \lambda^{-n-1} a^n)b.$$

So the Laurent series $\sum_{n=0}^{\infty} \lambda^{-n-1} a^n b$ converges for $|\lambda| > r_V(a;\, b)$. This proves the second inequality. □

4.3. REMARK. In [2] Lemma 2.2 Gray asserts that $\tilde{r}(T;\, x) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$ and in [5] Proposition 2.5 Vasilescu states that $\tilde{r}_V(T;\, x) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$. Both assertions are not correct as we shall show now. Again, let W be as in 2.4, 2.7 and 3.6. Let $x := \sum_{k=0}^{\infty} 2^{-\frac{1}{2}(k+1)} f_k$, then $\|x\| = 1$ and:

$$\limsup_{n\to\infty} \|W^n x\|^{1/n} =$$
$$= \limsup_{n\to\infty} \|\textstyle\sum_{k=n}^{\infty} 2^{-\frac{1}{2}(k+1)} f_{k-n}\|^{1/n} =$$
$$= \limsup_{n\to\infty} (\textstyle\sum_{k=n}^{\infty} 2^{-(k+1)})^{1/2n} = \tfrac{1}{2}\sqrt{2}.$$

In 2.7 we have found $\tilde{\sigma}(W;\, x) \cup S(W) = \sigma(W) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$, hence

$$\limsup_{n\to\infty} \|W^n x\|^{1/n} = \tfrac{1}{2}\sqrt{2} < 1 = \tilde{r}_V(W;\, x).$$

Since $\tilde{\sigma}(W;\, x) \subset \sigma(W;\, I) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ (by 1.3, 3.2 and 3.6) and $\tilde{r}(W;\, x) \leq \limsup_{n\to\infty} \|W^n x\|^{1/n} = \tfrac{1}{2}\sqrt{2}$, the local spectrum $\tilde{\sigma}(W;\, x)$ has to be empty and $\tilde{r}(W;\, x) = 0$. Hence $\tilde{r}(W;\, x) < \limsup_{n\to\infty} \|W^n x\|^{1/n}$, indeed.

5. LOCAL SPECTRAL MAPPING THEOREMS

A local operational calculus may be developed in a more or less standard way; see [1] Chapter 5. We restrict ourselves here to some remarks on local versions of the spectral mapping theorem. In [2] Theorem 5.2 it is asserted that for a bounded operator T on X and for a function F analytic on a neighbourhood of $\sigma(T)$ the equality $\tilde{\sigma}(F(T);\, x) = F(\tilde{\sigma}(T;\, x))$ holds for all $x \in X$, $x \neq 0$. There should be an additional assumption, among other reasons because the proof uses the identity $\bigcup_{x\in X} \tilde{\sigma}(T;\, x) = \sigma(T)$, which is not generally true as we have seen in 3.6. Recently, Vasilescu [6] has proved the equality $\tilde{\sigma}(F(T);\, x) = F(\tilde{\sigma}(T;\, x))$ together with $F(S(T)) = S(F(T))$ for functions which are analytic on a neighbourhood of $\sigma(T)$ and non constant on every component of their domains.

In [1] the assumption of non constancy is dropped; instead of it s.v.e.p. is assumed. The result is:

THEOREM. Let $a, b \in \mathbb{A}$; let F be analytic on a neighbourhood of $\sigma(a)$ and let a have s.v.e.p. at b. Then

$$\sigma(F(a);\ b) = F(\sigma(a;\ b)).$$

In case a does not have s.v.e.p. at b we can prove only the following inclusions:

$$F(\sigma(a;\ b)) \subset \sigma(F(a);\ b) \subset F(\sigma(a;\ b) \cup S(a)).$$

It seems interesting to weaken the conditions on F so far that F has to be analytic on a neighbourhood of $\sigma(a;\ b)$ only. If a then has s.v.e.p. at b one can study

$$F(a;\ b) := \frac{1}{2\pi i} \int_{(\sigma(a;\ b))} F(\xi) u(\xi, a;\ b) d\xi,$$

where $u(\ , a;\ b)$ is the local resolvent on $\varrho(a;\ b)$ and the integration path goes once around $\sigma(a;\ b)$ and lies entirely inside the domain of analyticity of F. If a does not have s.v.e.p. at b there may not be an integration path around $\sigma(a;\ b)$ which lies entirely inside the domain of a local resolvent, since there may not be a local resolvent on $\varrho(a;\ b)$.

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