A Note on Asymptotic Expansions for Markov Chains Using Operator Theory

J. L. JENSEN

Department of Theoretical Statistics, Institute of Mathematics, University of Aarhus, Aarhus, Denmark

We consider asymptotic expansions for sums $S_n$ on the form $S_n = f_0(X_0) + f(X_1, X_0) + \cdots + f(X_n, X_{n-1})$, where $X_i$ is a Markov chain. Under different ergodicity conditions on the Markov chain and certain conditional moment conditions on $f(X_i, X_{i-1})$, a simple representation of the characteristic function of $S_n$ is obtained. The representation is in term of the maximal eigenvalue of the linear operator sending a function $g(x)$ into the function $x \mapsto E(g(X_i)\exp[itf(X_i, x) | X_{i-1} = x])$.

1. INTRODUCTION

In this note we extend some results of Nagaev [5] concerning a representation of the characteristic function of a sum $S_n$ on the form

$$S_n = f_0(X_0) + \sum_{i=1}^{n} f(X_i, X_{i-1}).$$

(1.1)

Here $X_0, X_1, \ldots$ is a homogeneous Markov chain on the measure space $(E, \mathcal{E})$ with initial distribution $\pi$ and $f$ and $f_0$ are measurable functions. Nagaev used the ergodicity condition

$$\sup_{A, x, y} |P^{(k)}(A, x) - P^{(k)}(A, y)| < 1 \quad \text{for some } k \geq 1,$$

(1.2)

where $P^{(k)}(\cdot, x)$ is the $k$-step transition probability. Furthermore, $f$ were assumed to satisfy the conditional moment condition

$$\sup_{x_{i-1}} E(|f(X_i, x_{i-1})|^s | x_{i-1}) < \infty$$

(1.3)

for some $s \geq 3$. Considering a number of linear operators on the Banach
space of bounded functions on $E$, Nagaev derived the basic representation
\[ E(e^{itS_n}) = \lambda(t)^n \pi(t) P_1(t) \psi + \pi(t) P^n(t) P_2(t) \psi \] (1.4)
for $|t| < c$, say. For explanation see Section 2. The important thing here is
that the second term in (1.4) is exponentially small in $n$, uniformly for
$|t| < c$, and that the first term has a structure which makes it easy to derive
asymptotic expansions for the distribution of $S_n$.

Here we extend the range of applicability of (1.4) by considering instead
of (1.3) the condition
\[ \sup_{x_{i-r-1}} E(|f(X_i, X_{i-1})|^p |x_{i-r-1}) < \infty, \] (1.5)
where $p \geq 1$ is fixed. In most cases where $f$ depends on both $x_i$ and $x_{i-1}$,
(1.3) will not be satisfied whereas (1.5) will. A simple example is one in
which the $X_i$'s are independent and the variables $f(X_i, X_{i-1})$ thus are
1-dependent. Another extension is obtained by relaxing the ergodicity
condition (1.2). Assuming the existence of a stationary distribution $P_1$ we
treat the transition function $P(A, x)$ and $P_1$ as operators on the space $L^p$
(see Section 4 for a precise explanation). The new ergodicity condition then
states that the operator norm $\|P^n - P_1\|$ is exponentially small in $n$.

All the conditions that we impose imply the condition in [1]. Rosenblatt
[8] may be consulted for showing that the ergodicity conditions used here
imply that the sequence $X_0, X_1, \ldots$ is strongly mixing with an exponentially
small mixing coefficient. The work of Götze and Hipp [1] therefore gives an
asymptotic expansion for the distribution of $S_n$. However, it is not possible
to obtain the representation (1.4) from that paper and to show that the
coefficients of the expansion will be of the form of a constant times $n^{-k/2}$
for a suitable value of $k$. The representation (1.4) was used in [4] to obtain
asymptotic expansions for sums on the form (1.1) under much weaker
mixing conditions than used here.

In Section 2 we describe the setup used in [5] with a few extensions
concerning differentiability of the functions appearing in (1.4). In Section 3
we consider the extension given in (1.5) and Section 4 deals with the weaker
ergodicity condition described above. Finally, in Section 5 we briefly
discuss how the results may be generalized to continuous time Markov
processes. An appendix is included which gives the necessary facts concern-
ing linear operators on Banach spaces.

2. NAGAEV'S METHOD

In this section we describe briefly the setup and proofs in [5]. This will
then make it easy to discuss extensions in the following sections.
We consider a Markov chain $X_0, X_1, X_2, \ldots$ on some measure space $(E, \mathcal{E})$ with homogeneous transition function $P(A, x)$,

$$P(A, x) = P(X_i \in A | X_{i-1} = x).$$

The initial distribution of $X_0$ is denoted by $\pi$. The transition function is assumed to satisfy the ergodicity condition

$$\sup_{A, x, y} \|P^{(k)}(A, x) - P^{(k)}(A, y)\| < 1 \quad \text{for some } k \geq 1, \quad (2.1)$$

where $P^{(k)}$ is the $k$-step transition probability. It is easy to see that this implies the existence of a stationary distribution $P_1$, that is, $P_1$ satisfies the equation

$$P_1(A) = \int P(A, x) P_1(dx), \quad (2.2)$$

with the property

$$\sup_{A, x} \|P^{(n)}(A, x) - P_1(A)\| \leq \frac{1}{2} \gamma^p \quad (2.3)$$

for some $p < 1$ and $\gamma > 0$.

We will derive an expansion of the characteristic function of a sum $S_n$ of the form

$$S_n = f_0(X_0) + \sum_{i=1}^{n} f(X_i, X_{i-1}).$$

We consider the case where $f$ is one dimensional, but the generalization to the multi-dimensional case involves only a change in notation. Let $B$ be the Banach space of bounded functions on $E$,

$$B = \left\{ g: E \to \mathbb{C} \left| \|g\| = \sup_{x} |g(x)| < \infty \right. \right\},$$

and define the operators $P$, $P_1$, and $P(t)$ from $B$ to $B$ by

$$Pg = \int g(x) P(dx, \cdot), \quad P_1g = \int g(x) P_1(dx)$$

and

$$P(t)g = \int g(x) e^{itf(x, \cdot)} P(dx, \cdot).$$

In terms of these operators (2.2) and (2.3) become

$$P_1P = P_1 = PP_1 \quad \text{and} \quad \|P^n - P_1\| \leq \gamma^p. \quad (2.4)$$
We now find the resolvents of $P$ and $P(t)$ (for a definition see the Appendix).

**Lemma 2.1.** For $z \neq 1$ and $|z| > \rho$ the resolvent of $P$ is given by $-R(z)$, where

$$R(z) = \frac{1}{z - 1} P_1 + \sum_{n=0}^{\infty} \frac{1}{P^n - 1} \frac{1}{z^{n+1}}.$$  \hspace{1cm} (2.5)

**Proof.** From (2.4) it is seen that $R(z)$ is well defined and also we obtain

$$R(z)(P - zI) = -P_1 + \sum_{n=0}^{\infty} \frac{1}{P^n - P_1} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{P^n - P_1} \frac{1}{z^n}$$

$$= -I = (P - zI)R(z). \quad \Box$$

To establish the resolvent of $P(t)$ we introduce the assumption

$$\sup \int f(x, y) P(dx, y) = \sup E(f(x_i, x_{i-1}) | x_{i-1}) \leq M_1 < \infty. \hspace{1cm} \text{(2.6)}$$

**Lemma 2.2.** Let $M_2 = \sup \{|R(z)| | |z - 1| > (1 - \rho)/6 \text{ and } |z| > \rho + (1 - \rho)/6\}$. Then for $|t| < (M_1 M_2)^{-1}$ and $|z| > \rho + (1 - \rho)/6$, $|z - 1| > (1 - \rho)/6$ we may define

$$R(z, t) = \sum_{n=0}^{\infty} R(z)[(P(t) - P)R(z)]^k$$  \hspace{1cm} (2.7)

and $-R(z, t)$ is the resolvent of $P(t)$.

**Proof.** Note that $||R(z)|| \to 0$ for $|z| \to \infty$ so that $M_2 < \infty$. From (2.6) we have

$$|[P(t) - P]g(y)| = \left| \int g(x)(e^{itf(x, y)} - 1)P(dx, y) \right|$$

$$\leq t M_1 \|g\|,$$

implying that

$$||P(t) - P|| \leq |t|M_1. \hspace{1cm} \text{(2.8)}$$

From the definition of $M_2$ it then follows that $R(z, t)$ is well defined. Finally

$$R(z, t)(P(t) - zI) = R(z, t)[(P(t) - P) + (P - zI)]$$

$$= -I = (P(t) - zI)R(z, t). \quad \Box$$
From Lemma 2.2 it follows that the spectrum of $P(t)$ is inside the two circles given by
\[ I_1 = \{z \mid |z - 1| = (1 - \rho)/3\} \quad \text{and} \quad I_2 = \{z \mid |z| = \rho + (1 - \rho)/3\}. \]
From Theorem A.1 we therefore find that
\[ P_l(t) = \frac{1}{2\pi i} \int_{I_l} R(z, t) \, dz, \quad l = 1, 2 \]
are parallel projections onto the subspaces $B_1(t)$ and $B_2(t)$, say, where $B = B_1(t) \oplus B_2(t)$.

**Lemma 2.3.** Let $c_1 = \min\{(2M_1M_2)^{-1}, (2M_1M_2^2)^{-1}\}$. Then for $|t| < c_1$ the space $B_1(t)$ is one dimensional.

**Proof.** From (2.5) it appears that
\[ P_1 = \frac{1}{2\pi i} \int_{I_1} R(z) \, dz \]
and then
\[ P_1(t) - P_1 = \frac{1}{2\pi i} \int_{I_1} \sum_{k=1}^{\infty} R(z)[(P(t) - P)R(z)]^k. \quad (2.9) \]
Dominating the norm of the individual terms we get
\[ \|P_1(t) - P_1\| \leq M_2|t| M_1 M_2/(1 - |t| M_1 M_2) < 1 \quad (2.10) \]
for $|t| < c_1$. Let $\psi \in B$ be the constant function with
\[ \psi(x) \equiv 1. \]
Since $P_1\psi = \psi$ it is seen from (2.10) that $B_1(t) \neq \{0\}$. Let $h_1$ and $h_2$ be two elements of $B_1(t)$ and define $c$ by $P_1h_1 = cP_1h_2$ or $P_1(h_1 - ch_2) = 0$. Since $h_i \in B_1(t)$ we also have that $P_1(t)(h_1 - ch_2) = h_1 - ch_2$ and so
\[ [P_1(t) - P_1](h_1 - ch_2) = h_1 - ch_2. \]
The bound (2.10) then implies that $h_1 = ch_2$, that is, $B_1(t)$ is one dimensional. \(\square\)

We denote the eigenvalue corresponding to $B_1(t)$ by $\lambda(t)$. Since, from (2.10), $P_1(t)\psi$ is nonzero we can write
\[ P(t)P_1(t)\psi = \lambda(t)P_1(t)\psi. \quad (2.11) \]
We may now obtain our basic representation of the characteristic function of $S_n$,

$$E(e^{itS_n}) = \pi(t) P^n(t)\psi = \pi(t) P^n(t)[P_1(t) + P_2(t)]\psi = \lambda(t)^n \pi(t) P_1(t)\psi + \pi(t) P^n(t)P_2(t)\psi$$

(2.12)

for $|t| < c_1$, where $\pi(t)$ is the functional

$$\pi(t) g = \int g(x)e^{it\phi(x)}\pi(dx).$$

(2.13)

**Lemma 2.4.** For $|t| < c_1$ we have the bound

$$|\pi(t) P^n(t)P_2(t)\psi| \leq |t| \left(\frac{1 + 2\rho}{3}\right)^n.$$

**Proof.** From (A.1) and (2.7) we have

$$\pi(t) P^n(t)P_2(t)\psi = \pi(t) \frac{1}{2\pi i} \int_{I_2} z^n \sum_{k=0}^{\infty} R(z)[(P(t) - P)R(z)]^k \psi dz.$$

The term with $k = 0$ is $\int [z^n/(z - 1)]\psi dz = 0$ and the rest may be dominated by

$$\left(\frac{1 + 2\rho}{3}\right)^n |t| M_1 M_2 \frac{M_2}{1 - |t|M_1 M_2} \leq |t| \left(\frac{1 + 2\rho}{3}\right)^n$$

for $|t| < c_1 = \min\{(2M_1 M_2)^{-1}, (2M_1^2 M_2)^{-1}\}$. □

It is clear from the representation (2.12) and the bound in Lemma 2.4 that we are in a situation that looks very much like the summation of independent identically distributed observations. To use the general theory of asymptotic expansions developed in [9] we therefore only need to establish certain expansions for the basic functions $\lambda(t)$ and $\pi(t)P_1(t)\psi$. To expand $\lambda(t)$ we assume instead of (2.6) that

$$\sup_y \int |f(x, y)|^2P(dx, y) \leq M_3 < \infty$$

(2.14)

for some $s \geq 3$.

**Lemma 2.5.** There exist constants $K$ and $0 < c_2 < c_1$ such that for $|t| < c_2$ and $l \leq s$ we have

$$\frac{d^l}{dt^l}\lambda(t) = \sum_{j=0}^{s-1} \frac{1}{(j-l)!}t^{j-l}\lambda_j + \omega K|t|^{s-l},$$

where $|\omega| \leq 1$. Here $\lambda_j$, $j = 0, \ldots, s - 1$, are constants with $\lambda_0 = 1$. 
Proof. Integrating with respect to $P_1(dx)$ on both sides of (2.11) we have $\lambda(t) = P_1P(t)P_1(t)\psi/P_1P_1(t)\psi$ and using the definitions of $P_1(t)$ and $R(z, t)$ we find

$$P_1P(t)\delta P_1(t)\psi = 1 + \frac{1}{2\pi i} \int_{l_1} z^\delta \sum_{1}^\infty P_1R(z)[(P(t) - P)R(z)]^k \psi \, dz$$

(2.15)

for $\delta = 0$ and $1$. Here we have used that $R(z)\psi = \psi/(z - 1)$. When taking the $l$th derivative we see that we must consider terms of the form

$$\frac{d^l}{dt^l}[(P(t) - P)R(z)]^k = \sum_{l_i, k} \prod_{i,j} P_{ij}(t) R(z),$$

(2.16)

where $\sum_k i_j = l$, $|I_{l,k}| = k^l$, $P_0(t) = P(t) - P$ and for $m \geq 1$

$$P_m(t)g = i^m \int g(x)f(x, \cdot) e^{itf(x, \cdot)} P(dx, \cdot).$$

(2.17)

Since $\|P_m(t)\| \leq M_3^{m/s}$ we get from (2.8) the bound $k^l M_3^{l/s} (|t| M_1)^{k_0} M_2^k$ for the term (2.16). Here $k_0 = \sum_k 1\{i_j = 0\} \geq k - l$. For the sum over $k$ from $s$ to infinity in the derivative of (2.15) we thus get a bound on the form $K_1 |t|^{s-l}$ for $|t| < c_1$ and some constant $K_1$. For $k < s$ we expand $P_m(t)$ in the form

$$P_m(t) = i^m \left[ \sum_{s-m-1}^s \frac{(it)^j}{j!} H_{j+m} + \frac{|t|^{s-m}}{(s-m)!} R_s \right]$$

(2.18)

for $m \leq s$. Here $H_0 \equiv 0,$

$$H_j g = \int g(x)f(x, \cdot) P(dx, \cdot)$$

(2.19)

with $\|H_j\| \leq M_3^{j/s}$ and $\|R_s\| \leq M_3$. Using (2.18) in the sum from 1 to $s - 1$ in the derivative of (2.15) some of the resulting terms will have powers of $t$ greater than $s$, but we may simply replace some of the $t$'s by $c_1$. In this way we get an expansion for the derivatives of (2.15) and this in turn gives an expansion for the derivatives of $\lambda(t)$. $\Box$

To expand $\pi(t)P_1(t)\psi$ to order $s - 3$ we assume that

$$\int |f_0(x)|^{s-2}\pi(dx) \leq M_4 < \infty.$$  

(2.20)
LEMMA 2.6. There exist constants $K$ and $0 < c_2 < c_1$ such that for $|t| < c_2$ and $l \leq s - 2$ we have

$$\frac{d^l}{dt^l} \pi(t) P_1(t) \psi = \sum_{j=0}^{s-3} \frac{1}{(j-l)!} t^{j-l} \beta_j + \omega K |t|^{s-2-l},$$

where $|\omega| \leq 1$. Here $\beta_j$, $j = 0, \ldots, s - 3$, are constants with $\beta_0 = 1$.

Proof. As in (2.15) we write

$$\pi(t) P_1(t) \psi = 1 + (\pi(t) - \pi) \psi + \frac{1}{2\pi i} \int_{\Gamma_1} \pi(t) \sum_{j=0}^{\infty} R(z) [(P(t) - P) R(z)]^j \psi \, dz.$$  \hspace{1cm} (2.21)

The proof is now as in the proof of Lemma 2.5. $\Box$

LEMMA 2.7. There exist constants $K$ and $0 < c_2 < c_1$ such that for $|t| < c_2$ and $l \leq s - 2$ we have

$$\left| \frac{d^l}{dt^l} \pi(t) P^n(t) P_2(t) \psi \right| \leq K \left( \frac{1 + 2 \rho}{3} \right)^n.$$

Proof. As in (2.15) we write

$$\pi(t) P^n(t) P_2(t) \psi = \frac{1}{2\pi i} \int_{\Gamma_2} z^n \pi(t) \sum_{j=0}^{\infty} R(z) [(P(t) - P) R(z)]^j \psi \, dz.$$  \hspace{1cm} (2.22)

The proof is now as in the proof of Lemma 2.5. $\Box$

Remark. If instead of (2.6) we assume that

$$\sup_y \left| \int e^{s(x,y)} P(dx, y) \right| < \infty$$

for $|z| < A$ we may obtain the representation (2.12) with $t$ replaced by $z$ for $|z| < A_2 < A$ (see [6]) and with $\lambda(z)$ analytical. Since

$$\lim_{n \to \infty} \frac{1}{n} \ln E(e^{zS_n}) = \lambda(z)$$

for $|z| < A_2 < A_1$, say, the eigenvalue $\lambda(z)$ becomes important for large
deviations results. A treatment of this for finite Markov chains may be found in [2] and in a more general setting in [3].

3. RELAXING THE CONDITIONS ON THE CONDITIONAL MOMENTS

Nagaev assumed the condition (2.6) on \( f \) in order to obtain the basic relation (2.12) for the characteristic function of \( S_n \). We now relax this assumption and will instead use

\[
\sup_{x, y} E(\{f(\text{Xi}, \text{Xi} - 1) \mid \text{xi}_1, \ldots, \text{xi}_n\}) \leq M_1 < \infty \tag{3.1}
\]

for some \( \nu \geq 1 \). We may rewrite the condition as

\[
\|P^*H_1\| \leq M_1,
\]

where \( \bar{H}_1 \) is given like \( H_1 \) in (2.19) with \( f \) replaced by \( |f| \). We then have the following equivalent of Lemmas 2.2 and 2.3.

**Lemma 3.1.** There exists a constant \( K \) such that for \( |z - 1| > (1 - \rho)/6 \), \(|z| > \rho + (1 - \rho)/6\) we have

\[
\|\left[ (P(t) - P) \right]^{r+1} \| \leq K |t| \tag{3.2}
\]

Furthermore, there exists a \( c_1 > 0 \) such that for \( |t| < c_1 \) the resolvent \( R(z, t) \) is well defined for \( |z - 1| > (1 - \rho)/6 \) and \(|z| > \rho + (1 - \rho)/6\) and \( B_1(t) \) is one dimensional.

**Proof.** Write \( R(z) \) in the form

\[
R(z) = \sum_{0}^{\nu-1} \frac{1}{z^{n+1}} P^n + \left( \frac{1}{z - 1} + \sum_{1}^{\nu} \frac{1}{z^n} \right) P_1 + \left( \sum_{0}^{\infty} \frac{P^n - P_1}{z^{n+1+r}} \right) P^r
\]

\[
= \sum_{0}^{\nu-1} \frac{1}{z^{n+1}} P^n + Q_0(z) P_1 + Q_1(z) P^r, \tag{3.3}
\]

where \(|Q_0(z)|, ||Q_1(z)|| \to 0\) for \(|z| \to \infty\). We then get

\[
\|\left[ (P(t) - P) R(z) \right]^{r+1} \|
\leq M_2 |t| \left\| \left( \prod_{1}^{\nu} \left( P(t) - P \right) \left[ \sum_{0}^{\nu-1} \frac{P^n}{z^{n+1}} + Q_0(z) P_1 + Q_1(z) P^r \right] \right) \bar{H}_1 \right\|
\leq M_2 2^r |t| \left\| \left( \prod_{1}^{\nu} \left( \sum_{0}^{\nu-1} \frac{P^n}{z^{n+1}} + Q_0(z) P_1 + Q_1(z) P^r \right) \right) \bar{H}_1 \right\|,
\]
When evaluating the product from 1 to $\nu$ we find that all the terms will contain $P^*\tilde{H}_1$ and the bound (3.2) therefore follows from (3.1). Here we use that $P_1 = P_1P^*$.

It is obvious from (3.2) that $R(z, t)$ is well defined for $|t| < K^{-1}$.

To prove that $B_1(t)$ is one dimensional we proceed as in the proof of Lemma 2.3. Let $h_1, h_2 \in B_1(t)$ and $P_1(h_1 - ch_2) = 0$. Denote $h_1 - ch_2$ by $h$. We then want to show that $||(P_1(t) - P_1)h|| \leq K_1|t||h||$, for some constant $K_1$, which implies that $h = 0$. In (2.9) the sum from $\nu + 1$ to infinity applied on $h$ is bounded by $K_2|t||h||$ according to (3.2). Since $P_1h = 0$ we find

$$R(z)h = \sum_{0}^{\infty} \left( P^n - P_1 \right) \frac{1}{z^{n+1}} h = \psi(z)h$$

with $\psi(z)$ analytical at $z = 1$. Using (3.3) we must consider terms of the form

$$\left( \prod_{1}^{k} \left[ \sum_{0}^{\nu - 1} \frac{P^n}{z^{n+1}} + Q_0(z)P_1 + Q_1(z)P^* \right] (P(t) - P) \right) \psi(z)h$$

(3.4)

for $k \leq \nu$. When evaluating the product all those terms where $P_1$ or $P^*$ appear may be bounded by $K_3|t||h||$ according to (3.1). Here we replace the first $P(t) - P$ by $\tilde{H}_1$ and the remaining by $2P$. The remaining terms in the product are of the form

$$\left( \prod_{1}^{k} \frac{P^n_{n_j}}{z^{n_j+1}} (P(t) - P) \right) \psi(z)h,$$

(3.5)

which gives a zero contribution when integrated along $I_1$ as in (2.9). □

Lemma 3.1 allows us to derive the representation (2.12) of the characteristic function of $S_n$. We now show that the last term in (2.12) is exponentially small.

**Lemma 3.2.** There exist constants $K$ and $0 < c_2 < c_1$ such that for $|t| < c_2$ and $n > \nu^2$ we have

$$|\pi(t)P^n(t)P_2(t)\psi| \leq K|t| \left( \frac{1 + 2\rho}{3} \right)^n.$$

**Proof.** Using the formula (2.22) we proceed as in the last part of the proof of Lemma 3.1. Thus we split the sum into $k \geq \nu + 1$ and $k \leq \nu$. For $k \leq \nu$ we have expressions similar to (3.4) and (3.5) with $\psi(z)h$ replaced by $z^n\psi/(z - 1)$ and the contribution from (3.5) is again zero when $n > \nu^2$ and the term is integrated along $I_2$. □
To Taylor-expand the eigenvalue $\lambda(t)$ we assume instead of (2.14)
\[
\sup_{x_{i-\nu-1}} E \left[ (f(X_i, X_{i-1}))^2 | x_{i-\nu-1} \right] \leq M_3 < \infty
\] (3.6)
for some $s \geq 3$. Equivalently we write $\|P^s \tilde{H}_s\| \leq M_3$ with $\tilde{H}_s$ given like $H_s$ in (2.19) with $f$ replaced by $|f|$.

**Lemma 3.3.** Under the condition (3.6) the results of Lemma 2.5 hold.

**Proof.** Using that $P_t R(z) = P_t/(z - 1)$, $R(z)\psi = \psi/(z - 1)$ and (3.3) we consider instead of (2.16) terms of the form
\[
\sum_{i_k} P_i \left( \prod_{j=1}^{\nu-1} P_{j}(t) \left[ \sum_0^n P_{n+1} Q_0(z) P_1 + Q_1(z) P^* \right] \right) P_{i_k}(t) \psi.
\] (3.7)

In the proof of Lemma 2.5 we could bound each individual term due to (2.14). Here, however, we have to "terminate" the individual terms with $P^*$ or $P_1 P^*$ when using (3.6). We must therefore give a special treatment to terms of the form
\[
P^r \left( \prod_{j=1}^{m-1} P_j(t) P_{n_j} \right) P_{i_m}(t) \psi
\] (3.8)
with $\sum_i i_j \leq l \leq s$ and $0 \leq n_j \leq \nu - 1$. When expanding $P_j(t)$ as in (2.18) we get the terms
\[
i^{\Sigma i_j} (it)^{\Sigma r_j} P^r \left( \prod_{i=1}^{m-1} \frac{1}{r_j!} H_{i_j+r_j} P_{n_j} \right) \frac{1}{r_{m-1!}} H_{i_m+r_m} \psi,
\] (3.9)
where $i_j + r_j \leq s - 1$ and $\Sigma r_j \leq s - l - 1$. Since $\Sigma(i_j + r_j) \leq s - 1$ we get from Hölder's inequality the bound $(\prod_{i=1}^{m-1} r_{j!/}|t|^{\Sigma i_j} M_3^{\Sigma(i_j+r_j)/s})$ for (3.9). The remainder terms, when expanding (3.8), may be bounded by expressions on the form
\[
P^r \left( \prod_{i=1}^{m_0} \tilde{H}_{i_j} P_{n_j} \right) \left( \prod_{m_0+1}^{m-1} \frac{|t|^{r_j}}{r_j!} \tilde{H}_{i_j+r_j} P_{n_j} \right) \frac{|t|^{r_m}}{r_{m-1!}} \tilde{H}_{i_m+r_m} \psi
\]
with $0 \leq r_j \leq s - i_j$, and $\Sigma r_j = s - l$. Hölder's inequality then gives the bound $(\prod_{j=1}^{m-1} r_{j!/}|t|^{s-j}/M_3)$. We have now been able to treat the terms (3.7) for any $k$. We use this for $k \leq s(\nu + 1)$. For $k > s(\nu + 1)$ we can use (3.2) at least $s - l$ times. We may then complete the proof as in the proof of Lemma 2.5. $\square$
Finally, to expand $\pi(t)P_t(t)\psi$ we use the same condition (2.20) as in Section 2. However, we also need a condition stating that the initial distribution acts as a "terminator" as $P_1$ did in (3.7). The condition is

$$\int E[(f(X_k, X_{k-1}))^{2-2|x_0|}] \pi(dx_0) < \infty$$

(3.10)

for $k = 1, \ldots, \nu - 2$. The proof of the following lemma now follows the line of proof of Lemma 3.3 and we therefore omit it.

**Lemma 3.4.** Under the conditions (3.6), (3.10), and (2.20) the results of Lemmas 2.6 and 2.7 hold.

### 4. $L^p$-Ergodicity Condition

In Sections 2 and 3 we considered $P$, $P_1$, and $P(t)$ as operators on the space of bounded functions on $E$. We now want to consider these operators as acting on the space $L^p$ of functions integrable with respect to the stationary distribution. Our setup is then that $X_0, X_1, \ldots$ is a Markov chain on $(E, \mathcal{E})$ with homogeneous transition function $P(A, x)$, which we assume to have a stationary distribution $P_1$. The space $L^p$ is

$$L^p = \left\{ g: E \to \mathbb{C} \; \left| \|g\| = \left\{ \int |g(x)|^p P_1(dx) \right\}^{1/p} < \infty \right\}$$

where $1 \leq p \leq \infty$. This is a Banach space and the norm of a linear operator $T$ is defined in the usual way,

$$\|T\| = \sup_{\|g\| \leq 1} \|Tg\|.$$

Instead of the ergodicity condition (2.4) we assume the $L^p$-ergodicity condition

$$\|P^n - P_1\| \leq \gamma p^n$$

(4.1)

for some $\rho < 1$. Let us note in passing that the Riesz convexity theorem gives that both the $L^1$- and $L^\infty$-ergodicity conditions imply the $L^p$-condition for $1 < p < \infty$ and that for $1 < p < \infty$ the $L^p$-conditions are equivalent.

It is now possible, using (4.1), to repeat the derivations of Sections 2 and 3 essentially without changes. Remember that the operators $\tilde{H}_m$ are defined by

$$\tilde{H}_m g = \int g(x)|f(x, \cdot)|^m P(dx, y).$$
The equivalent of the assumption (2.6) is then

\[ \| \tilde{H}_1 \| \leq M_1 < \infty. \quad (4.2) \]

Before being able to derive the basic representation (2.12) of the characteristic function of \( S_n \) we need to assume that the initial distribution \( \pi \) considered as an operator on \( L^p \) is bounded, that is,

\[ \| \pi \| = \sup_{\| g \| \leq 1} | \pi g | = \sup_{\| g \| \leq 1} \left| \int g(x) \pi(dx) \right| < \infty. \quad (4.3) \]

This is of course true if \( \pi \) is the stationary distribution \( P \). If \( \pi \) is a point distribution (4.3) is not necessarily true, but then we may consider a new Markov chain starting at \( X_1 \) and with \( \pi(dx_1) = P(dx_1, x_0) \). We then require (4.3) to be true for this \( \pi \). Assume in the following that \( \| \pi \| = 1 \).

The equivalents of the moment conditions (2.14) and (2.20) are

\[ \| \pi_m \| \leq M_3 < \infty \quad \text{and} \quad \| \pi_{s-2} \| < \infty, \quad (4.4) \]

where \( \pi_m \) is the functional

\[ \pi_m g = \int g(x) \pi_0(x) \pi(dx). \]

**Lemma 4.1.** Under the conditions (4.2) and (4.3) the results of Lemmas 2.1 to 2.4 hold. Under the further condition (4.4) the results of Lemmas 2.5 to 2.7 hold.

**Proof.** To repeat the proofs in Section 2 we only need to note that (4.2) implies that \( \| P(t) - P \| \leq |t| M_1 \) and that (4.4) implies that \( \| \tilde{H}_k \| \leq M_3^{k/s} \) for \( k < s \), using Hölder’s inequality. \( \square \)

The extensions of Section 3 follow in the same way for the \( L^p \) case considered here. Condition (3.1) becomes

\[ \| P^r \tilde{H}_1 \| \leq M_1 \quad (4.5) \]

and the moment conditions (3.6) and (3.10) take the forms

\[ \| P^r \tilde{H}_s \| \leq M_3 \quad \text{and} \quad \| \pi P^k \tilde{H}_{s-2} \| < \infty \quad (4.6) \]

for \( k = 0, \ldots, \nu - 1 \).

**Lemma 4.2.** Under the conditions (4.3) and (4.5) the results of Lemmas 3.1 and 3.2 hold. If furthermore the second part of (4.4) and (4.6) are satisfied the results of Lemmas 3.3 and 3.4 hold.
Proof. The proofs are as in Section 3 except for the use of Hölder's inequality in the proof of Lemma 3.3. Here we must use Hölder's inequality to give bounds on the norm \( \| \cdot \| \) of the operators appearing in the proof of Lemma 3.3. \( \square \)

5. CONTINUOUS TIME MARKOV PROCESSES

We now briefly describe how the above results may be extended to give expansions for additive functionals of a continuous time Markov process.

Let the Markov process be \( X_t \) and the additive functional be denoted by \( H(0, t) \). Since

\[
H(0, t) = H(0, 1) + H(0, 2) + \cdots + H(n - 1, n) + H(n, t),
\]

where \( n \) is the integer part of \( t \), we will consider the following basic setup. Let \( (X_i, Y_i), i = 1, \ldots, n, \) be a homogeneous Markov chain with the property that the conditional distribution of \( (X_i, Y_i) \) given \( (X_{i-1}, Y_{i-1}) \) is the same as the conditional distribution of \( (X_i, Y_i) \) given \( X_i \). Also let \( Z_i \) be a random variable such that the conditional distribution of \( Z_i \) given \( \mathbf{X} \) equals the conditional distribution of \( Z_i \) given \( X_i \). Here then \( Y_i \) corresponds to \( H(i - 1, i) \) above and \( Z_i \) corresponds to \( H(n, t) \). Since \( X_t \) itself is a Markov chain we may define the operators \( P \) and \( P_1 \) as in Section 2. The operator \( P(u) \) is now defined by

\[
[P(u)g](x_0) = E(g(X_1)e^{iuY_1}|x_0).
\]

Furthermore we need the function \( \psi_i(u) \) given by

\[
[\psi_i(u)](x) = E(e^{iuZ_i}|X_i = x).
\]

Instead of the basic representation (1.4) we then get

\[
E(e^{iuH(0,t)}) = \lambda(u)\pi(u)P_1(u)\psi_i(u) + \pi(u)P^n(u)P_2(u)\psi_i(u),
\]

where \( \lambda(u) \) is the largest eigenvalue of the operator \( P(u) \) and \( P_1(u) \) and \( P_2(u) \) are the projections as in Section 2.

Finally, we note that expansions for the distribution function of \( H(0, t) \) may also be obtained under the weaker mixing conditions used in [4] by using the variables \( (X_i, Y_i) \). The stopping times used in [4] should then be defined in terms of the Markov chain \( X_i \).

APPENDIX: SPECTRAL THEORY

The description in this appendix of linear transformations of a Banach space has been taken from [7, Chap. XI].
Let $B$ be a Banach space with norm $\| \cdot \|$ and $T$ a bounded linear transformation of $B$. We denote also the norm of an operator by $\| \cdot \|$, i.e.,

$$\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|.$$  

The resolvent set $\rho(T)$ of $T$ is defined as

$$\rho(T) = \{z \in \mathbb{C} \mid (T - zI)^{-1} \text{ exists}\}$$

and the spectrum $\sigma(T)$ is the complement of $\rho(T)$,

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

For $z \in \rho(T)$ the inverse of $T - zI$ is called the resolvent and denoted by $R_z$. If $\zeta \in \rho(T)$ then $z \in \rho(T)$ for $|z - \zeta| < \|R_\zeta\|^{-1}$ and

$$R_z = R_\zeta + (z - \zeta)R_\zeta^2 + (z - \zeta)^3R_\zeta^3 + \cdots.$$  

If $|z| > \|T\|$ then $z \in \rho(T)$ and

$$R_z = \frac{-1}{z} I - \frac{1}{z^2} T - \frac{1}{z^3} T^2 - \cdots.$$  

In fact $z \in \rho(T)$ if $|z| > \|T^n\|^{1/n}$ for some $n \geq 1$ and $\sigma(T)$ is a closed non-empty set in $\{z \mid |z| \leq \inf_n \|T^n\|^{1/n}\}$. Let us also note the relation

$$R_z - R_\zeta = (z - \zeta)R_\zeta R_z R_\zeta$$

for any two points of $\rho(T)$.

We shall need the following important decomposition theorem. Let $\sigma(T) = \sigma \cup \overline{\sigma}$, where $\sigma$ and $\overline{\sigma}$ are disjoint and isolated. Let $I_o$ be a closed rectifiable curve in $\rho(T)$ which is the boundary of an open bounded region $D$ with the property that $\sigma = \sigma(T) \cap D$.

**Theorem A.1.** The space $B$ may be decomposed into the vector sum of two linearly independent subspaces $\mathcal{M}$ and $\mathcal{N}$, where

$$T(\mathcal{M}) \subseteq \mathcal{M}, \quad T(\mathcal{N}) \subseteq \mathcal{N}$$

and

$$\sigma(T|_{\mathcal{M}}) = \sigma, \quad \sigma(T|_{\mathcal{N}}) = \overline{\sigma}.$$  

The parallel projection of $B$ onto $M$ in the direction of $\mathcal{N}$ is equal to

$$P_o = \frac{1}{2\pi i} \int_{I_o} R_z \, dz.$$  

Furthermore, $P_\sigma = 1$ and $P_\bar{\sigma} = 0$ if and only if $\sigma$ coincides with $\sigma(T)$ and $\bar{\sigma}$ is empty. □

Finally we have the formulas

\begin{equation}
T^n P_\sigma = -\frac{1}{2\pi i} \int_{I_\sigma} z^n R_z \, dz
\end{equation}

and

\begin{equation}
R^n_a P_\sigma = -\frac{1}{2\pi i} \int_{I_\sigma} (z - a)^{-n} R_z \, dz,
\end{equation}

where $a$ is outside the region $D$ determining the curve $I_\sigma$.

**REFERENCES**