A \( q \)-Analogue of Mahler Expansions, I

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We examine a \( q \)-analogue of Mahler expansions for continuous functions in \( p \)-adic analysis, replacing binomial coefficient polynomials \( \binom{x^n}{n} \) with a \( q \)-analogue \( \binom{x^n}{n}_q \) for a \( p \)-adic variable \( q \) with \(|q|_p<1\). Mahler expansions are recovered at \( q=1 \) and we consider the \( p \)-adic \( q \)-Gamma function \( \Gamma_p(q) \) of Koblitz relative to its \( q \)-Mahler expansion.

Key Words: \( q \)-analogue; \( p \)-adic functions; Mahler expansions.

1. INTRODUCTION

Let \( \mathbb{Z}_p \) be the \( p \)-adic integers, \( \mathbb{Q}_p \) the \( p \)-adic rationals, and \( K \) a field extension of \( \mathbb{Q}_p \) which is complete with respect to a nonarchimedean absolute value \( | \cdot |_p \), normalized by \(|p|_p=1/p\).

About 40 years ago, Mahler introduced in [18] an expansion for continuous functions from \( \mathbb{Z}_p \) to \( K \) using special polynomials. Specifically, he observed that the \( n \)-th binomial coefficient polynomial

\[
\binom{x^n}{n} = \frac{x(x-1) \cdot \ldots \cdot (x-n+1)}{n!}
\]

sends \( \mathbb{Z}_p \) to \( \mathbb{Z}_p \) (it sends \( \mathbb{Z} \) to \( \mathbb{Z}/\mathbb{Z}_p \), then use continuity), so \(|\binom{x^n}{n}|_p \leq 1 \) for all \( x \in \mathbb{Z}_p \). Therefore for any sequence \( c_n \in K \) with \( \lim_{n \to \infty} c_n = 0 \), the series

\[
f(x) = \sum_{n \geq 0} c_n \binom{x^n}{n}
\]

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defines a continuous function \( Z_p \to K \). Mahler proved every continuous function from \( Z_p \) to \( K \) arises uniquely in this way, with

\[
    c_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k), \quad \sup_{x \in Z_p} |f(x)|_p = \max_{n \geq 0} |c_n|_p.
\]

The \( c_n \) are called the Mahler coefficients of \( f \) and the series \( \sum c_n x^n \) is called the Mahler expansion of \( f \).

In this paper a \( q \)-analogue of the Mahler expansion is studied, where \( q \) is a \( p \)-adic variable.

To set up the framework for our ideas, first we recall the philosophy of \( q \)-analogues over \( \mathbb{R} \) and \( \mathbb{C} \). For a complex number \( q \) other than 1, define the \( q \)-analogue of a positive integer \( n \) to be

\[
    (n)_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.
\]

As \( q \to 1 \), \( (n)_q \to n \), and this is the hallmark of a \( q \)-analogue: the limit as \( q \to 1 \) recovers the classical object. There are \( q \)-analogues of most functions in classical analysis [9]. For example, the geometric series

\[
    (1 - z)^{-a} = \sum_{n \geq 0} \frac{a(a+1) \cdots (a+n-1)}{n!} z^n
\]

for \( |z| < 1 \) and \( a \in \mathbb{C} \) has the \( q \)-analogue

\[
    1 + \frac{q^a - 1}{q - 1} z + \frac{(q^a - 1)(q^a + 1 - 1)}{(q - 1)(q^2 - 1)} z^2 + \cdots = \prod_{n \geq 0} \frac{1 - q^{a+n} z}{1 - q^a z},
\]

where the infinite product converges for \( |q| < 1 \). The analytic treatment of \( q \)-series in \( \mathbb{C} \) usually assumes \( |q| < 1 \) or \( 0 < q < 1 \). However, many results make sense in a formal way, allowing \( q \) to be viewed as an indeterminate. The study of \( q \)-analogues has connections with a number of areas of mathematics, such as partitions, modular functions, and quantum groups.

The Mahler expansion in \( p \)-adic analysis uses binomial coefficient polynomials \( \binom{x}{n}, x \in Z_p \). For \( q \in K \) with \( |q - 1|_p < 1 \) (the \( p \)-adic substitute for the condition \( |q| < 1 \) in \( \mathbb{C} \)), we will use \( q \)-analogues \( \binom{\cdot}{\cdot}_q \). These are exponential functions of \( x \in Z_p \) if \( q \) is not a root of unity, and are locally polynomials in \( x \) if \( q \) is a root of unity. In particular, \( \binom{\cdot}{1}_q = (\cdot)_q \). The \( q \)-analogue of Mahler’s theorem is
Theorem. For a complete extension field $K/Q_p$ and $q \in K$ with $|q - 1|_p < 1$, every continuous function $f: Z_p \to K$ has a unique expansion

$$f(x) = \sum_{n \geq 0} c_{n, q} \left( \frac{x}{q^{n}} \right),$$

where $c_{n, q} \in K$ and $c_{n, q} \to 0$ as $n \to \infty$. Furthermore,

$$c_{n, q} = \frac{n}{\prod_{k=0}^{n} \left( \frac{n}{k} \right)} (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k),$$

$$\sup_{x \in Z_p} |f(x)|_p = \max_{n \geq 0} |c_{n, q}|_p.$$

About 20 years ago, van Hamme [23] proved the $p$-adic analogue of a result of F. H. Jackson on real $q$-series, thereby giving explicit polynomial approximations for continuous functions on certain compact-open subsets $V_q$ of $Z_p$. The subset and the approximating polynomials depend on a parameter $q \notin Z^*_p$ which cannot be a root of unity. A. Verdoodt has continued this work. The point of view of van Hamme and Verdoodt is largely compatible with the one presented in Section 3 after a change of variables, although our approach, unlike theirs, permits a passage to the limit as $q \to 1$ to recover Mahler’s theorem at $q = 1$.

The structure of the paper is as follows. In Section 2 we review some properties of $q$-analogues, where $q$ will be treated mostly as an indeterminate. In Section 3 we let $q$ be a $p$-adic variable and discuss the $q$-analogue of Mahler’s theorem. Four proofs are given, having individual advantages. Because this paper may be of interest to people who work in $p$-adic analysis but not in $q$-series, and vice versa, we give extra details in Sections 2 and 3 for results that are well known to those familiar with one of these areas but not the other. In Section 4 we discuss properties of $q$-Mahler expansions. One aspect which is not apparent in the classical case is that the role of the $p$-adic logarithm in classifying differentiability in terms of $q$-Mahler expansions. In Section 5 we discuss the $q$-Mahler expansion of the $p$-adic $q$-Gamma function of Koblitz.

Here is a brief list of notation:

- $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \ldots\}$.
- $Z_p$ is the ring of $p$-adic integers.
- $Q_p$ is the field of $p$-adic numbers.
- $\zeta$ denotes a root of unity.
- $\Phi_n$ is the $n$th cyclotomic polynomial.
For a function $f$ on $\mathbb{Z}_p$, $(E_y f)(x) = f(x + y)$ is the shift by $y$. In particular, $(E_1 f)(x) = f(x + 1)$.

Let $(K, |\cdot|)$ be a complete extension field of $\mathbb{Q}_p$ with $|p| = 1/p$. The set of continuous functions from $\mathbb{Z}_p$ to $K$ will be denoted $C(\mathbb{Z}_p, K)$ and topologized by the sup-norm $|f|_{\sup} := \sup_{x \in \mathbb{Z}_p} |f(x)|$. (We only consider $p$-adic absolute values, so we write $|\cdot|$ rather than $|\cdot|_p$.)

A function $\mathbb{Z}_p \to K$ is called analytic if it is given by a single power series that converges on $\mathbb{Z}_p$. It is called locally analytic if it is locally expressible by a power series around each point of $\mathbb{Z}_p$.

2. A REVIEW OF $q$-FORMALISM

Here we recall the features of $q$-analogues that are needed for our purposes, generally insofar as $q$ can be treated as an indeterminate. Some remarks will be made about specializing $q$, especially at roots of unity. The focus will be on properties of $q$-binomial coefficients and $q$-difference operators.

For an integer $n$ and an indeterminate $q$, the $q$-analogue of $n$ is

$$(n)_q := \frac{q^n - 1}{q - 1}.$$  

For example, $(0)_q = 0$, $(1)_q = 1$, $(2)_q = 1 + q$, $(-1)_q = -1/q$.

When $n \geq 1$, $(n)_q = 1 + q + \cdots + q^{n-1}$ is a polynomial in $\mathbb{Z}[q]$.

For any integers $m$ and $n$,

$$(-n)_q = -\frac{1}{q^n} (n)_q, \quad (n)_{1/q} = \frac{1}{q^{n-1}} (n)_q, \quad (mn)_q = (m)_q (n)_q. \quad (2.1)$$

Specializing $q = 1$, $(n)_q$ becomes $n$.

The $q$-factorials are

$$(n)_q! := \begin{cases} 1, & n = 0; \\ (n)_q(n-1)_q \cdots (1)_q, & n \geq 1. \end{cases}$$

For example, $(1)_q! = 1$, $(2)_q! = 1 + q$, $(3)_q! = 1 + 2q + 2q^2 + q^3$, and

$$(n)_{1/q}! = \frac{1}{q^{n(n-1)/2}} (n)_q!.$$

$$\quad (2.2)$$
The $q$-binomial coefficient for nonnegative integers $m$ and $n$ with $m \geq n$ is

$$\binom{m}{n}_q := \frac{(m)_q!}{(n)_q! (m-n)_q!}$$

$$= \frac{(m)_q (m-1)_q \cdots (m-n+1)_q}{(n)_q!}$$

$$= \frac{(q^m-1)(q^{m-1}-1) \cdots (q^{m-n+1}-1)}{(q^n-1)(q^{n-1}-1) \cdots (q-1)}.$$

We use the second or third expression to extend the definition of $\binom{m}{n}_q$ to any integer $m$. These functions go back to Gauss [10, p. 16], so they are also called Gaussian coefficients.

The first few $q$-binomial coefficients are

$$\binom{m}{0}_q = 1, \quad \binom{m}{1}_q = (q^m-1)/(q-1), \quad \binom{m}{2}_q = (q^m-1)(q^{m-1}-1)/(q^2-1)(q-1).$$

For $m \geq n$, $\binom{m}{n}_q = \binom{m-n}{n}_q$, and (as a rational function in $q$) $\binom{m}{n}_q = 0$ precisely when $0 \leq m < n$. The $q$-binomial coefficient may vanish in other cases numerically, e.g., $\binom{4}{2}_q = (1+q^2)(1+q+q^2)$, so $\binom{4}{2}_q = 0$.

The following result is essentially due to Gauss [10, p. 17].

**Theorem 2.1.** For fixed integers $m \geq n \geq 0$, $\binom{m}{n}_q \in \mathbb{Z}[q]$ with degree $n(m-n)$.

**Proof.** The degree follows from the definition, once we know $\binom{m}{n}_q$ is a polynomial in $q$.

We give Gauss’ proof that $\binom{m}{n}_q \in \mathbb{Z}[q]$ and then an alternate proof that seems to be new.

The Pascal’s triangle recursion for binomial coefficients generalizes (for all $m$ in $\mathbb{Z}$) to

$$\binom{m}{n}_q = \binom{m-1}{n-1}_q + q^n \binom{m-1}{n}_q = q^{m-n} \binom{m-1}{n-1}_q + \binom{m-1}{n}_q \tag{2.3}$$

(when $m \geq n$, replace $n$ by $m-n$ to obtain either recursion from the other), and iterating the second recursion gives

$$\binom{m+n+1}{n+1}_q = q^n \binom{m+n}{n}_q + \binom{m+n}{n+1}_q = \sum_{k=0}^{m} q^k \binom{k+n}{n}_q.$$

So $\binom{m}{n}_q \in \mathbb{Z}[q]$ by induction on $n$ (and actually all the coefficients are non-negative).
As an alternate proof, the irreducible factors of the rational function \( \binom{m}{n}_q \) are cyclotomic polynomials. The multiplicity of the \( j \)th cyclotomic polynomial \( \Phi_j(q) \) as a factor of \( (n)_q \), so its multiplicity as a factor of \( \binom{m}{n}_q \) is \( \left\lfloor \frac{m}{j} - \left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{m-n}{j} \right\rfloor \right\rfloor \), which is 0 or 1. This shows for \( m \geq n \) not only that \( \binom{m}{n}_q \) is a polynomial in \( q \), but that its irreducible factors are all simple factors and \( \Phi_j(q) \) is a factor precisely when the units' digit of \( m \) in base \( j \) is less than the units' digit of \( n \) in base \( j \). I thank Ira Gessel for a simplification to the original form of this alternate proof.

Further identities for all \( m \in \mathbf{Z} \) (and \( k \geq j \geq 0 \)) are

\[
\binom{m}{n}_q = \frac{(m)_q}{n)_q} \cdot \frac{(m-1)_q}{n-1)_q}, \quad \binom{m}{n}_q = \frac{1}{q^{m(m-1)}(n)_q}, \quad \binom{m}{j}_q = \frac{(m)_q}{(m-j)_q} \cdot \frac{(j)_q}{(k-j)_q},
\]

\[
\binom{-m}{n}_q = (-1)^n q^{-n(n-1)/2} \cdot \binom{m+n-1}{n}_q
\]

\[
= (-1)^n q^{-n(n+1)/2} \cdot \binom{m+n-1}{n+1}_q. \tag{2.5}
\]

For example, \( (-1)_q = (-1)^n q^{-n(n+1)/2} \). By (2.5), for \( m > 0 \) \( \binom{-m}{n}_q \) is a polynomial in \( 1/q \) with degree \( n(n-1)/2 + mn \) whose coefficients are non-zero integers with sign \( (-1)^n \).

The next result is a \( q \)-analogue of the binomial theorem, the \( q \)-binomial theorem. It goes back to Cauchy [4, p. 46, Eq. 18].

**Theorem 2.2.** For \( m \geq 1 \),

\[
(1 + T)(1 + qT) \cdots (1 + q^{m-1}T) = \prod_{i=0}^{m-1} (1 + q^iT) = \sum_{k=0}^{m} \binom{m}{k}_q q^{k(k-1)/2} T^k.
\]

Equivalently, for commuting variables \( X \) and \( Y \),

\[
(X + Y)(X + qY) \cdots (X + q^{m-1}Y) = \prod_{i=0}^{m-1} (X + q^iY) = \sum_{k=0}^{m} \binom{m}{k}_q q^{k(k-1)/2} X^{m-k} Y^k.
\]
Following Cauchy [4, p. 51], let \( h(T) = \prod_{i=0}^{m-1} (1 + q^i T) = \sum_{k=0}^{m-1} a_k T^k \). Then \( (1 + T) h(qT) = h(T)(1 + qT) \). Equating coefficients of equal powers of \( T \),

\[
ak_k = \frac{q^{m-k-1}}{q^k - 1} a_{k-1} = \frac{q^{m-k+1} - 1}{q^k - 1} q^k - 1 a_{k-1},
\]

so \( a_k = (\frac{m}{k}) q^{k(1-1/2)} \).

In particular,

\[
(X-1)(X-q) \cdots (X-q^{m-1}) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{\frac{k}{2}} q^{k(1-1/2)} X^{m-k}, \tag{2.6}
\]

Equation (2.6). Actually, the idea of replacing \( T \) by \( qT \) to express \( q \)-products as \( q \)-series goes back to Euler [7, Chap. XVI, Sects. 306, 307].

The \( q^{k(1-1/2)} \) term that arises in the \( q \)-binomial theorem can be removed from explicit appearance. Define the \( n \)th \( q \)-power of a polynomial \( f(T) \) to be \( f^{(0; q)} = 1 \) and \( f^{(n; q)} := f(T) f(qT) \cdots f(q^{n-1} T) \) for \( n \geq 1 \). Then the \( q \)-binomial theorem becomes

\[
(1 + T)^{m; q} = \sum_{k=0}^{m} \binom{m}{k} q^{(k; q)} T^k.
\]

We can consider \( q \)-deformed powers of a polynomial in several variables by singling out one variable, e.g., in two variables

\[
f(X, Y)^{(n; q)} := f(X, Y) f(X, qY) \cdots f(X, q^{n-1} Y).
\]

This will appear later in the case of \( (X \pm Y)^{(n; q)} \), whose value at \( X = x, Y = y \) will be written with abuse of notation as \( (x \pm y)^{(n; q)} \). For example,

\[
(x + 0)^{(m; q)} = x^m, \quad (0 + y)^{(m; q)} = q^{m(1-1/2)} y^m, \quad \binom{m}{k} q = \binom{q^{m-1}}{k}{q^{-1/2}}.
\]

The \( q \)-Vandermonde formula for \( \binom{m+n}{k}_q \) is proven as for ordinary binomial coefficients.

**Theorem 2.3.** For \( m_1, m_2 \geq 0, \) \( (m_1 + m_2)_q = \sum_{j=0}^{k} \binom{m_1}{j}_q (m_2)_q q^{j(k-j)} \).

Note the asymmetric roles of \( j \) and \( k-j \) in the exponent of \( q \) on the right side.
Proof. Compare the coefficient of $T^k$ on both sides of

\[
\prod_{i=0}^{m_1 + m_2 - 1} (1 + q^i T) = \prod_{i=0}^{m_2 - 1} (1 + q^i T) \prod_{i=0}^{m_1 - 1} (1 + q^i q^m T).
\]

By a specialization argument, Theorem 2.3 is true for all integers $m_1$ and $m_2$, possibly negative.

The following simple fact will be used when we let $q$ vary $p$-adically.

**Theorem 2.4.** For $m, n \geq 0$, \((m \choose n)_q \in (q_1 - q_2) \mathbb{Z}[q_1, q_2].\)

Proof. For all $i \geq 0$, \((k \choose l)_q \in \mathbb{Z}[q_1, q_2].\)

We now discuss the value of \((m \choose n)_q\) for $m \geq n$ when $q$ is specialized to various numbers.

When $q = 1$, \((m \choose n)_1 = (m \choose n)\) counts the number of $n$ element subsets of an $m$ element set. When $q$ is a prime power, \((m \choose n)_q\) counts the number of $n$-dimensional subspaces of an $m$-dimensional vector space over the field of size $q$.

This suggests the possibility of proving identities for $q$-binomial coefficients by letting $q$ run through (infinitely many) prime powers and interpreting the identity as a combinatorial statement in linear algebra over finite fields. See [11] for this approach.

We now consider the case when $q$ is specialized to a root of unity. For $\zeta$ a root of unity of order $b$ and $n < b$, the value of \((m \choose n)_\zeta\) can be computed directly from the definition, since $(n)_\zeta \neq 0$. The next theorem reduces the evaluation of all \((m \choose n)_\zeta\) to the case when $n < b$.

**Theorem 2.5.** Let $\zeta$ be a root of unity of order $b$.

(i) For integers $k$ and $l$ with $l \geq 0$, \((bk \choose bl)_\zeta = (k \choose \zeta)_\zeta)\).

(ii) For integers $k$ and $l$ with $l \geq 0$ and $0 \leq r, s < b$, \((bk+rs \choose bl+rs)_\zeta = (bk \choose bl)_\zeta (r \choose s)_\zeta\).

In particular, if $n < b$ and $m_1 \equiv m_2 \mod b$, then \((m \choose n)_\zeta = (m \choose n)_\zeta\).

Proof. (i)

\[
(bk \choose bl)_q = \prod_{j=0}^{b-1} \frac{q^{bk} - j - 1}{q^{bl} - j - 1} = \prod_{j=0}^{b-1} \frac{q^{bk} - j - 1}{q^{bl} - j - 1} \cdot \prod_{j=0}^{b-1} \frac{q^{bk} - j - 1}{q^{bl} - j - 1}.
\]

At $q = \zeta$, the right side becomes \(\prod_{i=0}^{b-1} (k - i)/(l - i) = (k \choose \zeta).\)
First we show \( (bk + a)_q = (bk + a - 1)_q \) when \( a \) is not divisible by \( b \). Setting \( m = bk + a, n = bl \), and \( q = \zeta \) in the equation \((m)_q = ((m)_q / (m - n)_q)(n - 1)_q\), we get what we want. So the theorem is true for \( s = 0 \). For \( s \neq 0 \),

\[
\frac{(bk + r)(bk + r - 1)_q \cdots (bk + r - s + 1)_q}{(bl + s)_q (bl + s - 1)_q \cdots (bl + 1)_q} (bk + r - s).
\]

None of the terms \((bl + j)_q\) appearing in the denominator vanishes at \( q = \zeta \), so we can evaluate and find

\[
\frac{(bk + r)}{bl + s}_q = \frac{(r)(r - 1)_q \cdots (r - s + 1)_q}{(s)(s - 1)_q \cdots (1)} (bk + r - s) = \frac{r}{s} (bk + r - s)_q.
\]

**Corollary 2.6.** Let \( \zeta \) be a root of unity of order \( b \) and \( n \in \mathbb{N} \). For \( m \) running through a fixed residue class mod \( b \), \((m)_q\) is a polynomial in \( m \).

**Proof.** By Theorem 2.5(ii), \((m)_q\) is a polynomial in \([m/b] = (m - r)/b\) and \( r \) is fixed.

**Examples.**

\[
\begin{align*}
(19)_{-1} &= (18 + 1)_{-1} = (9)_{-1} = 36, \\
(17)_{1} &= (16 + 1)_{1} = (4)_{1} = 0, \\
(-5)_{i} &= (-8 + 3)_{i} = (3)_{i} = -2i.
\end{align*}
\]

The periodicity of \((m)_q\) in \( m \) mod \( b \), stated at the end of Theorem 2.5, can also be verified by computing \((m + b)_q - (m)_q \) with the \( q \)-Vandermonde formula.

Theorem 2.5 (and an extension to \( q \)-multinomial coefficients) can be proven by group actions [21].

For a root of unity \( \zeta \) of order \( b \), that \( (\zeta)_q = 0 \) for \( 1 \leq n \leq b - 1 \) can be seen without Theorem 2.5, since the numerator of \((\zeta)_q\) vanishes at \( q = \zeta \) while the denominator does not, or (using Theorem 2.2) since
\[
\prod_{j=0}^{b-1} (1 + \zeta^j T) = 1 - (-T)^b.\]
Stated in terms of the \(b\)th cyclotomic polynomial \(\Phi_b(q)\), this vanishing becomes
\[
\left( \begin{array}{c} b \\ n \end{array} \right)_q \equiv 0 \mod \Phi_b(q)
\]  
(2.7)
when \(1 \leq n \leq b - 1\), which is also clear from the second proof of Theorem 2.1. Specializing (2.7) at \(q = 1\), we recover the familiar integer congruence \(\left( \begin{array}{c} p^n \\ n \end{array} \right) \equiv 0 \mod p\) when \(b = p^N\) is a power of a prime \(p\). Since
\[
\Phi_{p^N}(q) = \frac{q^{p^N} - 1}{q^{p^N-1} - 1} = (p)_{q^{p^N-1}},
\]
when \(b = p^N\) (2.7) can be written as \(\left( \begin{array}{c} p^n \\ n \end{array} \right)_q \equiv 0 \mod (p)_{q^{p^N-1}}\).

The \(q\)-analogue of the exponential series was introduced by Jackson [13].

\[
E_q(X) := \sum_{n \geq 0} \frac{X^n}{(n)_q!}
\]
(In the literature, the notation \(E_q(X)\) may denote a slightly different series.)

Jackson’s \(q\)-version of \(e^{x+y} = e^{x}e^{y}\) comes from (2.2) and the \(q\)-binomial theorem,

\[
E_q(X) E_{1/q}(Y) = \sum_{n \geq 0} \frac{(X + Y)(X + qY) \cdots (X + q^{n-1}Y)}{(n)_q!}
= \sum_{n \geq 0} \frac{(X + Y)^{(n, q)}}{(n)_q!}
\]  
(2.8)
In particular,
\[
E_q(X)^{-1} = E_{1/q}(-X) = \sum_{n \geq 0} (-1)^n q^{n(n-1)/2} \frac{X^n}{(n)_q!}
\]  
(2.9)
We now discuss \(q\)-difference operators. Powers \(\Lambda^n\) of the difference operator \(\Lambda\), where \((\Lambda h)(x) = h(x + 1) - h(x)\) (here and in the rest of this section, \(x\) is an integer variable), play a role in Mahler expansions which will be taken over in the \(q\)-analogue by a sequence of operators \(\Lambda_q^n\) first introduced by Jackson [14, p. 256; 15, p. 145].
The powers of $A$ behave nicely on binomial coefficients, namely

$$A^m \binom{x}{n} = \begin{cases} \binom{x}{n-m} & \text{if } m \leq n; \\ 0 & \text{if } m > n. \end{cases}$$

The $q$-analogue of powers of $A$ arise naturally by considering differences of $q$-binomial coefficients.

First, note that in analogy with $A(2^x) = (n+1)_q$, 

$$A \left( \binom{x}{n} \right)_q = q^{x+1-n} \binom{x}{n-1}_q. $$

Then, guided by the equation $A^2(x) = A(\binom{x+1}{n}) - A(\binom{x}{n-2})$, we compute

$$A \left( \binom{x+1}{n} \right)_q = q^{x+2-n} \binom{x+1}{n-1}_q,$$

so we're naturally led to calculate not $A(\binom{x+1}{n})_q - A(\binom{x}{n})_q$ but

$$A \left( \binom{x+1}{n} \right)_q - qA \left( \binom{x}{n} \right)_q = q^{x+2-n} \left( \binom{x+1}{n-1}_q - \binom{x}{n-1}_q \right)$$

$$= q^{x+2-n} \binom{x}{n-2}_q.$$

Let $(Eh)(x) = h(x+1)$ be the shift operator, so we've computed

$$(E-I) \left( \binom{x}{n} \right)_q = q^{x+1-n} \binom{x}{n-1}_q,$$

$$(E-I)(E-q) \left( \binom{x}{n} \right)_q = q^{x+2-n} \binom{x}{n-2}_q.$$

Of course $n \geq 1$ and $n \geq 2$ for these respective equations.

Experience with $q$-deformed products as in the $q$-binomial theorem now makes the following definition natural: $A^*_n := (E-I)^{\binom{n}{0}} = A^{(n-q)}$. In full, this says

$$A^*_n := \begin{cases} \frac{I}{I}, & n = 0; \\ (E-I)(E-q) \cdots (E-q^{n-1}), & n \geq 1, \end{cases}$$
so
\[
A_q^m \binom{x}{n} = \begin{cases} 
q^{m(x+m-n)} \binom{x}{n-m}_q, & \text{if } m \leq n; \\
0, & \text{if } m > n.
\end{cases}
\] (2.10)

In particular, \(A_q^m \binom{x}{n}_q |_{x=0} = \delta_{m0}\). The appearance of a function of \(x\) on the right side of (2.10), outside the \(q\)-binomial coefficient, can be removed by using an alternate \(q\)-difference operator,
\[
(D_m^q f)(x) := q^{-m} (A_q^m f)(x).
\]

Then
\[
D_q^m \binom{x}{n} = \begin{cases} 
q^{-m(n-m)} \binom{x}{n-m}_q, & \text{if } m \leq n; \\
0, & \text{if } m > n.
\end{cases}
\]

By (2.6),
\[
(A_q^n f)(x) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{k(k-1)/2} f(x+n-k).
\] (2.11)

The shift \(E\) commutes with multiplication by \(q\), so \(A_q^n\) and \(A_q^m\) commute, but \(A_q^n A_q^m \neq A_q^{n+m}\). To give a formula for \(A_q^{n+m}\) in terms of \(A_q^n\) and \(A_q^m\),
\[
A_q^{n+m} = (E - q^{n+m-1}) \cdots (E - q^n) A_q^m
\]
\[
= \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-1)/2} (-q^n)^k E^{n-k} A_q^m
\]
by the \(q\)-binomial theorem, so
\[
(A_q^{n+m} f)(x) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{k(k-1)/2} q^{nk} (A_q^m f)(x+n-k)
\]
\[
= q^{n(x+n)} (A_q^m f)(x),
\]
where \( g(x) = q^{-n} x (A^m_q f)(x) \). This can be written more conveniently in terms of the \( D_{q^n} \):

\[
D_{q^n}^{n + n'} = q^{mn} D_n^n D_{q^n}^{n'}.
\]

(2.12)

For \( n \in \mathbb{Z} \), let \( \mathcal{U}_n(x) = q^n x \) (this depends on \( q \)), so \( A_n^m = \mathcal{U}_n D_n^m \) and \( E^k \mathcal{U}_n = q^{kn} \mathcal{U}_n E^k \). When \( q = 1 \) the need for \( \mathcal{U}_n \) is not apparent. The notation \( \mathcal{U}_n \) comes from a similar function \( U_n \) used by Verdoodt [25]. Her paper will be discussed in Section 4.

The effort to directly relate \( A_n^m A_q^n \) with \( A_n^m + n^q \) led to a concise multiplicative relation (2.12) among the \( D_{q^n} \)'s rather than among the \( A_q^n \)'s. We now use (2.12) to give a formula for \( A_n^m A_q^n \) as a linear combination of various \( A_q^n \), so the \( q \)-difference operators are a basis of the algebra they generate (they have no linear relations by (2.10)).

**Theorem 2.6.** For \( m, n \geq 0 \),

\[
A_n^m A_q^n = \sum_{j=0}^{m} \binom{m}{j}_q (q^n - 1)(q^n - q) \cdots (q^n - q^{m-j-1}) A_q^{n+j} = \sum_{i+j=m+n} \binom{m}{i}_q \binom{n}{j}_q (q^n - 1)^{i+j} A_q^i
\]

Proof. By the \( q \)-binomial theorem,

\[
A_n^m A_q^n = \sum_{k=0}^{m} \binom{m}{k}_q (-1)^{m-k} q^{(m-k)(m-k-1)/2} E^k A_q^n.
\]

To get a formula for \( E^k A_q^n \), we use the following identity: for all \( k \geq 0 \),

\[
a^k = \sum_{i=0}^{k} \binom{k}{i}_q (a-1)(a-q) \cdots (a-q^{i-1}) = \sum_{i=0}^{k} \binom{k}{i}_q (a-1)^{i+1}.
\]

This is dual to (2.6), or arises naturally from consideration of \( q \)-Mahler expansions in Section 3 (i.e., from the \( q \)-difference calculus), so we won’t stop to motivate it here. Setting \( a = E \),

\[
E^k = \sum_{i=0}^{k} \binom{k}{i}_q A_q^i.
\]

(2.13)
Thus

\[
E^k A_q^n = q^k \mathcal{U}_q E^k \mathcal{D}_q^n
= q^k \mathcal{U}_q \sum_{i=0}^k \binom{k}{i}_q \mathcal{D}_q^i \mathcal{D}_q^n
\]

by (2.13)

\[
= \sum_{i=0}^k \binom{k}{i}_q q^{n(k-i)} \mathcal{U}_q \mathcal{D}_q^{n+i}
\]

by (2.12)

\[
= \sum_{i=0}^k \binom{k}{i}_q q^{n(k-i)} A_q^{n+i}
\]

so

\[
A_q^m A_q^n = \sum_{k=0}^m \sum_{i=0}^k (-1)^{m-k} q^{(m-k)(m-k-1)/2} q^{n(k-i)} \binom{m}{k}_q \binom{k}{i}_q A_q^{n+i}
\]

by (2.6)

\[
= \sum_{i=0}^m \binom{m}{i}_q (q^n - 1)^{m-i} q^{n+i} A_q^{n+i}
\]

by (2.6)

\[
= \sum_{i=0}^m \binom{m}{i}_q \binom{n}{i}_q (q^n - 1)^{(m-i)} A_q^{n+i}
\]

\[
(1 - 1) q (q^n - 1) A_q^{n+1} A_q^{n+2}.
\]

The case \(m = 1\) of Theorem 2.6 is essentially the recursive definition

\[
A_q^{n+1} = (E - q) A_q^n.
\]

Once the formula in Theorem 2.6 is found, it can also be proven by induction on \(m\), without using noncommuting operators \(E^k\) and \(\mathcal{U}_q\), as the polynomial identity

\[
(X - 1)^{(m, q)} (X - 1)^{(n, q)} = \sum_{i=0}^m \binom{m}{i}_q (q^n - 1)^{(m-i, q)} (X - 1)^{(n+i, q)}
\]

\[
= \sum_{i=0}^m \binom{m}{i}_q (q^n - 1)^{(m-i, q)} (X - 1)^{(n, q)} (X - q^n)^{(i, q)}.
\]

Dividing by \((X - 1)^{(n, q)}\), we get an identity which is a special case of the generalized \(q\)-binomial theorem [11, p. 252].
The \( q \)-analog of the formula

\[
A^*(fg) = \sum_{k=0}^{n} \binom{n}{k} (A^k f)(A^{n-k} E^k g)
\]

is

\[
A_q^*(fg) = \sum_{k=0}^{n} \binom{n}{k} q^k (A_q^k f)(A_q^{n-k} E_q^k g).
\] (2.14)

In the inductive verification of this, use (for \( r \leq n \))

\[
(E - q^n)(FG) = (E - q^n) F \cdot EG + q^n F \cdot (E - q^{n-1}) G
\]

with \( F = A_q^k f, G = A_q^{n-k} E_q^k g \), and \( r = k \).

3. \( p \)-adic features of \( q \)-formalism

In Section 2, the emphasis was on \( q \) as an indeterminate. Here it will be on \( q \) as a \( p \)-adic variable, i.e., as an element of a complete valued field \( K \) containing \( \mathbb{Q}_p \). (We do not assume \( q \in \mathbb{Q}_p \).) As we will have no use for the archimedean absolute value function, the absolute value on \( K \) will be denoted simply as \( | \cdot | \), and \( \text{ord} \) is the corresponding additive valuation:

\[
|z| = \left( \frac{1}{p} \right)^{\text{ord}(z)}. \]

The valuation ring \( \{ z \in K : |z| \leq 1 \} \) will be denoted \( \mathcal{O}_K \), with maximal ideal \( m_K \). We normalize the absolute value so \( |p| = 1/p \).

For the benefit of readers outside of number theory, we recall some facts about power functions and roots of unity in \( p \)-adic fields.

**Lemma 3.1.** (i) The roots of unity in \( K \) which reduce to 1 in the residue field \( \mathcal{O}_K/m_K \) are exactly the \( p \)th power roots of unity in \( K \).

(ii) If \( \zeta \) is a root of unity of order \( p^N > 1 \), then

\[
|\zeta - 1| = \left( \frac{1}{p} \right)^{1/p^{N-1}(p-1)} \geq \left( \frac{1}{p} \right)^{1/(p-1)}.
\]

The roots of unity in \( K \) are a discrete set.

(iii) For \( q \in K \), the sequence \( \{1, q, q^2, q^3, \ldots\} \) can be extended to a continuous function \( q^x \) for \( x \in \mathbb{Z}_p \) if and only if \( |q-1| < 1 \), in which case

\[
q^x = \sum_{n \geq 0} (q-1)^n \binom{x}{n}, \quad |q^x - 1| \leq |q-1| < 1.
\]

(iv) If \( |q-1| < 1 \), then \( q^x = 1 \) for \( x \neq 0 \) if and only if \( q \) is a root of unity of order \( p^N \) and \( x \in p^N \mathbb{Z}_p \).

**Proof.** (i) The residue field \( \mathcal{O}_K/m_K \) has characteristic \( p \). Since \( X^a - 1 \) has distinct roots in characteristic \( p \) when \( a \) is prime to \( p \), a root of unity
ζ in \( K \) of order \( ap^b \) with \( a > 1 \) and \((a, p) = 1)\) has \( \zeta^p \not\equiv 1 \mod m_K \), so \( \zeta \not\equiv 1 \mod m_K \). Since the only \( p \)th power root of unity in characteristic \( p \) is 1, if \( \zeta^p = 1 \) in \( K \), then in the residue field of \( K \) we have \( \zeta^p \equiv 1 \mod m_K \), so \( \zeta \equiv 1 \mod m_K \).

(ii) We have

\[
\prod_{i=1 \atop (p, i) = 1}^{p^N} (1 - \zeta^i) = \Phi_{p^n}(1) = p,
\]

so

\[
p = (1 - \zeta)^{p^{N-1}(p-1)} \prod_{i=1 \atop (p, i) = 1}^{p^N} \frac{1 - \zeta^i}{1 - \zeta},
\]

and for \( i \) prime to \( p \), the ratio \( (1 - \zeta^i)/(1 - \zeta) = 1 + \zeta + \cdots + \zeta^{i-1} \) is congruent in the residue field of \( K \) to \( i \not\equiv 0 \mod m_K \), so this ratio has absolute value 1, hence \( 1 - \zeta \) has the indicated size.

For two distinct roots of unity \( \zeta \) and \( \zeta' \) in \( K \), either \( \zeta \equiv \zeta' \mod m_K \), or \( \zeta/\zeta' \equiv 1 \mod m_K \), and then \( |\zeta - \zeta'| = |\zeta/\zeta' - 1| \geq (1/p)^{(p-1)} \), so the roots of unity in \( K \) are a (bounded) discrete set.

(iii) For “if,” we have for any \( m \in \mathbb{N} \) that

\[
q^m = (1 + q - 1)^m = \sum_{n=0}^{m} (q - 1)^n \binom{m}{n}.
\]

Since \( (q - 1)^n \to 0 \), the continuous function

\[
q^x = \sum_{n \geq 0} (q - 1)^n \binom{x}{n}
\]

on \( \mathbb{Z}_p \) is the \( p \)-adic interpolation of \( \{q^m\}_{m \geq 0} \). For “only if,” \( q^x \to q^0 = 1 \) as \( N \to \infty \), so \( |q| = 1 \) and as in (i) we conclude \( |q - 1| < 1 \).

(iv) Let \( x = p^n u \) with \( u \) a unit in \( \mathbb{Z}_p \). Then \( q^m \equiv 1 \) if and only if \( q^m \equiv 1 \), by taking the \((1/u)^{\text{th}}\) power.

Applying (iii) to \( q \)-analogues, \( (m)_q = (q^m - 1)/(q - 1) \) for \( m \in \mathbb{Z} \) extends to a continuous function \( (x)_q \) for \( x \in \mathbb{Z}_p \) if and only if \( |q - 1| < 1 \), in which case the extension to \( \mathbb{Z}_p \) is

\[
(x)_q = \begin{cases} 
\frac{q^x - 1}{q - 1}, & \text{if } q \not\equiv 1; \\
x, & \text{if } q \equiv 1.
\end{cases}
\]

and by (iii), \( (x)_q \equiv x \mod m_K \). In particular, if \( x \in \mathbb{Z}^\times_p \), then \( (x)_q \in \mathbb{C}^\times_K \).
For \( q \neq 1 \), \((x)_q\) is a nonvanishing function unless, by (iv), \( q \) is a nontrivial root of unity of order \( p^N \), in which case \((x)_q = (j)_q\) where \( x \equiv j \mod p^N \) and \( 0 \leq j \leq p^N - 1 \).

We now define the \( q \)-analogue of binomial coefficient functions.

For \(|q-1|<1\), \( \binom{n}{m}_q \) has a continuous extension from \( m \in \mathbb{Z} \) to \( x \in \mathbb{Z}_p \), given by

\[
\binom{x}{n}_q = \frac{(x)_q (x-1)_q \cdots (x-n+1)_q}{(n)_q!},
\]

provided \((n)_q! \neq 0\), i.e., \( q \) is not a nontrivial \( p \)th power root of unity of order \( \leq n \).

If \(|q-1|<1\) and \( q \) is a root of unity of order \( p^N \), Corollary 2.6 implies \( (\cdot)_q \) is a polynomial function of \( x \) on cosets of \( p^N \mathbb{Z}_p \). For \( x = p^N y + r \) and \( n = p^N l + s \) where \( 0 \leq r, s < p^N \), Theorem 2.5(ii) extends by continuity to

\[
\binom{x}{n}_q = \binom{y}{l}_q \binom{r}{s}_q.
\]

For example, if \( p = 2 \), then

\[
\binom{x}{2l-1}_q = \begin{cases} 
\binom{x/2}{l}, & \text{if } x \equiv 0 \mod 2; \\
\binom{(x-1)/2}{l}, & \text{if } x \equiv 1 \mod 2.
\end{cases}
\]

\[
\binom{x}{2l+1}_q = \begin{cases} 
0, & \text{if } x \equiv 0 \mod 2; \\
\binom{(x-1)/2}{l}, & \text{if } x \equiv 1 \mod 2.
\end{cases}
\]

So \( (\cdot)_q \) is an exponential function of \( x \) (a polynomial in \( q^x \)) if \( q \) is not a root of unity and is locally a polynomial in \( x \) if \( q \) is a root of unity.

By Theorem 2.1, \(|(\cdot)_q| \leq 1\) for all \( x \in \mathbb{Z}_p \), with equality if \( x = n \).

The difference operators \( D^n_q \) and \( T^n_q \) make sense on functions of a \( p \)-adic integer variable \( x \), and Eqs. (2.10) and (2.11) remain true when \( x \) is any \( p \)-adic integer.

By continuity, Theorems 2.3 and 2.4 become
Theorem 3.1. If $|q - 1| < 1$, then for all $x, y \in \mathbb{Z}_p$, $(x^q y^{q(k - j)})_q = \sum_{j=0}^{k} \binom{x}{j}_q q^k y^{q(k - j)}$.

Theorem 3.2. If $x \in \mathbb{Z}_p$ and $|q_1 - 1| < 1$, $|q_2 - 1| < 1$, then $|\binom{x}{q_1} - \binom{x}{q_2}| \leq |q_1 - q_2|$. So $\binom{x}{q} = \lim_{q \to q_1} \binom{x}{q}$. In particular, formulas involving $q$-binomial coefficients when $q$ is a root of unity can be computed first at non-roots of unity and then pass to a limit.

For example, let $1 \leq k \leq p^r$ with $k = p^r j$ and $k'$ prime to $p$. For $|q - 1| < 1$ with $q$ not a root of unity,

$$\left(\frac{p^r}{k}\right)_q \left(\frac{p^r - 1}{k - 1}\right)_q = \frac{(p^r - 1)_q}{(k)_q} \frac{1}{(k')_q} \left(\frac{p^r - 1}{k - 1}\right)_q.$$ 

In $\mathbb{C}_K/\mathbb{Q}_K$, $\left(\frac{p^r - 1}{k - 1}\right)_q \equiv (-1)^{k - 1}$ and $(k')_q \equiv k' \neq 0$, so

$$\left|\frac{p^r}{k}\right)_q = \left|\frac{p^r - 1}{(k')_q}\right|. \quad (3.2)$$

By continuity in $q$, (3.2) is also true when $q$ is a root of unity. Alternatively, (3.1) could be used instead for a direct calculation when $q$ is a root of unity.

We now discuss the $q$-analogue of Mahler expansions.

Theorem 3.3 (q-Mahler Theorem). For $q \in K$ with $|q - 1| < 1$, every continuous function $f : \mathbb{Z}_p \to K$ has a unique representation in the form

$$f(x) = \sum_{n \geq 0} c_{n,q} \binom{x}{n}_q,$$

where $c_{n,q} \in K$ and $\lim_{n \to \infty} c_{n,q} = 0$. A formula for $c_{n,q}$ is

$$c_{n,q} = (A^n_q f)(0)$$

$$= \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{k(k - 1)/2} f(n - k)$$

$$= \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{n-k(n-k-1)/2} f(k).$$

We will give four proofs of Theorem 3.3 below.

In Theorem 3.3, we call $c_{n,q}$ the $n$th $q$-Mahler coefficient of $f$ and $\sum c_{n,q} \binom{n}{q}_q$ the $q$-Mahler expansion of $f$. The terms “Mahler coefficient” and
'Mahler expansion' will refer to the case $q = 1$. The formula for $c_n, q$ in Theorem 3.3 will be called the $q$-Mahler Inversion Formula.

The formula for $c_n, q$ follows from computing $(A^*_q f)(0)$ using (2.10). Replacing $f$ by $((E+y)f)(x) = f(x + y)$, we have $\lim_{y \to 0} (A^*_q f)(iy) = 0$ for all $y \in \mathbb{Z}_p$. Like the case $q = 1$, this limit turns out to be uniform in $y$, and in fact there is some uniformity in $q$ as well (which is not apparent by looking only at the case $q = 1$). Such uniformities will arise from two of the proofs of Theorem 3.3.

**Example.** For $|a - 1| < 1$ and $|q - 1| < 1$,

$$a^* = \sum_{n \geq 0} (a-1)(a-q)\cdots(a-q^{n-1}) (x^n)_q = \sum_{n \geq 0} (a-1)^{(n, q)} (x^n)_q. \quad (3.3)$$

This could also be proven in a style similar to that of Lemma 3.1(iii).

For example, using the $q$-binomial theorem, the sequence $(1 + t)^{(m; q)}$ extends continuously from $m \in \mathbb{N}$ to $x \in \mathbb{Z}_p$ if and only if $|t| < 1$, when

$$(1 + t)^{(x; q)} = \sum_{n \geq 0} q^n (n-1)! \cdot \binom{x}{n}_q.$$

For any $x, y \in \mathbb{Z}_p$, $(1 + t)^{(x + y; q)} = (1 + t)^{(x; q)} (1 + q^n t)^{(y; q)}$. Setting $y = -x$ yields

$$((1 + t)^{(x; q)})^{-1} = (1 + q^n t)^{(-x; q)}.$$

For example, computing $(1 + q^n t)^{(-m; q)}$ in two ways for $m \geq 1$, we have

$$\frac{1}{(1 + t)(1 + qt)\cdots(1 + q^{m-1}t)} = \sum_{n \geq 0} q^n (n-1)! \cdot \binom{x}{n}_q$$

which is due to Cauchy [4, Eq. 19, p. 46] as an identity over the complex numbers.

**Warning.** For $|a - 1| < 1$, writing $a = 1 + t$, it seems reasonable to define $a^{(m; q)} = (1 + t)^{(m; q)}$ in the sense of the above example. However, although $|a^{(m; q)} - 1| < 1$ and $(1 + T)^{(m; q)} = ((1 + T)^{(m; q)} / (1 + T))^{(m; q)}$ (which implies (2.1) by looking at the coefficient of $T$), it is false that $a^{(m; q)} = (a^{(m; q)})^{(m; q)}$,
even when $m = n = 2$. A correct way to state the $q$-version of $(1 + T)^m = ((1 + T)^n)^m$ so that it is valid to specialize the variable is

$$(1 + T)^{(mn; q)} = (1 + T)^{(m; q^n)}(1 + q T)^{(n; q^m)} \cdots (1 + q^{m-1} T)^{(n; q^n)}.$$ 

Our first proof of Theorem 3.3 will deduce the result from the known case $q = 1$. Recall that a countable set of vectors $\{e_n\}_{n \geq 0}$ in a $K$-Banach space $(V, \| \cdot \|)$ (we assume the norm on $V$ is nonarchimedean: $\|v + w\| \leq \max(\|v\|, \|w\|)$) is called an orthonormal basis if every $v \in V$ has a unique representation in the form $v = \sum c_n e_n$ where $c_n \to 0$ and $\|v\| = \max |c_n|$. Mahler’s theorem says the functions $(^*_{q})$ are an orthonormal basis of $C(\mathbb{Z}_p, K)$, topologized by the sup-norm.

The following standard lemma shows that a small perturbation of an orthonormal basis is still an orthonormal basis. The ideas in the proof are taken from [3, Proposition 2, Sect. 1.1.4, Proposition 4, Sect. 2.7.2].

**Lemma 3.2.** Let $K$ be a complete nonarchimedean nontrivially valued field and $V$ be a $K$-Banach space with an orthonormal basis $\{e_n\}_{n \geq 0}$. If $e' \in V$ with $\sup_{n \geq 0} \|e_n - e'_n\| < 1$, then $\{e'_n\}$ is an orthonormal basis of $V$.

**Proof.** Step 1. $\|\sum_{n=0}^{N} c_n e'_n\| = \max_{0 \leq n \leq N} |c_n|$. Let $\varepsilon = \sup_{n \geq 0} \|e_n - e'_n\| < 1$. Writing

$$\sum_{n=0}^{N} c_n e'_n = \sum_{n=0}^{N} c_n (e'_n - e_n) + \sum_{n=0}^{N} c_n e_n,$$

the first sum has size at most $\varepsilon \max |c_n|$.

Step 2. The $K$-linear span (finite linear combinations) of the $e'_n$ is dense in $V$. Let $W$ be this span. For $v \in V$, let $v = \sum_{n \geq 0} c_n e_n$. Choose $N$ so $|c_n| \leq \varepsilon \|v\|$ for $n \geq N + 1$. Then

$$v - \sum_{n=0}^{N} c_n e'_n = \sum_{n=0}^{N} c_n (e'_n - e_n) + \sum_{n \geq N+1} c_n e_n$$

has norm $\leq \varepsilon \|v\|$. Assume $W$ is not dense, so there is $v \in V$ such that $a = \inf_{w \in W} \|v - w\| > 0$. Since $a/\varepsilon > 0$, there is $w \in W$ such that $0 < \|v - w\| < a/\varepsilon$. From above, there is $w' \in W$ such that

$$\|v - w'\| \leq \varepsilon \|v - w\| < a,$$

a contradiction.
Step 3. \{e'_n\} is an orthonormal basis.

By Step 1, it suffices to show for each \(v \in V\) that 
\(v = \sum c_n e'_n\) for some sequence \(c_n \to 0\) in \(K\).

Choose \(w_1 \in W\) such that 
\(\|v - w_1\| \leq 1/2\). Choose \(w_2 \in W\) such that 
\(\|v - w_1 - w_2\| \leq 1/4\). Continuing, choose \(w_m \in W\) such that 
\(\|v - w_1 - \cdots - w_m\| \leq 1/2^m\). Then \(\|w_m\| \to 0\) and 
\(v = \sum w_m\). Writing \(w_m = \sum b_{m,n} e'_n\), we have 
\(b_{m,n} = 0\) for \(n\) large and \(|b_{m,n}| \leq \|w_m\|\) by Step 1. Thus

\[v = \sum_n \left( \sum_m b_{m,n} e'_n \right) = \sum_n \left( \sum_m b_{m,n} \right) e'_n,\]

where the interchange of the double sum is justified by \([12, \text{Lemma 4.1.3}]\).

Here is a first proof of Theorem 3.3.

**Proof.** By Mahler's theorem, \(\{(x)\}_{x \in \mathbb{Z}_p}\) is an orthonormal basis of 
\(C(\mathbb{Z}_p, K)\). For all \(n \geq 0\), Theorem 3.2 implies

\[\left| \binom{x}{n}_q - \binom{y}{n}_q \right|_{\sup} \leq |q - 1| < 1.\]

Therefore we are done by Lemma 3.2.

This proof of Theorem 3.3 is succinct, but depends on already having the result in the case \(q = 1\). The same argument would deduce the result for all \(q\) with \(|q - 1| < 1\) if we had it for any one such \(q\).

By a similar idea, since \(\{(x)\}_{x \in \mathbb{Z}_p}\) is an orthonormal basis of 
\(C(\mathbb{Z}_p \times \mathbb{Z}_p, K)\), topologized by the sup-norm, so is \(\{(x)_{q_1(\cdot)_{q_2}}\}\) for fixed \(q_1, q_2 \in K\) with \(|q_1 - 1|, |q_2 - 1| < 1\). There is a similar extension to 
\(C(\mathbb{Z}_p, K)\) for any \(r \geq 1\).

Since \(A_q(x) = A_q(x^r)(0)\), by the \(q\)-Mahler theorem we have 
\(\lim_{r \to \infty} A_q(x) = A_q(x^r)(0)\) for each \(x \in \mathbb{Z}_p\). However, this limit is actually uniform in \(x\). To see this we give a second proof of the \(q\)-Mahler theorem, one which will not assume Mahler's theorem already. It will show directly that 
\(\lim_{r \to \infty} A_q(x) = 0\) in \(C(\mathbb{Z}_p, K)\).

First we record a lemma. It gives some properties of the size of \((x)_q\). Extending (2.1) from \(Z\) to \(\mathbb{Z}_p\), if \(|q - 1| < 1\) then 
\((xy)_q = (x)_q(y)_q\) for \(x, y \in \mathbb{Z}_p\). In particular, for \(n \in \mathbb{N}\) and \(u \in \mathbb{Z}_p^\times\),

\[(p^n u)_q = (p^n)_q(u)_q.\] (3.4)
Lemma 3.3. Let $|q - 1| < 1$.

(i) If $x = p^n u$ with $u \in \mathbb{Z}_p^*$, $|(x)_q| = |(p^n)_q|$.

(ii) $|(p^n)_q| \leq \prod_{i=0}^{n-1} \max(|q^i - 1|, 1/p) \leq \max(|q - 1|, 1/p)^{n-1} < 1$.

(iii) If $|q - 1| < (1/p)^{1/(p-1)}$, then $|(x)_q| = |x|$ for all $x \in \mathbb{Z}_p$.

Proof. (i) Use (3.4), recalling $(u)_{q^*} \equiv u \not\equiv 0 \mod m_K$.

(ii) By (2.1),

$$
(p^n)_q = (p)_q (p)_q \cdots (p)_q^{p-1},
$$

so it suffices to show for $|q - 1| < 1$ that $|(p)_q| \leq \max(|q - 1|, 1/p)$. In $\mathcal{O}_K/(q-1, p)$,

$$(p)_q = \Phi_p(q) \equiv (q - 1)^{p-1} \equiv 0.$$

(iii) By (i), we only need to show the result for $x = p^n$. Moreover, by (3.5) and $|p^{n-1} - 1| \leq |q - 1| < (1/p)^{1/(p-1)}$, it suffices to show the result for $x = p$. Since

$$
(p)_q = \frac{q^p - 1}{q - 1} = \sum_{k=1}^{p} \left(\begin{array}{c} p \\ k \end{array}\right) (q - 1)^{k-1}
$$

and each term in the sum except the one for $k = 1$ has size less than $1/p$, we're done. [22, Theorem 32.4]

From (ii), (3.2) can be weakened to

$$
\left| \binom{p^r}{k}_q \right| \leq \max(|q - 1|, 1/p)^{j-1},
$$

where we recall $1 \leq k \leq p^r$, $j = \text{ord}(k)$.

We now give a second proof of Theorem 3.3. The idea is taken from the proof of Mahler’s theorem in [22, Exercise 52.E].
Proof. Since $|A_q^n + 1 f|_{sup} \leq |A_q^n f|_{sup}$, it suffices to show $\lim_{n \to \infty} A_q^n f = 0$. We have

$$(A_q^n f)(x) = \sum_{k=0}^{\ell} \left( \begin{array}{c} p^n \\ k \\ q \end{array} \right) (-1)^{p^n-k} q^{(p^n-k)(p^n-k-1)/2} f(k + x)$$

$$= \sum_{k=0}^{\ell} \left( \begin{array}{c} p^n \\ k \\ q \end{array} \right) (-1)^{p^n-k} q^{(p^n-k)(p^n-k-1)/2} (f(k + x) - f(x)).$$

The $k = 0$ term vanishes, so by (3.6)

$$|A_q^n f|_{sup} \leq \max_{i+j=r} \max_{i+j=r} |q - 1|, 1/p^i \rho_j(f),$$

where $\rho_j(f) = \sup_{|x-y| < 1/p^j} |f(x) - f(y)|$. The terms indexed by $i$ and $j$ are both uniformly bounded above, and each tends to zero for large values of the index.

Not only does this show $\lim_{n \to \infty} (A_q^n f)(x) = 0$ uniformly in $x$, but also (for fixed $\delta \in (0, 1)$) uniformly in $q$ for $|q - 1| \leq \delta < 1$.

For the third proof of the $q$-Mahler theorem, we extend a periodicity property of ordinary binomial coefficients to $q$-binomial coefficients: for any $N \geq 1$ and all $n < p^N$,

$$a \equiv b \mod p^N \Rightarrow \left( \begin{array}{c} a \\ n \end{array} \right) \equiv \left( \begin{array}{c} b \\ n \end{array} \right) \mod p.$$  

For $q$-binomial coefficients, the same result is true provided $N$ is taken large enough depending on $q$.

**Lemma 3.4.** Let $|q - 1| < 1$. For $N$ large, depending on $q$, if $x \equiv y \mod p^N \mathbb{Z}_p$ and $n < p^N$ then

$$\left| \left( \begin{array}{c} x \\ n \end{array} \right)_{q} - \left( \begin{array}{c} y \\ n \end{array} \right)_{q} \right| \leq \frac{1}{p}.$$  

More precisely, this is true if $1/(p^N - 1) < \text{ord}(q - 1)$.

**Proof.** By Theorem 2.5,

$$m_1 \equiv m_2 \mod p^N \Rightarrow \left( \begin{array}{c} m_1 \\ n \end{array} \right) - \left( \begin{array}{c} m_2 \\ n \end{array} \right) \in \Phi_{p^N(q)} \mathbb{Z}_q.$$
So by continuity,
\[ \left| \binom{x}{n} - \binom{y}{n} \right| \leq |\Phi_{p^n}(q)| = |(p)_{q^n-1}|. \]

For \( N \) large, \(|q^{p^n-1} - 1| < (1/p)^{(p-1)}\), so \((p)_{q^n-1}\) has size \(|p| = 1/p\) by Lemma 3.3(iii).

Let’s be more precise about how large \( N \) has to be. For any \( N \),
\[ \Phi_{p^n}(q) = \prod_{\zeta \neq 1} (q - \zeta). \]

There are \( p^{N-1}(p-1) \) terms in the product. When \( 1/p^{N-1}(p-1) < \text{ord}(q-1) \), then \(|q-1| < |\zeta-1| = (1/p)^{(p-\gamma_p(p-1)}\) for all such \( \zeta \) by Lemma 3.1(ii), so all the terms have the same size and therefore
\[ |\Phi_{p^n}(q)| = \frac{1}{p}. \]

If we work modulo \((q-1, p)\), then for \( x \equiv y \mod p^N \) and \( n < p^N \),
\( \binom{x}{n} \equiv \binom{y}{n} \equiv \binom{z}{n} \) modulo \( q^\gamma_p \), so without needing \( N \) to be large, we have
\[ |(\zeta_1)_{q^n-1}| = \max(|q-1|, 1/p). \]

Now we give a third proof of Theorem 3.3. Like the second, it does not require prior knowledge at \( q = 1 \). It is based on the proof in [17, pp. 99–100].

**Proof.** Let
\[ L: \{ (c_n)_{n \geq 0}: c_n \in K, c_n \to 0 \} \to C(\mathbb{Z}_p, K) \]
by \((c_n) \mapsto \sum_{n \geq 0} c_n \binom{x}{n}_q\). This is \( K \)-linear and continuous, where the domain and range are both topologized by the appropriate sup-norm. We want to show \( L \) is onto. By scaling it suffices to show the restriction \( L: B \to C(\mathbb{Z}_p, K) \) is onto, where
\[ B = \{ (c_n): |c_n| \leq 1, c_n \to 0 \}. \]

By completeness of \( B \) and continuity of \( L \), it is enough to show that for any \( f \in C(\mathbb{Z}_p, K) \), there is some \( s \in B \) such that \(|f - L(s)| \leq |p|\). (Then apply the result to \( g = (f - L(s))/p \) to get \( s' \in B \) such that \(|f - L(s + ps')| \leq |p^2|\), etc.) That is, we want to show surjectivity of the map
\[ \{ (c_n): c_n \in K/p, c_n = 0 \text{ for large } n \} \to C(\mathbb{Z}_p, K/p) \]
given by
\[(c_n) \mapsto \sum_{n \geq 0} c_n \binom{x}{n} \mod p. \tag{3.7}\]

Note that the quotient topology on $\mathcal{O}_K/p$ is the discrete topology. Thus
\[C(\mathbb{Z}_p, \mathcal{O}_K/p) = \bigcup_{N \geq 1} \text{Maps}(\mathbb{Z}_p/p^N\mathbb{Z}_p, \mathcal{O}_K/p). \tag{3.8}\]

The union in (3.8) can be taken over just large integers. Lemma 3.4 suggests that at least for large $N$ (depending on $q$), $f \in C(\mathbb{Z}_p, \mathcal{O}_K/p)$ factors through $\mathbb{Z}_p/p^N\mathbb{Z}_p$ when its $n$th $q$-Mahler coefficient vanishes for $n \geq p^N$, thus suggesting the more precise surjectivity of
\[\{(c_n)_{n=0}^{p^N-1} : c_n \in \mathcal{O}_K/p\} \rightarrow \text{Maps}(\mathbb{Z}_p/p^N\mathbb{Z}_p, \mathcal{O}_K/p) \tag{3.9}\]
given by (3.10) with the sum over $0 \leq n \leq p^N - 1$. (Note that by Lemma 3.4, \((\cdot)_q\) mod $p$ is well-defined on $\mathbb{Z}_p/p^N\mathbb{Z}_p$ for $N$ large and $n < p^N$.) The surjectivity (even bijectivity) of (3.9) follows from the argument that $q$-Mahler coefficients are unique.

We could have worked in $\mathcal{O}_K/(q-1, p)$ and not needed to use only large $N$ at the end of the proof.

Here’s a fourth proof of Theorem 3.3, which like the second will yield some uniformity statements in $q$.

**Proof.** Define the numbers $c_n = c_n,q$ as in the statement of Theorem 3.3, so
\[f(m) = \sum_{n \geq 0} c_n \binom{m}{n} \tag*{for all nonnegative integers $m$. We thus only need to show that $|c_n| \to 0$. To do this we adapt Bojanic's argument in [2].}

Bojanic’s proof uses two different formulas for $(A^nf)(m)$. First,
\[(A^nf)(m) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k+m).\]
Writing $(A^nf)(m) = (A^nE^m f)(0)$, we also have
\[E^m = (I + A)^m = \sum_{j=0}^{m} \binom{m}{j} A^j \Rightarrow (A^nf)(m) = \sum_{j=0}^{m} \binom{m}{j} (A^{n+j}f)(0). \]
For the $q$-analogue of these, (2.11) gives
\[
(q^n f)(m) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k+m),
\]
while the equation $E_m = \sum_{j=0}^{m} \binom{m}{j}_q q^{m-j}(E - q^j f(q))$ gives
\[
(q^n f)(m) = \sum_{j=0}^{m} \binom{m}{j}_q q^{m-j} (A^*_q f)(j)(0).
\]
Equating these formulas for $(A^*_q f)(m)$ and isolating the $j = m$ term,
\[
c_{n+m} = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k+m)
- \sum_{j=0}^{m-1} \binom{m}{j}_q q^{m-j} n c_{n+j}.
\]
With this formula we show $|c_n| \to 0$.
The $j = 0$ term is $q^n c_n = q^n \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k)$, so
\[
c_{n+m} = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k+m)
- \sum_{j=1}^{m-1} \binom{m}{j}_q q^{m-j} n c_{n+j}.
\]
Scaling, we may assume $|f(x)| \leq 1$ for all $x \in \mathbb{Z}_p$, so $|c_n| \leq 1$ for all $n$.

Let $m = p^r$, for $r$ to be determined. Then
\[
|c_{n+p^r}| \leq \max_{0 \leq k \leq n, \ 1 \leq j \leq p^r-1} \left| f(k+p^r) - q^{p^r} f(k) \right| \left| c_{n+j} \right|.
\]
For such $j$, $|\binom{p^r}{j}_q| \leq |\Phi_p(q)|$ by (2.7).
Choose $\varepsilon > 0$. For large $r$, depending on $f$,
\[
|x - y| \leq \frac{1}{p^r} \Rightarrow |f(x) - f(y)| \leq \varepsilon.
\]
Thus $f(k+p^r) - q^{p^r} f(k) = f(k+p^r) - f(k) + f(k)(1 - q^{p^r})$, where the first term has size at most $\varepsilon$, while the second is at most $|q - 1| \cdot \max(|q-1|, 1/p^r)$, which is $\leq \varepsilon$ for $r$ large (depending on $q$).
By the proof of Lemma 3.4, \(|\Phi_{p'}(q)| = 1/p\) for all large \(r\), depending on \(q\). So there is a large \(r\) such that for all \(n \geq 0,\)
\[
|c_{n+p'}| \leq \max_{1 \leq j \leq p'-1} (\varepsilon, (1/p) |c_{n+j}|)
\]
\[
\leq \max(\varepsilon, 1/p).
\]
Thus \(|c_n| \leq \max(\varepsilon, 1/p)\) for \(n \geq p'\). Replacing \(n\) by \(n + p'\) gives, for all \(n \geq 0,\)
\[
|c_{n+2p'}| \leq \max_{1 \leq j \leq p'-1} (\varepsilon, (1/p) |c_{n+p' + j}|)
\]
\[
\leq \max(\varepsilon, 1/p^2).
\]
So
\[
|c_n| \leq \max(\varepsilon, 1/p^2)
\]
for \(n \geq 2p'\). Repeating this \(s - 1\) more times gives
\[
|c_n| \leq \max(\varepsilon, 1/p^s)
\]
for \(n \geq sp'\). Choosing \(s\) so large that \(1/p^s \leq \varepsilon\) we have \(|c_n| \leq \varepsilon\) if \(n \geq sp'\). ⊓⊔

Since the functions \(E^q f\) are equicontinuous, this proof shows \(\lim_{n \to \infty} A_q^n f = 0\) uniformly in \(q\) for \(|q - 1| \leq \delta < 1\).

For the reader who knows about \(q\)-derivatives, a second way to obtain the two formulas for \((A_q^n f)(m)\) in the proof above is to make the proposed equality of these two expressions a universal polynomial identity, and to establish it by \(q\)-differentiating the equation
\[
\sum_{k \geq 0} f(k) \frac{X^k}{(k)_q} = E_q(X) \sum_{n \geq 0} c_n \frac{X^n}{(n)_q}
\]
m times, dividing by \(E_q(X)\), and then equating coefficients of \(X^n\).

Although the \(q\)-Mahler expansion is treated above for a single function \(f \in C(\mathbb{Z}_p, K)\), we will look in Section 5 at an example of a family of functions \(f_q \in C(\mathbb{Z}_p, K)\) that depends continuously on \(q\) and consider the expansion of \(f_q\) relative to the \(q\)-Mahler basis.

4. PROPERTIES OF \(q\)-MAHLER EXPANSIONS

We now go through properties of \(q\)-Mahler expansions that are analogous to properties of Mahler expansions. Throughout this section, \(|q - 1| < 1\).
First, note that \( \{q \in K : |q - 1| < 1\} \) is a multiplicative group, unlike the parameter set that arises for \( q \)-series over \( \mathbb{C} \), the open unit disk. So we can also consider 1/q-Mahler expansions.

**Theorem 4.1.** Let \( |q - 1| < 1 \), \( f \in C(\mathbb{Z}_p, K) \) with \( q \)-Mahler coefficients \( c_{n,q} \). Then

(i) \( \sup_{x \in \mathbb{Z}_p} |f(x)| = \max_{n \geq 0} |c_{n,q}| \).
(ii) \( f(x + 1) = \sum_{n \geq 0} (q^n c_{n,q} + c_{n+1,q} x^n)_q \).
(iii) \( f(x + y) = \sum_{n \geq 0} (A_q^n f)(y)(y^n)_q \).
(iv) \( f(-x) = q^{-n(n+1)/2} \sum_{n \geq 0} c_{n,q} (-1)^n q^{n(n-1)/n} x^n \).
(v) \( (x)_q f(x) = \sum_{n \geq 1} (n)_q (c_{n,q} + q^n c_{n-1,q}) x^n \).

**Proof.** Part (i) follows from the \( q \)-Mahler Inversion Formula, or from the first proof of Theorem 3.3.

Part (ii) is a special case of part (iii) or can be done on its own. For part (iii), note \( f(x + y) = (Ef)(x) \) and the \( n \)th \( q \)-Mahler coefficient of \( Ef \) is \( (A_q^n Ef)(0) = (A_q^n f)(y) \).

Part (iii) can also be proven by using the \( q \)-Vandermonde formula and an interchange of a double sum, which is Mahler’s original method at \( q = 1 \).

For part (iv), use (2.5). Note that the expansion given in (iv) is related to a 1/q-Mahler expansion, which can be explicitly computed using Theorem 3.1.

For part (v), use \( (n)_q (q)_q = (x)_q (x+1)_q \).

In light of (iii), \( (A_q^n f)(y) \) should be called the \( n \)th \( q \)-Mahler coefficient of \( f \) at \( y \).

As with Mahler expansions, a function \( \mathbb{Z}_p \to K \) with a pointwise representation as \( \sum c_{n,q} x^n \) must be continuous, since \( c_{n,q} \to 0 \) by looking at \( x = -1 \).

Let’s see how the difference operators act on \( q \)-Mahler expansions. For \( q = 1 \),

\[
A^n \left( \sum_{j \geq 0} c_j \binom{x}{j} \right) = \sum_{j \geq 0} c_{m+j} \binom{x}{j},
\]

but for general \( q \), (2.10) implies

\[
A_q^n \left( \sum_{j \geq 0} c_{j,q} \binom{x}{j}_q \right) = \sum_{j \geq 0} c_{m+j,q} q^{m(n-1)} \binom{x}{j}_q,
\] (4.1)
which is not a \( q \)-Mahler expansion, because of the term \( q^m x \). So using the operator \((D_q^m f)(x) = q^{-m}(A_q^m f)(x)\), we can write this instead as
\[
\sum_{j \geq 0} c_{j,q} x_j = \sum_{j \geq 0} c_{m+j,q} q^{-m} x_j.
\]

The formula in part (v) of Theorem 4.1 can be extended to \((A_q^m f)(x)\), computing the \( n \)th \( q \)-Mahler coefficient by (2.14) and (4.1) for \( n \geq m\),
\[
A_q^n \left( \binom{x}{m_q} f(x) \right)(0) = \sum_{k=0}^{n} \binom{n}{k} \left( A_q^k \left( \binom{x}{m_q} (A_q^{n-k} f)(k) \right) \right) (0) = \sum_{k=0}^{m} q^{(m-k)} \binom{m}{k} c_{n-k,q}.
\]

We now discuss the relation between differentiability and \( q \)-Mahler expansions. When \( q = 1 \), Mahler shows in [18, Theorem 3; 19] that \( f \in C(\mathbb{Z}_p, K) \) is differentiable at \( y \) if and only if \( \lim_{m \to \infty} (A_q^m f)(y)/m = 0 \) and then
\[
f'(y) = \sum_{m \geq 1} \frac{(A_q^m f)(y)}{m} \left( -1 \right)^{m-1}.
\]

The extension of this result to general \(|q-1| < 1\) involves the \( p \)-adic logarithm, whose properties we will summarize for the convenience of readers outside of number theory. These readers should notice in particular part (iv) below, which says the \( p \)-adic logarithm is locally an isometry.

**Lemma 4.1.** (i) The series \( \log_p (1 + z) = \sum_{n \geq 1} (-1)^{n-1} z^n/n \) converges at \( z \in K \) if and only if \(|z| < 1\).

(ii) If \(|u_1 - 1|, |u_2 - 1| < 1\), then \( \log_p (u_1 u_2) = \log_p (u_1) + \log_p (u_2) \).

(iii) For \(|q - 1| < 1\), \( \lim_{x \to 0} ((q^x - 1)/x) = \log_p q \).

(iv) If \(|u - v| < (1/p)^{1/(p-1)}\), then \( |\log_p u - \log_p v| = |u - v| \).

(v) \( \log_p u = 0 \) if and only if \( u \) is a \( p \)th power root of unity in \( K \).

(vi) If \(|\zeta - 1| < 1 \) and \( \zeta^m = 1 \), then \( \lim_{q \to \zeta} ((\log_p q)/(q^m - 1)) = 1/m \).

**Proof.** (i) \(|z|^n \leq |z^n/n| \leq n |z|^n \).

(ii) See [12, Proposition 4.5.3].

(iii) For \( x \neq 0 \), \( (q^x - 1)/x = \sum_{n \geq 1} ((q - 1)^n/n)(q_n^x - 1) \) and \( (q - 1)^n/n \to 0 \)
by (i).
By (ii) we may take \( v = 1 \). The first term of the series for \( \log_p u \) is \( u - 1 \). For \( u \neq 1 \), all the remaining terms have size less than \( |u - 1| \) since for \( n \geq 2 \), the unique minimum of \( |n|^{1/(n-1)} = (1/p)^{\text{ord}_n(n-1)} \) occurs at \( n = p \).

(v) For any integer \( r \), \( \log_p u = 0 \) if and only if \( \log_p (u^r) = 0 \). For \( r \) large, \( |u^r - 1| < (1/p)^{1/(p-1)} \), and by (iv) the only \( z \) with \( |z - 1| < (1/p)^{1/(p-1)} \) and \( \log_p z = 0 \) is \( z = 1 \).

(vi) \( (\log_p q)^{1/(q-1)} = \log_p (q/\zeta)/((q/\zeta)^m - 1) \) and \( \lim_{m \to 1} (\log_p u)/(u^m - 1) = 1/m \) since \( \lim_{n \to 1} (\log_p u)/(u - 1) = 1 \) from the definition of \( \log_p u \).

**Lemma 4.2.** Let \( g : \mathbb{Z}_p \to K \) be continuous on \( \mathbb{Z}_p - \{ -1 \} \), with \( g(x) = \sum_{n \geq 0} c_n(x) \) for \( x \neq -1 \). Then \( g \) is continuous at \(-1\) if and only if \( c_n = 0 \), in which case \( g(-1) = \sum_{n \geq 0} c_{-1}(n) \).

**Proof.** The “if” direction is clear. For “only if,” continuity of \( g \) at \(-1\) is the same as continuity of \( g \) on \( \mathbb{Z}_p \), by our hypothesis. Letting \( x \) run through the nonnegative integers, we see by the \( q \)-Mahler Inversion Formula that \( c_n \) is the \( n \)-th \( q \)-Mahler coefficient of \( g \), so we’re done by Theorem 3.3.

Here is the test for differentiability with \( q \)-Mahler expansions. Compare with formulas for the derivative in (4.2).

**Theorem 4.2.** Let \( f \in C(\mathbb{Z}_p, K) \).

(i) When \( q \) is not a (nontrivial) root of unity, \( f \) is differentiable at \( x \in \mathbb{Z}_p \) if and only if \( \lim_{m \to \infty} (A^m_q f)(x)/(m)_q = 0 \), in which case

\[
(f'(x) = \frac{\log_p q}{q-1} \sum_{m \geq 1} \frac{(A^m_q f)(x)}{(m)_q} \frac{1}{q^m q} (1 - q^{m(m-1)/2}).
\]

(ii) When \( q \) is a root of unity of order \( p^N (N \geq 0) \), \( f \) is differentiable at \( x \in \mathbb{Z}_p \) if and only if \( \lim_{m \to \infty} (A^m p^N f)(x)/(p^N)_q = 0 \), in which case

\[
(f'(x) = -\sum_{l \geq 1} \frac{(A^l_q p^N f)(x)}{p^N l} (-1)^{l-1}.
\]

**Proof.** (i) For \( h \neq 0 \), \( f(x + h) = f(x) + \sum_{m \geq 1} (A^m_q f)(x)(h/m)_q \) by Theorem 4.1. Therefore

\[
\frac{f(x + h) - f(x)}{h} = \sum_{m \geq 1} \frac{(A^m_q f)(x)}{m}_q \frac{1}{h} (h/m)_q = \frac{(h)}{h} \sum_{m \geq 1} \frac{(A^m_q f)(x)}{m}_q (h - 1) (m)_q. \tag{4.4}
\]
Since \((h_q/h = (q^h - 1)/(h(q-1)))\) is continuous at all \(h \in \mathbb{Z}_p \setminus \{0\}\) and its limit as \(h \to 0\) is \((\log_q q)/(q-1) \neq 0\) (even if \(q = 1\)), the function \((h_q/h)\) is continuous and nowhere vanishing. So by Lemma 4.2 (with \(h - 1\) as the variable), \(f'(x)\) exists if and only if \((A_q^m f(x))/(m_q) \to 0\) and then \(f'(x)\) has the indicated form.

(ii) We consider only suitably small \(h\), say \(h = p^N z\) for \(z \in \mathbb{Z}_p\). For \(z \neq 0\),

\[
\frac{f(x + p^N z) - f(x)}{p^N z} = \sum_{m \geq 1} (A_q^m f(x)) \frac{1}{p^N z} \binom{p^N z}{m}^q,
\]

and by (3.1),

\[
\binom{p^N z}{m}^q = \begin{cases} 
\frac{z}{(m/p^N)}, & \text{if } p^N | m; \\
0, & \text{if } p^N \nmid m,
\end{cases}
\]

so

\[
\frac{f(x + p^N z) - f(x)}{p^N z} = \sum_{l \geq 1} (A_q^l f(x)) \frac{1}{p^N z} \binom{z}{l}^q
= \sum_{l \geq 1} (A_q^l f(x)) \binom{z - 1}{l - 1}^q.
\]

Apply Lemma 4.2 (for \(q = 1\)) with \(z - 1\) as the variable.

Let’s unify both parts of this theorem. For \(q\) not a root of unity, \((\log_q q)/(q - 1)(m_q) = (\log_q q)/(q^{m-1} - 1)\), while Lemma 4.1(vi) shows that for \(q\) a root of unity, \((\log_q q)/(q^{m-1} - 1)\) equals \(1/m\) when \(q = 1\) and equals 0 otherwise. Moreover, if \(q\) is a root of unity of order \(p^N\), then \((p^{l-1})_q = (-1)^{l-1}\) for \(l \geq 1\). So for any \(|q - 1| < 1\), a root of unity or not, \(f\) is differentiable at \(x\) if and only if \(\lim_{m \to \infty} (A_q^m f(x))(\log_q q)/(q^{m-1} - 1) = 0\), in which case

\[
f'(x) = \sum_{m \geq 1} (A_q^m f(x)) \frac{\log_q q}{q^{m-1} - 1} \binom{-1}{m-1}_q.
\]

In particular,

\[
f'(0) = \sum_{m \geq 1} c_{m,q} \frac{\log_q q}{q^{m-1} - 1} \binom{-1}{m-1}_q.
\]
When \( f(x) = \sum c_n(x^n) \) is differentiable and \( f' \) is continuous, Mahler [18, Theorem 4] gives the Mahler expansion for \( f' \),
\[
f'(x) = \sum_{n \geq 0} \left( \sum_{j \geq 1} \frac{c_{n+j}}{j} (-1)^{j-1} \right) \binom{x}{n}.
\] (4.3)

For the \( q \)-analogue, we use the following \( q \)-analogue of [22, Proposition 47.4],
\[
p^k \ll n < p^{k+1} = \binom{x}{n} - \binom{y}{n} \ll p^k |x - y|.
\]

**Lemma 4.3.** Let \( n \geq 1, p^k \ll n < p^{k+1} \).

(i) When \( q \) is not a (nontrivial) root of unity,
\[
\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq \frac{1}{[(p^k)_q]} |(x)_q - (y)_q|
\leq \frac{1}{[(p^k)_q]} \max(|q-1|, 1/p^{\text{ord}(x-y)}).
\]

(ii) When \( q \) is a root of unity of order \( p^N (N \geq 0) \) and \( x \equiv y \mod p^N \),
\[
\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \ll p^k |x - y|.
\]

**Proof.** (i) Let \( x = y + z \), so by Theorem 3.1,
\[
\binom{x}{n}_q - \binom{y}{n}_q = \sum_{j=1}^n \binom{z}{j}_q \binom{z-1}{j-1}_q \binom{y}{n-j}_q q^{j(y+j-n)},
\]

hence
\[
\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq \max_{1 \leq j \leq n} \left| \binom{z}{j}_q \right| = \max_{m \leq k} \frac{1}{[(p^m)_q]} |(x)_q - (y)_q|.
\]

(ii) The difference vanishes if \( n < p^k \), so we may assume \( n \geq p^N \), i.e., \( k \geq N \). Let \( x \equiv y \equiv r \mod p^N \), \( 0 \leq r \leq p^N - 1 \). Write \( x = p^N x' + r, y = p^N y' + r, n = p^N l + s, 0 \leq s \leq p^N - 1 \), so \( p^k - N \leq l < p^{k+1} - N \). Then \( \binom{x}{n}_q - \binom{y}{n}_q = \binom{z}{j}_q - \binom{z}{j}_q \), so (knowing the case \( q = 1 \) already)
\[
\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \ll \left| \binom{x'}{l}_q - \binom{y'}{l}_q \right| \ll p^{k-N} |x' - y'| = p^k |x - y|.
\]
If \(|q-1| < (1/p)^{1/(p-1)}\), then part (i) reduces to \(|(\tilde{z}_{x,q}) - (\tilde{z}_{y,q})| \leq p^k |x-y|\), which (for \(q \in \mathbb{Z}\)) is a special case of [8, Theorem 4.5].

Here is the \(q\)-analogue of the Mahler expansion of \(f'\) when \(f'\) is continuous, extending (4.3).

**Theorem 4.3.** Let \(f(x) = \sum_{n \geq 0} c_{n,q}(\tilde{z}_{x})\) be a continuous function from \(\mathbb{Z}_p\) to \(K\) with a continuous derivative. The \(q\)-Mahler expansion of \(f'\) is

\[
f'(x) = \sum_{n \geq 0} \left( n c_{n,q} \log_p q + \sum_{j \geq 1} c_{n+j,q} \log_p q \left( q^{-j} \right) \chi_{n/q} \right) x^n.
\]

**Proof.** Apply \(\lim_{m \to \infty} (A_{q}^m f)(x)(\log_p q)/(q^m - 1) = 0\) at \(x = 0, 1, 2, \ldots\) to see \(\lim_{m \to \infty} c_{n+m,q}(\log_p q)/(q^m - 1) = 0\) for all \(n \in \mathbb{N}\).

For \(y \neq 0\),

\[
\frac{f(x + y) - f(x)}{y} = \sum_{n \geq 0} \left( \left( A_{q}^n f \right)(0) - c_{n,q} \right) \left( \frac{y}{n/q} \right).
\]

By (4.1),

\[
\frac{(A_{q}^n f)(y) - c_{n,q}}{y} = c_{n,q} \left( \frac{q^n - 1}{y} \right) + \sum_{j \geq 1} c_{n+j,q} q^n q^{-j} \left( \frac{1}{y} \right) \left( \frac{y}{j/q} \right).
\]

How does each term behave as \(y \to 0\)? The first term tends to \(c_{n,q} \log_p q = nc_{n,q} \log_p q\). For the other terms,

\[
q^n q^{-j} \left( \frac{1}{y} \right) \left( \frac{y}{j/q} \right) = q^n q^{-j} \left( \frac{1}{y} \right) \left( \frac{y}{j/q} \right) q^{-j} = \log_p q \left( \frac{1}{j/q} \right) q^{-j}.
\]

This calculation is valid only if \(q^j \neq 1\), but the result is true if \(q^j = 1\) by using (3.1). So we expect

\[
f'(x) = \sum_{n \geq 0} \left( nc_{n,q} \log_p q + \sum_{j \geq 1} c_{n+j,q} \log_p q q^{-j} \chi_{n/q} \right) \left( \frac{x}{n/q} \right).
\]
However, though we know $\lim_{n \to \infty} c_{n + j, q}(\log_p q)/(q^j - 1) = 0$ for each $n$, so the putative $q$-Mahler coefficients of $f$ in (4.4) do make sense, we don’t yet know

$$\lim_{n \to \infty} \sum_{j \geq 1} c_{n + j, q} \frac{\log_p q}{q^j - 1} \binom{-1}{j - 1} q^{-m} = 0,$$

so convergence of the infinite series over $n$ in (4.4) is not clear. To get around this, we use the idea of Mahler from his proof of Theorem 4.3 at $q = 1$, namely by the hypothesis of continuity of $f$ it suffices to verify (4.4) when $x = m \in \mathbb{N}$. In this case the sum over $n$ becomes finite,

$$\frac{f(m + y) - f(m)}{y} = \sum_{n=0}^{m} c_{n, q} \binom{q^m - 1}{y} + \sum_{j=1} c_{n + j, q} q^{n(j - 1)} \binom{1}{j - 1} \binom{1}{n}.$$  

The outer sum is finite, so to verify termwise evaluation of $\lim_{y \to 0} \forall$ we need to do is check

$$\lim_{y \to 0} c_{n + j, q} \frac{1}{y} \binom{y}{j} = \frac{\log_p q}{q^j - 1} \binom{-1}{j - 1}$$

uniformly in $j$ (but perhaps not in $q$ or $n$).

**Case 1.** $q$ is a root of unity of order $p^N$, so $\lim_{j \to \infty} c_{n + j, q}j = 0$, as $j$ runs through multiples of $p^N$.

If $q^j \neq 1$, then $(\binom{y}{j})_q = 0$ for $|y| \leq 1/p^N$.

If $q^j = 1$, say $j = p^Nj'$, then

$$\lim_{y \to 0} c_{n + j, q} \frac{1}{y} \binom{y}{j} = \lim_{z \to 0} c_{n + j', q} \frac{1}{j'} \binom{z - 1}{j' - 1}.$$  

We consider the difference

$$\frac{1}{j} \binom{z - 1}{j - 1} - \frac{1}{j} \binom{-1}{j - 1} = \frac{c_{n + j', q}}{j} \left( \binom{z - 1}{j' - 1} - \binom{-1}{j' - 1} \right).$$

Choose a power of $p$, say $p^s$, such that $|c_{n + j', q}| \leq \delta$ for $j \geq p^s$ (and $p^N | j$).

For $j < p^s$, $(\binom{z - 1}{j' - 1}) - (\binom{-1}{j' - 1})$ has size at most $p^{s-1} |z|$ by Lemma 4.3.

Therefore

$$\lim_{y \to 0} c_{n + j, q} \frac{1}{y} \binom{y}{j} = \frac{\log_p q}{q^j - 1} \binom{-1}{j - 1}$$

uniformly in $j$.  

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Case 2. $q$ is not a root of unity.
So $\log_p q \neq 0$, hence $\lim_{j \to \infty} c_{n+j,q}(q^j - 1) = 0$.

Since
\[
c_{n+j,q} \frac{1}{y} \binom{y}{j_q} - c_{n+j,q} \frac{\log_p q}{q^j - 1} \binom{-1}{j_q} = \frac{c_{n+j,q}}{q^j - 1} \left( \frac{q^j - 1}{y} \binom{y-1}{j_q} - \log_p q \binom{-1}{j_q} \right)
\]
\[
+ \frac{c_{n+j,q}}{q^j - 1} \log_p q \left( \binom{y-1}{j_q} - \binom{-1}{j_q} \right),
\]
we need to show that
\[
\lim_{y \to 0} \frac{c_{n+j,q}}{q^j - 1} \left( \binom{y-1}{j_q} - \binom{-1}{j_q} \right) = 0
\]
uniformly in $j$. For $\delta > 0$, choose $p'$ so $|c_{n+j,q}(q^j - 1)| \leq \delta$ for $j \geq p'$. For $j < p'$, Lemma 4.3 implies
\[
\left| \binom{y-1}{j_q} - \binom{-1}{j_q} \right| \leq \frac{1}{[(p'^{j-1})]_q} \max(|q-1|, 1/p')^{\text{ord}(y)},
\]
which is $\leq \delta$ for $\text{ord}(y)$ large enough.

So for $f'$ continuous and $q$ not a root of unity,
\[
f'(x) = \frac{\log_p q}{q-1} \sum_{n \geq 0} (q-1)^n c_{n,q} + \sum_{j \geq 1} \frac{c_{n+j,q}}{j_q} (-1)^{j-1} q^{-j(j-1)/2-p} \binom{x}{n}_q,
\]
while for $q$ a root of unity of order $p^N$,
\[
f'(x) = \sum_{n \geq 0} \left( \sum_{j \geq 1} \frac{c_{n+j,q}}{j_q} (-1)^{j(p^N-1)} \binom{x}{n}_q. \right.
\]

The Mahler expansion characterizes analyticity: $\sum c_n(x)$ is analytic if and only if $c_n/n! \to 0$ [22, Theorem 54.4]. For example, the function $q^x$ is
an analytic function of $x$ if and only if $|q-1|<(1/p)^{1/(p-1)}$, in which case its $m$th Taylor coefficient at $x=0$ is $(\log_p q)^m/m!$. For other $q$, $|q^p-1|<(1/p)^{1/(p-1)}$ for $r$ large, so $(x)_q$ is locally analytic.

To describe analyticity in terms of $q$-Mahler expansions, we only consider $|q-1|<(1/p)^{1/(p-1)}$, since this is the region of $q$ where the functions $(x)_q$ are all analytic. For such $q$, $|(x)_q|=|x|$. In particular, $|n!|=|(n)_q|!$.

**Lemma 4.4.** Let $a_1, b_1, ..., a_m, b_m \in K$ with $|a_j|, |b_j| \leq 1$. Then

$$|a_1a_2 \cdots a_n - b_1b_2 \cdots b_n| \leq \max |a_j - b_j|.$$

**Proof.** In $C_K((a_1 - b_1, ..., a_n - b_n)$, $a_1 \cdots a_n \equiv b_1 \cdots b_n$. □

**Theorem 4.4.** For $|q-1|<(1/p)^{1/(p-1)}$, $\sum c_n(x)_q$ is analytic if and only if $c_n/(n)_q! \to 0$.

**Proof.** As with the first proof of Theorem 3.3, we'll get the result for general $q$ from the case $q=1$ by Lemma 3.2.

Let $A(Z_p, K) = \{ f(x) = \sum a_n x^n : a_n \in K, a_n \to 0 \}$ be the analytic functions from $Z_p$ to $K$. It is a $K$-Banach space under the norm $\|f\| = \max |a_n|$. (This norm does not generally coincide with the sup-norm over $Z_p$, e.g., $\|x^r - x\| = 1$, but $\|x^r - x\|_{\text{sup}} = 1/p$.)

Writing

$$\sum a_n x^n = \sum b_n x(x-1) \cdots (x-n+1) = \sum n! b_n \binom{x}{n},$$

we see $a_n - b_n \in Z[[b_{n+1}, b_{n+2}, ...]]$, so $\max |a_n| = \max |b_n|$. Therefore the norm in $A(Z_p, K)$ of an analytic function written as $\sum c_n(x)_q$ is $\max |c_n/(n)_q|!$.

In other words, the functions $n!(x)_q = x(x-1) \cdots (x-n+1)$ are an orthonormal basis of $A(Z_p, K)$.

The theorem amounts to showing the functions $(n)_q! (x)_q (x-1)_q \cdots (x-n+1)_q$ are an orthonormal basis of $A(Z_p, K)$. To show this we compare these functions to $n!(x)$ in order to use Lemma 3.2. By Lemma 4.4, it suffices to find an $\varepsilon < 1$ such that $\|(x-f)_q - (x-f)\| \leq \varepsilon$ for all $f \in N$.

Well,

$$(x-f)_q - (x-f) = \left(\frac{\log_p q}{q-1} - 1\right) (x-f) + \frac{\log_p q}{q-1} \sum_{r \geq 2} \frac{(\log_p q)^{r-1}}{r!} (x-f)^r.$$

(4.5)
We want a uniform upper bound < 1 on the Taylor coefficients. (The definition of the norm on \( A(\mathbb{Z}_p, K) \) is based on a Taylor expansion around 0, but recentering the series at \( j \) does not affect the maximum size of the Taylor coefficients.)

The coefficient of \( x^j \) on the right side of (4.5) is
\[
\frac{\log q}{q-1} = \sum_{n \geq 2} \frac{(q-1)^{n-1}}{n} (-1)^{n-1}.
\]

Note \( |(q-1)^{n-1}/n| \leq |(q-1)^{n-1}/n!| \). By Lemma 4.4(iv), the coefficients of the higher powers of \( x^j \) in (4.5) have size
\[
\left| \frac{\log q}{q-1} \cdot \left( \frac{\log q}{q-1} \right)^{r-1} r! \right| = \left| (q-1)^{r-1} r! \right|.
\]

So provided \( \sup_{r \geq 2} |(q-1)^{r-1}/r!| < 1 \), we're done. Letting \( s_p(r) \) be the sum of the base \( p \) digits of \( r \),
\[
\left| \frac{(q-1)^{r-1}}{r!} \right| = |q-1|^{r-1} p^{(r-1)(p-1)} \leq |q-1| p^{1/(p-1)}.
\]

**Corollary 4.1.** For \( |q-1| < (1/p)^{1/(p-1)} \) and \( |t| < 1 \), \( (1 + t)^{(x,q)} \) is analytic on \( \mathbb{Z}_p \) if and only if \( |t| < (1/p)^{1/(p-1)} \).

We now connect the work here with that of van Hamme and Verdooldt. They consider the following. Let \( a, q \in \mathbb{Z}_p^* \), perhaps \( q \not\equiv 1 \mod p \), and assume \( q \) is not a root of unity. Let \( V_q \) denote the closure of the set \( \{aq^n \}_{n \geq 0} \) in \( \mathbb{Z}_p \). It is a compact subset of \( \mathbb{Z}_p \), and open since \( q \) is not a root of unity. As \( q \to 1 \), \( V_q \) “shrinks” to \( \{ a \} \). In [23], van Hamme proves every continuous function \( f: V_q \to \mathbb{Q}_p \) has the form
\[
f(x) = \sum_{n \geq 0} \frac{(D_q^n f)(a)}{(n)_q!} (x-a)^{(n,q)}
\] (4.6)
for \( x \in V_q \), where \( (D_q^n f)(x) := (f(qx) - f(x))/(qx - x) \) is the \( q \)-derivative, \( D_q^n \) its \( n \)th iterate. Note that the domain \( V_q \) of the function depends on \( q \) and \( a \). Having \( (n)_q! \) in the denominator of (4.12) keeps \( q \) away from roots of unity.

When \( q \not\equiv 1 + p \mathbb{Z}_p \) and is not a root of unity, (4.6) is essentially a \( q \)-Mahler expansion. Indeed, in this case the elements of \( V_q \) have the form \( x = aq^r \) for unique \( y \in \mathbb{Z}_p \), in which case
(D_q^n f)(a) = (D_q^n f)(a) - (aq^n - a)^n

= (D_q^n f)(a) - a^n(q - 1)^n

\times \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})}{(q^n - 1)(q^n - 1) \cdots (q - 1)}

= (D_q^n f)(a) - a^n(q - 1)^n q^{(n-1)/2} \binom{y}{n_q}.

This last expression has an alternate form by [23, Lemma 3],

(D_q^n f)(a) = a^n(q - 1)^n q^{(n-1)/2} \sum_{k=0}^{n} (-1)^k q^{k(n-1)/2} \binom{n}{k_q} f(aq^{n-k}).

This goes back to Jackson [14, Eq. 12].

Letting \( g(y) = f(aq^y) \) be the pullback of \( f \) to a continuous function on \( \mathbb{Z}_p \), van Hamme’s expansion (4.6) becomes

\[ g(y) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} (-1)^k q^{k(n-1)/2} \binom{n}{k_q} g(n-k) \right) \binom{y}{n_q}, \]

which is the \( q \)-Mahler expansion of \( g \). But \( q \)-Mahler expansions do allow \( q \) to be a root of unity, as well as to lie outside of \( \mathbb{Q}_p \), though subject to the restriction \( |q - 1| < 1 \). In [6], a \( q \)-analogue of Mahler expansions will be described for \( q \in \mathbb{K}, |q| = 1 \), that will reduce to van Hamme’s expansion when \( q \in \mathbb{Z}_p^* \) and \( q \) is not a root of unity.

In [24, Theorem 3], van Hamme gives a remainder formula for the Mahler expansion. For a complete extension field \( K_\mathbb{Q}_p \) and a continuous function \( f: \mathbb{Z}_p \to K \) with Mahler coefficients \( c_n \),

\[ f(x) = c_0 + c_1 \binom{x}{1} + \cdots + c_n \binom{x}{n} + A^{n+1} f^* \binom{.}{n}, \]

where \( f^* \) is a modified convolution of continuous functions that we now recall. For two continuous functions \( g \) and \( h \) from \( \mathbb{Z}_p \) to \( K \), let \( g \ast h: \mathbb{Z}_p \to K \) be the \( p \)-adic interpolation to \( \mathbb{Z}_p \) of the function \( \mathbb{N} \to K \) given by \( n \mapsto \sum_{k=0}^{n} g(k) h(n-k) \). (For a proof that this sequence interpolates, see [22, Exercises 34.E, 52.J; 24, Lemma 1].) The operation \( * \) is an associative, commutative multiplication on \( C(\mathbb{Z}_p, K) \) and \( |g \ast h|_{\text{sup}} \leq |g|_{\text{sup}} |h|_{\text{sup}} \). By definition, \( (g \ast h)(x) := (g \ast h)(x-1). \) Since \( A^{n+1} f^* \to 0 \) in \( C(\mathbb{Z}_p, K) \), (4.7) is a Mahler expansion with remainder.

Here is the \( q \)-Mahler expansion with remainder.
Theorem 4.5. Choose \( q \in K \) with \( |q - 1| < 1 \) and \( f \in C(\mathbb{Z}_p, K) \). Letting \( c_{0, q}, c_{1, q}, \ldots \) be the \( q \)-Mahler coefficients of \( f \),

\[
f(x) = c_{0, q} + c_{1, q} \left( \frac{x}{1} \right)_q + \cdots + c_{n, q} \left( \frac{x}{n} \right)_q + \Delta_{q}^{n+1} f \left( \frac{x}{n} \right)_q.
\]

Our proof below will be a translation of Verdooft’s ideas in [25], where she proves a version of this expansion with remainder for functions on the sets \( V_q \). To simplify the comparison with [25], we write the variable in \( \mathbb{Z}_p \) as \( y \).

For \( y \in \mathbb{Z}_p \), set \( \mathcal{U}_n(y) = q^n y \), so \( \mathcal{U}_0(y) = q^y. \) (The functions \( \mathcal{U}_n = \mathcal{U}_n, q \) were already used in Section 3.)

Lemma 4.5. For any \( n \geq 0 \), \( f = f(0) \mathcal{U}_n + (E - q^n) f \ast \mathcal{U}_n. \)

Proof. We evaluate the right hand side at \( y = m \in \mathbb{Z}_+ \),

\[
(E - q^n) f \ast \mathcal{U}_n(m) = \sum_{i=0}^{m-1} (f(i+1) - q^i f(i)) q^{n(m-i)} = \sum_{i=0}^{m-1} f(i+1) q^{n(m-i)} - \sum_{i=0}^{m-1} f(i) q^{n(m-i)} = f(m) - f(0) q^m.
\]

Lemma 4.6. For all \( n \),

\[
\mathcal{U}_{n+1} \ast \left( \frac{m}{n} \right)_q = \left( \frac{n+1}{m+1} \right)_q.
\]

Proof. Using the first recursion in (2.3),

\[
\left( \frac{m}{n+1} \right)_q = \left( \frac{m-1}{n} \right)_q + q^{n+1} \left( \frac{m-1}{n+1} \right)_q = \left( \frac{m-1}{n} \right)_q + q^{n+1} \left( \frac{m-2}{n} \right)_q + q^{2n+1} \left( \frac{m-2}{n+1} \right)_q = \sum_{i=0}^{m-1} \left( \frac{m-1-i}{n} \right)_q q^{n+1} = \mathcal{U}_{n+1}(y) \ast \left( \frac{y}{n} \right)_q \quad \text{at} \quad y = m - 1
\]

\[
= \mathcal{U}_{n+1}(y) \ast \left( \frac{y}{n} \right)_q \quad \text{at} \quad y = m.
\]
Now we prove Theorem 4.5.

Proof: Writing \( g \ast^* h \) for \((g \ast^* h)(y)\) in order to cut down on parentheses,

\[
f(y) = f(0) \ast y + (E - I) f \ast y
\]

\[
= f(0) + Af \ast y
\]

\[
= f(0) + ((Af)(0) \ast y + (E - q)Af \ast y) \ast y
\]

by Lemma 4.5

\[
= f(0) + (Af)(0)\ast y + A_q^2 f \ast y
\]

by Lemma 4.6.

Assuming

\[
f(y) = f(0) + (A_q f)(0)\left(\frac{y}{1}\right) + \cdots + (A_q^n f)(0)\left(\frac{y}{n}\right) + A_q^{n+1} f \ast y
\]

apply Lemma 4.5 at \( n + 1 \) with the function \( A_q^{n+1} f \), and then use Lemma 4.6.

It is left to the reader to extend the \( q \)-Mahler expansion and some properties of it in this section to the case when \( K \) is a complete field of characteristic \( p \), or a complete commutative \( Z_p \)-algebra.

In addition to the \( q \)-numbers and \( q \)-binomial coefficients we have used, the study of quantum groups has focused attention on the \( q \)-analogues

\[
[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + \frac{1}{q^{n-3} + \cdots + \frac{1}{q^{n-2} = \frac{1}{q^{n-1}} (n)_q},
\]

\[
[n]_q! := [n]_q [n-1]_q \cdots [1]_q = \frac{1}{q^{(n-1)/2}} (n)_q!,
\]

and

\[
\binom{m}{n}_q := \frac{[m]_q [m-1]_q \cdots [m-n+1]_q}{[n]_q!} = \frac{1}{q^{(m-n)/2}} \binom{m}{n}_q.
\]

The extra property these have is invariance when \( q \) is replaced by \( 1/q \).
All the properties of $\binom{m}{n}_q$ have analogues for $\binom{mn}{n}_q$ such as

$$[-n]_q = [-n]_q, \quad [n]_{1q} = [n]_q, \quad [mn]_q = [m]_q[n]_q,$$

$$\binom{m}{n}_q \in \mathbb{Z}[q, 1/q],$$

$$\binom{-m}{n}_q = (\pm 1)^m \binom{m+n-1}{n}_q,$$

$$\binom{m_1 + m_2}{k}_q = \sum_{i+j=k} \binom{m_1}{i}_q \binom{m_2}{j}_q q^{m_2i - m_1j}.$$

That $\binom{n}{m}_q$ is related to $\binom{m}{n}_q^2$ means there is a different formula for $\binom{mn}{n}_q$ in the case when $\zeta$ is an odd or even order root of unity.

For $|q-1| < 1$, we get a continuous extension $\binom{n}{m}_q = (1/q^{m(n-m)}) \binom{n}{m}_q$, and $\binom{n}{m}_q - \binom{m}{n}_q \in [q-1]$, so the functions $\binom{n}{m}_q$ form an orthonormal basis of $C(\mathbb{Z}_q, K)$.

It is left to the reader to formulate all the results of this paper so far in this context. As an example of some differences, let $\delta_q(X) = \sum X^n \binom{n}{n}_q$.

Then $\delta_{1q}(X) = \delta_q(X)$ and $\delta_q(X) \delta_q(Y)$ equals

$$\sum_{n \geq 0} \frac{1}{[n]_q!} \left( \sum_{m=0}^{\infty} \binom{n}{m}_q X^{n-m} Y^m \right) = \sum_{n \geq 0} \frac{(X+Y/q)^{n-1}(X+Y/q^{n-3}) \cdots (X+q^{n-1}Y)}{[n]_q!},$$

where powers of $q$ in consecutive terms of the product on the right hand side differ by two.

Set

$$(X+Y)^{[n]}_q := (X+Y/q^{n-1})(X+Y/q^{n-3}) \cdots (X+q^{n-1}Y)$$

$$= \sum_{k=0}^{n} \binom{n}{k}_q X^{n-k} Y^k,$$

so $\delta_q(X) \delta_q(Y) = \sum_{n \geq 0} (X+Y)^{[n]}_q \binom{n}{n}_q !$ and $(X+Y)^{[m+n]}_q = (X+qY)^{[m]}_q (X+Y/q^{n-1})^{[n]}_q$. Note $(X-Y)^{[n]}_q \neq 0$ if $n$ is even. In particular, $\delta_q(X) \delta_q(-X) \neq 1$, and there doesn’t seem to be a simple formula for the coefficients of $\delta_q(X)^{-1}$. For example,
\[
E_q(X)^{-1} = 1 - X + \frac{q^2 - q + 1}{q^2 + 1} X^2 - \frac{q^6 - 2q^5 + 2q^4 - q^3 + 2q^2 - 2q + 1}{(1 + q^2)(1 + q^2 + q^4)} X^3 + \cdots,
\]
and the numerator of the coefficient of \(X^3\) is irreducible in \(\mathbb{Z}[q]\).

We define polynomials \(\mu_n(q)\) by \(E_q(X)^{-1} = \sum_{n \geq 0} \mu_n(q) X^n/[n]_q^!\), using the notation \(\mu\) by analogy with combinatorial inversion formulas. Then

\[
f(x) = \sum_{n \geq 0} C_{n,q} \left[ \frac{x}{n} \right]_q \Rightarrow C_{n,q} = \sum_{k=0}^n \binom{n}{k}_q \mu_{n-k}(q) f(k).
\]

5. THE \(p\)-ADIC \(q\)-GAMMA FUNCTION

To illustrate the possibility of using \(q\)-Mahler expansions with a family of functions depending continuously on a parameter, we consider Morita's \(p\)-adic Gamma function \(\Gamma_p\) and its \(q\)-analogue \(\Gamma_{p,q}\) as defined by Koblitz.

For a nonnegative integer \(n\), Morita [20] defines

\[
\Gamma_p(n+1) := (-1)^{n+1} \prod_{1 \leq j \leq n} j = (-1)^{n+1} \frac{n!}{p^{\text{ord}_p(n!/p)}}.
\]

for \(n \geq 1\) and \(\Gamma_p(1) = -1\). Morita's proof that \(\Gamma_p\) is \(p\)-adically continuous is based on congruence properties of the sequence \(\{\Gamma_p(n+1)\}\). For our treatment here, it is Barsky's proof [1] of the continuity which is of primary interest. Barsky's method is based on the identity

\[
\sum_{n \geq 0} \frac{(-1)^n \Gamma_p(n+1)}{n!} X^n = (1 + X + \cdots + X^{p-1}) e^{X/p}, \quad (5.1)
\]

which implies that the Mahler coefficients \(\tau_p(n)\) (say) of the sequence \(\Gamma_p(n+1)\) satisfy

\[
\sum_{n \geq 0} \frac{(-1)^n \tau_p(n)}{n!} X^n = (1 + X + \cdots + X^{p-1}) e^{X+pX/p}.
\]
Writing \( e^{X+X^n/p} = \sum_{n \geq 0} (b_{p,n}/n!) X^n \), estimates of Dwork [17, p. 320] imply \( b_{p,n} \to 0 \) \( p \)-adically as \( n \to \infty \), so \( t_p(n) \to 0 \) as \( n \to \infty \). Therefore \( \Gamma_p \) extends continuously from \( \mathbb{N} \) to \( \mathbb{Z}_p \).

We recall Dwork’s proof that \( b_{p,n} \to 0 \). Multiply \( \exp(X+X^n/p) \) by the additional terms \( \exp(X^{p^j}/p^j) \) for \( j \geq 2 \) and then remove them:

\[
e^{X+X^n/p} = \exp \left( \sum_{j \geq 0} \frac{X^{p^j}}{p^j} \right) \prod_{j \geq 2} e^{-X^{p^j}/p^j}.
\]

(5.3)

We want to show \( \exp(X+X^n/p) \) is in the space of \( p \)-adic divided power series \( \sum c_n X^n/n! \) where \( c_n \to 0 \). Such series form the Leopoldt space. It is a Banach algebra when we norm such series by sup \( |c_n| \). Since \( \exp(\sum j \geq 0 X^{p^j}/p^j) \) is the Artin–Hasse series, which has \( \mathbb{Z}_p \)-coefficients, it is a Leopoldt series. (Any series with bounded coefficients is a Leopoldt series.) By a direct calculation, \( \exp(\pm X^n/p) \) is a Leopoldt series and \( 1 \) in the Leopoldt norm as \( j \to \infty \). So by completeness the right side of (5.3) is a Leopoldt series. Thus \( b_{p,n} \to 0 \).

For \( |q-1| < 1 \), the \( q \)-analogue \( \Gamma_{p,q} \) of \( \Gamma_p \) is defined by Koblitz [16] by

\[
\Gamma_{p,q}(n+1) := (-1)^{n+1} \prod_{1 \leq j \leq n \atop (p,j)=1} \frac{q^j-1}{q^j} = (-1)^{n+1} \prod_{1 \leq j \leq n \atop (p,j)=1} (1 + q + \cdots + q^{j-1})
\]

for \( n \geq 1 \) and \( \Gamma_{p,q}(1) = -1 \). For fixed \( q \) with \( 0 < |q-1| < 1 \), Koblitz shows that the sequence \( \Gamma_{p,q}(n+1) \) \( p \)-adically interpolates to \( \mathbb{Z}_p \) by comparing \( \Gamma_{p,q} \) with \( \Gamma_p \), whose continuity is already known. There are alternate proofs of the interpolation for \( \Gamma_{p,q} \) (cf. [5]), but we would like to have available a proof of the interpolation based on Barsky’s method, proceeding as follows.

For any integer \( j \), \( (j)_q \equiv (j) \mod q_1 - q_2 \), so \( |\Gamma_{p,q}(n+1) - \Gamma_{p,q}(n+1)| \leq |q_1 - q_2| \). Thus \( p \)-adic interpolation of \( \Gamma_{p,q}(n+1) \) for general \( q \) will follow from that for a dense set of \( q \). So we may suppose \( q \) is not a root of unity, making \( (n)_q \) nonzero for all \( n \).

In this case, which we may assume we are in from now on,

\[
\Gamma_{p,q}(n+1) = \frac{(-1)^{n+1} (n)_q!}{\prod_{k \leq [n/p]} (p)_q} = \frac{(-1)^{n+1} (n)_q!}{(p)_q^{[n/p]} ([n/p])_q!}.
\]
Following Barsky, we consider
\[
\sum_{n \geq 0} \frac{(-1)^n \Gamma_{p,q}(n+1)}{(n)_q!} X^n = \sum_{n \geq 0} \frac{1}{(p)^{[n/p]}([n/p])!} X^n
\]
\[
= \sum_{r=0}^{n-1} \sum_{m \geq 0} \frac{1}{(p)^m (m)_q!} X^{p^m + r}
\]
\[
= (1 + X + \cdots + X^{p-1}) \sum_{m \geq 0} \frac{(X^r/p)_q)^m}{(m)_q!}
\]
\[
= (1 + X + \cdots + X^{p-1}) E_{q}(X^r/p)_q.
\]

Let \( \tau_{p,q}(n) \) be the \( n \)-th \( q \)-Mahler coefficient of the sequence \( \Gamma_{p,q}(n+1) \). We want to show \( \tau_{p,q}(n) \to 0 \) as \( n \to \infty \). Continuing with the above calculations, we obtain
\[
\sum_{n \geq 0} \frac{(-1)^{n+1} \tau_{p,q}(n)}{(n)_q!} X^n = (1 + X + \cdots + X^{p-1}) E_q(-X)^{-1} E_{q}(X^r/p)_q
\]
\[
= (1 + X + \cdots + X^{p-1}) E_{1,q}(X) E_{q}(X^r/p)_q.
\]

Comparing this with (5.2) shows the \( q \)-analogue of \( e^{X+X^r/p} \) is apparently
\[
E_{1,q}(X) E_{q}(X^r/p)_q = (E_{1,q}(X) E_{1,q}(X)) \cdot E_{q}(X^r/p)_q.
\]

By the \( q \)-Mahler theorem, the existence of a \( p \)-adic interpolation for \( \Gamma_{p,q}(n+1) \) is thus equivalent to the fact that, when we write
\[
E_{1,q}(X) E_{q}(X^r/p)_q = \sum_{n \geq 0} b_{p,q,n} \frac{X^n}{(n)_q!},
\]
the sequence \( b_{p,q,n} \) tends to 0 as \( n \to \infty \). This suggests looking at a \( q \)-Leopoldt space, namely the \( q \)-divided power series \( \sum c_n X^r/p)_q \) where \( c_n \to 0 \). By a direct calculation for \( j \geq 2 \), \( E_{q}(X^r/p)_q \) is a unit in the \( q \)-Leopoldt space, so carrying out a \( q \)-version of Barsky’s argument comes down to checking that a \( q \)-analogue of the Artin-Hasse series,
\[
E_{1,q}(X) \prod_{j \geq 1} E_{q}(X^r/p)_q,
\]
is in the \( q \)-Leopoldt space. (Since \( E_{1,q}(X) E_{1,q}(X) \) is a unit in the \( q \)-Leopoldt space, we can replace \( E_{1,q}(X) \) with \( E_{q}(X) \) in (5.4) without affecting the property of being or not being a \( q \)-Leopoldt series.)
Here we are left with a gap, as we do not see how to establish (5.4) is a $q$-Leopoldt series without referring to the preexisting fact that $\Gamma_{p,q}(n+1)$ interpolates. Is there a method of analyzing (5.4) without using anything about $\Gamma_{p,q}$, and ideally also not relying on the case $q = 1$ first? It may be possible to carry out this task more easily when $|q-1| < (1/p)^{1/(p-1)}$, but ultimately there should be an argument valid for $|q-1| < 1$.

REFERENCES

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