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Kernels of minimum size gossip schemes☆

Roger Labahn

Universität Rostock, Fachbereich Mathematik, 18051 Rostock, Germany

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Abstract

The main part of gossip schemes are the kernels of their minimal orders. We give a complete characterization of all kernels that may appear in gossip schemes on simple graphs with a minimum number of calls. As consequences we prove several results on gossip schemes, e.g. the minimum number of rounds of a gossip scheme with a minimum number of calls is computed. Moreover, in the new context we give proofs of known results, e.g. the well-known four-cycle theorem.

In the last part, we deal with order theoretic questions for such kernel posets. After describing all p-grid-kernels in terms of permutations and subsets, isomorphism is investigated and they are enumerated. Then we compute the order dimension and the jump number of all possible kernels, and finally, we show how to determine the numbers of their linear extensions.

1. Introduction

One of the main results of [16] is the description of the structure of gossip schemes according to a decomposition of the corresponding minimal order. It turned out that the most interesting part is the so-called kernel of the information flow. While the remaining parts are structurally easy and well-understood, we have only very small insight into all possible structures in kernels. Basically, the main known facts are lower (see [16]) and upper (see [11]) bounds on the size of kernels. On the other hand, the wide variety of different gossip schemes is generated by different kernels, i.e. for a better understanding of information flows in general, we should start investigating kernels more detailed. Of course, one of the most interesting tasks is to recover the structure of kernels in information flows of minimum size. In the literature, two

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different parameters have been studied: the number of calls, or the number of rounds in a parallelized scheme. We refer to [6, 9] for surveys of known results.

In the present paper we give a complete characterization of all kernels which can appear in an information flow on graphs with a minimum number of calls. This continues [16] in the case of simple graphs as underlying networks. Moreover, several consequences of the structural characterization are given. After listing known facts along with some technical statements in Section 2, we will prove a conjecture of [16] in Section 3: Every kernel with minimum number of elements has as many minimal as maximal elements. Section 4 contains the structure analysis of such kernels. Finally, applications are listed in Section 5, e.g. we show how the four-cycle theorem (which was one of the most exciting results in the early research on gossiping) follows easily, calculate how many rounds such an information flow needs in minimum, and construct minimum size graphs corresponding to kernels.

The last two sections deal with the posets themselves. Section 6 presents the complete characterization and enumeration of p-grid-kernels up to isomorphism. Finally, in Section 7 we compute important order invariants related to linear extensions (jump number, dimension), and enumerate the linear extensions.

2. Preliminaries

We list some facts proved in [16]. Note that we deal with simple graphs rather than hypergraphs. Moreover, for the purposes of this paper, some notation has been changed.

The *kernel* of an information flow is a poset $K_{<}$, i.e. a set K of elements for which a reflexive, antisymmetric, and transitive relation $<$ is defined such that the following properties are satisfied:

(K1) Every element of $K_{<}$ has at most 2 immediate predecessors and at most 2 immediate successors.

(K2) Every minimal element precedes every maximal element.

(K3) If for a given element x, every minimal (maximal) element preceeds (succeeds) x, then x is a maximal (minimal) element.

We use the operators min and max to denote the sets of all minimal and all maximal elements of a fixed poset $P_{<}$, respectively, and the operators pre and suc to denote the sets of all immediate predecessors and all immediate successors of a fixed element, respectively. For these we will also use the shorter terms *lower cover*, respectively, *upper cover* instead. The sets of all predecessors or successors of an element x are $\underline{x}$ or $\bar{x}$, respectively, and more generally, $\underline{N}$ and $\bar{N}$ denote the lower, respectively, upper ideal generated by a set N of elements. If there is no danger of ambiguity, P is used to denote the ground set P and the poset $P_{<}$ simultaneously. Similarly, subsets stand for a set as well as for the poset induced on it. Moreover, we will simultaneously work with the poset and its Hasse diagram $H(P)$ which can be considered as a digraph with each edge directed to the top. In this context, we also speak about *in-degrees*, $\deg^{-}$, or *out-degrees*, $\deg^+$, of the vertices resp. elements. Finally, let $\lessdot$ denote the *covering relation* of $P_<$, i.e. for $x, y \in P$, we have $x \lessdot y$ iff $x < y$ and there is no $z \in P$ with $x < z < y$.

If $|\min K| = p$ and $|\max K| = q$ then we call K a *(p,q)-kernel.* Clearly, in this case the inverse order $K_<^{\smile} = K_>$ is a (q,p)-kernel. Hence, throughout the paper we may assume w.l.o.g. that $p \leqslant q$.

From [16] we know that every poset K satisfying (K1) and (K2) contains at least $2(|\min K| + |\max K|) - 4$ elements. In particular, every (p,q)-kernel K contains at least $2(p+q) - 4$ elements, i.e. there are not less than $p + q - 4$ nonextremal elements which form the *inner* kernel $K_0 := K \backslash (\min K \cup \max K)$. We call a (p,q)-kernel *L-minimum* if it has exactly $2(p+q) - 4$ elements. These are important because every information flow with a minimum number of calls, denoted by L in that context, must have a (p,q)-kernel with $2(p+q) - 4$ elements for some integers p, q. Since it is known that there are no L-minimum $(1,q)$-kernels, throughout the paper we assume $p \geqslant 2$. Note that together with (K2) this implies $\min K \cap \max K = \emptyset$ for any kernel K considered here.

An important tool in our proofs is to find kernels in certain posets. This follows the general approach introduced in [16] to define the reduced minimal order of an information flow. In any poset, P, satisfying (K1) and (K2) pick all elements which preceed every maximal element of P, and let P' be the set of all maximal among them. Similarly, let P'' be the set of all those elements succeeding each element of P' which are minimal with respect to the latter property. Now, the *kernel of P* is the poset induced on $\ker P := \overline{P'} \cap \underline{P''}$, i.e. $\ker P$ collects all elements 'between' the antichains P' and P'' including themself. Obviously, $P' = \min \ker P$ and $P'' = \max \ker P$. By similar proofs as in [16] it immediately follows from the above definition:

Proposition 2.1. $\ker P$ *satisfies* (K1)–(K3), *and* $|\ker P| = |P| - (|\underline{\min \ker P}| + |\overline{\max \ker P}|) + (|\min \ker P| + |\max \ker P|)$.

Next we show a technical statement later used frequently.

Proposition 2.2. *Let K be any kernel. For any antichain $M \subseteq K$, $|\underline{M}| \geqslant 2|\underline{M} \cap \min K| - |M|$ and $|\bar{M}| \geqslant 2|\bar{M} \cap \max K| - |M|$. Moreover, in case of equality, the Hasse diagram of $\underline{M}$, resp., $\bar{M}$ consists of $|M|$ components each being a binary tree.*

Proof. We count the edges in the Hasse diagram $H(\underline{M})$ twice: $\sum_{x \in \underline{M}} \deg^- x \leqslant 2(|\underline{M}| - |\underline{M} \cap \min K|)$ because of (K1), and $\sum_{x \in \underline{M}} \deg^+ x \geqslant (|\underline{M}| - |M|)$ because every element except the generators has a successor. This immediately implies the first inequality. If equality holds then every nonmaximal element has exactly one successor, i.e. each maximal element generates a tree component, and every nonminimal element has exactly two predecessors, i.e. the trees are binary. The proof is analog for $\bar{M}$. $\square$

3. Existence of L-minimum kernels

For any $p \geqslant 2$, L-minimum (p,p)-kernels were constructed in [16]. As the first part of our characterization results, in the present section we prove the nonexistence of L-minimum (p,q)-kernels for $p \neq q$.

The crucial statement is the following:

(∗) *If K is an L-minimum (p,q)-kernel then for all $x \in K_0$, $|\operatorname{pre} x| = 2$.*

We first show that this is sufficient for the main result.

Lemma 3.1. (a) *Let K be kernel. Then for all $x \in \min K$, $|\operatorname{suc} x| = 2$, and for all $x \in \max K$, $|\operatorname{pre} x| = 2$.*

(b) *Assume (∗) holds, and let K be any L-minimum (p,q)-kernel. Then $p = q$, and for all $x \in K_0$, $|\operatorname{suc} x| = |\operatorname{pre} x| = 2$.*

Proof. Let $x \in \max K$ be fixed arbitrarily. Clearly, $1 \leqslant |\operatorname{pre} x| \leqslant 2$. Assume, $\operatorname{pre} x = \{y\}$. Then by (K2), $\underline{y} \cap \min K = \underline{x} \cap \min K = \min K$, i.e. $\min K \subseteq \underline{y}$ which contradicts (K3) because $y \notin \max K$. Hence, $\operatorname{pre} x$ contains at least 2 elements. If $x \in \min K$, the proof of (a) is similar.

To see (b), let us count the edges of the Hasse diagram of K twice. This gives

$$2(|K| - p) = \sum_{x \in K} \deg^- x = \sum_{x \in K} \deg^+ x \leqslant 2(|K| - q) \leqslant 2(|K| - p).$$

Hence, we must have equality, i.e. $p = q$ and $\sum_{x \in K} \deg^+ x = 2(|K| - q)$. Since $\deg^+ x = |\operatorname{suc} x| \leqslant 2$ for all elements, the latter equality implies $|\operatorname{suc} x| = 2$ for all $x \notin \max K$. □

Thus, after having proved (∗), we will have the following result.

Theorem 3.1. *An L-minimum $(p,1)$-kernel exists iff $p = q$.*

3.1. *Proof of* (∗)

We use double induction over q, $q \geqslant 2$, and p, $2 \leqslant p \leqslant q$. For $q = 2$, we know $p = 2$, and it is easy to see that there is only one L-minimum $(2,2)$-kernel the Hasse diagram of which is shown in Fig. 1. Since here $K_0 = \emptyset$, (∗) holds in this case.

For the rest of the proof, let $q \geqslant 3$ be arbitrarily fixed and suppose that (∗) holds for all L-minimum kernels with $2, 3, \ldots,$ or $q - 1$ maximal elements. Then additionally, by Lemma 3.1 it is known that those kernels must have as many minimal as maximal elements, and all indegrees and outdegrees are either 0 or 2. These statements are called the q-induction hypothesis.

In [16] (Theorem 6.2) it was proved that an L-minimum kernel with 2 minimal (resp. maximal) and more than 2 maximal (resp. minimal) elements cannot exist.

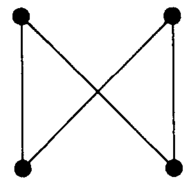


Fig. 1. The L-minimum (2, 2)-kernel.

Therefore, the asserted implication (∗) holds for $p = 2$ and our fixed $q \geqslant 3$. This starts the induction on p. To carry out the induction step, we fix any p, $3 \leqslant p \leqslant q$, and assume that (∗) also holds for all L-minimum kernels with q maximal and $2, 3, \ldots,$ or $p - 1$ minimal elements. This is called the p-induction hypothesis, and by Lemma 3.1 it only means the nonexistence of such kernels.

For the rest of the proof let any L-minimum (p, q)-kernel K be fixed. Suppose there is an element $v \in K_0$ with exactly one immediate predecessor $w \in K$. Then clearly, $\underline{v} = \underline{w} \cup \{v\}$ and $\underline{v} \cap \min K = \underline{w} \cap \min K$. Set $r := |\underline{v} \cap \min K|$, and note that by (K3), $1 \leqslant r \leqslant p - 1$. Hence, $\min K \backslash \underline{v} \neq \emptyset$, and this set generates a nonempty upper ideal, $P := \min K \backslash \underline{v}$.

Proposition 3.1. *P satisfies* (K1) *and* (K2). *Moreover, $P = K \backslash \underline{v}$, and $|P| = 2(p + q - r) - 4$.*

Proof. Obviously, $P \subseteq K$ satisfies (K1). Since (K2) holds for K, every element of $\min P = \min K \backslash \underline{v} \neq \emptyset$ precedes every element of $\max K$. Hence, $\max P = \max K$, and P also satisfies (K2). Consequently, $|P| \geqslant 2(|\min P| + |\max P|) - 4 = 2(p - r + q) - 4$. On the other hand, $P \cap \underline{v} = \emptyset$, i.e. $|P| \leqslant |K| - |\underline{v}|$. By definition of the L-minimality and by Proposition 2.2 we have $|K| = 2(p + q) - 4$ and $|\underline{v}| = |\underline{w}| + 1 \geqslant 2r$, respectively. Therefore, $|P| \leqslant 2p + 2q - 2r - 4$, which forces equality in all estimates. In particular, $|P| = |K| - |\underline{v}|$, that is, P and $\underline{v}$ are complementary in K. □

Now, let us consider the kernel $\ker P$. Because all elements of P, which precede each element of $\max P = \max K$, belong to $\min K$ we know that $\min \ker P = \min K \backslash \underline{v}$. Let $s := |\max \ker P|$.

Proposition 3.2. $\ker P$ *is L-minimum $(p - r, s)$-kernel, and $s \leqslant q$.*

Proof. By Proposition 2.1, $\ker P$ is $(p - r, s)$-kernel, which implies $|\ker P| \geqslant 2(p - r + s) - 4$. On the other hand, in this particular situation Proposition 2.1 yields

$$|\ker P| = |P| - |\overline{\max \ker P}| + |\max \ker P|.$$

We use Proposition 2.2 to get $|\overline{\max\ker P}| \geqslant 2q - s$, and put this and Proposition 3.1 into the above equation:

$$|\ker P| \leqslant 2(p+q-r) - 4 - 2q + s + s = 2(p-r+s) - 4.$$

Hence, $|\ker P| = 2(p-r+s) - 4$, and moreover we must have equality in all estimates. Particularly, $2q - s = |\overline{\max\ker P}| \geqslant |\max\ker P| = s$ which implies $q \geqslant s$. $\square$

If $s = q$ then $p - r < p$ and the p-induction hypothesis applies to $\ker P$. For $p - r \leqslant s < q$, we may apply the q-induction hypothesis to $\ker P$, and finally for $s < p - r$, we apply it to its inverse order $(\ker P)^-$. As shown in the beginning of Section 3, $(*)$ implies $s = p - r \geqslant 2$, and all elements of $\ker P$ except the minimal resp. maximal have exactly two lower resp. upper covers which also belong to $\ker P$. Roughly spoken, in the Hasse diagram $\ker P$ is a part which is connected to the rest of K only from $\max\ker P$ upwards. The main consequence of this is that no element of $\ker P$ has a predecessor not in $\ker P$, i.e. $\ker P$ is the lower ideal generated by $\max\ker P$. Hence, for the upper ideal $Q := \underline{v} \cap \min K$, $Q \cap \ker P = \emptyset$ because Q is generated by elements not belonging to $\ker P$.

Proposition 3.3. *Q satisfies* (K1) *and* (K2). *Moreover,* $Q = K \setminus \ker P$, *and* $|Q| = 2(p+q) - 8$.

Proof. (K1) is obvious. Since $\underline{v} \cap \min K \neq \emptyset$ and (K2) holds for K, $\max Q = \max K$ and Q also satisfies (K2). Therefore, $|Q| \geqslant 2(r+q) - 4$. On the other hand, $|Q| \leqslant |K| - |\ker P| = 2q + 4r - 2p$, and both inequalities together imply $p - r \leqslant 2$. But we already know $p - r \geqslant 2$, i.e. equality holds in all estimates above. In particular, $r = p - 2$, $Q = K \setminus \ker P$, and $|Q| = 2(p+q) - 8$. $\square$

Remark. Note that by $p - r = 2$, $\ker P$ is the unique $(2,2)$-kernel drawn in Fig. 1.

Moreover, $2(p+q) - 8 \geqslant |Q| \geqslant (p-2) + q + 1$ because $\min Q \cup \max Q \subseteq Q$, and $v \in Q$ but v belongs neither to $\max Q$ nor to $\min Q$. Hence, $p + q \geqslant 7$ and $q \geqslant 4$.

As we did before with respect to P, we now continue with considering the corresponding kernel $\ker Q$. Again we know $\min\ker Q = \min Q = \underline{v} \cap \min K$. First we prove a helpful statement.

Proposition 3.4. *For any $x \in Q \setminus \underline{v}$ with $\min Q \subseteq \underline{x}$, $x \in \max K$.*

Proof. By Proposition 3.1, $x \in P$, but by Proposition 3.3, $x \notin \ker P$. Hence, $x \in \overline{\max\ker P}$ which implies $\min\ker P = \min P \subseteq \underline{x}$. Consequently, $\min K = \min P \cup \min Q \subseteq \underline{x}$, and $x \in \max K$ by (K3). $\square$

Proposition 3.5. $\ker Q$ *is L-minimum* $(p-2, q-2)$*-kernel.*

Proof. Because all elements of Q, which precede each element of $\max Q = \max K$, belong to $\min K$, we know that $\min\ker Q = \min Q = \underline{v} \cap \min K$, and $|\min\ker Q| = p-2$.

It remains to determine $\max\ker Q$. By Proposition 3.1, $|\underline{w}| = |\underline{v}| - 1 = 2r - 1$. Hence, in Proposition 2.2 equality holds, and we know that the Hasse diagram of $\underline{w}$ is a binary tree. Therefore, w and v are the only elements of $\underline{v}$ succeeding all elements of $\underline{w} \cap \min K = \min Q$, but v is not minimal with respect to this property since $w < v$. Consequently, $\max\ker Q \cap \underline{v} = \{w\}$.

This implies that $(\max\ker Q\backslash\underline{v}) \cap \bar{w} = \emptyset$. Clearly, every element of $\max K$ succeeds every element of $\min Q$, but by Proposition 3.4, outside of $\underline{v}$ there are no smaller elements with that property. Altogether we have $\max\ker Q\backslash\underline{v} = \max K\backslash\bar{w}$.

Now, let $W := \bar{w}\backslash\{w, v\}$. Since $\underline{v} \cap W = \emptyset$, every element of W satisfies the assumptions of Proposition 3.4, i.e. $W \subseteq \max K$. Consequently, $\max K\backslash\bar{w} = \max K\backslash W$, and W consists of at most 3 elements: two upper covers of v and the remaining upper cover of w.

This finally yields

$$\max\ker Q = (\max K\backslash W) \cup \{w\}, \qquad \ker Q = Q\backslash(W \cup \{v\})$$

and

$$|\max\ker Q| = |\max K| - |W| + 1 = q - |W| + 1,$$

$$|\ker Q| = |Q| - |W| - 1 = 2(p+q) - 9 - |W|.$$

But on the other hand, $|\ker Q| \geqslant 2(q - 2 + |\max\ker Q|) - 4 = 2(2q - |W| - 1) - 4$. Consequently, $|W| \geqslant 2(q-p) + 3 \geqslant 3$, i.e. $|W| = 3$ as well as $|\max\ker Q| = q-2$ and $|\ker Q| = 2(p+q) - 12$. $\square$

As a consequence of Proposition 3.3 we got $q \geqslant 4$, and an element $x \in \max(\ker Q)\backslash\{w\}$ can be chosen. Moreover, by the results of [16], an L-minimum $(1, q-2)$-kernel does not exist. Hence, $p - 2 \geqslant 2$, and we may apply the q-induction hypothesis to $\ker Q$ which, in particular, says that every nonminimal element of $\ker Q$ must have both its lower covers also in $\ker Q$. Consequently, $\underline{x} \subseteq \ker Q$, and $\underline{x} \cap \min K$ contains at most the $p-2$ elements of $\min Q$. By (K2), $x \notin \max K$. But on the other hand, $x \in \max\ker Q\backslash\{w\} \subseteq \max K$. This contradiction implies that — contrary to our assumption — the element $v \in K_0$ cannot have exactly one lower cover in K. Thus, the induction, the proof of $(*)$, and the proof of Theorem 3.1 are complete.

3.2. Consequences

The results of the previous section confirm what was conjectured in [16] already. There are some remarkable consequences for the structure of gossip schemes with as few calls as possible. Note that the following facts also hold for the more general situation of (X, Y)-complete information flows considered in [16]. Here pairwise different items of information each generated in exactly one vertex of a subset X of vertices in a given graph $G = (V, E)$ are all to be conveyed to every vertex of a subset Y of vertices whereby information transmission follows the usual rules of the 'classical' gossip model. As shown in [16] such process requires at least $|X| + |Y| - 4$ calls. Clearly, for $X = Y = V$, the original case is contained. In the following we use terms and notation defined in [16].

Corollary 3.1. *For any (X, Y)-complete information flow on simple graphs which has exactly $|X| + |Y| - 4$ calls, the numbers of irredundant F^-- and F-calls are equal. Moreover, even the sets of F^-- and F-points are equal.*

Proof. The kernel of the minimal order of all calls of the given information flow is L-minimum because the number of calls is as small as possible. The irredundant F^-- or F-calls are its minimal or maximal elements, respectively. Hence, their numbers are equal by Theorem 3.1.

Assume now there is a vertex v being F- but not F^--point, i.e. v participates in a call which is maximal in the kernel, K, of the reduced minimal order, but v does not participate in any of the minimal calls. Hence, the smallest call in K which v takes part in is nonminimal but cannot have two lower covers because there is no smaller call containing v, too. This contradicts $(*)$ or Lemma 3.1(a). Analogously, there is no vertex being F^--point but not F-point. □

The approach of [16] yielded a structural description of (X, Y)-complete gossip schemes. Now we are able to improve this if the number of calls is minimum. Then necessarily the information flow must have the following structure:

Pick $p \geqslant 4$ 'master' vertices in $X \cap Y$ each of which in Phase 1, collects one block of items whereby every item generated by a vertex of X belongs to exactly one block. Phase 2 is a complete gossiping with $2p - 4$ calls among the master vertices, i.e. after Phase 2, all master vertices know all necessary information. Finally, in Phase 3 this is sent to every vertex of Y. The structure of Phases 1 and 3 is 'tree-like' as described in [16] but it was not known that Phase 2, the kernel, must have this very special structure.

Remarks. (1) We obtained that (X, Y)-complete gossiping with $|X| + |Y| - 4$ calls is possible only when $|X \cap Y| \geqslant 4$. This was proved before in [17, 19].

(2) In [11] we tried to maximize the number of calls. The gossip scheme presented there based on the idea to introduce many calls with only one lower cover in the kernel.

Corollary 3.2. *For any (X, Y)-complete information flow on simple graphs which has exactly $|X| + |Y| - 4$ calls, the reduced minimal order and the minimal order coincide.*

Proof. As shown above, in the minimal order no element of the inner kernel is lower or upper cover of an element not belonging to the kernel. Therefore, no relation can be omitted. □

Order-theoretically, this means that already in the minimal order, every saturated maximum chain contains both an irredundant F^-- and an irredundant F-call, i.e. the antichains of these elements are cut-sets.

By Lemma 3.1 we know that in our situation, no edge in the Hasse diagram of the kernel can be omitted without destroying property (K2). This means that here the kernel of the minimal order, or equivalently, the gossip scheme is not only minimal with respect to the calls but also with respect to relations. This does not remain true in general and in the whole minimal order there might well be redundant relations. To omit these was the basic idea which led to the definition of the reduced minimal order in [16].

4. Structure of L-minimum kernels

Because of Theorem 3.1, we may use *p-kernel* instead of L-minimum (p,p)-kernel for the rest of the paper.

Let L-minimum kernels K and K' be called *audiomorphic*, $K \cong_a K'$, iff the corresponding inner kernels K_0 and K'_0 are isomorphic in the usual order-theoretic sense, $K_0 \cong K'_0$. To see the meaning of this notion, let $K \cong_a K'$ but even such that $K \backslash \max K \cong K' \backslash \max K'$. In the underlying information flow, before the final round of the kernel, exactly the same blocks of items were built both in K and K'. Hence, the only difference is how these blocks are put together during the irredundant F-calls. The Hasse diagrams of K and K' can be made identical by interchanging edges leading to maximal elements. We leave this distinction out of account to give an easy classification of all kernels by describing the classes corresponding to the audiomorphism equivalence between them, i.e. describing the inner kernels. Note that $|K| = 4p-4$ and $|K_0| = 2p-4$.

Let any $p \geqslant 2$ and any p-kernel K be fixed. We start with technical statements.

Proposition 4.1. *For any* $x \in K_0$, $|\bar{x} \cap K_0| \geqslant |\bar{x} \cap \max K| - 1$ *and* $|\underline{x} \cap K_0| \geqslant |\underline{x} \cap \min K| - 1$. *For any* $x \in \min K$ *or* $x \in \max K$, $|\bar{x} \cap K_0| \geqslant p-2$ *or* $|\underline{x} \cap K_0| \geqslant p-2$, *respectively.*

Moreover, in case of equality, the Hasse diagram of $\underline{x}$ *or* $\bar{x}$ *is a binary tree rooted in* x.

Proof. The statements immediately follow from Proposition 2.2 by setting $M = \{x\}$. If $x \in \min K$, then additionally note that $\bar{x} \cap \max K = \max K$ and $x \notin K_0$, i.e. $|\bar{x} \cap K_0| \geqslant 2p-1-p-1 = p-2$.

The proof for $x \in \max K$ is analog. □

As mentioned before (see Section 3.1 and Fig. 1) there is a uniquely determined 2-kernel. Hence, let $p \geqslant 3$ in the following. Now we distinguish two types of kernels:

Type 1: None of the maximal elements covers a minimal element.

Type 2: There is a maximal element covering a minimal element.

4.1. *Type 1 kernels*

The Hasse diagram $H(K_0)$ contains $|K_0| = 2p - 4$ vertices and by Lemma 3.1, $2 \cdot |K_0| - 2|\min K| = 2p - 8$ edges because in Type 1 kernels, every edge starting in $\min K$ ends in an element of K_0. Hence, this case is possible only if $p \geqslant 4$, and a component of $H(K_0)$ must be a tree. Therefore, we find an element $v \in K_0$ with $\deg^- v + \deg^+ v \leqslant 1$ in $H(K_0)$. It is extremal in K_0, w.l.o.g. let v be maximal in K_0. In K, v has exactly two lower covers, c_1, c_2, and two upper covers, d_1, d_2, by Lemma 3.1. By the above setting, $\operatorname{suc} v = \{d_1, d_2\} \subset \max K$ and $|\operatorname{pre} v \cap K_0| \leqslant 1$, i.e. w.l.o.g. we may assume $c_1 \in \min K$. Finally, let w be the second upper cover of c_1. Note that $w \in K_0$ because K is of Type 1. Moreover, by the basic property (K3), $\max K \backslash \bar{w} \neq \emptyset$ and w.l.o.g. we may assume $d_1 \notin \bar{w}$. Then $\underline{d}_1 \cap \bar{w} = \emptyset$ and $|\underline{d}_1 \cap K_0| + |\bar{w} \cap K_0| \leqslant |K_0| = 2p - 4$.

Proposition 4.2. $|\bar{w} \cap K_0| = p - 3$.

Proof. From Proposition 4.1 we know $|\bar{w} \cap K_0| \geqslant |\bar{w} \cap \max K| - 1$. But on the other hand, $\max K = \bar{c}_1 \cap \max K = (\bar{v} \cup \bar{w}) \cap \max K = \{d_1, d_2\} \cup (\bar{w} \cap \max K)$. Thus, $|\bar{w} \cap \max K| \geqslant p - 2$ and $|\bar{w} \cap K_0| \geqslant p - 3$. But $|\underline{d}_1 \cap K_0| \geqslant p - 2$ by Proposition 4.1. This yields $|\bar{w} \cap K_0| \leqslant p - 2$. Now the assertion is proved if we can show that $|\bar{w} \cap K_0| < p - 2$.

Assume, $|\bar{w} \cap K_0| = p - 2$. Then $|\underline{d}_1 \cap K_0| = p - 2$ and $(\underline{d}_1 \cap K_0) \cup (\bar{w} \cap K_0) = K_0$. Moreover, for $\underline{d}_1$, in Proposition 4.1 equality holds and we know that $H(\underline{d}_1)$ is a binary tree with all elements of $\min K$ as its leaves. Therefore, any element $c \in \min K$ has its second upper cover neither in $\underline{d}_1$ nor in $\max K$, i.e. in $\bar{w} \cap K_0$. On the other hand, w may have two, but every other element of $\bar{w} \cap K_0$ at most 1 lower cover in $\min K$. Altogether this yields the contradiction $p = |\min K| \leqslant |\bar{w} \cap K_0| + 1 = p - 1$. $\square$

Note that according to the first part of the previous proof, $\bar{w} \cap \max K = \max K \backslash \{d_1, d_2\}$. Therefore, although we prove statements on d_1, they hold for d_2, too.

Proposition 4.3. *For* $i = 1, 2$, $|\underline{d}_i \cap K_0| = p - 2$.

Proof. Clearly, $|\underline{d}_1 \cap K_0| \leqslant 2p - 4 - |\bar{w} \cap K_0| = p - 1$. Assume $|\underline{d}_1 \cap K_0| = p - 1$, i.e. again $(\underline{d}_1 \cap K_0) \cup (\bar{w} \cap K_0) = K_0$. As above, one can see that the elements of $\bar{w} \cap K_0$ together cover of at most $|\bar{w} \cap K_0| + 1 = p - 2$ minimal elements of K. Consequently, there are two elements in $\min K$ covered from above by elements of

$\underline{d}_1 \cap K_0$ only. Hence, in $H(\underline{d}_1)$, $\sum_{x \in \underline{d}_1} \deg^+ x \geqslant |\underline{d}_1 \cap K_0| + |\min K| + 2 = 2p + 1$, but $\sum_{x \in \underline{d}_1} \deg^- x = 2|\underline{d}_1 \cap K_0| + 2 = 2p$. This contradiction completes the proof. □

Corollary 4.1. (a) *For $i = 1, 2$, there is exactly one element $u_i \in K_0$ with $u_i \notin \underline{d}_i \cup \bar{w}$.*
(b) *For any $x \in \bar{w} \cap K_0$, $\operatorname{pre} x \cap \min K \neq \emptyset$, and $\operatorname{pre} w \subset \min K$, $\operatorname{pre} u_i \subset \min K$.*

Proof. (a) Note that $|\underline{d}_1 \cap K_0| + |\bar{w} \cap K_0| = 2p - 5 = |K_0| - 1$.

(b) By Propositions 4.3 and 4.1, $H(\underline{d}_1)$ is tree, and as in the proof of Proposition 4.2 this implies that every element of $\min K$ has one upper cover in $\underline{d}_1$ and the other one in $(\bar{w} \cap K_0) \cup \{u_1\}$. Therefore, u_1 must cover 2, and the elements of $\bar{w} \cap K_0$ must cover $p - 2$ elements of $\min K$. Since $|\bar{w} \cap K_0| = p - 3$, w itself covers two, and every other element in $\bar{w} \cap K_0$ exactly one of them. □

Now we distinguish two cases in each of which finally, we will precisely know the kernel.

Lemma 4.1. *If $\underline{d}_1 \cap K_0 \neq \underline{d}_2 \cap K_0$, then $p = 4$, and K is isomorphic to the poset shown in Fig. 2.*

Proof. Since $\underline{d}_1 \cap \bar{w} = \emptyset$ and v is common lower cover of d_1 and d_2, both the second lower covers of d_1 and d_2 must be different elements $z_1, z_2 \in K_0 \backslash \bar{w}$. From Proposition 4.2, we have $p - 2 = |\underline{d}_i \cap K_0| = |\underline{v} \cap K_0| + |\underline{z}_i \cap K_0|$. Hence, $|\underline{z}_1 \cap K_0| = |\underline{z}_2 \cap K_0|$, i.e. z_1 and z_2 are incomparable. Therefore, $z_2 \in K_0$ but $z_2 \notin \underline{d}_1$ $z_2 \notin \underline{w}$. By Corollary 4.1, $z_2 = u_1$, $z_1 = u_2$, and $\operatorname{pre} z_i \subset \min K$. For $i = 1, 2$, because $H(\underline{d}_i)$ is a tree, $\underline{v} \cap \underline{z}_i = \emptyset$ which together with $\min K \subseteq \underline{d}_i$ implies that $\underline{z}_i \cap \min K = \min K \backslash \underline{v}$. Hence, $\operatorname{pre} z_1 = \operatorname{pre} z_2$ consists of two elements $c_3, c_4 \in \min K$.

Next, let $\underline{d}'_i$ be the remaining upper cover of z_i, $i = 1, 2$. Clearly, $d'_i \neq d_{3-i}$, and $d'_i \neq z_{3-i}$ because z_1 and z_2 are incomparable. Moreover, $d'_i \notin \underline{d}_i$ since otherwise $z_i < d'_i < d_i$ and z_i would not be covered by d_i. Finally, we know that every element of $\bar{w} \cap K_0$ covers at least one minimal element. But $d'_i \in \bar{w} \cap K_0$ implies $d'_i > z_i$, $d'_i > w$, and $\operatorname{pre} d'_i \cap \min K = \emptyset$. Hence, $d'_i \notin \bar{w} \cap K_0$. Altogether, we got $d'_i \in \max K \backslash \{d_1, d_2\}$.

Now, for the minimal element $c_3 \in \min K$, we have $\max K = \bar{c}_3 \cap \max K = (\bar{z}_1 \cup \bar{z}_2) \cap \max K = \{d_1, d'_1, d_2, d'_2\}$. Consequently, $p = 4$. This immediately gives $|K_0| = 4$ and $K_0 = \{v, w, z_1, z_2\}$, as well as $\min K = \{c_1, c_2, c_3, c_4\}$. Then necessarily $c_2 \lessdot w \lessdot d'_i, i = 1, 2$, which finally gives the poset shown in Fig. 2. □

Lemma 4.2. *If $\underline{d}_1 \cap K_0 = \underline{d}_2 \cap K_0$, then $p = 4$, and K is isomorphic the poset shown in Fig. 3 or the inverse of that shown in Fig. 2.*

Proof. Here we find an element $z \neq v$ with $z \lessdot d_i$, $i = 1, 2$. Let us consider $K' := K \backslash \{d_1, d_2, v, z\}$. Let c_3, c_4 be the lower covers of z. For this poset, $\min K' = \min K$ because neither v nor z are minimal in K. Since $H(\underline{d}_1)$ is a tree rooted in d_1, every element of $\underline{d}_1 \cap K_0$ has an upper cover which does not

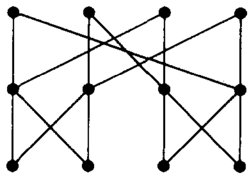


Fig. 2. The 'twisted' 4-FFT-kernel.

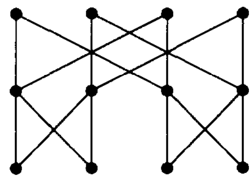


Fig. 3. The 4-FFT-kernel.

belong to $\underline{d}_1$ or $\underline{d}_2$. Hence, every element of $K' \cap K_0$ has an upper cover in K', i.e. $\max K' = \max K \setminus \{d_1, d_2\}$ and $K'_0 = K_0 \setminus \{v, z\}$. Moreover, since $\operatorname{suc} v = \operatorname{suc} z = \{d_1, d_2\}$ even for K', every minimal element precedes every maximal element. Therefore, K' satisfies (K1) and (K2) and we may consider $\ker K'$.

Obviously, $\max \ker K' = \max K'$ which in Proposition 2.1 gives

$$|\ker K'| = |K'| - |\underline{\min \ker K'}| + |\min \ker K'|.$$

Proposition 2.2 yields

$$\begin{aligned} |\underline{\min \ker K'}| &\geqslant 2|\underline{\min \ker K'} \cap \min K'| - |\min \ker K'| \\ &= 2|\min K| - |\min \ker K'|, \end{aligned}$$

and together with $|K'| = |K| - 4 = 4p - 8$ and $|\min K| = p$ in the above equation:

$$|\ker K'| \leqslant 2p - 8 + 2|\min \ker K'|.$$

Since $|\max \ker K'| = |\max K'| = |\max K| - 2 = p - 2$, this means

$$|\ker K'| \leqslant 2(|\max \ker K'| + |\min \ker K'|) - 4.$$

But the converse inequality is known from [16]. Hence, $\ker K'$ is even L-minimum, i.e. $\ker K'$ is $(p-2)$-kernel!

By the results of Section 3 we know that every nonmaximal element of $\ker K'$ must have two upper covers in $\ker K'$. This does not hold for the lower covers of $v, c_{1,2}$, and for those of $z, c_{3,4}$. Consequently for $i = 1,2,3,4, c_i \notin \ker K'$ which means $\bar{c}_i \supseteq \max \ker K' = \max K \setminus \{d_1, d_2\}$. Clearly, $c_i < v < d_{1,2}$ or $c_i < z < d_{1,2}$. Altogether, we got $\max K \subseteq \bar{c}_i$ which implies $c_i \in \min K$ since K satisfies (K3). But then $\min K = \underline{d}_1 \cap \min K = (\underline{v} \cup \underline{z}) \cap \min K = \{c_1, c_2, c_3, c_4\}$. Consequently, $p = 4$, and $|K_0| = 4$. Besides v, z, and w we find another element, u, in K_0, and besides d_1 and d_2 two more elements $d'_1, d'_2 \in \max K$, which necessarily cover w and u.

If $c_2 < w$ and $c_3, c_4 < u$, then K is isomorphic to the poset shown in Fig. 3. Otherwise, let w.l.o.g. $c_3 < w$ and $c_2, c_4 < u$. Then K is the inverse poset to the one shown in Fig. 2. $\square$

The analysis of Type 1 kernels finally yielded only two nonisomorphic 4-kernels which form an audiomorphism class because the inner kernel always is an 4-element antichain. The most interesting kernel is the one shown in Fig. 3. It represents the well-known and often used 'standard' gossip scheme on the 3-dimensional cube with 12 calls and 3 rounds. According to the Hasse diagram of this, we call the kernels of this audiomorphism class 4-FFT-kernels.

4.2. Type 2 kernels

It is easy to check that there is only one 2-kernel which was shown already in Fig. 1. So, let us assume $p \geqslant 3$. In Type 2 kernels we have $c_1 \in \min K$ and $d_1 \in \max K$ with $c_1 \lessdot d_1$. Let v_1 and w_1 be the remaining upper cover of c_1 and lower cover of d_1, respectively. By (K2), $\max K = \bar{c}_1 \cap \max K = \{d_1\} \cup (\bar{v}_1 \cap \max K)$, but by (K3), $\bar{v}_1 \neq \max K$. Hence, $\bar{v}_1 \cap \max K = \max K \backslash \{d_1\}$. This means $|\bar{v}_1 \cap \max K| = p - 1 \geqslant 2$ which implies $v_1 \notin \max K$. Then by Proposition 4.1, $|\bar{v}_1 \cap K_0| \geqslant p - 2$. Analogously, $\underline{w}_1 \cap \min K = \min K \backslash \{c_1\}$, $|\underline{w}_1 \cap \min K| = p - 1$, $w_1 \notin \min K$, and $|\underline{w}_1 \cap K_0| \geqslant p - 2$.

Because $d_1 \notin \bar{v}_1$, v_1 does not precede w_1, i.e. $\bar{v}_1 \cap \underline{w}_1 = \emptyset$. Therefore, $2p - 4 \leqslant |\bar{v}_1 \cap K_0| + |\underline{w}_1 \cap K_0| = |(\bar{v}_1 \cup \underline{w}_1) \cap K_0| \leqslant |K_0| = 2p - 4$, and we know that even equality holds above: $|\bar{v}_1 \cap K_0| = |\underline{w}_1 \cap K_0| = p - 2$ and $K_0 \subset \bar{v}_1 \cup \underline{w}_1$. Moreover, in this case Proposition 4.1 says that $H(\bar{v}_1)$ and $H(\underline{w}_1)$ are binary trees with $p - 1$ leaves each.

Each of the $p - 1$ elements of $\min K \backslash \{c_1\}$ has exactly one upper cover in $\underline{w}_1$ and the second one in $\bar{v}_1$. But on the other hand, any of the $p - 2$ elements in $\bar{v}_1 \cap K_0$ can have at most one lower cover in $\min K \backslash \{c_1\}$. Consequently, we find $c_2 \in \min K \backslash \{c_1\}$ not covered by any element of $\bar{v}_1 \cap K_0$, i.e. there is $d_2 \in \max K \backslash \{d_1\} = \bar{v}_1 \cap \max K$ with $c_2 \lessdot d_2$. Let v_2 and w_2 be the remaining upper cover of c_2 and lower cover of d_2, respectively. Note that the above statements about $\bar{v}_1$ and $\underline{w}_1$ hold analogously for $\bar{v}_2$ and $\underline{w}_2$. For any elements $x, y \in K$, let $[x, y]$ denote the interval $\bar{x} \cap \underline{y}$.

Proposition 4.4. *$[v_1, w_2]$ and $[v_2, w_1]$ are chains of length $p - 2 - |\underline{w}_1 \cap \underline{w}_2 \cap K_0|$.*

Proof. Since $c_2 \in \underline{w}_1$ and $w_1 \in K_0$, there is an upper cover of c_2 belonging to $\underline{w}_1$. But $d_2 \notin \underline{w}_1$, i.e. $v_2 \in \underline{w}_1$. Because $H(\underline{w}_1)$ is a tree, $[v_2, w_1]$ has to be a chain.

Moreover we know $\underline{w}_1 \cap K_0 = \underline{w}_1 \cap (\bar{v}_2 \cup \underline{w}_2) \cap K_0 = (\bar{v}_2 \cap \underline{w}_1 \cap K_0) \cup (\underline{w}_1 \cap \underline{w}_2 \cap K_0)$. Here, $\bar{v}_2 \cap \underline{w}_2 \neq \emptyset$ implies $(\bar{v}_2 \cap \underline{w}_1 \cap K_0) \cap (\underline{w}_1 \cap \underline{w}_2 \cap K_0) = \emptyset$. Therefore, $|[v_2, w_1]| = |\bar{v}_2 \cap \underline{w}_1 \cap K_0| = |\underline{w}_1 \cap K_0| - |\underline{w}_1 \cap \underline{w}_2 \cap K_0|$. By $|\underline{w}_1 \cap K_0| = p - 2$, the assertion follows immediately for $[v_2, w_1]$. The proof for $[v_1, w_2]$ is analog. $\square$

Note that w_1 and w_2 are incomparable because $\min K \subseteq \underline{w}_1 \cup \underline{w}_2$ but neither $\min K \subseteq \underline{w}_1$ nor $\min K \subseteq \underline{w}_2$ by (K3). Analogously, v_1 and v_2 are incomparable.

Proposition 4.5. $\underline{w}_1 \cap \underline{w}_2 \cap K_0 = \emptyset$.

Proof. Assume the contrary, and let v be a maximal element of $\underline{w}_1 \cap \underline{w}_2 \cap K_0$. By (K3), we find an element $d_0 \in \max K$ which does not succeed v. Because for $i = 1, 2$,

$v \leqslant w_i < d_i$, $d_0 \notin \{d_1, d_2\}$. Hence, $d_0 \in \bar{v}_1 \cap \bar{v}_2$, and we may choose a minimal element w among all elements in $\bar{v}_1 \cap \bar{v}_2$ which precede d_0. Clearly, $v \neq w_i$ and $w \neq v_i$ for $i = 1, 2$. Now let elements $c \in \underline{v} \cap \min K$ and $d \in \bar{w} \cap \max K$ as well as $x \in [c, d]$ be fixed arbitrarily. Note that $[c, d] \neq \emptyset$ because $c < d$ by (K2).

Claim 1. *If* $x \in \underline{w}_1 \cup \underline{w}_2$ *then* $x < v$.

Proof. Assume $x \in \underline{w}_1$. Then $c < v < w_1$ and $c \leqslant x \leqslant w_1$, i.e. x and v must be comparable because $H(\underline{w}_1)$ is a tree.

For suppose, $v < x$. Then $x \in \underline{w}_1 \backslash \underline{w}_2 = \underline{w}_1 \cap \bar{v}_2$ since v is maximal in $\underline{w}_1 \cap \underline{w}_2$. If $x < w$ then $v < x < w \leqslant d_0$ in contradiction to our construction. If $w \leqslant x$ then $x \in \bar{v}_1 \cap \underline{w}_1$ but we already know that $\bar{v}_1 \cap \underline{w}_1 = \emptyset$. If, otherwise, x and w are incomparable then $v_2 < x \leqslant d$ and $v_2 < w \leqslant d$ which contradicts the fact that $H(\bar{v}_2)$ is a tree, too. So altogether we have $x \leqslant v$, and analogously, $x \in \underline{w}_2$ implies $x \leqslant v$.

Assume now that $x = v$, and let x' be that upper cover of v also belonging to $[c, d]$. Because $v < w_1$ and $v < w_2$, but v is maximal in $\underline{w}_1 \cap \underline{w}_2$, either $x' \in \underline{w}_1 \backslash \underline{w}_2$ or $x' \in \underline{w}_2 \backslash \underline{w}_1$. Consequently, $v < x'$ and $x' \in \underline{w}_1$ or $x' \in \underline{w}_2$, and application of the above proof ideas to x' instead of x leads to a contradiction. Hence, $x \neq v$, which completes the proof of Claim 1. □

Claim 2. *If* $x \in \bar{v}_1 \cap \bar{v}_2$ *then* $x > w$.

Proof. Because $v_2 < x \leqslant d$ and $v_2 < w \leqslant d$, x and w must be comparable to ensure that $[v_2, d]$ is a chain in the tree $H(\bar{v}_2)$. But $x < w \leqslant d_0$ would contradict the minimality of w in $\bar{v}_1 \cap \bar{v}_2 \cap \underline{d}_0$. Hence, $w \leqslant x$.

If $x = w$, then again due to the minimality of w, the lower cover x' of x which also belongs to $[c, d]$ is in either $\bar{v}_1 \backslash \bar{v}_2 \subset \underline{w}_2$ or $\bar{v}_2 \backslash \bar{v}_1 \subset \underline{w}_1$. In contradiction, Claim 1 implies $x' < v \in \underline{w}_1 \cap \underline{w}_2$, i.e. $x' \notin \bar{v}_1 \cup \bar{v}_2$. Therefore, necessarily $x > w$, and the proof of Claim 2 is complete. □

In the above, let $x = d = d_0$. Then from Claim 2 we know that $d_0 > w$, i.e. $w \notin \max K$ and $|\bar{w} \cap \max K| \geqslant 2$.

Now, we finally consider the poset K' the Hasse diagram of which is the subgraph of $H(K)$ on the elements belonging to $(\underline{v} \backslash \{v\}) \cup (\bar{w} \backslash \{w\})$. Note that in general this approach does not yield the induced poset. Because for $i = 1, 2$, $K_0 \subseteq \underline{w}_i \cup \bar{v}_i$, Claims 1 and 2 mean that chains from any element of $\min K' = \underline{v} \cap \min K$ to any element of $\max K' = \bar{w} \cap \max K$ completely belong to K'. Hence, K' satisfies (K2), and, obviously, (K1).

Because $H(\underline{v})$ and $H(\bar{w})$ are binary trees with $|\underline{v} \cap \min K|$ resp. $|\bar{w} \cap \max K|$ leaves, $|K'| = |\underline{v}| + |\bar{w}| - 2 = 2|\underline{v} \cap \min K| + 2|\bar{w} \cap \max K| - 4 = 2|\min K'| + 2|\max K'| - 4$, i.e. K' even is of minimum size with respect to (K1) and (K2).

We consider $\ker K'$. Application of Propositions 2.1 and 2.2, and $\underline{\min \ker K'} \cap \min K = \min K'$, $\overline{\max \ker K'} \cap \max K = \max K'$ yield $|\ker K'| \leqslant |K'| - (2(|\min K'| + |\max K'|) + 2(|\min \ker K'| + |\max \ker K'|)$. Furthermore, we obtain $|\ker K'| \leqslant 2(|\min \ker K'| + |\max \ker K'|) - 4$, by putting in the above equation for $|K'|$. This means, equality must hold and $\ker K'$ is even L-minimum, too.

Because $|\bar{w}\backslash\{w\}| = 2|\max K'| - 2$, no element of $\bar{w}\backslash\{w\}$ precedes each element of $\max K$. Hence, $\min \ker K' \subseteq \underline{v}\backslash\{v\}$. But these elements have exactly one upper cover in K' which contradicts our basic result about L-minimum kernels in Section 3. Hence, the assumption on the existence of v is false. □

Lemma 4.3. *In every p-kernel K of Type* 2, K_0 *is the union of two disjoint chains of length* $p-2$. *K is audiomorphic to the poset shown in Fig.* 4.

Proof. For $p \geqslant 3$, by Propositions 4.4 and 4.5, two disjoint chains of length $p-2$ belong to K_0, and there are no more elements because $|K_0| = 2p-4$. Obviously, the case $p = 2$ discussed earlier, now can be formally included.

It can easily be checked that the poset shown in Fig. 4 indeed is a p-kernel of this type. □

The analysis of Type 2 kernels yielded one audiomorphism class for each $p \geqslant 2$. Due to the pattern of the most regular example, we call these *p-grid*-kernels. Actually, the NOHO gossip schemes studied in [22] already do have such kernels, and we used a certain generalization in [16].

For later purposes, we add a conclusion concerning the structure of p-grid-kernels. The notation of the proof of Proposition 4.4 is used.

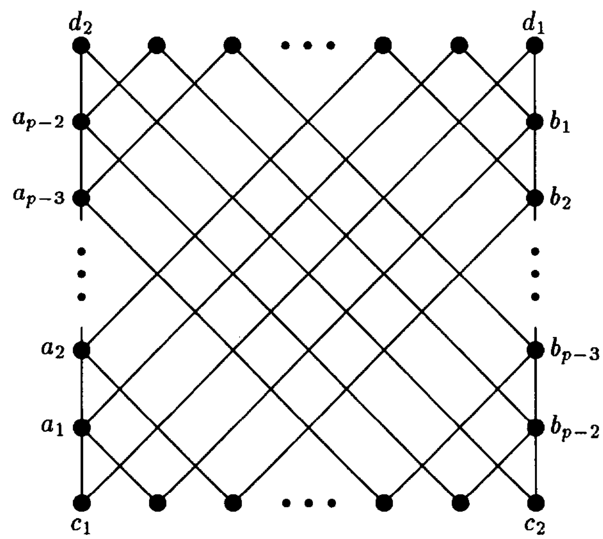


Fig. 4. The standard p-grid-kernel.

Corollary 4.2. *For every $c \in \min K \backslash \{c_1, c_2\}$ and every $d \in \max K \backslash \{d_1, d_2\}$,* $\operatorname{suc} c \subseteq K_0$, *and* $\operatorname{pre} d \subseteq K_0$, *respectively. Furthermore, each of both chains in K_0 contains exactly one upper cover c resp. lower cover of d.*

Proof. For $i = 1, 2$ it holds: Because c is incomparable with c_i but $c < d_i$ by (K2), we have $c < w_i$. Consequently, $\operatorname{suc} c \cap \underline{w}_i \neq \emptyset$ and $\operatorname{suc} c$ consists of one element of each chain $\underline{w}_i \backslash \min K$. The proof for the maximal elements is analog. $\square$

This enables us to give an easy description of p-grid-kernels: Let the elements of K_0 be denoted by a_i resp. b_i $(1 \leqslant i, j \leqslant p-2)$ such that $c_1 \lessdot v_1 = a_1 \lessdot a_2 \lessdot \cdots \lessdot a_{p-2} = w_2 \lessdot d_2$ and $c_2 \lessdot v_2 = b_{p-2} \lessdot \cdots \lessdot b_2 \lessdot b_1 = w_1 \lessdot d_1$. (see Fig. 4) For every a_i, we find a unique $c \in \min K \backslash \{c_1, c_2\}$ with $c \lessdot a_i$, and by Corollary 4.2, a unique b_j with $c \lessdot b_j$. Thus, a permutation ϱ_K on $\{1, \ldots, p-2\}$ is well-defined by $\varrho_K(i) = j$ iff a_i and b_j share a common lower cover. Analogously, let another permutation σ_K be defined by $\sigma_K(i) = j$ iff a_i and b_j have a common upper cover. The following property of those permutations will be used frequently.

Proposition 4.6. *For $i = 1, 2, \ldots, p-2$, $\varrho_K(i) \geqslant i-1$ and $\sigma_K(i) \leqslant i+1$.*

Proof. Let $c \in \min K$ be the uniquely determined common lower cover of a_i and $b_{\varrho_K(i)}$. Because $\max K \subseteq \bar{c} = \bar{a}_i \cup \bar{b}_{\varrho_K(i)}$, we have $p \leqslant |\bar{a}_i \cap \max K| + |\bar{b}_{\varrho_K(i)} \cap \max K| = p - i + \varrho_K(i) + 1$, and $\varrho_K(i) \geqslant i - 1$. The proof for σ_K is analog. $\square$

Clearly, any p-grid-kernel is characterized by a pair of permutations. For $\varrho_K = \sigma_K = id$, we get the 'standard' p-grid-kernel shown in Fig. 4. All possible pairs will be given in Section 6. Note finally that throughout the rest of the paper, we will always use the above notation for the elements of a p-grid-kernel.

5. Applications of the structure results

In the following we present some results on gossiping with minimum number of calls proved basing on our knowledge about the structure of L-minimum kernels.

5.1. The four-cycle theorem

In [1] it was shown that for $n \geqslant 4$, a graph on n vertices allows to gossip with the minimum number of calls, $2n-4$, iff it contains a 4-cycle. While it is easy to see that the existence of a 4-cycle is sufficient, necessity was a longstanding conjecture, see e.g. [8]. The paper [12] which actually was at the beginning of our investigations of the

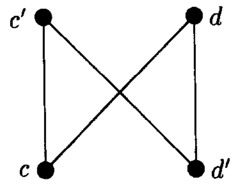


Fig. 5. The 2 kernel.

order background of gossiping contains a shorter proof of the above fact. Now we are able to conclude it even easier from the kernel structure.

Lemma 5.1. *Every p-kernel contains* 4 *calls between exactly* 4 *different vertices of the underlying graph which induce the* 2-*kernel as a subordering.*

Proof. We denote the calls of a 2-kernel as shown in Fig. 5. Since the calls c and d' are incomparable they have no participant in common, i.e. we may assume that vertices 1 and 2 take part in c while vertices 3 and 4 take part in d'. By the definition of the minimal order, c' and d must share a common participant with each of c and d'. Hence, only 1, 2, 3, or 4 are involved in c' and d. This proves our assertion for 2-kernels itself as well as for 4-FFT-kernels because those contain the structure of Fig. 5 in their Hasse diagram.

Let us consider any given p-grid-kernel K, and let $c \in \min K$ and $d \in \max K$ be such that $c < d$. Then by definition and Lemma 3.1, c and d share precisely one common participant. So, w.l.o.g. let 0 and 1 or 1 and 2 be the vertices taking part in c or d, respectively. By Corollary 3.1, there is $c' \in \max K$ which involves 0. Let vertex α be the second participant of c'. Similarly, let d' be the minimal call involving vertex 2, and β be the remaining participant of this call. By definition of the minimal ordering, $c < c'$ and $d' < d$, and c and d' as well as c' and d are incomparable. Furthermore, $d' < c'$ by (K2).

If even c' covers d', then $\alpha = \beta$, i.e. c, c', d, d' use the 4 vertices $0, 1, 2$, and α and induce a 2-kernel. Otherwise, d' has an upper cover, v, preceding c'. By Corollary 4.2 this belongs to one of the two chains $\bar{c} \cap K_0$ or $\underline{d} \cap K_0$, for suppose, $v \in \bar{c} \cap K_0$. Then $v \notin \underline{d} \cap K_0$, i.e. v and d are incomparable and vertex 2 is no participant of v. Consequently, d' and v share β as a common participant. Since $v < c'$, v belongs to the uniquely determined saturated chain from c to c' in the tree $H(\bar{c})$. On the other hand, c and c' are connected by the chain of all calls in which vertex 0 takes part in. Thus, 0 has to be the remaining participant of v. Altogether, c, v, d, d' involve vertices $0, 1, 2, \beta$ and induce a 2-kernel. If $v \in \underline{d} \cap K_0$, vertices 2 and α take part in v and our assertion is proved similarly. □

It can very easily be checked that in any of the above situations the 4 edges along which information is passed during the 4 specified calls form a 4-cycle. This completes our proof of the four-cycle theorem.

5.2. Multigraph gossiping

This section deals with a slightly modified model for gossiping (see [3] for a survey): Instead eventually assigning several numbers to an edge of a simple graph, we allow multiple edges but each edge can be used at most once. This is just an ordinary edge-labeling of a multigraph, and an important question to answer is which multigraphs can be labeled such that the labeling represents a complete gossip scheme. The remarkable results is the following.

Theorem 5.1 (Burosch et al. [4]). *A multigraph $G=(V,E)$ with $|V|=n\geqslant 4$ permits gossiping if and only if it is of one of the following types:*

(1) *G contains the union of two spanning trees which have at most one common edge.*

(2) *G contains the union of three pairwise edge-disjoint subgraphs: a cycle C of length at least 4, and two spanning forests F_1, F_2 each consisting of exactly 4 components whereby for $i=1,2$, every component of F_i shares a common vertex with C.*

It is easy to see that gossiping on multigraphs of the first type requires $2n-3$ calls. Here the kernel of the corresponding minimal order is just a single call, see [16] for a discussion of this situation. Therefore, it is interesting to characterize those multigraphs which allow to gossip in the minimum number of calls, $2n-4$. In view of the above and the four-cycle theorem, the following result of [7] does not surprise. It can also be concluded from [4].

Theorem 5.2 (Göbel et al. [7]). *A multigraph $G=(V,E)$ with $|V|=n\geqslant 4$ permits gossiping within $2n-4$ calls if and only if it contains the union of three pairwise edge-disjoint subgraphs: a cycle C of length exactly 4, and two spanning forest F_1, F_2 each consisting of exactly 4 components whereby for $i=1,2$, every component of F_i shares a common vertex with C.*

In the following, we discuss a proof of Theorem 5.2. Sufficiency is easy to see by the old standard idea: Use the edges of F_1 to bring all items to one of the vertices of C; this requires $n-4$ calls. After carrying out two rounds each of two parallel calls along the edges of C, every vertex in C knows everything. Finally, use the edges of F_2 to bring the collected block of all items to any other vertex; this again requires $n-4$ calls. Note that this procedure uses the 2-kernel.

The hard part is to prove necessity. From [16] we know that we may restrict ourselves to consider the kernel because all calls before resp. after it always form tree-components, i.e. forests, attached to a vertex which participates in a call of the kernel. So what really remains to be proved is the following.

Lemma 5.2. *Let K be an arbitrarily fixed p-kernel. There are four calls along the edges of a 4-cycle C, and the remaining calls generate two disjoint forests F_1, F_2 which consist of four components each intersecting C in a vertex.*

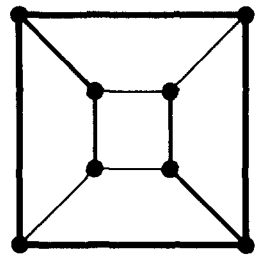


Fig. 6.

Proof. For K being a 4-FFT-kernel, the underlying gossip scheme is for $2p = 8$ vertices within 3 rounds and 12 calls. It is known [13] that this can be achieved on the 3-dimensional cube or on the 'twisted cube' only (see Fig. 7). In Fig. 6 we show an appropriate decomposition for the cube which can be carried over to the twisted cube easily. Now let K be a p-grid-kernel. We continue the proof of Lemma 5.1 with the same notation. It was proved there that we always find an induced 2-kernel involving c and d and one call, x, of $\bar{c}\backslash\{c,d\}$ (v or c') and one call, y, of $\underline{d}\backslash\{c,d\}$ (v or d'). Let C be the 4-cycle generated by these calls.

Let us consider $\bar{c}$ resp. $\underline{d}$. Because $\max K \subseteq \bar{c}$ and $\min K \subseteq \underline{d}$, both involve all vertices. Hence, the calls of $\bar{c}$ resp. $\underline{d}$ generate connected graphs on $2p$ vertices with $2p - 1$ edges, i.e. trees T_c and T_d. After deleting c and d and the corresponding edges $\{0,1\}$ and $\{1,2\}$, each of them splits into 3 components and moreover, the arising forests F_c and F_d now are edge-disjoint.

Finally, we consider F_c. The vertices 1 and 2 form one component of F_c each because neither 1 nor 2 is involved in any of the calls of $\bar{c}\backslash\{c,d\}$. The third component of F_c must contain the edge with x uses, i.e. deletion of x produces one more component and finally we have a forest F_1. Each of its four components has a common vertex with C. Note that deletion of y does not affect F_1 because $y \notin \bar{c}$.

Analogously, after deleting all edges of C, F_d becomes a forest F_2 with the same properties as F_1 has. Clearly, F_c and F_d are edge-disjoint, too. □

Lemma 5.2 completes our proof of Theorem 5.2. Note that these results also apply to simple graphs: A simple graph allows to gossip such that every edge is used at most once iff it satisfies the condition of Theorem 5.2.

5.3. *Minimum time*

Let any complete information flow on $n \geqslant 4$ vertices with $2n - 4$ calls be given. In the following we investigate how many rounds this gossip scheme must have at least. In [13] we gave estimates to this, in particular, an algorithm was presented how to complete gossiping within $2\lceil \log_2 n \rceil - 3$ rounds. Already [18] asserts that this is best possible but we will discuss at the end of the present section why the proof contained in that paper cannot be accepted.

We use terminology and results of [16], and consider the full minimal order, P, throughout this section. Remember that the number of rounds, T, is at least as large as the length of the longest chain in P. By $\underline{M}_P$ or $\bar{M}^P$ we denote the lower or upper ideal generated by the set M in P.

Theorem 5.3. *Any gossip scheme on $n \geqslant 4$ vertices with $2n-4$ calls has at least $2\lceil \log_2 n \rceil - 3$ rounds.*

Proof. Let K be the kernel of the corresponding minimal order.

Case 1. K *is* 4-FFT-*kernel*: From [16] we know that in P, below $\min K$ there are at least $n-8$ elements, i.e. $|\underline{\min K}_P| \geqslant n-4$. On the other hand, $|\underline{\min K}_P| \leqslant 4\cdot(2^t-1)$ where t denotes the maximum length of an ascending chain in P ending in $\min K$. Hence, $t \geqslant \lceil \log_2 n \rceil - 2$. Analogously, we find a chain of length $\lceil \log_2 n \rceil - 2$ beginning in an element of $\max K$. Altogether, we know that P contains a chain of length at least $2\cdot(\lceil \log_2 n \rceil - 2) + 1 = 2\lceil \log_2 n \rceil - 3$, because one additional element of K_0 is included.

Case 2. K *is p-grid-kernel*: Let the elements of the kernel be denoted as shown in Fig. 4, and for simplicity set $a_0 := c_1$, $a_{p-1} := d_2$, $b_0 := d_1$, $b_{p-1} := c_2$. Moreover, let i be the smallest index for which $|\underline{a_i}_P| \geqslant n/2 - 1$. Clearly, such an index exists because we know from [16] that $|\underline{a_{p-1}}_P| = p - 1 + |\underline{\min K}_P| = p - 1 + n - p = n - 1$. Since $\underline{a_i}_P$ is a binary tree, we find an ascending chain of length at least $\lceil \log(n/2 - 1 + 1) \rceil = \lceil \log_2 n \rceil - 1$ leading to a_i in this lower ideal.

For suppose, $i = 0$. Then analogously, $|\overline{a_0}^P| = n-1$, and there is an ascending chain of length at least $\lceil \log_2 n \rceil$ in this upper ideal which begins in a_0. Consequently, in the whole minimal order we find a chain of length at least $2\lceil \log_2 n \rceil - 2$ passing through a_0, i.e. $T \geqslant 2\lceil \log_2 n \rceil - 2$. So, throughout the rest of this proof, we may assume $i \geqslant 1$.

Now, let j be the largest index for which $\{\varrho_K(i), \varrho_K(i+1), \dots, \varrho_K(p-2)\} \subseteq \{j, j+1, \dots, p-2\}$. Because the latter set must contain at least $p-i-1$ elements, we have $j \leqslant i$. On the other hand, Proposition 4.6 implies $j \geqslant i-1$. By the definition of j, $\min K \subseteq \underline{a_{i-1}}_P \cup \underline{b_j}_P$. Consequently, $|\underline{a_{i-1}}_P \cup \underline{b_j}_P| = (i-1) + (p-j-1) + |\underline{\min K}_P| = (p+i-j-2) + n - p \geqslant n-2$. But by the definition of i, $|\underline{a_{i-1}}_P| < n/2 - 1$, i.e. $|\underline{b_j}_P| > n/2 - 1$, and we find an ascending chain up to b_j with length at least $\lceil \log_2 n \rceil - 1$ in $\underline{b_j}_P$, too.

By Proposition 4.6, $\varrho_K^{-1}(j) \leqslant j+1 \leqslant i+1$. If $\varrho_K^{-1}(j) < i$, then already $\{j+1, \dots, p-2\}$ would contain all of $\varrho_K(i), \dots, \varrho_K(p-2)$ which contradicts the maximality of j. Hence, $i \leqslant \varrho_K^{-1}(j) \leqslant i+1$.

If $\varrho_K^{-1}(j) = i$, then a_i and b_j have a common lower cover in $\min K$ and necessarily $\max K \subseteq \overline{a_i}^P \cup \overline{b_j}^P$. If, otherwise, $\varrho_K^{-1}(j) = i+1$, then analogously $\max K \subseteq \overline{a_{i+1}}^P \cup \overline{b_j}^P$. Because $\overline{a_{i+1}}^P \subset \overline{a_i}^P$, we always have $\max K \subseteq \overline{a_i}^P \cup \overline{b_j}^P$ and $|\overline{a_i}^P \cup \overline{b_j}^P| = (p-i-1) + j + |\overline{\max K}^P| = p + j - i - 1 + n - p \geqslant n-2$ by $j \geqslant i-1$. Therefore, $\overline{a_i}^P$ or $\overline{b_j}^P$ must contain at least $n/2 - 1$ elements which implies that in one of these upper ideals, there is an ascending chain of length at least $\lceil \log_2 n \rceil - 1$ starting in a_i or b_j, respectively. So altogether, through one of these elements it passes a chain of length at least $2(\lceil \log_2 n \rceil - 1) - 1 = 2\lceil \log_2 n \rceil - 3$. □

For any of the above cases, there indeed are gossip schemes achieving all required lower bounds. In the following drawings we present both the minimal order (left) and the graph (right). In the minimal ordering, calls are marked by the vertices involved in, while in the graph, edges are marked by the round the incident vertices make a call in.

Fig. 7 shows the 'twisted cube' (see [22]) and the well-known gossiping in $L = 2n - 4 = 12$ calls and $T = 2\lceil \log_2 n \rceil - 3 = 3$ rounds. This has a 4-FFT-kernel.

Fig. 8 shows the situation in the NOHO-gossiping for $n = 10$, $L = 2n - 4 = 16$, and $T = 2\lceil \log_2 n \rceil - 3 = 5$ which was constructed firstly in [22]. In our context, it turns out that the standard 5-grid-kernel is the corresponding kernel.

The above examples can be extended to larger values of n by attaching trees to the vertices used in the kernel. So in our first example, extending each of the 8 vertices to a minimal broadcast tree of size $\lfloor n/8 \rfloor$ or $\lceil n/8 \rceil$ will easily produce a gossiping on

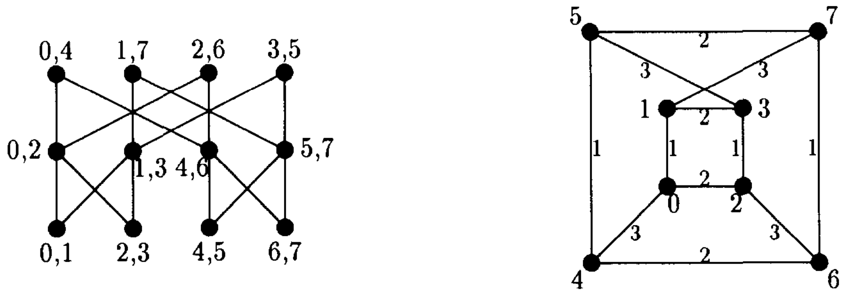


Fig. 7.

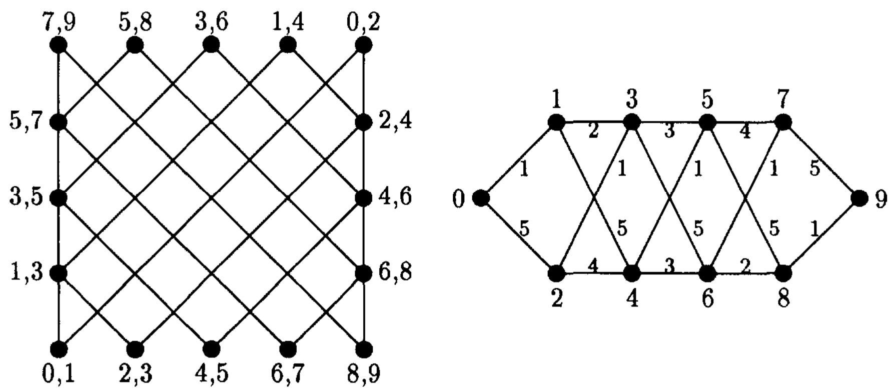


Fig. 8.

n vertices with $2n-4$ calls and $2\lceil \log_2 n \rceil - 3$ rounds. This construction is contained in [18], too. But for some values of n, the same can be reached by a suitable extension of 5- or 3-grid kernels. Both are shown in Fig. 9 for $n = 12$ or $n = 24$. Note firstly, that on the left-hand side, now the minimal order contains more calls than only the kernel. We rearranged the placement slightly such that calls belonging to the same round are on the same level. Calls outside the kernel are marked by $\circ$.

Moreover note that from the last example for every $n \leqslant \frac{3}{4} \cdot 2^{\lceil \log_2 n \rceil}$, one can get an information flow with $2n-4$ calls and $2\lceil \log_2 n \rceil - 3$ rounds containing the 3-grid-kernel. Combining the ideas of both examples yields the same with a p-grid-kernel where p is odd and $p \leqslant 2\lceil \log_2 n \rceil - 3$.

Remark. The key lemma of [18] asserts that any graph on n vertices that allows to gossip with $2n-4$ calls and $2\lceil \log_2 n \rceil - 3$ rounds must contain the 3-dimensional cube with minimal broadcast trees of size $\lfloor n/8 \rfloor$ or $\lceil n/8 \rceil$ attached to each of its vertices. As our examples show this is only one special case out of many essentially different possibilities. Besides the trivial counterexample of the 'twisted cube' which is

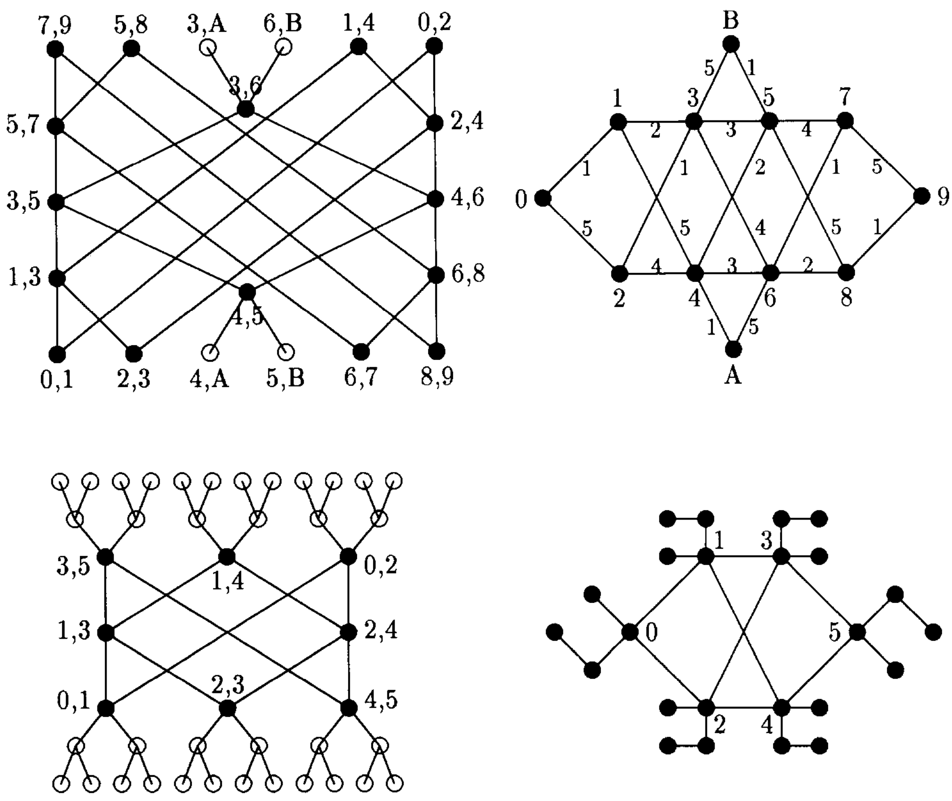


Fig. 9.

nonisomorphic to the cube, all gossip schemes containing grid-kernels are of completely different structure. From the point of view of minimum number of calls, [18] only covers the one exceptional 'sporadic' type, 4-FFT-kernel, while the 'regular' type, p-grid-kernel, does not appear at all. Hence, the bound on the number of rounds has not really been proved there.

5.4. *Minimum size graphs for kernels*

As the examples in the previous section show, there are several graphs which we can construct a gossip scheme on that has a p-kernel as minimal order. For fixed p, we are interested in those with minimum number of edges among them. Clearly, this number cannot exceed the number of calls, $4p-4$. On the other hand, it might well be possible to use certain edges more than once which could result in smaller graphs.

The 4-FFT-kernels represent gossip schemes on 8 vertices within 3 rounds. It is well-known [13, 15] that in this case, 12 edges are required. Moreover, the 3-dimensional cube and the 'twisted cube' are the only 8-mgg's, i.e. graphs of minimum size that allow gossiping between 8 vertices in minimum number of rounds. In our context, this result can also be verified easily by checking all appropriate assignments of edges to the Hasse diagrams shown in Figs. 2 and 3. Note that no edge may be used twice, and consequently, 4-FFT-kernels are the worst realization for minimizing the graph size.

Next, we investigate p-grid-kernels. Let a graph $G=(V,E)$ and an information flow with minimal order isomorphic to a p-grid-kernel K be given. Then clearly, $n=|V|=2p$ and $L=2n-4$.

Lemma 5.3. (a) $|E| \geqslant 3p-2$.
(b) *If* $|E|=3p-2$, *then* K *is the standard* p*-grid-kernel.*

Proof. (a) Let c and d be calls of round t_1 and t_2, respectively, which use the same edge $\{u,v\} \in E$, and $t_1 < t_2$ w.l.o.g. In the p-grid-kernel, there are two different ascending chains from c to d, namely those consisting of all calls including u and all calls including v between rounds t_1 and t_2. Thus, in the Hasse diagram we must have a cycle through c and d.

Now, let the elements of K be denoted as in Section 4 (see Fig. 4). We know that $H(\bar{c}_i)$ and $H(\underline{d}_i)$ all are trees, i.e. there is no cycle in any of them. Consequently, $c \notin \bar{c}_i$ and $d \notin \underline{d}_i$ $(i=1,2)$. But then, necessarily $c \in \min K \setminus \{c_1, c_2\}$ and $d \in \max K \setminus \{d_1, d_2\}$. This implies that any edge can be used at most twice, that is in the first and last round, and that there are at most $|\min K \setminus \{c_1, c_2\}| = p-2$ of such edges. Hence, $|E| \geqslant L-(p-2) = 3p-2$.

(b) With the above notation, let a_i and b_j be the upper covers of c. Then $\bar{c} = \bar{a}_i \cup \bar{b}_j \cup \{c\}$, and d must belong to both $\bar{a}_i$ and $\bar{b}_j$. Because $\max K \subseteq \max \bar{a}_i \cup \max \bar{b}_j$, this means $|\max \bar{a}_i| + |\max \bar{b}_j| \geqslant |\max \bar{a}_i \cup \max \bar{b}_j| + 1 \geqslant p+1$. Since $|\max \bar{a}_i| = p-i$ and $|\max \bar{b}_j| = j+1$, this amounts to $p-i+j+1 \geqslant p+1$ or $j \geqslant i$.

If $|E| = 3p - 2$ this must hold for every of the $p - 2$ elements in $\min K \backslash \{c_1, c_2\}$, i.e. the permutation ϱ_K which describes the lower part of K is such that for any $i \in \{1, \dots, p-2\}$, $\varrho_K(i) \geqslant i$. Hence, ϱ_K — and analogously σ_K for the upper part — leaves every i fixed. As mentioned at the end of Section 4, these permutations produce the standard p-grid-kernel. □

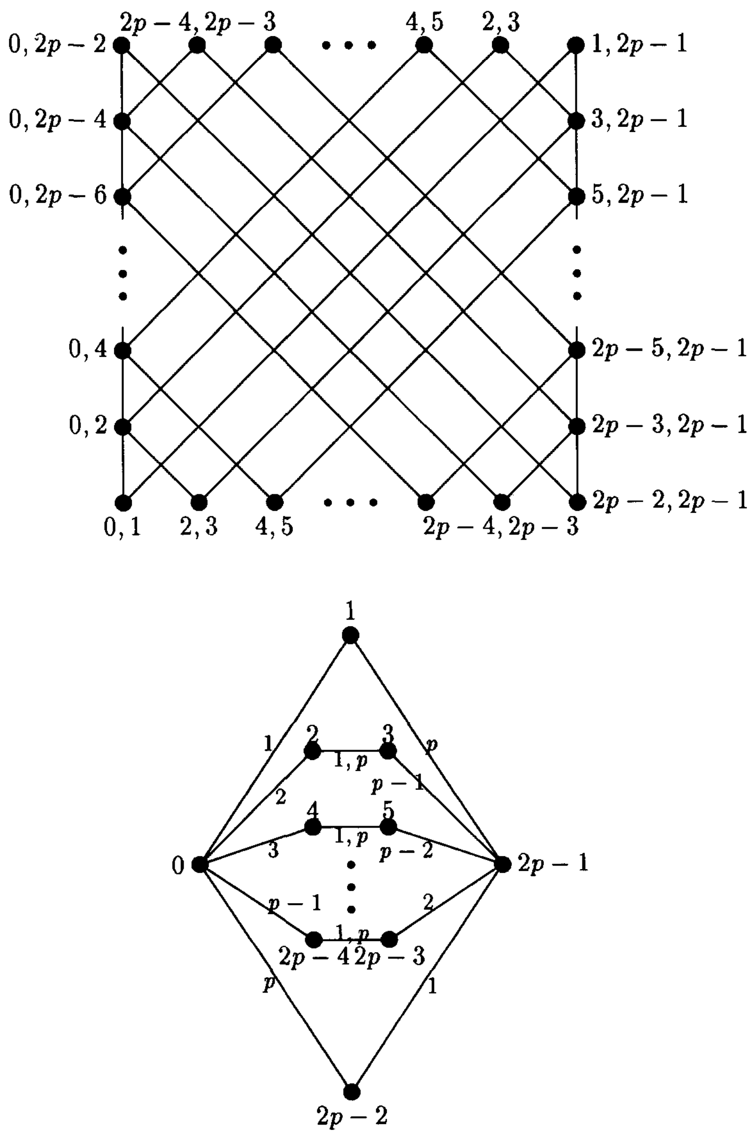


Fig. 10. Minimum size graphs for p-grid-kernels.

Now, up to isomorphism, it is uniquely determined which edges the calls of K must take place: w.l.o.g. we start with calls on $\{2i, 2i+1\}$ for $i = 0, \ldots, p-1$ in round one, in particular, let c_1 be on $\{0, 1\}$ and c_2 on $\{2p-2, 2p-1\}$. Moreover, let w.l.o.g. 1 or $2p-2$ be the common participant of c_1 and d_1 or c_2 and d_2, respectively, and for $i = 1, \ldots, p-2$, let $2i$ participate in a_i while $2i+1$ participates in b_i. Then necessarily 0 or $2p-1$ is the remaining participant of all a_i and d_2 or b_i and d_1, respectively. The other maximal calls are along $\{2i, 2i+1\}$ for $i = p-2, \ldots, 1$ again. Fig. 10 shows both the kernel where each call is labeled by the participating vertices, and the arising graph.

6. p-grid-kernels

Throughout this section, let any $p \geqslant 3$ be fixed. At the end (see Section 6.4), we illustrate all investigations for $p = 5$. By the results of Section 4, a p-grid-kernel K is a poset satisfying the conditions (K1)–(K3) with p minimal and p maximal elements, and such that its inner kernel K_0 consists of two chains of length $p-2$. Let $a_1 \lessdot a_2 \lessdot \cdots \lessdot a_{p-2}$ and $b_{p-2} \lessdot b_{p-3} \lessdot \cdots \lessdot b_1$ be those chains. Then by Corollary 4.2 and Proposition 4.6, K is determined by two permutations $\varrho = \varrho_K$ and $\sigma = \sigma_K$ describing which a_i and b_j share a common lower or upper cover, respectively. These permutations satisfy for $i = 1, \ldots, p-2$:

(P1) $\varrho(i) \geqslant i-1$.

(P2) $\sigma(i) \leqslant i+1$.

Conversely, given two permutations ϱ and σ, a poset of the discussed pattern is well determined: Take two chains $c_0 \lessdot a_1 \lessdot a_2 \lessdot \cdots \lessdot a_{p-2} \lessdot d_{p-1}$ and $c_{p-1} \lessdot b_{p-2} \lessdot b_{p-3} \lessdot \cdots \lessdot b_1 \lessdot d_0$ together with the relations $c_0 \lessdot d_0$ and $c_{p-1} \lessdot d_{p-1}$. Then add $p-2$ more minimal elements c_i, $i = 1, \ldots, p-2$, and let c_i be covered by a_i and $b_{\varrho(i)}$. Analogously, add $p-2$ more maximal elements d_i, $i = 1, \ldots, p-2$, and let d_i cover a_i and $b_{\sigma(i)}$. By this construction, (K1) and (K3) are satisfied, but we obtain a p-grid-kernel only iff (K2) holds, too.

Unfortunately, on the one hand, given any permutation ϱ satisfying (P1) we will always get a p-grid-kernel for $\sigma = id$ and vice versa. But on the other hand, not all possible pairs of permutations ϱ and σ satisfying (P1), (P2) describe a kernel because the important condition (K2) may be violated. Therefore, it remains to determine all pairs (ϱ, σ) corresponding to kernels. Having this we will be able to solve the isomorphism problem and to enumerate p-grid-kernels up to isomorphism.

6.1. Characterization

To every permutation ϱ on $\{1, \ldots, p-2\}$ satisfying (P1), we assign a subset $f(\varrho) \subseteq \{1, \ldots, p-3\}$ by $f(\varrho) := \{i \in \{1, \ldots, p-3\}\colon \varrho(i+1) \geqslant i+1\}$. Note that because of (P1), $\varrho(i+1) \geqslant i+1$ iff $\varrho(\{i+1, \ldots, p-2\}) = \{i+1, \ldots, p-2\}$ iff $\varrho(\{1, \ldots, i\}) = \{1, \ldots, i\}$ whereby as usual $\varrho(X) := \{\varrho(i)\colon i \in X\}$ for any

$X \subseteq \{1, \ldots, p-2\}$. Hence, in $f(\varrho)$ we have all elements $i \neq p-2$ for which ϱ sends $\{1, \ldots, i\}$ to itself.

Lemma 6.1. *f is a bijective mapping from the set $\mathscr{P}_{p-2}$ of all permutations on $\{1, \ldots, p-2\}$ satisfying* (P1) *onto the set $\mathscr{B}_{p-3}$ of all subsets of $\{1, \ldots, p-3\}$.*

Proof. Let $X = \{x_1, \ldots, x_m\} \subseteq \{1, \ldots, p-3\}$ be chosen arbitrarily, and $0 =: x_0 < x_1 < x_2 < \cdots < x_m < x_{m+1} := p-2$. The product of cycles $\varrho = \varrho(X) := \prod_{j=1}^{m+1} (x_j x_j - 1 \cdots x_{j-1} + 1)$ belongs to $\mathscr{P}_{p-2}$ and it holds $f(\varrho) = \{x_1, \ldots, x_m\} = X$. Hence, f is surjective, and it remains to prove that the preimage $\varrho(X)$ is always uniquely determined.

For some $\varrho \in \mathscr{P}_{p-2}$, assume $f(\varrho) = X$, and let any $i \in \{1, \ldots, p-2\}$ be fixed. Moreover, let j be the index with $x_{j-1} < i \leqslant x_j$. For $i = x_{j-1} + 2, \ldots, x_j$, it holds $\varrho(i) = i - 1$, because $i - 1 \notin f(\varrho)$, i.e. $\varrho(i) < i$, and (P1). But $\varrho(\{x_{j-1} + 1, \ldots, x_j\}) = \{x_{j-1} + 1, \ldots, x_j\}$ because $\{1, \ldots, x_{j-1}\}$ and $\{1, \ldots, x_j\}$ are mapped onto itself by ϱ. Hence, $\varrho(x_{j-1} + 1) = x_j$, and the restriction of ϱ to $\{x_{j-1} + 1, \ldots, x_j\}$ is the cycle $(x_j x_j - 1 \cdots x_{j-1} + 1)$. This immediately gives $\varrho = \varrho(X)$. $\square$

Similarly, to any permutation σ satisfying (P2), we assign the set $f(\sigma^{-1}) \in \mathscr{B}_{p-3}$. Note that this is a bijection, too, because $\sigma^{-1} \in \mathscr{P}_{p-2}$. Now, we are able to give the characterization of all admissible pairs of permutations.

Theorem 6.1. *Permutations ϱ and σ satisfying* (P1) *and* (P2) *determine a p-grid-kernel iff $f(\varrho) \cup f(\sigma^{-1}) = \{1, \ldots, p-3\}$.*

Proof. We have to investigate those permutations for which the corresponding poset satisfies (K2). Due to the construction, (K2) holds iff for $i, j = 1, \ldots, p-2$, $c_i < d_j$ (1). But by definition, $c_i \prec a_i \leqslant a_j \prec d_j$ iff $i \leqslant j$, and this holds independently from ϱ and σ. Therefore, instead of (1), we may equivalently require that for $j \leqslant i - 1$, $c_i \prec b_{\varrho(i)} \leqslant b_{\sigma(j)} \prec d_j$ or $\varrho(i) \geqslant \sigma(j)$ (2). Since $\varrho(i) \geqslant i - 1$ and $\sigma(j) \leqslant j + 1 \leqslant i$ by (P1), (P2), (2) is violated only iff for some $i \in \{1, \ldots, p-2\}$, $\varrho(i) = i - 1$ and $\sigma(i-1) = i$, or $\varrho(i) = \sigma^{-1}(i) = i - 1$. Hence, (2) is equivalent to the condition that for any $i = 1, \ldots, p-2$, $\varrho(i) = i - 1$ implies $\sigma^{-1}(i) \geqslant i$, i.e. for $i = 2, \ldots, p-2$, $i - 1 \notin f(\varrho) \Rightarrow i - 1 \in f(\sigma^{-1})$ which immediately implies the assertion. $\square$

Theorem 6.1 gives a convenient possibility for handling p-grid-kernels in terms of subsets of $\{1, \ldots, p-3\}$, that is in the Boolean lattice $\mathscr{B}_{p-3}$. In particular, we now can easily check whether a given pair of permutations determines a p-grid-kernel.

In the following, we use that any p-grid-kernel K can be represented by a uniquely determined pair of sets $X = X_K$, $Y = Y_K \in \mathscr{B}_{p-3}$ with $X \cup Y = \{1, \ldots, p-3\}$, or a uniquely determined pair of permutations $\varrho = \varrho_K$, $\sigma = \sigma_K$ with $\varrho, \sigma^{-1} \in \mathscr{P}_{p-2}$ and $f(\varrho) = X$, $f(\sigma^{-1}) = Y$.

6.2. Isomorphism

Two kernels K, K' are called *isomorphic* iff the poset K' is order-isomorphic to either the poset K or its inverse order K^-. The latter case is included in our definition, because K^- obviously belongs to the inverse gossip scheme which is not essentially different from the original one. For different pairs of permutations and sets, the corresponding p-grid-kernels eventually are isomorphic. Therefore, we should determine all possible isomorphisms between p-grid-kernels and characterize all the isomorphism classes in terms of permutations and subsets.

For this, let $\mathscr{M}_{p-3} := \{(X, Y) \in \mathscr{B}_{p-3}^2 : X \cup Y = \{1, \dots, p-3\}\}$, i.e. the set of all pairs of sets which correspond to p-grid-kernels. Then, let α and β be bijections of $\mathscr{M}_{p-3}$ onto itself defined for all $(X, Y) \in \mathscr{M}_{p-3}$ by

$$\alpha(X, Y) := (Y, X), \qquad \beta(X, Y) := (X^-, Y^-),$$

whereby for any set $X \in \mathscr{B}_{p-3}$, $X^- := \{i \in \{1, \dots, p-3\} : p-2-i \in X\} \in \mathscr{B}_{p-3}$. Note that indeed $\alpha(X, Y)$, $\beta(X, Y) \in \mathscr{M}_{p-3}$ because $X^- \cup Y^- = (X \cup Y)^- = \{1, \dots, p-3\}^- = \{1, \dots, p-3\}$.

Proposition 6.1. *For any permutation $\varrho \in \mathscr{P}_{p-2}$, $f(\varepsilon\varrho^{-1}\varepsilon) = f(\varrho)^-$ whereby the permutation ε is defined by $\varepsilon(i) := p-1-i$, $i = 1, \dots, p-2$.*

Proof. By definition, $i \notin f(\varepsilon\varrho^{-1}\varepsilon)$ iff $i + 1 > \varepsilon\varrho^{-1}\varepsilon(i+1) = p - 1 - \varrho^{-1}(p-1-i-1)$ iff $\varrho^{-1}(p-2-i) > p-2-i$. By (P1), $\varrho^{-1}(p-2-i) \leqslant p-1-i$, i.e. the above holds only iff $\varrho(p-1-i) = p-2-i$. Therefore conversely, $i \in f(\varepsilon\varrho^{-1}\varepsilon)$ iff $\varrho(p-1-i) \geqslant p-1-i$ which equivalently means that $p-2-i \in f(\varrho)$ or $i \in f(\varrho)^-$. $\square$

Remark. Via the bijection $f : \mathscr{P}_{p-2} \to \mathscr{B}_{p-3}$, α and β generate a mapping between pairs of permutations. Clearly, $(\varrho, \sigma) \overset{\alpha}{\mapsto} (\sigma^{-1}, \varrho^{-1})$, and $(\varrho, \sigma) \overset{\beta}{\mapsto} (\varepsilon\varrho^{-1}\varepsilon, \varepsilon\sigma^{-1}\varepsilon)$ by Proposition 6.1.

Obviously, α and β are mappings of order 2, i.e. $\alpha^2 = \beta^2 = id$, and $\alpha\beta = \beta\alpha$. Hence, the generated group $H = \langle \alpha, \beta \rangle$ is Klein's four-group. We consider the action of H on $\mathscr{M}_{p-3}$. The following theorem shows that the orbits exactly are the types of p-grid-kernels corresponding to the above isomorphism.

Theorem 6.2. *For fixed $p \geqslant 2$, two p-grid-kernels K and K' are isomorphic iff (X_K, Y_K) and $(X_{K'}, Y_{K'})$ belong to the same orbit of $\mathscr{M}_{p-3}$ under the action of H.*

Proof. Let $X = X_K$, $Y = Y_K$, $X' = X_{K'}$, $Y' = Y_{K'}$ for short, and let the elements and corresponding permutations of K, K' be denoted as above where $'$ is used to specify everything associated with K'.

Assume firstly, that (X, Y) and (X', Y') belong to the same orbit. It suffices to consider the generates of H, i.e. the cases $(X', Y') = \alpha(X, Y)$ and $(X', Y') = \beta(X, Y)$.

Case 1.1. $X' = Y$, $Y' = X$: Then $\varrho' = \sigma^{-1}$, $\sigma' = \varrho^{-1}$, and the following mapping is an order isomorphism between K^- and K' which in fact just turns the kernel upside down and reflects left and right:

$$a_i \mapsto b'_i, \quad b_i \mapsto a'_i \qquad (i = 1, \dots, p-2),$$

$$c_i \mapsto d'_{\varrho(i)}, \quad d_i \mapsto c'_{\sigma(i)} \qquad (i = 0, \dots, p-1).$$

Case 1.2. $X' = X^-$, $Y' = Y^-$: Then $\varrho' = \varepsilon\varrho^{-1}\varepsilon$, $\sigma' = \varepsilon\sigma^{-1}\varepsilon$, and the following mapping is an order isomorphism between K and K' which in fact just reflects the left- and right-hand side of the kernel:

$$a_i \mapsto b'_{p-1-i}, \quad b_i \mapsto a'_{p-1-i} \qquad (i = 1, \dots, p-2),$$

$$c_i \mapsto c'_{p-1-\varrho(i)}, \quad d_i \mapsto d'_{p-1-\sigma(i)} \qquad (i = 0, \dots, p-1).$$

Now assume conversely that K and K' are isomorphic kernels.

Case 2.1. $K \cong K'$: The two chains in K_0 must be assigned to the two chains in K'_0. If $a_i \mapsto a'_i$ for $i = 1, \dots, p-2$, then $b_i \mapsto b'_i$, i.e. $\varrho = \varrho'$, $\sigma = \sigma'$, and $(X', Y') = (X, Y)$. If otherwise $a_i \mapsto b'_{p-1-i}$ for $i = 1, \dots, p-2$, then $b_i \mapsto a'_{p-1-i}$. Hence in K', b'_{p-1-i} and $a'_{p-1-\varrho(i)}$ have a common lower cover, i.e. $\varrho'(p-1-\varrho(i)) = p-1-i$ or $\varrho'(j) = p-1-\varrho^{-1}(p-1-j) = \varepsilon\varrho^{-1}\varepsilon(j)$ for $j = 1, \dots, p-2$. Analogously, $\sigma' = \varepsilon\sigma^{-1}\varepsilon$ which by Proposition 6.1 implies that $(X', Y') = (X^-, Y^-) = \beta(X, Y)$.

Case 2.2. $K^- \cong K'$: Here the order of the chains in the inner kernel has to be reversed. If for $i = 1, \dots, p-2$, $a_i \mapsto a'_{p-1-i}$ and $b_i \mapsto b'_{p-1-i}$, then $\sigma'(p-1-i) = p-1-\varrho(i)$ because the common lower cover of a_i and $b_{\varrho(i)}$ implies that a'_{p-1-i} and $b'_{p-1-\varrho(i)}$ share a common upper cover. Hence for $j = 1, \dots, p-2$, $\sigma'(j) = p-1-\varrho(p-1-j) = \varepsilon\varrho\varepsilon(j)$. Because $\varepsilon^2 = id$, $f(\sigma'^{-1}) = f(\varepsilon\varrho\varepsilon) = X^-$. Analogously, $f(\varrho') = f(\varepsilon\sigma\varepsilon) = f(\sigma^{-1})^- = Y^-$, i.e. $(X', Y') = (Y^-, X^-) = \alpha\beta(X, Y)$. If otherwise for $i = 1, \dots, p-2$, $a_i \mapsto b'_{p-1-i}$ and $b_i \mapsto a'_{p-1-i}$, then we analogously get by composing the above mappings that $(X', Y') = (Y, X) = \alpha(X, Y)$. $\square$

Altogether we know that for any p-grid-kernel K the isomorphic kernels correspond to (X, Y), (Y, X), (X^-, Y^-), or (Y^-, X^-). This is a very easy criterion for isomorphy.

6.3. *Enumeration*

Theorem 6.2 allows to compute the number of isomorphism classes of p-grid-kernels as the number of orbits in $\mathcal{M}_{p-3}$ under the action of H. For this, we use the standard Pólya type approach.

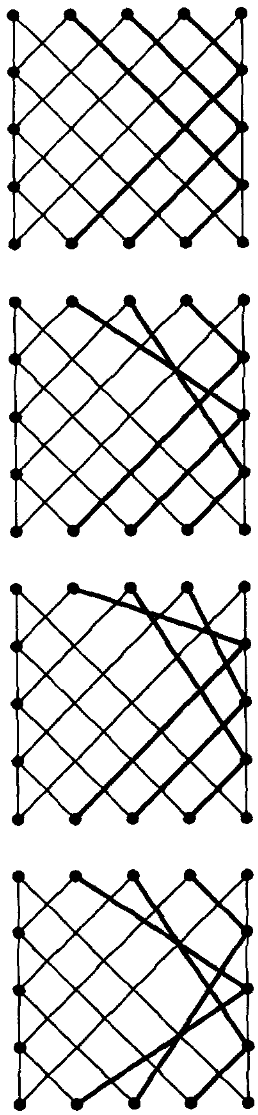
Theorem 6.3. *For fixed $p \geqslant 3$, there are*

$$\tfrac{1}{4}(3^{p-3} + 3^{\lceil (p-3)/2 \rceil} + 3^{\lfloor (p-3)/2 \rfloor} + 1)$$

pairwise nonisomorphic p-grid-kernels.

Proof. By Burnside's lemma, H generates $(1/|H|)\sum_{\gamma\in H}|\mathscr{F}_\gamma|$ orbits in $\mathscr{M}_{p-3}$ where for any $\gamma \in H$, $\mathscr{F}_\gamma := \{(X, Y) \in \mathscr{M}_{p-3}\colon \gamma(X, Y) = (X, Y)\}$, i.e. $\mathscr{F}_\gamma$ contains the fixpoints of the mapping γ. In the following, we determine the size of the 4 fixpoint sets.

Clearly, $\mathscr{F}_{id} = \mathscr{M}_{p-3}$. By definition, for any fixed $X \in \mathscr{B}_{p-3}$, $(X, Y) \in \mathscr{M}_{p-3}$ iff $Y \in \mathscr{B}_{p-3}$ belongs to the interval $[\{1,\dots,p-3\}\backslash X, \{1,\dots,p-3\}]$. Since this is a subcube of dimension $|X|$, $|\mathscr{F}_{id}| = |\mathscr{M}_{p-3}| = \sum_{X\in\mathscr{B}_{p-3}}|[\{1,\dots,p-3\}\backslash X, \{1,\dots,p-3\}]| = \sum_{X\in\mathscr{B}_{p-3}}2^{|X|} = \sum_{k=0}^{p-3}\binom{p-3}{k}2^k = 3^{p-3}$.



$\sigma = (1)(2)(3)$ $Y = \{1,2\}$

$(\{1,2\},\{1,2\})$

$\varrho = (1)(2)(3)$ $X = \{1,2\}$

$\sigma = (1)(23)$ $Y = \{1\}$

$(\{1,2\},\{1\})$
$(\{1\},\{1,2\})$
$(\{1,2\},\{2\})$
$(\{2\},\{1,2\})$

$\varrho = (1)(2)(3)$ $X = \{1,2\}$

$\sigma = (123)$ $Y = \emptyset$

$(\{1,2\},\emptyset)$
$(\emptyset,\{1,2\})$

$\varrho = (1)(2)(3)$ $X = \{1,2\}$

$\sigma = (1)(23)$ $Y = \{1\}$

$(\{1\},\{2\})$
$(\{2\},\{1\})$

$\varrho = (12)(3)$ $X = \{2\}$

Fig. 11. Isomorphy types of 5-grid-kernels.

Similarly, we handle β and $\alpha\beta$. Consider any $(X, Y) \in \mathscr{F}_\beta$. Because $\beta(X, Y) = (X^-, Y^-) = (X, Y)$ iff $X = X^-$, $Y = Y^-$, both X and Y are uniquely determined by the respective intersection X', Y' with $\{1, \dots, \lceil (p-3)/2 \rceil\}$. Moreover in this situation, $X \cup Y = \{1, \dots, p-3\}$ iff $X' \cup Y' = \{1, \dots, \lceil (p-3)/2 \rceil\}$. Hence, $|\mathscr{F}_\beta| = \sum_{X' \in \mathscr{B}_{\lceil (p-3)/2 \rceil}} |[\{1, \dots, \lceil (p-3)/2 \rceil\} \setminus X', \{1, \dots, \lceil (p-3)/2 \rceil\}]| = 3^{\lceil (p-3)/2 \rceil}$.

Next, let $(X, Y) \in \mathscr{F}_{\alpha\beta}$. Because $\alpha\beta(X, Y) = (Y^-, X^-) = (X, Y)$ iff $X = Y^-$, $Y = X^-$ and the latter conditions are equivalent, pairs in $\mathscr{F}_{\alpha\beta}$ bijectively correspond to sets $X \in \mathscr{B}_{p-3}$ with $X \cup X^- = \{1, \dots, p-3\}$. Let $X' := X \cap \{1, \dots, \lfloor (p-3)/2 \rfloor\}$ and $X'' := X \cap \{\lceil (p-3)/2 \rceil + 1, \dots, p-3\}$. Then $X \cup X' = \{1, \dots, p-3\}$ iff $X' \cup X''^- = \{1, \dots, \lfloor (p-3)/2 \rfloor\}$ and $\lceil (p-3)/2 \rceil \in X$ if p is even. Hence, $|\mathscr{F}_{\alpha\beta}| = \sum_{X' \in \mathscr{B}_{\lfloor (p-3)/2 \rfloor}} |[\{1, \dots, \lfloor (p-3)/2 \rfloor\} \setminus X', \{1, \dots, \lfloor (p-3)/2 \rceil\}]| = 3^{\lfloor (p-3)/2 \rfloor}$.

Finally, $\alpha(X, Y) = (Y, X) = (X, Y)$ iff $X = Y$. But $(X, X) \in \mathscr{M}_{p-3}$ iff $X \cup X = X = \{1, \dots, p-3\}$, i.e. $|\mathscr{F}_\alpha| = 1$. Putting all into Burnside's lemma immediately gives the assertion. $\square$

6.4. Example

This is for illustrating the above investigations in a summarized form for $p = 5$. In Fig. 11 we show the Hasse-diagrams of the 4 pairwise nonisomorphic 5-grid-kernels together with their corresponding permutations ϱ, σ represented as a product of pairwise disjoint cycles, the sets $X, Y \in \mathscr{B}_2$ and their images under the action of H, i.e. the orbits of $\mathscr{M}_2$. Note that indeed, all $3^2 = 9$ possible pairs $(X, Y) \in \mathscr{M}_2$ appear.

7. Linear extensions of p-kernels

In the present section, we shall compute some important numbers related to a poset P. All of them use *linear extensions*, i.e. linear (total) orderings containing P. Let $\mathscr{L}(P)$ denote the set of all of them. Then we are interested in the *number of linear extensions* $l(P) := |\mathscr{L}(P)|$, the *order dimension* $\dim P := \min |\mathscr{L}|$ where the minimum is extended over all subsets $\mathscr{L} \subseteq \mathscr{L}(P)$ with $\bigcap_{L \in \mathscr{L}} L = P$, and the *jump number* $s(P) := \min_{L \in \mathscr{L}(P)} |\{(x, y) \in P^2 \colon x \lessdot_L y \text{ and } (x, y) \notin P\}|$. We refer to [20] for a general introduction.

Due to our notion of isomorphy between kernels we shall consider the *standard* 4-FFT-kernel (Fig. 14), the *twisted* 4-FFT-kernel (Fig. 12), the *standard* p-grid-kernels (i.e. $\varrho = \sigma = id$), and the remaining (nonstandard) p-grid-kernels (i.e. $(\varrho, \sigma) \neq (id, id)$).

7.1. Jump number

We start with a general estimate. In the following, we call any pair (x, y) counted in $s(P)$ a *jump from x to y*.

Lemma 7.1. *For any p-kernel K, $s(K) \geqslant 2p - 2$.*

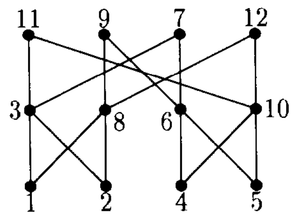


Fig. 12. Minimum jump number for the twisted 4-FFT-kernel.

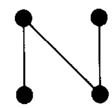


Fig. 13. The N-poset.

Proof. Clearly, for any $L \in \mathscr{L}(K)$, and for all but one minimal (maximal) elements, there is a jump to (from) it. By the general property (K2), all of those jumps are different because any jump to a minimal element has to precede any jump from a maximal one. Hence, we have at least $2(p-1)$ jumps. $\square$

This easy lower bound gives the final answer in all but one case.

Theorem 7.1. *Let K be any p-kernel. Then*

$$s(K) = \begin{cases} 2p-1 & \textit{if } K \textit{ is isomorphic to the standard } 4\textit{-FFT-kernel}, \\ 2p-2 & \textit{otherwise}. \end{cases}$$

Proof. Let K be any p-grid-kernel with all notation as used in Section 6, i.e. $c_0 < a_1 < \cdots < a_{p-2} < d_{p-1}$, $c_{p-1} < b_{p-2} < \cdots < b_1 < d_0$, $c_0 < d_0$, $c_{p-1} < d_{p-1}$, and $c_i < a_i < d_i$, $c_i < b_{\varrho(i)}$, $b_{\sigma(i)} < d_i$ for $i = 1, \ldots, p-2$. The chain $c_0c_1a_1c_2a_2 \cdots c_{p-1}d_{p-1}b_{p-2}d_{\sigma^{-1}(p-2)}b_{p-3}d_{\sigma^{-1}(p-3)} \cdots b_1d_{\sigma^{-1}(1)}d_0$ is an extension of K with the $2(p-1)$ jumps (c_i, c_{i+1}) $(i = 0, \ldots, p-2)$, (d_{p-1}, b_{p-2}), $(d_{\sigma^{-1}(i)}, b_{i-1})$ $(i = p-2, \ldots, 2)$, $(d_{\sigma^{-1}(1)}, d_0)$. Together with Lemma 7.1, this proves the assertion for p-grid-kernels.

Fig. 12 shows the 'twisted' 4-FFT-kernel, and the labeling gives a linear extension of it having exactly $2 \cdot 4 - 2 = 6$ jumps.

Finally, for the standard 4-FFT-kernel, we follow the method of [2]. Note that in this case, K is an *N-free poset*, i.e. there is no induced subordering, isomorphic to the one shown in Fig. 13.

Then it is known that K has a so-called *root*, i.e. there is a poset $\hat{K}$ such that the Hasse-diagram of K is the line-digraph of the Hasse-diagram of $\hat{K}$. Both are shown in Fig. 14.

In this particular situation, the jump number of K is known to be one less than the difference between the number of edges in $H(\hat{K})$ and the number of non-extremal elements of $\hat{K}$: $s(K) = 12 - 4 - 1 = 7$. $\square$

7.2. Dimension

We refer to [10, 23] for more details about dimension theory and an extensive list of references. Clearly, the dimension of any of the p-kernels is at least 2 because they are

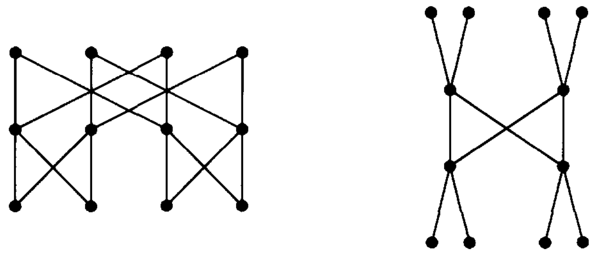


Fig. 14. 4-FFT-kernel (left) and its root (right).

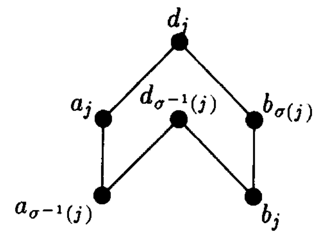


Fig. 15. 3-irreducible poset in nonstandard p-grid-kernels.

not just chains. To decide whether the dimension exceeds 2, we must check the existence of an induced subordering belonging to the known list of 3-irreducible posets, i.e. posets of dimension 3 with no proper subordering of dimension 3.

Theorem 7.2. *Let K be any p-grid-kernel. Then*

$$\dim K = \begin{cases} 2 & \text{if } K \text{ is standard,} \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Firstly, let K be nonstandard, and $\sigma = \sigma_K \neq id$. Note that for $\varrho = \varrho_K \neq id$, the proof is similar just by inverting everything.

Now, let j be the largest i with $\sigma(i) \neq i$. Then $\sigma(j) < j$ and $\sigma^{-1}(j) < j$. Consequently, $d_j, a_j, b_{\sigma(j)}, d_{\sigma^{-1}(j)}, a_{\sigma^{-1}(j)}, b_j$ induce the subordering shown in Fig. 15. From [20] we know that this is 3-irreducible, i.e. $\dim K \geqslant 3$.

It remains to list linear extensions the intersection of which equals the given p-grid-kernel K. Let L_1, L_2 be the chains

$$c_0 c_1 a_1 c_2 a_2 \cdots c_{p-1} d_{p-1} b_{p-2} d_{\sigma^{-1}(p-2)} b_{p-3} d_{\sigma^{-1}(p-3)} \cdots b_1 d_{\sigma^{-1}(1)} d_0$$

and

$$c_{p-1} c_{\varrho^{-1}(p-2)} b_{p-2} c_{\varrho^{-1}(p-3)} b_{p-3} \cdots c_{\varrho^{-1}(1)} b_1 c_0 d_0 a_1 d_1 a_2 d_2 \cdots a_{p-2} d_{p-2} d_{p-1},$$

respectively. If $\varrho = \sigma = id$ then clearly $L_1 \cap L_2 = K$, and $\dim K = 2$ is shown. Otherwise, the pairwise incomparability of all minimal resp. maximal elements is not yet guaranteed in $L_1 \cap L_2$. So, let L_3 be the chain

$$c_{p-1}c_{p-2}\cdots c_1c_0a_1\cdots a_{p-2}b_{p-2}\cdots b_1d_{p-1}d_{p-2}\cdots d_1d_0.$$

Then all c's or all d's are conversely ordered in L_1 and L_3 or in L_2 and L_3, respectively. Hence, $L_1 \cap L_2 \cap L_3 = K$ which proves $\dim K = 3$ for nonstandard p-grid-kernels. □

Theorem 7.3. *Let K be any 4-FFT-kernel. Then*

$$\dim K = \begin{cases} 3 & \text{if } K \text{ is standard,} \\ 4 & \text{otherwise.} \end{cases}$$

Proof. In Figs. 12 and 14, we denote the elements of K by $a, b, \ldots, k, l$ levelwise bottom-up, and each level from left to right.

If K is standard then the 6 elements a, c, e, g, i, j induce a subordering isomorphic to the 3-irreducible poset shown in Fig. 15, i.e. $\dim K \geqslant 3$, and it holds equality because K is the intersection of the 3 chains $a\,b\,e\,f\,c\,d\,h\,j\,l\,g\,i\,k$, $d\,c\,g\,h\,b\,a\,f\,l\,j\,e\,k\,i$, and $a\,b\,c\,d\,e\,g\,i\,k\,f\,h\,j\,l$.

Now, let K be twisted. In general, it is known (see e.g. [10]) that the dimension does not exceed the *width* of the poset, i.e. the maximum size of an antichain. For 4-FFT-kernels, it is easy to see that the latter value is 4, and hence, $\dim K \leqslant 4$. For suppose, there are 3 linear extensions L_1, L_2, L_3 — referred to by $<_1$, $<_2$, $<_3$ — such that $L_1 \cap L_2 \cap L_3 = K$. Because of symmetries, we assume w.l.o.g. that in L_1, e is the smallest element among $\{e, f, g, h\}$, and moreover that $h <_1 g$. Then $a <_1 e <_1 h$ but because a and h are incomparable in K, there it must be another linear extension, say L_2, where $h <_2 a$.

The scheme in Fig. 16 shows that $e <_1 f$, $h <_1 g$, $h <_2 a$, and $e <_1 g$ imply $g <_i l$ for $i = 1, 2, 3$. Therefore, $g < l$ in $L_1 \cap L_2 \cap L_3$ which contradicts the incomparability of g and l in K. Note that in the scheme, a junction of two implications means that for two incomparable elements x, y, $x < y$ in two linear extensions forces $y < x$ in the third one. □

7.3. Enumeration

Firstly, we explain our approach to count linear extensions in general. For this, let P be any poset. By $\mathscr{I}(P)$ we denote the set of all *lower ideals* in P, i.e. any $I \in \mathscr{I}(P)$ is a subset $I \subseteq P$ such that $x \in I$, $y \leqslant x$ imply $y \in I$ for all $x, y \in P$. Moreover, for any $I \in \mathscr{I}(P)$, the poset induced on $P \backslash I$ — we denote it by $P - I$ — is an *upper ideal*, i.e. $x \in I$, $y \geqslant x$ imply $y \in I$ for all $x, y \in P$. The following statements are of general interest.

Lemma 7.2. *Let k, $0 < k < |P|$, be fixed arbitrarily. Then*

$$l(P) = \sum_{I \in \mathscr{I}(P),\, |I| = k} l(I) \cdot l(P - I).$$

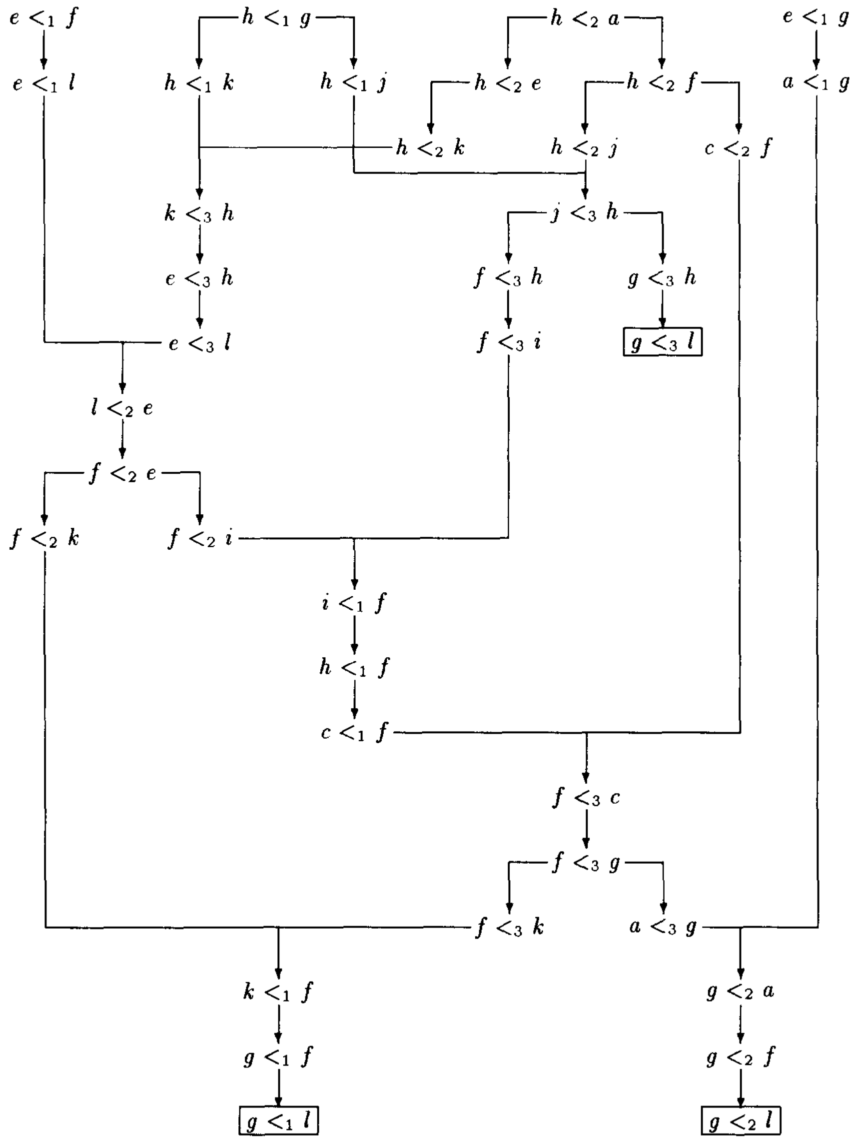


Fig. 16. Implications for the proof of Theorem 7.3.

Proof. For any linear extension L of P, the first k elements form a lower ideal I of size k in P, and the remaining elements clearly are the upper ideal $P - I$. Hence, L induces a linear extension of both I and $P - I$. This gives a mapping

$$\alpha : \mathscr{L}(P) \to \bigcup_{I \in \mathscr{I}(P), |I| = k} \mathscr{L}(I) \times \mathscr{L}(P - I).$$

Conversely, given any lower ideal of size k in P and linear extensions of I and $P - I$, concatenating them yields their uniquely determined pre-image under α. Hence, α is bijection, and the assertion follows immediately, because all the $\mathscr{L}(I) \times \mathscr{L}(P - I)$ are pairwise disjoint. $\square$

For general P, Lemma 7.2 cannot be expected to yield $l(P)$ easily because computing $l(I)$ or $l(P - I)$ might be as hard as the original problem. We only use it in the special cases $k = |P| - 1$ or $k = |P|/2$:

For $k = |P| - 1$, $P - I$ is just one maximal element of $|P|$, i.e. $l(P) = \sum_{x \text{ maximal}} l(P - \{x\})$ (see [5]).

For kernels K, there are reasonably few types of lower or upper ideals of size $|K|/2$ which moreover have an easy structure because we know from [12, 16] that any lower (upper) ideal of size $|K|/2$ in a kernel K contains every minimal (maximal) but none of the maximal (minimal) elements.

The next statement will be used for counting the linear extensions of those ideals. Here as usual, for posets P, Q on disjoint sets of elements, the *sum* (*direct sum*, *disjoint union*) $P + Q$ is the union of the two relations defined on the union of both ground sets. This result can be found in [21].

Lemma 7.3. *Let $P_1, \ldots, P_k$ be posets on pairwise disjoint sets of elements. Then*

$$l(P_1 + \cdots + P_k) = \frac{(|P_1| + \cdots + |P_k|)!}{|P_1|! \cdots |P_k|!} \cdot l(P_1) \cdots l(P_k).$$

Proof. Any linear extension of $P_1 \cup \cdots \cup P_k$ is uniquely determined by choosing arbitrary linear extensions of $P_1, \ldots, P_{k-1}$, and P_k independently, and by choosing which of its $|P_1| + \cdots + |P_k|$ elements is to be taken from $P_1, \ldots, P_{k-1}$, or P_k. The latter choice is a permutation with repetitions for any choice of the linear extensions of $P_1, \ldots, P_k$. $\square$

7.3.1. 4-FFT-kernels

An easy case analysis yields that in 4-FFT-kernels (standard or twisted), any lower or upper ideal with 6 elements is order isomorphic to one of the posets drawn in Fig. 17 or its inverse.

By Lemma 7.3 we immediately get

$$l(I_1) = \frac{6!}{4!1!1!} \cdot 4 \cdot 1 \cdot 1 = 120, \qquad l(I_3) = \frac{6!}{3!3!} \cdot 2 \cdot 2 = 80$$

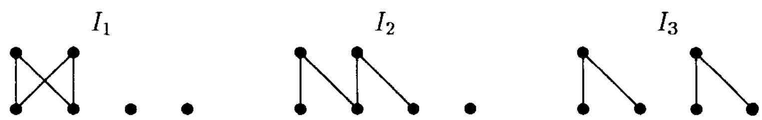
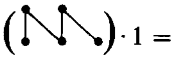


Fig. 17. 6-ideals of 4-FFT-kernels.

and

$l(I_2) = \frac{6!}{5!1!} \cdot l\Big($  $\Big) \cdot 1 = 6 \cdot 2 \cdot l\Big($  $\Big)$

by Lemma 7.2 for $k = |P| - 1$, and finally

$$l(I_2) = 12 \cdot \frac{4!}{3!1!} \cdot 2 \cdot 1 = 96.$$

Theorem 7.4. *Let K be any 4-FFT-kernel. Then*

$$l(K) = \begin{cases} 51200 & \text{if } K \text{ is standard,} \\ 49920 & \text{otherwise.} \end{cases}$$

Proof. If K is standard, then the $\binom{4}{2} = 6$ different lower ideals I of size 6 in K are such that twice $I \cong I_1$ and $K - I \cong I_3^-$, twice $I \cong I_3$ and $K - I \cong I_1^-$, and twice $I \cong I_3$ and $K - I \cong I_3^-$. Hence, $l(K) = 2(2 \cdot 120 \cdot 80 + 80 \cdot 80) = 51200$. If K is twisted, then we have $I \cong I_3$ and $K - I \cong I_2^-$ four times, and $I \cong I_1$ and $K - I \cong I_3^-$ twice, i.e. $l(K) = 4 \cdot 80 \cdot 96 + 2 \cdot 120 \cdot 80 = 49920$. □

7.3.2. p-grid-kernels

For the remaining, let K be any p-grid-kernel for an arbitrarily fixed $p \geqslant 3$, and let all elements be denoted as in Section 6. In particular, K is determined by the permutations $\varrho = \varrho_K$, $\sigma = \sigma_K$ or by the sets $X = X_K = f(\varrho)$, $Y = Y_K = f(\sigma^{-1})$. Any lower ideal of size $|K|/2 = 2p - 2$ consists of the p minimal elements of K and $p - 2$ elements taken upwards from the two chains of K_0. We get all such ideals by including the $r = 0, \ldots,$ or $p - 2$ smallest elements from the a_i's and the remaining $p - 2 - r$ smallest elements from the b_j's. If $I_{p,r} = \{c_0, \ldots, c_{p-1}, a_1, \ldots, a_r, b_{r+1}, \ldots, b_{p-2}\}$ is this ideal, then by Lemma 7.2, $l(K) = \sum_{r=0}^{p-2} l(I_{p,r}) \cdot l(K - I_{p,r})$.

Lemma 7.4. *Let $J_{p,r}$ and $J_{p,r,s,t}$ be the posets shown in Fig. 18. Then*

$$I_{p,r} = \begin{cases} J_{p,r} & \text{if } r \in X \cup \{0, p-2\}, \\ J_{p,r,s,t} & \text{if } r \in \{1, \ldots, p-3\} \backslash X, \end{cases}$$

whereby in the second case, t is the smallest element in $X \cup \{p - 2\}$ larger than r and $s = \varrho^{-1}(t)$.

Proof. If $r \in X$ then by definition, $\varrho(\{1, \ldots, r\}) = \{1, \ldots, r\}$, i.e. none of $c_1, \ldots, c_r$ is covered by any of $b_{r+1}, \ldots, b_{p-2}$. Hence, $\varrho^{-1}(\{r + 1, \ldots, p - 2\}) = \{r + 1, \ldots, p - 2\}$, and $I_{p,r} \cong J_{p,r}$. The cases $r = 0$ and $r = p - 2$ are obvious.

Now, assume $1 \leqslant r \leqslant p - 3$ but $r \notin X$. Then $\varrho(r + 1) = r$, i.e. c_{r+1} is maximal in $I_{p,r}$ because $a_{r+1} \notin \{a_1, \ldots, a_r\}$ and $b_{\varrho(r+1)} \notin \{b_{r+1}, \ldots, b_{p-2}\}$. But for $i = r + 2, \ldots, p - 2$, $\varrho(i) \geqslant r + 1$ and $b_{\varrho(i)} \in I_{p,r}$. Hence c_{r+1} is the only element of $\min K$ without an upper cover in $I_{p,r}$.

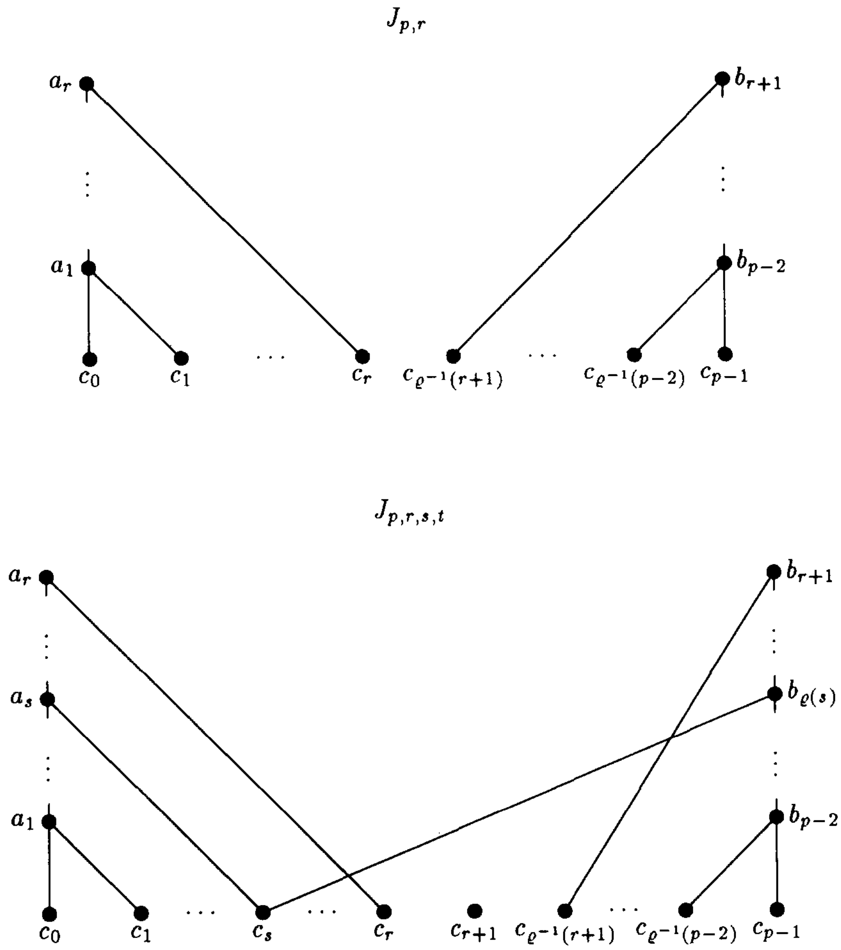


Fig. 18. $2p-2$-ideals of p-grid-kernels.

This immediately implies that there is exactly one element $c_s \in \min K$ having both its upper covers, a_s and $b_{\varrho(s)}$, in $I_{p,r}$. Consequently, $s \leqslant r$ and $\varrho(s) \geqslant r+1$. But then $\varrho^{-1}(\varrho(s)) = s < \varrho(s)$ which by $\varrho^{-1}(i) \leqslant i+1$ for all i means that $\varrho^{-1}(\{1,\dots,\varrho(s)\}) = \{1,\dots,\varrho(s)\}$, and $\varrho(s) \in X \cup \{p-2\}$. Finally assume that there is an t' with $r+1 \leqslant t' < \varrho(s)$ and $t' \in X$. Then $\varrho(\{1,\dots,t'\}) = \{1,\dots,t'\}$ and because of $s \leqslant r < t'$ we would get $\varrho(s) \leqslant t'$ in contradiction to the above. Therefore, altogether we know that $t := \varrho(s)$ is the smallest element of $X \cup \{p-2\}$ larger than r. $\square$

Note that the above characterization works for the upper ideal $K - I_{p,r}$ in an analogous way with respect to Y instead of X. In particular by Theorem 6.1, if $r \notin X$ then $r \in Y$, i.e. whenever $I_{p,r}$ is of the more complicated second type, $K - I_{p,r} \cong J^-_{p,r}$ is easy.

It remains to enumerate the linear extensions of both types of ideals.

Lemma 7.5. *For* $0 \leqslant r \leqslant p-2$,

$$l(J_{p,r}) = 2^{p-2} \cdot \binom{2p-2}{2r+1} \cdot r!(p-2-r)!.$$

Proof. Let $J'_{p,r}$ be the poset shown in Fig. 19. Then by Lemmas 7.2 and 7.3,

$$l(J'_{p,r}) = \frac{(2r)!}{(2r-1)! \cdot 1!} \cdot l(J'_{p,r-1}) \cdot 1 = 2r \cdot l(J'_{p,r-1}).$$

Because $l(J'_{p,0}) = 1$, this amounts to $l(J'_{p,r}) = 2^r \cdot r!$. But $J_{p,r} = J'_{p,r} + J'_{p,p-2-r}$. Hence, Lemma 7.3 yields

$$\begin{aligned} l(J_{p,r}) &= \frac{(2p-2)!}{(2r+1)!(2p-2r-3)!} \cdot 2^r r! \cdot 2^{p-2-r}(p-2-r)! \\ &= 2^{p-2} \cdot \binom{2p-2}{2r+1} \cdot r!(p-2-r)!. \end{aligned}$$

This completes the proof. □

Finally, we compute $l(J_{p,r,s,t})$ for arbitrary values of $r \in \{1, \dots, p-3\}$, $1 \leqslant s \leqslant r$, and $r+1 \leqslant t \leqslant p-2$.

Lemma 7.6.

$$\begin{aligned} l(J_{p,r,s,t}) = 2^{p-2}(p-1)! \Bigg[&\sum_{i=0}^{r-s} \frac{\binom{t-s-1-i}{t-r-1}\binom{2(p-t+s-1+i)}{2p-2t-3}}{(p-t+s-1+i)\binom{p-t+s-2+i}{p-t-2}} \\ &+ \sum_{i=0}^{t-r-1} \frac{\binom{t-s-1-i}{r-s}\binom{2(p-t+s-1+i)}{2s-1}}{(p-t+s-1+i)\binom{p-t+s-2+i}{s-1}} \Bigg]. \end{aligned}$$

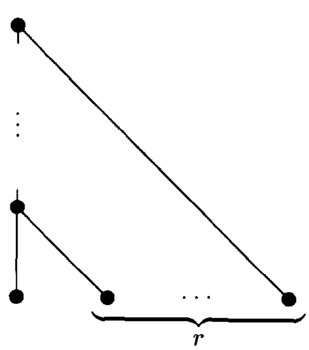


Fig. 19. The poset $J'_{p,r}$.

Proof. For the sake of brevity, let $u := s - 1, v := p - 2 - t, x := r - s, y := t - r - 1$. Then $0 \leqslant u, v, x, y$, and $u + v + x + y = p - 4$. For arbitrarily fixed u, v with $0 \leqslant u, v \leqslant p - 4$, we consider $l(J_{p,r,s,t})$ as a function $f(x,y)$. By Lemmas 7.2 and 7.3, the following recurrence formulas hold:

(1) $f(x,y) = 2(u + v + x + y + 3) \cdot [f(x-1,y) + f(x,y-1)]$ if $x, y \geqslant 1$,

(2) $f(x,0) = 2(u + v + x + 3) \cdot [f(x-1,0) + l(J_{u+v+x+3,v})]$ if $x \geqslant 1$,

(3) $f(0,y) = 2(u + v + y + 3) \cdot [l(J_{u+v+y+3,u}) + f(0,y-1)]$ if $y \geqslant 1$,

(4) $f(0,0) = 2(u + v + 3) \cdot [l(J_{u+v+3,u}) + l(J_{u+v+3,v})]$.

Starting with the values known by Lemma 7.5, standard techniques show that

$$f(x,y) = 2^{u+v+x+y+2}(u + v + x + y + 3)!$$

$$\times \left[\sum_{i=0}^{x} \frac{\binom{x+y-i}{y}\binom{2(u+v+i+2)}{2v+1}}{(u+v+i+2)\binom{u+v+i+1}{v}} + \sum_{i=0}^{y} \frac{\binom{x+y-i}{x}\binom{2(u+v+i+2)}{2u+1}}{(u+v+i+2)\binom{u+v+i+1}{u}} \right]$$

satisfies all of the recurrences. In terms of p, r, s, t, this is the assertion.

To see the technical part, for $z \in \{u, v\}$, let

$$a_i(z) := \frac{\binom{2(u+v+i+2)}{2z+1}}{(u+v+i+2)\binom{u+v+i+1}{z}}.$$

Then for $x, y \geqslant 1$,

$$2(u + v + x + y + 3)[f(x-1,y) + f(x,y-1)]$$

$$= 2^{u+v+x+y+2}(u + v + x + y + 3)!$$

$$\times \left[\sum_{i=0}^{x-1} a_i(v)\binom{x+y-i-1}{y} + \sum_{i=0}^{y} a_i(u)\binom{x+y-i-1}{x-1} \right.$$

$$\left. + \sum_{i=0}^{x} a_i(v)\binom{x+y-i-1}{y-1} + \sum_{i=0}^{y-1} a_i(u)\binom{x+y-i-1}{x} \right]$$

$$= 2^{u+v+x+y+2}(u + v + x + y + 3)!$$

$$\times \left[\sum_{i=0}^{x} a_i(v)\binom{x+y-i}{y} + \sum_{i=0}^{y} a_i(u)\binom{x+y-i}{x} \right]$$

$$= f(x,y)$$

because $\binom{x-1}{x} = \binom{y-1}{y} = 0$.

The proof of (2) and (3) is analog, and to see (4), the same steps have to be done. We show (2):

$$2(u+v+x+3)[f(x-1,0)+l(J_{u+v+x+3,v})]$$
$$=2^{u+v+x+2}\cdot\left\{(u+v+x+3)!\left[\sum_{i=0}^{x-1}a_i(v)+a_0(u)\right]\right.$$
$$\left.+(u+v+x+3)\binom{2(u+v+x+2)}{2v+1}\cdot v!(u+x+1)!\right\}$$
$$=2^{u+v+x+2}(u+v+x+3)!$$
$$\times\left[\sum_{i=0}^{x-1}a_i(v)+a_0(u)+\frac{\binom{2(u+v+x+2)}{2v+1}}{(u+v+x+2)\binom{u+v+x+1}{v}}\right]$$
$$=2^{u+v+x+2}(u+v+x+3)!\left[\sum_{i=0}^{x}a_i(v)+a_0(u)\right]$$
$$=f(x,0).$$

This completes the proof. □

Now, given any p-grid-kernel by the two sets X and Y, we know the structure of any $I_{p,r}$ or $K-I_{p,r}$ by Lemma 7.4 as well as the numbers of their linear extensions by Lemma 7.5 or Lemma 7.6. Hence, $l(K)=\sum_{r=0}^{p-2}l(I_{p,r})\cdot l(K-I_{p,r})$ can be computed.

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