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Versal unfoldings for linear retarded functional differential equations

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Abstract

We consider parametrized families of linear retarded functional differential equations (RFDEs) projected onto finite-dimensional invariant manifolds, and address the question of versality of the resulting parametrized family of linear ordinary differential equations. A sufficient criterion for versality is given in terms of readily computable quantities. In the case where the unfolding is not versal, we show how to construct a perturbation of the original linear RFDE (in terms of delay differential operators) whose finite-dimensional projection generates a versal unfolding. We illustrate the theory with several examples, and comment on the applicability of these results to bifurcation analyses of *nonlinear* RFDEs.

1. Introduction

Differential equations are used to model a very wide variety of phenomena. Frequently, these differential models contain several parameters which are often varied or "tuned" to describe more accurately the phenomenon under study. Thus, there is considerable interest, from both a pure and an applied point of view, to understand how the properties of solutions of a parametrized family of differential equations are affected by variation of the parameters. This philosophy is at the core of bifurcation theory.

One large class of differential equations which are particularly important in applications are *retarded functional differential equations* (RFDEs) [12,13], which

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includes the class of ordinary differential equations (ODEs), the class of delay differential equations, as well as certain types of integro-differential equations, among others. These equations are used to model various phenomena in fields ranging from mathematical biology [3,11,15,16] to industrial processes [18], and to atmospheric science [19]. The theory for both linear and nonlinear RFDEs is rather well developed [12,13]. Essentially, these equations behave like abstract ODEs on an infinite-dimensional (Banach) phase space. Thus, many results which are known for ODEs on finite-dimensional spaces have analogs in the context of RFDEs. For example, in the neighborhood of an equilibrium point of a nonlinear RFDE, there exists local invariant manifolds (stable, unstable and center manifolds) which are tangent to the corresponding invariant subspaces of the linearized equations about the equilibrium point, and on which the flow near the equilibrium is either exponentially attracting (stable manifold), exponentially repelling (unstable manifold), or nonhyperbolic (center manifold). In the context of bifurcation theory, the center manifold reduction of the flow is important, since this is where bifurcations (qualitative changes in the flow) take place as parameters are varied. In applications, there have been many studies (see for example [2]) of specific RFDEs where stability and bifurcations of equilibria were investigated using the center manifold reduction theory developed in [12,13]. Another example of the similarity between ODEs and RFDEs is in [6,7], where the theory of Poincaré–Birkhoff normal forms was extended to RFDEs.

One aspect of the bifurcation theory of RFDEs which, surprisingly, has not yet been developed is that of extending Arnold's theory [1] on versal unfoldings of matrices to the case of parameter-dependent linear RFDEs. This is the purpose of our present paper. Versal unfoldings of RFDEs for certain singularities have been computed for particular classes of RFDEs in the study of restrictions on the possible flows on a center manifold see [4,9,17]. These unfoldings are computed using the normal form theory of Faria and Magalhães [7]. However, no attempt is made to give a systematic treatment of unfolding of linear RFDEs.

We adopt the following strategy. We begin with a linear RFDE $\dot{x}(t) = \mathscr{L}_0(x_t)$ (where $\mathscr{L}_0$ is a bounded linear functional operator) whose semiflow restricted to a finite-dimensional subspace is defined by the matrix B. Using a versal unfolding of B, we explicitly construct a parametrized family $\mathscr{L}(\alpha)$ of bounded linear functional operators whose finite-dimensional restricted semiflow is defined by a versal unfolding $\mathscr{B}(\alpha)$ of the matrix B. In comparison, the realization of linear ODEs by linear RFDEs obtained by Faria and Magalhães [8] provides an existence result. Using the Hahn–Banach theorem they show the following. For any finite-dimensional matrix B, a necessary and sufficient condition for the existence of a bounded linear operator $\mathscr{L}_0$ from $C([-\tau,0],\mathbb{R}^n)$ into $\mathbb{R}^n$ with infinitesimal generator having spectrum containing the spectrum of B is that n be larger than or equal to the largest number of Jordan blocks associated with each eigenvalue of B. While some of the proofs in our paper have a flavor similar to the ones in [8], our results cannot be deduced from realizability results.

Recall that for a given $c \times c$ matrix B with complex entries, a *p-parameter unfolding* of B is a C^∞ p-parameter family of matrices $\mathscr{B}(\alpha)$ such that $\mathscr{B}(\alpha_0) = B$ for some $\alpha_0 \in \mathbb{C}^p$. The unfolding $\mathscr{B}(\alpha)$ is said to be a *versal unfolding* of B if for all q-parameter unfoldings $A(\beta)$ of B (with $A(\beta_0) = B$), there exists a C^∞ mapping $\phi : \mathbb{C}^q \to \mathbb{C}^p$ and a C^∞ family of invertible matrices $C(\beta)$ satisfying

$$\phi(\beta_0) = \alpha_0,$$

$$C(\beta_0) = I,$$

$$A(\beta) = C(\beta)\mathscr{B}(\phi(\beta))(C(\beta))^{-1}.$$

Thus, a versal unfolding of B is, up to similarity transformations, a general smooth perturbation of B. A versal unfolding is said to be *mini-versal* if the dimension of the parameter space is the smallest possible for a versal unfolding. Of course, the concept of versal unfolding of matrices is of importance not only to linear differential equations, but to nonlinear differential equations as well, since questions of stability and bifurcations of equilibria in nonlinear systems always involve an analysis of an associated linearized system. It is therefore important to understand the dependence of the associated matrix on system parameters (i.e. in the case where the resulting unfolding is not versal, there are restrictions on the movement of the eigenvalues, which may influence the possible range of dynamics).

In the space $\mathrm{Mat}_{c\times c}$ of $c \times c$ matrices with complex entries, let Σ denote the similarity orbit of the matrix B. We will use the following sufficient criterion for versality which can be found in [1,20].

Proposition 1.1. *Let $\mathscr{B}(\alpha)$ be a p-parameter unfolding of the matrix B. If*

$$\mathrm{Mat}_{c\times c} = T\Sigma_{\mathscr{B}(\alpha_0)} + D_\alpha\mathscr{B}(\alpha_0)\cdot\mathbb{C}^p,$$

where α_0 is such that $\mathscr{B}(\alpha_0) = B$ and $T\Sigma_{\mathscr{B}(\alpha_0)}$ denotes the tangent space to Σ at $\mathscr{B}(\alpha_0)$, then $\mathscr{B}(\alpha)$ is a versal unfolding of B. If the codimension of $T\Sigma_{\mathscr{B}(\alpha_0)}$ is equal to p, then the unfolding is mini-versal.

As motivation for our analysis in this paper consider a parameter-dependent nonlinear RFDE

$$\dot{x}(t) = F(x_t, \alpha) \tag{1.1}$$

which has a nonhyperbolic equilibrium, which we assume for the sake of simplicity to be at $x = 0$, $\alpha = 0$. As previously mentioned, a crucial step in the analysis of the bifurcations and stability of this equilibrium is to perform a parameter-dependent center manifold reduction of the RFDE [7] in order to obtain a parameter-dependent nonlinear c-dimensional ODE

$$\dot{z}(t) = Bz + G(z, \alpha), \tag{1.2}$$

where B is a $c \times c$ constant matrix, and G is *nonlinear in* z *and* α. The emphasis is important, since terms which are parameter dependent and linear in z are contained in the expression for G and they are the ones which unfold the matrix B. Thus, it is natural to investigate the versality of this unfolding, for reasons that we have previously mentioned. However, the potential difficulty here is that the reduction process for RFDEs to finite-dimensional invariant manifolds introduces restrictions on the possible types of nonlinearities G which can be achieved in (1.2) [8]. Thus, there is no a priori guarantee that a versal unfolding of the matrix B can be achieved in G by this reduction process, even if we have perhaps many parameters (e.g. more than the codimension of $T\Sigma_B$ in $\mathrm{Mat}_{c\times c}$) in the original RFDE. It is clear that only terms which are linear in x_t in the right-hand side of (1.1) contribute to terms which are linear in z in the right-hand side of (1.2). Thus, we restrict our attention to the case where (1.1) is a parametrized family of linear RFDEs.

We have two main results. The first gives a sufficient condition on the parametrized linear RFDE (1.1) which guarantees that the right-hand side of (1.2) is a versal unfolding of the matrix B. The second main result is twofold: first, we show that despite the previously mentioned restrictions on G in (1.2), it is always possible to realize a versal unfolding of B by a suitable choice of parameter-dependent RFDE in (1.1); and furthermore, we give a "canonical" method of computing such a parameter-dependent RFDE in terms of delay differential operators. The theory is then illustrated with several examples.

Although our original interest and motivation lies in bifurcation theory (i.e. B in (1.2) has all of its eigenvalues on the imaginary axis), there is no additional complication in considering a reduction of (1.1) to a general finite-dimensional invariant manifold (i.e. not necessarily a center manifold). Therefore, we develop the theory in this general context.

As is the case for Arnold's theory of versal unfoldings of matrices, we develop our theory by working in complex spaces; since the diagonalization theory is much simpler in this context. Versal unfoldings in the real case can be constructed by a decomplexification of the complex unfolding, as is done in [1] and which we illustrate in Section 7.

2. Reduction of linear RFDEs

Let $C_n = C([-\tau, 0], \mathbb{C}^n)$ be the Banach space of continuous functions from the interval $[-\tau, 0]$, into $\mathbb{C}^n$ $(\tau > 0)$ endowed with uniform norm. We are interested in the linear homogeneous RFDE

$$\dot{z}(t) = \mathscr{L}_0(z_t), \tag{2.1}$$

where $\mathscr{L}_0$ is a bounded linear operator from C_n into $\mathbb{C}^n$. We write

$$\mathscr{L}_0(\varphi) = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta),$$

where η is an $n \times n$ matrix-valued function of bounded variation defined on $[-\tau, 0]$. Let A_0 denote the infinitesimal generator of the semiflow generated by Eq. (2.1). Then it is well-known that the spectrum $\sigma(A_0)$ of A_0 is equal to the point spectrum of A_0, and $\lambda \in \sigma(A_0)$ if and only if λ satisfies the characteristic equation

$$\det \Delta(\lambda) = 0, \quad \text{where } \Delta(\lambda) = \lambda I_n - \int_{-\tau}^{0} d\eta(\theta) e^{\lambda\theta}, \tag{2.2}$$

where I_n is the $n \times n$ identity matrix. We suppose that $\Lambda \subset \mathbb{C}$ is a nonempty finite set of eigenvalues of A_0, with corresponding generalized c-dimensional eigenspace P. Using adjoint theory, it is known that we can write

$$C_n = P \oplus Q, \tag{2.3}$$

where Q is invariant under the semiflow of (2.1), and invariant under A_0.

Define $C_n^* = C([0, \tau], \mathbb{C}^{n*})$, where $\mathbb{C}^{n*}$ is the n-dimensional space of row vectors. We have the adjoint bilinear form on $C_n^* \times C_n$:

$$(\psi, \varphi)_n = \psi(0)\varphi(0) - \int_{-\tau}^{0} \int_{0}^{\theta} \psi(\xi - \theta)\, d\eta(\theta)\varphi(\xi)\, d\xi. \tag{2.4}$$

We let $\Phi = (\varphi_1, \dots, \varphi_c)$ be a basis for P, and $\Psi = \operatorname{col}(\psi_1, \dots, \psi_c)$ be a basis for the dual space P^* in C_n^*, chosen so that $(\Psi, \Phi)_n$ is the $c \times c$ identity matrix, I_c. In this case, we have $Q = \{\varphi \in C_n \colon (\Psi, \varphi)_n = 0\}$. We denote by B the $c \times c$ constant matrix such that $\dot{\Phi} = \Phi B$. The spectrum of B coincides with Λ. Using decomposition (2.3), any solution to (2.1) can be written as $z = \Phi x + y$, where $x \in \mathbb{C}^c$ and $y \in Q$ is a C^1 function. The dynamics of (2.1) on P are then given by

$$\dot{x} = Bx.$$

3. Parametrized families of linear RFDEs

Consider now a parametrized family of linear RFDEs of the form

$$\dot{z}(t) = \mathscr{L}(\alpha)(z_t), \tag{3.1}$$

where $\alpha \in \mathbb{C}^p$, $\alpha \mapsto \mathscr{L}(\alpha)$ is a C^∞ function with values in the space of bounded linear operators from C_n into $\mathbb{C}^n$, and $\mathscr{L}(\alpha_0) = \mathscr{L}_0$ is as in (2.1), for some $\alpha_0 \in \mathbb{C}^p$. In the sequel, we will assume that a translation has been performed in the parameter space $\mathbb{C}^p$ such that $\alpha_0 = 0$. Our outline and notation here follows closely that of [7]. We rewrite (3.1) as the system

$$\dot{z}(t) = \mathscr{L}_0(z_t) + [\mathscr{L}(\alpha) - \mathscr{L}_0](z_t),$$

$$\dot{\alpha}(t) = 0. \tag{3.2}$$

The solutions of this system are of the form $\tilde{z}(t) = (z(t), \alpha(t))^T \in \mathbb{C}^{n+p}$ (where the superscript T denotes transpose), the phase space is $\tilde{C} = C_{n+p} = C([-\tau, 0], \mathbb{C}^{n+p})$, and we write (3.2) as

$$\dot{\tilde{z}}(t) = \tilde{\mathscr{L}}_0 \tilde{z}_t + \tilde{F}(\tilde{z}_t), \tag{3.3}$$

where $\tilde{\mathscr{L}}_0((u,v)^T) = (\mathscr{L}_0(u), 0)^T$ and $\tilde{F}((u,v)^T) = ([\mathscr{L}(v(0)) - \mathscr{L}_0](u), 0)^T$, $u \in C_n$, $v \in C_p$.

Let Λ, P, Q, Φ, Ψ, $(\ ,\ )_n$ and B be as in the previous section. Define $\tilde{P} = P \times \mathbb{C}^p$, $\tilde{Q} = Q \times R$, where $R = \{v \in C_p\colon\ v(0) = 0\}$, and consider for bases of $\tilde{P}$ and $\tilde{P}^*$, respectively, the columns of the matrix $\tilde{\Phi}$ and the rows of the matrix $\tilde{\Psi}$,

$$\tilde{\Phi} = \begin{pmatrix} \Phi & 0 \\ 0 & I_p \end{pmatrix}, \quad \tilde{\Psi} = \begin{pmatrix} \Psi & 0 \\ 0 & I_p \end{pmatrix},$$

which satisfy $(\tilde{\Psi}, \tilde{\Phi})_{n+p} = I_{c+p}$. We have $\dot{\tilde{\Phi}} = \tilde{\Phi}\tilde{B}$, where $\tilde{B} = \mathrm{diag}(B, 0)$. It follows that we have an invariant splitting $\tilde{C} = \tilde{P} \oplus \tilde{Q}$.

Let BC_n denote the space of functions from $[-\tau, 0]$ to $\mathbb{C}^n$ which are uniformly continuous on $[-\tau, 0)$ and with a jump discontinuity at 0. If we define $X_0 : [-\tau, 0] \to \mathrm{Mat}_{n\times n}$ by

$$X_0(\theta) = \begin{cases} I_n, & \theta = 0, \\ 0, & -\tau \leqslant \theta < 0, \end{cases}$$

then the elements of BC_n can be written as $\xi = \varphi + X_0\mu$, with $\varphi \in C_n = C([-\tau, 0], \mathbb{C}^n)$ and $\mu \in \mathbb{C}^n$, so that BC_n is identified with $C_n \times \mathbb{C}^n$. In order to study (3.3), we need to consider the space $B\tilde{C} = BC_n \times BC_p$, which can be identified with $\tilde{C} \times \mathbb{C}^{n+p}$. Define $Y_0 : [-\tau, 0] \to \mathrm{Mat}_{p\times p}$ by

$$Y_0(\theta) = \begin{cases} I_p, & \theta = 0, \\ 0, & -\tau \leqslant \theta < 0. \end{cases}$$

Let $\pi : BC_n \to P$ denote the projection

$$\pi(\varphi + X_0\mu) = \Phi[(\Psi, \varphi)_n + \Psi(0)\mu],$$

where $\varphi \in C_n$ and $\mu \in \mathbb{C}^n$. We consider the projection $\tilde{\pi} : B\tilde{C} \to \tilde{P}$ given by

$$\tilde{\pi}((\varphi + X_0\mu, \psi + Y_0 v)^T) = \tilde{\Phi}\left[\left(\tilde{\Psi}, \begin{bmatrix} \varphi \\ \psi \end{bmatrix}\right)_{n+p} + \tilde{\Psi}(0)\begin{bmatrix} \mu \\ v \end{bmatrix}\right] = \begin{bmatrix} \pi(\varphi + X_0\mu) \\ \psi(0) + v \end{bmatrix}.$$

We now decompose $\tilde{z}$ in (3.3) according to the splitting

$$B\tilde{C} = \tilde{P} \oplus \ker \tilde{\pi},$$

with the property that $\tilde{Q}\subsetneq\ker\tilde{\pi}$, and get

$$\begin{bmatrix}\dot{x}\\ \dot{\alpha}\end{bmatrix}=\tilde{B}\begin{bmatrix}x\\ \alpha\end{bmatrix}+\tilde{\Psi}(0)\tilde{F}\left(\tilde{\Phi}\begin{bmatrix}x\\ \alpha\end{bmatrix}+\begin{bmatrix}y\\ w\end{bmatrix}\right),$$

$$\frac{d}{dt}\begin{bmatrix}y\\ w\end{bmatrix}=\tilde{A}_{\tilde{Q}^1}\begin{bmatrix}y\\ w\end{bmatrix}+(I-\tilde{\pi})\begin{bmatrix}X_0\\ Y_0\end{bmatrix}\tilde{F}\left(\tilde{\Phi}\begin{bmatrix}x\\ \alpha\end{bmatrix}+\begin{bmatrix}y\\ w\end{bmatrix}\right), \tag{3.4}$$

where $x\in\mathbb{C}^c$, $\alpha\in\mathbb{C}^p$, $y\in Q^1\equiv Q\cap C_n^1$, $w\in R^1\equiv R\cap C_p^1$ (C_n^1 and C_p^1 denote, respectively, the subsets of C_n and C_p consisting of continuously differentiable functions), and $\tilde{A}_{\tilde{Q}^1}$ is the operator from $\tilde{Q}^1\equiv\tilde{Q}\cap\tilde{C}^1=Q^1\times R^1$ into $\ker\tilde{\pi}$ defined by

$$\tilde{A}_{\tilde{Q}^1}\begin{bmatrix}\varphi\\ \psi\end{bmatrix}=\begin{bmatrix}\dot{\varphi}\\ \dot{\psi}\end{bmatrix}+\begin{bmatrix}X_0\\ Y_0\end{bmatrix}\left(\tilde{\mathscr{L}}_0\begin{bmatrix}\varphi\\ \psi\end{bmatrix}-\begin{bmatrix}\dot{\varphi}(0)\\ \dot{\psi}(0)\end{bmatrix}\right).$$

If $A_{Q^1}:Q^1\subset\ker\pi\to\ker\pi$ is defined by $A_{Q^1}\varphi=\dot{\varphi}+X_0\,[\mathscr{L}_0\,(\varphi)-\dot{\varphi}(0)]$, then (3.4) is equivalent to

$$\begin{bmatrix}\dot{x}\\ \dot{\alpha}\end{bmatrix}=\begin{bmatrix}Bx\\ 0\end{bmatrix}+\begin{bmatrix}\Psi(0)[\mathscr{L}(\alpha(0)+w(0))-\mathscr{L}_0](\Phi x+y)\\ 0\end{bmatrix},$$

$$\frac{d}{dt}\begin{bmatrix}y\\ w\end{bmatrix}=\begin{bmatrix}A_{Q^1}y\\ \dot{w}-Y_0\dot{w}(0)\end{bmatrix}+\begin{bmatrix}(I-\pi)X_0[\mathscr{L}(\alpha(0)+w(0))-\mathscr{L}_0](\Phi x+y)\\ 0\end{bmatrix}.$$

Since $w\in R$, it follows that $w(0)=0$, so that we get the following equations in $BC_n=P\oplus\ker\pi$

$$\dot{x}=Bx+\Psi(0)[\mathscr{L}(\alpha)-\mathscr{L}_0](\Phi x+y),$$

$$\frac{d}{dt}y=A_{Q^1}y+(I-\pi)X_0[\mathscr{L}(\alpha)-\mathscr{L}_0](\Phi x+y), \tag{3.5}$$

where $x\in\mathbb{C}^c$ and $y\in Q^1$.

4. Reduction to parameter-dependent invariant manifold

In this section, we show that (3.4) admits a local, semiflow invariant, $c+p$-dimensional manifold in $B\tilde{C}$, which is tangent at the origin to $\tilde{P}$, and such that the dynamics of (3.4) restricted to this manifold are linear in $x\in\mathbb{C}^c$.

We want the nontrivial part of our invariant manifold to be of the form

$$u=\Phi x+h(\alpha)x, \tag{4.1}$$

where $h:\mathbb{C}^p\to\mathrm{Mat}_{1\times c}(Q^1)$ is a C^∞ map, and $\mathrm{Mat}_{1\times c}(Q^1)$ denotes the space of $1\times c$ matrices whose elements are in the space Q^1 which has been defined in the previous section. Combining (4.1) and the first equation in (3.5) we obtain

$$\begin{aligned}\frac{du}{dt} &= (\Phi + h(\alpha))\dot{x} \\ &= (\Phi + h(\alpha))(Bx + \Psi(0)[\mathscr{L}(\alpha) - \mathscr{L}_0](\Phi + h(\alpha))x). \qquad (4.2)\end{aligned}$$

Using the second of Eqs. (3.5), we get that expression (4.1) represents a locally semiflow invariant manifold of (3.2) near the origin $(z, \alpha) = (0, 0)$ if

$$\begin{aligned}&(A_{Q^1} + (I - \pi)X_0[\mathscr{L}(\alpha) - \mathscr{L}_0])h(\alpha) + (I - \pi)X_0[\mathscr{L}(\alpha) - \mathscr{L}_0](\Phi) \\ &\quad - h(\alpha)(B + \Psi(0)[\mathscr{L}(\alpha) - \mathscr{L}_0](\Phi + h(\alpha))) = 0 \qquad (4.3)\end{aligned}$$

for all α in a neighborhood of 0.

Our main tool for proving this result is the implicit function theorem (IFT). Because of the smoothness properties required for the application of the IFT, we need to work with the C^1 norm instead of the uniform norm on the space Q^1. In Lemmas 5.1 and 5.2 of [6], it was shown that

$$\sigma(A_{Q^1}) = P\sigma(A_{Q^1}) = \sigma(A_0)\backslash\Lambda, \qquad (4.4)$$

where σ denotes the spectrum, and $P\sigma$ is the point spectrum. Now, if $\lambda \notin P\sigma(\mathscr{A})$, with $\mathscr{A}$ equal to A_0, A and A_{Q^1} respectively, then we have that the operators

$$\mathscr{A} - \lambda I : (D(\mathscr{A}), ||\cdot||_{C^1}) \to (\mathscr{X}, ||\cdot||_{\mathscr{X}})$$

are surjective with bounded inverse, where $\mathscr{X}$ is C_n, BC_n and $\ker \pi$, respectively. This follows from the surjectivity of the above operators, the fact that $D(\mathscr{A})$ are closed subsets of $(C^1, ||\cdot||_{C^1})$ and from the closed graph theorem.

Proposition 4.1. *Let* $(Q^1)^c = Q^1 \times \cdots \times Q^1$ *(c times) endowed with* C^1 *norm*

$$||(h^1, \dots, h^c)||_{(Q^1)^c} = \sum_{i=1}^{c} (|h^i|_C + |\dot{h}^i|_C)$$

(where $|\ |_C$ *denotes uniform norm), and let* $(\ker \pi)^c = \ker \pi \times \cdots \times \ker \pi$ *(c times) endowed with norm*

$$||(\psi^1 + X_0\alpha^1, \dots, \psi^c + X_0\alpha^c)||_{(\ker \pi)^c} = \sum_{i=1}^{c} (|\psi^i|_C + |\alpha^i|_{\mathbb{C}^n}).$$

Consider the nonlinear operator $N : \mathbb{C}^p \times (Q^1)^c \to (\ker \pi)^c$ *defined by*

$$\begin{aligned}N(\alpha, h) &= (A_{Q^1} + (I - \pi)X_0[\mathscr{L}(\alpha) - \mathscr{L}_0])h + (I - \pi)X_0\,[\mathscr{L}(\alpha) - \mathscr{L}_0](\Phi) \\ &\quad - h(B + \Psi(0)[\mathscr{L}(\alpha) - \mathscr{L}_0](\Phi + h)), \qquad (4.5)\end{aligned}$$

where the actions of A_{Q^1} *and* $\mathscr{L}(\alpha)$ *on* $h \in (Q^1)^c$ *are defined componentwise in the obvious way. Then, there exists a neighborhood* V *of* 0 *in* $\mathbb{C}^p$ *and a unique* C^∞ *mapping* $\alpha \mapsto h(\alpha)$ *from* V *into* $(Q^1)^c$ *such that* $h(0) = 0$ *and such that* $N(\alpha, h(\alpha)) = 0$ *for all* $\alpha \in V$.

Proof. It is clear that with the chosen topologies, N is a C^∞ mapping. Moreover, it is also clear that $N(0,0) = 0$. The partial Fréchet derivative $N_h(0,0)$ is the bounded linear operator from $(Q^1)^c$ into $(\ker \pi)^c$ defined by

$$N_h(0,0)v \equiv Jv = A_{Q^1}v - vB.$$

We suppose that we have chosen a system of coordinates such that the $c \times c$ matrix B is in Jordan canonical form

$$B = \begin{pmatrix} \lambda_1 & \sigma_1 & & & & \\ & \lambda_2 & \sigma_2 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \sigma_{c-1} \\ & & & & & \lambda_c \end{pmatrix},$$

where $\sigma_i = 1$ or $\sigma_i = 0$. Suppose that $v = (v^1, \dots, v^c) \in (Q^1)^c \backslash \{0\}$ is such that $Jv = 0$. Then this is equivalent to

$$A_{Q^1}(v^1, \dots, v^c) = (\lambda_1 v^1, \sigma_1 v^1 + \lambda_2 v^2, \dots, \sigma_{c-1} v^{c-1} + \lambda_c v^c),$$

which implies that one of the λ_i must be in the point spectrum of A_{Q^1}, which is a contradiction (see (4.4)). Therefore, $\ker J = \{0\}$.

We now show that J is surjective. We use an approach similar to the proof of Theorem 5.4 of [6]. Let $i \in \{1, \dots, c\}$, $\varphi \in Q^1$, and $\xi \in \ker \pi$ be such that $\xi = (A_{Q^1} - \lambda_i I)\varphi$. Define $v = (v^1, \dots, v^c) \in (Q^1)^c$ by $v^k = 0$, $k \neq i$, and $v^i = \varphi$. Then $f = Jv$ has the form $f = (f^1, \dots, f^c)$, where

$$f^k = \begin{cases} \xi & \text{if } k = i, \\ -\sigma_i \varphi & \text{if } k = i+1 \text{ and } i < c, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Let $g = (g^1, \dots, g^c) \in (\ker \pi)^c$. We now use an induction argument to show that there exists an $h = (h^1, \dots, h^c) \in (Q^1)^c$ which is such that $Jh = g$. We know from the remark following (4.4) that λ_1 is such that $A_{Q^1} - \lambda_1 I$ has a bounded inverse. Therefore, there exists $\varphi_1 \in Q^1$ such that $(A_{Q^1} - \lambda_1 I)\varphi_1 = g^1$. If we define $\mathscr{H}_1 = (\varphi_1, 0, \dots, 0) \in (Q^1)^c$, then we get from (4.6) that $(J\mathscr{H}_1)^1 = g^1$. Suppose now that $i \in \{1, \dots, c-1\}$ is such that there exists $\mathscr{H}_i \in (Q^1)^c$ satisfying

$$(J\mathscr{H}_i)^p = g^p, \quad \forall p \leqslant i. \tag{4.7}$$

Define $H^{i+1} = (J\mathscr{H}_i)^{i+1} \in \ker \pi$. Since $A_{Q^1} - \lambda_{i+1} I$ has a bounded inverse, there exists $\varphi_{i+1} \in Q^1$ such that $(A_{Q^1} - \lambda_{i+1} I)\varphi_{i+1} = g^{i+1} - H^{i+1}$. If we define $\mathscr{G}_{i+1} = (\zeta^1, \dots, \zeta^c) \in (Q^1)^c$ by $\zeta^k = 0$ if $k \neq i+1$, $\zeta^{i+1} = \varphi_{i+1}$, it follows from (4.6) that

$$(J\mathscr{G}_{i+1})^k = \begin{cases} g^{i+1} - H^{i+1} & \text{if } k = i+1, \\ -\sigma_i \varphi_{i+1} & \text{if } k = i+2 \text{ and } i+1 < c, \\ 0 & \text{otherwise.} \end{cases}$$

If we now set $\mathscr{H}_{i+1} = \mathscr{H}_i + \mathscr{G}_{i+1}$, we get from (4.7)

$$(J\mathscr{H}_{i+1})^p = \begin{cases} g^{i+1} & \text{if } p = i+1, \\ g^p & \text{if } p \leqslant i. \end{cases}$$

By induction on i, we have that (4.7) holds for $i = c$, and thus J is a surjection. It now follows that J has a bounded inverse, and we get the conclusion of the proposition by virtue of the implicit function theorem. $\square$

Proposition 4.2. *Let $h(\alpha)$ be the solution of $N = 0$ in* (4.5) *defined for all $\alpha \in V$, where V is some neighborhood of the origin in $\mathbb{C}^p$. Then the following set*

$$W = \{((x, \alpha), (y, w)) \in \mathbb{C}^{c+p} \times \tilde{Q}^1 \colon \ w = 0, \ y = h(\alpha)x, \ x \in \mathbb{C}^c, \ \alpha \in V\} \tag{4.8}$$

is a locally semiflow invariant C^∞ manifold for system (3.4), *and tangent to $\tilde{P}$. The* (*nontrivial*) *dynamics on this manifold reduce to*

$$\dot{x} = Bx + \Psi(0)[\mathscr{L}(\alpha) - \mathscr{L}_0](\Phi + h(\alpha))x. \tag{4.9}$$

Proof. It is obvious that W in (4.8) is tangent to $\tilde{P}$. The semiflow invariance of W follows from substitution of $y = h(\alpha)x$ into (3.5), and using the fact that $h(\alpha)$ is a solution to $N = 0$ in (4.5). $\square$

5. Sufficient condition for versality

From the previous section, we have seen that (3.1) admits an invariant manifold through the equilibrium solution $(z, \alpha) = (0, 0)$ on which the dynamics reduces to the c-dimensional parametrized linear system

$$\dot{x} = \mathscr{B}(\alpha)x, \tag{5.1}$$

where by (4.9) we have

$$\mathscr{B}(\alpha) = B + \Psi(0)[\mathscr{L}(\alpha) - \mathscr{L}_0](\Phi + h(\alpha)). \tag{5.2}$$

Note that $\mathscr{B}(0) = B$ and since $h(0) = 0$, we have

$$D_\alpha \mathscr{B}(0) = \Psi(0) D_\alpha [\mathscr{L}(\alpha)(\Phi)]|_{\alpha=0}.$$

Let $\mathrm{Mat}_{c\times c}$ denote the space of $c \times c$ matrices with complex entries. For $i, j \in \{1, \ldots, c\}$, let $E_{ij} \in \mathrm{Mat}_{c\times c}$ be the matrix whose elements are all 0 except the element in row i and column j, whose value is 1. For $U_1, U_2 \in \mathrm{Mat}_{c\times c}$, denote $[U_1, U_2] \equiv U_1 U_2 - U_2 U_1$.

Denote the linear mapping $\Theta : \mathrm{Mat}_{c\times c} \to \mathbb{C}^{c^2}$ by $\Theta(E_{ij}) = e_{(i-1)c+j}$, where e_ℓ denotes the row vector whose components are all 0 except the ℓth which is 1.

Theorem 5.1. *The parametrized family* (3.1) *generates a versal unfolding* $\mathscr{B}(\alpha)$ (*see* (5.2)) *of the matrix* $B = (\Psi, \dot{\Phi})_n$ *if the* $(c^2 + p) \times c^2$ *matrix*

$$S = \begin{pmatrix} \Theta([B, E_{11}]) \\ \Theta([B, E_{12}]) \\ \vdots \\ \Theta([B, E_{cc}]) \\ \Theta\left(\Psi(0) \dfrac{\partial}{\partial \alpha_1} [\mathscr{L}(\alpha)(\Phi)]\Big|_{\alpha=0}\right) \\ \vdots \\ \Theta\left(\Psi(0) \dfrac{\partial}{\partial \alpha_p} [\mathscr{L}(\alpha)(\Phi)]\Big|_{\alpha=0}\right) \end{pmatrix}$$

has rank c^2. *If, in addition,*

$$\dim \operatorname{span}(\Theta([B, E_{11}]), \Theta([B, E_{12}]), \ldots, \Theta([B, E_{cc}])) = c^2 - p, \tag{5.3}$$

then the versal unfolding $\mathscr{B}(\alpha)$ *is mini-versal.*

Proof. In $\mathrm{Mat}_{c\times c}$, the tangent space of the similarity orbit through B is given by [1]

$$\mathcal{O} = \{[B, X] \colon X \in \mathrm{Mat}_{c\times c}\}.$$

Thus, if the matrix S has rank c^2, we have

$$\mathrm{Mat}_{c\times c} = \mathcal{O} + \Psi(0) D_\alpha [\mathscr{L}(\alpha)(\Phi)]|_{\alpha=0} \cdot \mathbb{C}^p,$$

which implies that the mapping $\alpha \mapsto \mathscr{B}(\alpha)$ is transversal (mini-transversal if (5.3) holds) to the similarity orbit of B at $\alpha = 0$. The conclusion now follows from Proposition 1.1. $\square$

From the previous result, we are now led to define a notion of versal unfolding for RFDEs of type (3.1).

Definition 5.2. The parametrized family of RFDEs (3.1) is said to be a Λ-*versal unfolding* (respectively a Λ-*mini-versal unfolding*) for the RFDE (2.1) if the matrix $\mathscr{B}(\alpha)$ defined by (5.2) is a versal unfolding (respectively a mini-versal unfolding) for B.

6. Decomposition of $Mat_{c\times c}$ by $\boldsymbol{\Psi}(0)$

Consider now the following problem: given a linear homogeneous RFDE such as (2.1) and a set Λ of solutions to the characteristic equation (2.2), find a Λ-versal unfolding (3.1) for (2.1). It is certainly not immediately obvious that this problem admits a solution, since the structure of the right-hand side of (4.9) is severely restricted by the structure of the matrix $\Psi(0)$. To solve this problem, we will need to characterize the subspace of matrices in $\mathrm{Mat}_{c\times c}$ whose columns are in the range of $\Psi(0)$, and show that one can build a versal unfolding of the matrix B in (4.9) even in this restricted context. We start with the following.

Definition 6.1. We define $\mathscr{R}(\Psi(0))$ to be the set of all $c\times c$ matrices whose columns are in the range of the matrix $\Psi(0)$.

The main result we will need in order to construct the above-mentioned versal unfolding is the following:

Proposition 6.2. *Let* $\mathscr{T}:\mathrm{Mat}_{c\times c}\to\mathrm{Mat}_{c\times c}$ *be defined by* $\mathscr{T}(M)=[B,M]\equiv BM-MB$. *There exists a subspace* $\hat{\mathscr{W}}\subset\mathscr{R}(\Psi(0))$ *such that*

$$\mathrm{Mat}_{c\times c}=\mathrm{range}(\mathscr{T})\oplus\hat{\mathscr{W}}. \tag{6.1}$$

This entire section is devoted to proving this result, including giving an explicit construction of $\hat{\mathscr{W}}$. We will start by first proving several lemmas, and then proceed to the proof of Proposition 6.2.

6.1. Commutator of B

We assume that the matrix B is in the following Jordan block diagonal form:

$$B=\mathrm{diag}(B_1^1(\lambda_1),\ldots,B_{k_1}^1(\lambda_1),B_1^2(\lambda_2),\ldots,B_{k_2}^2(\lambda_2),\ldots,B_1^r(\lambda_r),\ldots,B_{k_r}^r(\lambda_r)), \tag{6.2}$$

where $\lambda_1,\ldots,\lambda_r$ are the distinct (without multiplicities) eigenvalues of B, and the Jordan block $B_\ell^j(\lambda_j)$ is $n_{j,\ell}\times n_{j,\ell}$ of the form $\lambda_j I_{n_{j,\ell}}+N_{n_{j,\ell}\times n_{j,\ell}}$, where $N_{n_{j,\ell}\times n_{j,\ell}}$ is the nilpotent matrix with 1's on the upper diagonal, and 0's everywhere else. Moreover, we assume that for each $j \in \{1, \dots, r\}$, we have

$$n_{j,1} \geqslant n_{j,2} \geqslant \cdots \geqslant n_{j,k_j}.$$

The first result we need is the following lemma, which can be found in [10]:

Lemma 6.3. *Let B^* denote the conjugate transpose of B, and let $M \in \mathrm{Mat}_{c\times c}$ be such that $[B^*, M] = 0$. Then M is of the form*

$$M = \mathrm{diag}(\mathscr{M}_1, \dots, \mathscr{M}_r), \tag{6.3}$$

where $\mathscr{M}_j$ is a $(n_{j,1} + \cdots + n_{j,k_j}) \times (n_{j,1} + \cdots + n_{j,k_j})$ matrix, partitioned into blocks with dimensions $n_{j,p} \times n_{j,q}$, and of the form illustrated in Fig. 1, where each oblique segment in each separate block denotes a sequence of equal entries, and all other entries are zero.

The second lemma we need is the following. The proof is left to the appendix.

Lemma 6.4. *Let B be as in (6.2), and consider the mapping $\mathscr{T}: \mathrm{Mat}_{c\times c} \to \mathrm{Mat}_{c\times c}$ given by $\mathscr{T}(M) = [B, M]$. Then $Y \in \mathrm{range}(\mathscr{T})$ if and only if Y is of the form*

$$Y = \begin{pmatrix} \mathscr{Y}_{1,1} & \mathscr{Y}_{1,2} & \cdots & \cdots & \mathscr{Y}_{1,r} \\ \mathscr{Y}_{2,1} & \mathscr{Y}_{2,2} & \cdots & \cdots & \mathscr{Y}_{2,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathscr{Y}_{r,1} & \mathscr{Y}_{r,2} & \cdots & \cdots & \mathscr{Y}_{r,r} \end{pmatrix}, \tag{6.4}$$

where $\mathscr{Y}_{p,q}$ is a $(n_{p,1} + \cdots + n_{p,k_p}) \times (n_{q,1} + \cdots + n_{q,k_q})$ matrix, with the only constraint being that for each $j \in \{1, \dots, r\}$, the sub-matrix $\mathscr{Y}_{j,j}$ is partitioned exactly as $\mathscr{M}_j$ (see Fig. 1), and is such that the sum of the elements in each given oblique segment is zero (however, in contrast to $\mathscr{M}_j$, the elements of $\mathscr{Y}_{j,j}$ which are not on the oblique segments are completely arbitrary, i.e. not necessarily zero).

We now define useful integers which give the number of columns of the matrix B corresponding to the first j eigenvalues. Let $N_0 = 0$, and

$$N_j = N_{j-1} + \sum_{\ell=1}^{k_j} n_{j,\ell}, \quad j = 1, \dots, r-1.$$

Using Lemma 6.4, we now define a set of matrices $\mathscr{S}$ which forms a basis for range($\mathscr{T}$). The set $\mathscr{S}$ consists of all $c \times c$ matrices Y which are partitioned as in (6.4) and whose elements are all zero except one element whose value is 1, and possibly one other element whose value is -1, satisfying the following constraints:

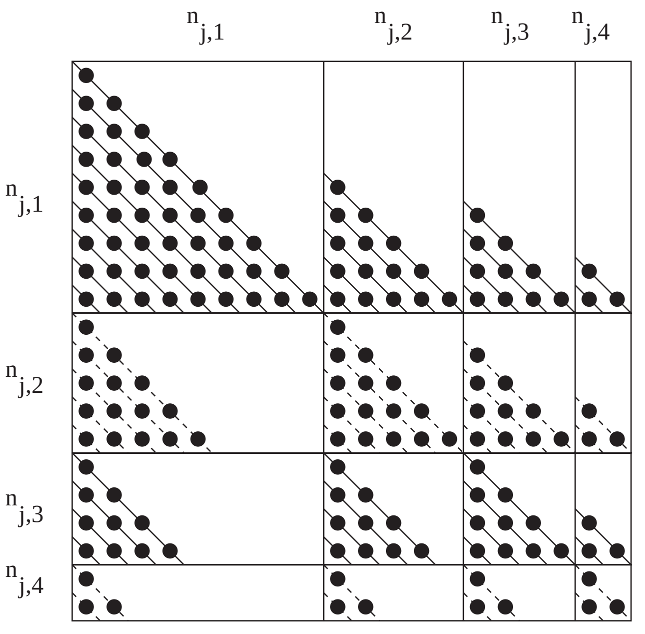

Fig. 1. Structure of a matrix $\mathscr{M}_j$. In this example, $k_j = 4$, $n_{j,1} = 9$, $n_{j,2} = 5$, $n_{j,3} = 4$, $n_{j,4} = 2$.

- if the element whose value is 1 is in the block $\mathscr{Y}_{k,\ell}$ where $k \neq \ell$, then it is the *only* nonzero element in the matrix Y,
- if the element whose value is 1 lies in the block $\mathscr{Y}_{j,j}$ for some $j \in \{1, \dots, r\}$, then:
  - if this element (whose value is 1) is not on any of the oblique segments of Fig. 1, then it is the *only* nonzero element in the matrix Y
  - if this element (whose value is 1) does lie on one of the oblique segments of Fig. 1, then it does not lie in any of the following rows:
$$N_{j-1} + n_{j,1},\ \ N_{j-1} + n_{j,1} + n_{j,2}, \dots, N_{j-1} + n_{j,1} + \cdots + n_{j,k_j}, \tag{6.5}$$
    and there is a '−1' at the bottom end of the oblique segment which contains the '1'.

Note that the row numbers given by (6.5) correspond to the last row of $n_{j,k_j} \times n_{j,k_j}$ blocks as shown in Fig. 1 for each j.

Now consider $\mathscr{W} \subset \mathrm{Mat}_{c \times c}$, where $\mathscr{W}$ is the subspace of matrices of the form
$$\mathrm{diag}(\omega_1, \dots, \omega_r),$$
where ω_j is a $(n_{j,1} + \cdots + n_{j,k_j}) \times (n_{j,1} + \cdots + n_{j,k_j})$ matrix, partitioned into blocks with dimensions $n_{j,p} \times n_{j,q}$, and of the form illustrated in Fig. 2. In this figure, the

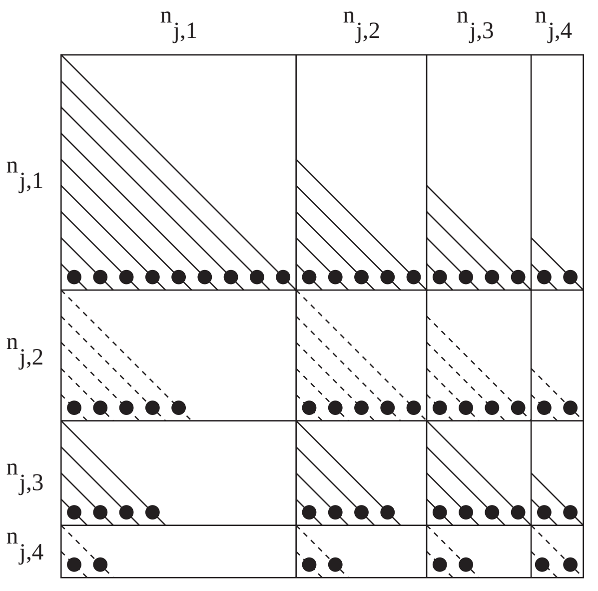

Fig. 2. Structure of a matrix ω_j. In this example, $k_j = 4$. The only elements which are not forced to be zero are those illustrated with a bold dot, at the bottom of each oblique segment.

only elements which are not forced to be zero are those at the bottom of each oblique segment, illustrated with a bold dot. It is well known [10] that

$$\dim(\mathscr{W}) = \sum_{j=1}^{r} \sum_{\ell=1}^{k_j} (2\ell - 1) n_{j,\ell}. \tag{6.6}$$

It follows from the definition of the basis $\mathscr{S}$ of range$(\mathscr{T})$ that

Lemma 6.5.

$$\mathrm{Mat}_{c\times c} = \mathrm{range}(\mathscr{T}) \oplus \mathscr{W}. \tag{6.7}$$

6.2. *Characterizing the range of $\Psi(0)$*

Before we can prove Proposition 6.2, we need to establish some properties about the subspace $\mathscr{R}(\Psi(0))$ defined in Definition 6.1.

Lemma 6.6. *The matrix $\Psi(0) = \mathrm{col}(\psi_1(0), \ldots, \psi_c(0))$ satisfies the following property: for each $j \in \{1, \ldots, r\}$, if $i \in \{N_{j-1} + n_{j,1}, N_{j-1} + n_{j,1} + n_{j,2}, \ldots, N_{j-1} + n_{j,1} + \cdots + n_{j,k_j}\}$ then*

$$\psi_i(0) \neq 0 \quad \textit{and} \quad \psi_i(0)\Delta(\lambda_j) = 0,$$

where the characteristic matrix $\Delta(\lambda)$ *is as in* (2.2). *Moreover*, *for each fixed* $j\in\{1,\dots,r\}$, *the set of row-vectors*

$$\{\psi_{N_{j-1}+n_{j,1}}(0),\psi_{N_{j-1}+n_{j,1}+n_{j,2}}(0),\dots,\psi_{N_{j-1}+n_{j,1}+\dots+n_{j,k_j}}(0)\}$$

is linearly independent in $\mathbb{C}^{n*}$.

Proof. Each row $\psi_i(\theta)$ of the matrix $\Psi(\theta)=e^{-B\theta}\Psi(0)$ must satisfy

$$\dot{\psi}_i(0)=-\int_{-\tau}^{0}\psi_i(-\theta)\,d\eta(\theta). \tag{6.8}$$

Since B has the form (6.2), it follows that for each $j\in\{1,\dots,r\}$, if $i\in\{N_{j-1}+n_{j,1},N_{j-1}+n_{j,1}+n_{j,2},\dots,N_{j-1}+n_{j,1}+\dots+n_{j,k_j}\}$ then the ith row of the matrix $e^{-B\theta}$ is zero except for the diagonal element which is equal to $e^{-\lambda_j\theta}$. It follows that for all $i\in\{N_{j-1}+n_{j,1},N_{j-1}+n_{j,1}+n_{j,2},\dots,N_{j-1}+n_{j,1}+\dots+n_{j,k_j}\}$ we have

$$\psi_i(\theta)=e^{-\lambda_j\theta}\psi_i(0),$$

which when substituted into (6.8) yields $\psi_i(0)\Delta(\lambda_j)=0$. Recall that the columns of $\Psi(\theta)$ form a basis of P^*, therefore the set

$$\{\psi_{N_{j-1}+n_{j,1}}(\theta),\psi_{N_{j-1}+n_{j,1}+n_{j,2}}(\theta),\dots,\psi_{N_{j-1}+n_{j,1}+\dots+n_{j,k_j}}(\theta)\}$$

is linearly independent in $C([0,\tau],\mathbb{C}^{n*})$, and it follows that

$$\{\psi_{N_{j-1}+n_{j,1}}(0),\psi_{N_{j-1}+n_{j,1}+n_{j,2}}(0),\dots,\psi_{N_{j-1}+n_{j,1}+\dots+n_{j,k_j}}(0)\}$$

is linearly independent in $\mathbb{C}^{n*}$. $\square$

Remark 6.7. For each $j\in\{1,\dots,r\}$, let

$$\{\psi_{N_{j-1}+n_{j,1}}(0),\psi_{N_{j-1}+n_{j,1}+n_{j,2}}(0),\dots,\psi_{N_{j-1}+n_{j,1}+\dots+n_{j,k_j}}(0)\}$$

be as in Lemma 6.6. Consider the linear mapping

$$\Pi_j:\mathbb{C}^n\to\mathbb{C}^{k_j}$$

defined by

$$\Pi_j(v)=\mathrm{col}(\psi_{N_{j-1}+n_{j,1}}(0),\psi_{N_{j-1}+n_{j,1}+n_{j,2}}(0),\dots,\psi_{N_{j-1}+n_{j,1}+\dots+n_{j,k_j}}(0))\cdot v.$$

Then it follows from Lemma 6.6 that Π_j is onto $\mathbb{C}^{k_j}$.

6.3. Proof of Proposition 6.2

We are now ready to prove Proposition 6.2. For purposes of clarity, we first prove it in the case where $r = 1$ in (6.2), and then we show how to generalize the arguments of this proof to the case $r > 1$. Before giving the proof, we first establish some useful notation.

In the case where $r = 1$ in (6.2), any matrix $M \in \mathrm{Mat}_{c\times c}$ can be partitioned as

$$M = \begin{pmatrix} \mathscr{M}_{1,1} & \mathscr{M}_{1,2} & \cdots & \cdots & \mathscr{M}_{1,k_1} \\ \mathscr{M}_{2,1} & \mathscr{M}_{2,2} & \cdots & \cdots & \mathscr{M}_{2,k_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathscr{M}_{k_1,1} & \mathscr{M}_{k_1,2} & \cdots & \cdots & \mathscr{M}_{k_1,k_1} \end{pmatrix}, \tag{6.9}$$

where $\mathscr{M}_{\xi,\vartheta}$ is $n_{1,\xi} \times n_{1,\vartheta}$, for $\xi, \vartheta \in \{1, \dots, k_1\}$. It is convenient to label the elements of M according to this partitioning as follows: for $\xi, \vartheta \in \{1, \dots, k_1\}$, $m \in \{1, \dots, n_{1,\vartheta}\}$ and $u \in \{1, \dots, n_{1,\xi}\}$, we denote by $M^{\xi,\vartheta,u,m}$ the element of the matrix M which lies in the block $\mathscr{M}_{\xi,\vartheta}$ in (6.9), at the intersection of row u and column $n_{1,\vartheta} - m + 1$ relative to this block.

Fix $\xi, \vartheta \in \{1, \dots, k_1\}$ and let $\mathscr{Q}(\xi, \vartheta)$ be the set of integers $m \in \{1, \dots, n_{1,\vartheta}\}$ such that column $n_{1,\vartheta} - m + 1$ intersects an oblique line in the block $\mathscr{M}_{\xi,\vartheta}$. In particular,

$$\mathscr{Q}(\xi, \vartheta) = \begin{cases} \{1, \dots, n_{1,\vartheta}\} & \text{if } n_{1,\xi} \geqslant n_{1,\vartheta}, \\ \{n_{1,\vartheta} - n_{1,\xi} + 1, \dots, n_{1,\vartheta}\} & \text{if } n_{1,\xi} < n_{1,\vartheta}. \end{cases}$$

Now, for all $\xi, \vartheta \in \{1, \dots, k_1\}$ and $m \in \mathscr{Q}(\xi, \vartheta)$, we define $\Omega_{\xi,\vartheta,m}$ as the $c \times c$ matrix whose entries are all zero except $\Omega_{\xi,\vartheta,m}^{\xi,\vartheta,n_{1,\xi},m} = 1$. It follows that the set of all matrices $\Omega_{\xi,\vartheta,m}$ thus defined is a basis for the subspace $\mathscr{W}$ in (6.7) (see Fig. 2).

Similarly, fix $\xi, \vartheta \in \{1, \dots, k_1\}$ and $m \in \{1, \dots, n_{1,\vartheta}\}$. Let $\mathscr{P}(\xi, \vartheta, m)$ consist of all $u \in \{1, \dots, n_{1,\xi} - 1\}$ such that row u has an intersection with an oblique line in the block (ξ, ϑ) at column $n_{1,\vartheta} - m + 1$. Of course, if $n_{1,\xi} = 1$, $m \notin \mathscr{Q}(\xi, \vartheta)$ or $m = \min \mathscr{Q}(\xi, \vartheta)$ then $\mathscr{P}(\xi, \vartheta, m) = \emptyset$.

For all $\xi, \vartheta \in \{1, \dots, k_1\}$, choose $m \in \mathscr{Q}(\xi, \vartheta)$ such that $\mathscr{P}(\xi, \vartheta, m)$ is nonempty and choose $u \in \mathscr{P}(\xi, \vartheta, m)$. Define the $c \times c$ matrix $E_{\xi,\vartheta,u,m}$ with only two nonzero entries as follows. Let $E_{\xi,\vartheta,u,m}^{\xi,\vartheta,u,m} = 1$, and $E_{\xi,\vartheta,u,m}^{\xi,\vartheta,n_{1,\xi},p(m,u)} = -1$, where $p(m,u) = m - n_{1,\xi} + u$. Note that $p(m,u)$ is always less than m. See Fig. 3 for an illustration of $\mathscr{Q}(\xi, \vartheta)$, $\mathscr{P}(\xi, \vartheta, m)$ and $p(m,u)$. Each of these matrices $E_{\xi,\vartheta,u,m}$ defined above is in range$(\mathscr{T})$ (Lemma 6.4).

Remark 6.8. It follows from the definitions of the matrices $\Omega_{\xi,\vartheta,m}$ and the matrices $E_{\xi,\vartheta,u,m}$ that if $\vartheta \in \{1, \dots, k_1\}$, $\xi \in \{1, \dots, k_1\}$, $m \in \{1, \dots, n_{1,\vartheta}\}$ and $u \in \mathscr{P}(\xi, \vartheta, m)$, then $p(m,u) \in \mathscr{Q}(\xi, \vartheta)$ and $E_{\xi,\vartheta,u,m} + \Omega_{\xi,\vartheta,p(m,u)}$ is a matrix whose only

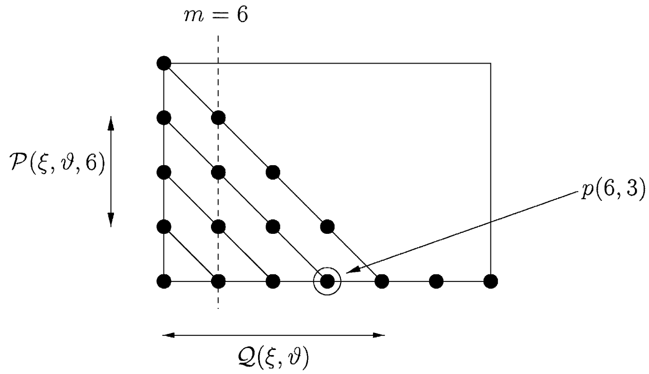

Fig. 3. Example of sets $\mathcal{Q}$, $\mathcal{P}$ and of a point $p(m,u)$ for a 5×7 matrix. Here $\mathcal{Q}=\{3,4,5,6,7\}$ and $\mathcal{P}=\{2,3,4\}$.

nonzero element is

$$(E_{\xi,\vartheta,u,m}+\Omega_{\xi,\vartheta,p(m,u)})^{\xi,\vartheta,u,m}=1.$$

We now proceed to the proof of Proposition 6.2 in the case where $r=1$.

Lemma 6.9. *Suppose B in* (6.2) *is such that* $r=1$. *Then Proposition* 6.2 *holds.*

Proof. We begin by defining a subspace of matrices contained in $\mathcal{R}(\Psi(0))$ and isomorphic to $\mathcal{W}$. We do this in the following way.

By virtue of Remark 6.7, let $v_1,\dots,v_{k_1}\in\mathbb{C}^n$ be such that $\Pi_1(v_\ell)$ is a k_1-dimensional vector whose only nonzero component is the ℓth component, whose value is 1. Now, consider the c-dimensional vector $\Psi(0)v_\ell$. We label the components of this vector using the integers $\xi\in\{1,\dots,k_1\}$ and $u\in\{1,\dots,n_{1,\xi}\}$ as follows: $(\Psi(0)v_\ell)^{\xi,u}$ is the uth component of $\Psi(0)v_\ell$ if $\xi=1$; and the $(n_{1,1}+\cdots+n_{1,\xi-1}+u)$th component of $\Psi(0)v_\ell$ if $\xi>1$. We then have

$$(\Psi(0)v_\ell)^{\xi,u}=\begin{cases}\gamma_{\xi,u,\ell} & \text{if } u\notin\{n_{1,1},\dots,n_{1,k_1}\},\\ 1 & \text{if } \xi=\ell \text{ and } u=n_{1,\xi},\\ 0 & \text{otherwise},\end{cases}\tag{6.10}$$

where the exact values of the coefficients $\gamma_{\xi,u,\ell}$ are not important.

Now for all $\xi,\vartheta\in\{1,\dots,k_1\}$ and $m\in\mathcal{Q}(\xi,\vartheta)$ define $R_{\xi,\vartheta,m}$ to be the $n\times c$ matrix whose $(n_{1,1}+\cdots+n_{1,\vartheta}-m+1)$th column is v_ξ, and all other columns are zero. We then define a linear mapping $\mathcal{E}:\mathcal{W}\to\mathcal{R}(\Psi(0))$ by the following action on the basis elements of $\mathscr{W}$:

$$\mathscr{E}(\Omega_{\xi,\vartheta,m}) = \Psi(0)R_{\xi,\vartheta,m}, \quad \xi,\vartheta \in \{1,\dots,k_1\},\ m \in \mathscr{Q}(\xi,\vartheta). \tag{6.11}$$

It is clear that $\mathscr{E}$ is an isomorphism between $\mathscr{W}$ and $\hat{\mathscr{W}} \equiv \mathscr{E}(\mathscr{W}) \subset \mathscr{R}(\Psi(0))$, since the set

$$\{\mathscr{E}(\Omega_{\xi,\vartheta,m}){:}\ \xi,\vartheta \in \{1,\dots,k_1\},\ m \in \mathscr{Q}(\xi,\vartheta)\}$$

is linearly independent in $\mathrm{Mat}_{c\times c}$.

Our strategy now is to show that

$$\mathscr{W} \subset \mathrm{range}(\mathscr{T}) + \hat{\mathscr{W}},$$

from which it follows from (6.7) that

$$\mathrm{Mat}_{c\times c} = \mathrm{range}(\mathscr{T}) + \hat{\mathscr{W}},$$

and since $\mathscr{W}$ and $\hat{\mathscr{W}}$ are isomorphic, we will then have

$$\mathrm{Mat}_{c\times c} = \mathrm{range}(\mathscr{T}) \oplus \hat{\mathscr{W}}. \tag{6.12}$$

Any matrix $Z \in \mathrm{Mat}_{c\times c}$ can be written as a sum of two matrices as is illustrated in Fig. 4, where the elements which are not on the oblique segments are all zero. It is clear from Lemma 6.4 that the second summand in Fig. 4 belongs to $\mathrm{range}(\mathscr{T})$. Thus, for any $Z \in \mathrm{Mat}_{c\times c}$, we define $\Gamma(Z)$ to be the first summand in this decomposition, as illustrated in Fig. 4. It immediately follows that for any $Z \in \mathrm{Mat}_{c\times c}$, we have $Z - \Gamma(Z) \in \mathrm{range}(\mathscr{T})$.

Fix a value of $\vartheta \in \{1,\dots,k_1\}$. For all $\xi \in \{1,\dots,k_1\}$, if $1 \in \mathscr{Q}(\xi,\vartheta)$ then $\Gamma(\Psi(0)R_{\xi,\vartheta,1}) = \Omega_{\xi,\vartheta,1}$, from which it follows that

$$\Omega_{\xi,\vartheta,1} \in \mathrm{range}(\mathscr{T}) + \hat{\mathscr{W}}, \quad \forall \xi \in \{1,\dots,k_1\} \text{ such that } 1 \in \mathscr{Q}(\xi,\vartheta).$$

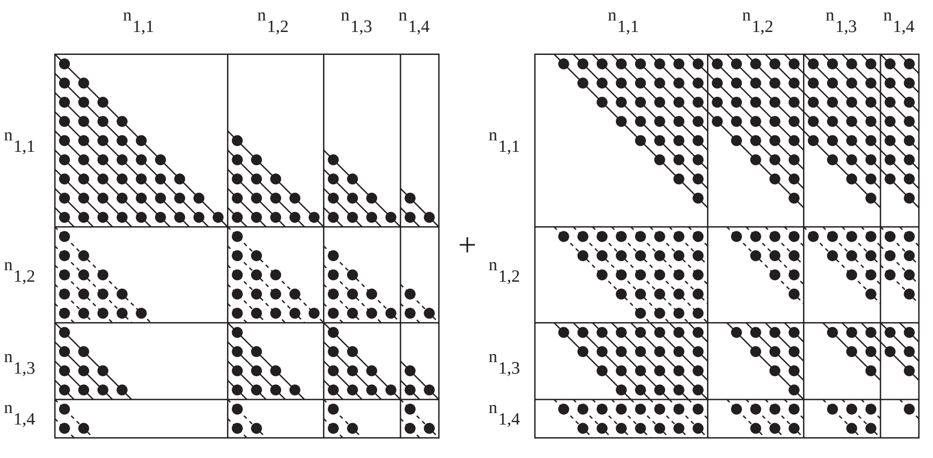

Fig. 4. Decomposition of $Z \in \mathrm{Mat}_{c\times c}$. In this example, $k_1 = 4$. The elements which are not on the oblique segments are all zero. The first summand is $\Gamma(Z)$. The second summand is in $\mathrm{range}(\mathscr{T})$.

If $n_{1,\vartheta} > 1$, let μ be an integer in the range $1, \dots, n_{1,\vartheta} - 1$ such that for all $m = 1, \dots, \mu$, we have

$$\Omega_{\xi,\vartheta,m} \in \operatorname{range}(\mathscr{T}) + \hat{\mathscr{W}}, \quad \forall \xi \in \{1, \dots, k_1\} \text{ such that } m \in \mathscr{Q}(\xi, \vartheta). \tag{6.13}$$

We claim that this implies that

$$\Omega_{\xi,\vartheta,\mu+1} \in \operatorname{range}(\mathscr{T}) + \hat{\mathscr{W}}, \quad \forall \xi \in \{1, \dots, k_1\} \text{ such that } \mu + 1 \in \mathscr{Q}(\xi, \vartheta). \tag{6.14}$$

From this claim, and the fact that the above arguments are independent of the particular choice of ϑ, it follows by induction that (6.12) holds. It thus remains to prove the claim.

Suppose that $\xi \in \{1, \dots, k_1\}$ is such that $\mu + 1 \in \mathscr{Q}(\xi, \vartheta)$. Then a simple computation shows that

$$\Gamma(\Psi(0)R_{\xi,\vartheta,\mu+1}) = \Omega_{\xi,\vartheta,\mu+1} + G_{\xi,\vartheta,\mu+1}, \tag{6.15}$$

where $G_{\xi,\vartheta,\mu+1}$ is a $c \times c$ matrix whose columns are all zero except possibly the $(n_{1,1} + \cdots + n_{1,\vartheta} - \mu)$th column, whose (χ, u) component is given by the formula

$$\begin{cases} \gamma_{\chi,u,\xi} & \text{if } u \in \mathscr{P}(\chi, \vartheta, \mu + 1), \\ 0 & \text{otherwise}, \end{cases}$$

where the coefficients $\gamma_{\chi,u,\xi}$ are as in (6.10). Now, $\Psi(0)R_{\xi,\vartheta,\mu+1} - \Gamma(\Psi(0)R_{\xi,\vartheta,\mu+1}) \in \operatorname{range}(\mathscr{T})$ implies that $\Gamma(\Psi(0)R_{\xi,\vartheta,\mu+1}) \in \operatorname{range}(\mathscr{T}) + \hat{\mathscr{W}}$. From Remark 6.8,

$$G_{\xi,\vartheta,\mu+1} = \sum_{\chi=1}^{k_1} \sum_{u \in \mathscr{P}(\chi,\vartheta,\mu+1)} \gamma_{\chi,u,\xi}(E_{\chi,\vartheta,u,\mu+1} + \Omega_{\chi,\vartheta,p(\mu+1,u)}) \tag{6.16}$$

from which it follows that (6.14) holds since $E_{\chi,\vartheta,u,\mu+1} \in \operatorname{range}(\mathscr{T})$ and $\Omega_{\chi,\vartheta,p(\mu+1,u)} \in \operatorname{range}(\mathscr{T}) + \hat{\mathscr{W}}$ from the fact that $p(\mu + 1, u) < \mu + 1$ and by the induction hypothesis. $\square$

Proof of Proposition 6.2. The essential idea here is to decompose the proof into r separate blocks, where we use the arguments of the proof of Lemma 6.9 on each of the blocks.

First, for all $j \in \{1, \dots, r\}$, we let $\mathscr{W}_j$ denote the subspace of $\mathscr{W}$ consisting of matrices whose columns are all zero except the columns between $N_{j-1} + 1$ and N_j inclusively.

As in the proof of Lemma 6.9, we construct a basis $\{\Omega_{j;\xi,\vartheta,u,m}\}$ of $\mathscr{W}_j$, and we note that the space $\mathscr{W}$ in (6.7) is equal to $\mathscr{W}_1 \oplus \cdots \oplus \mathscr{W}_r$. From Remark 6.7, for each $j \in \{1, \dots, r\}$ we can choose vectors $v_{j,1}, \dots, v_{j,k_j}$ which are such that the vector $\Pi_j(v_{j,\ell})$ has a 1 in row ℓ and 0's everywhere else. For each $j \in \{1, \dots, r\}$, we then construct matrices $R_{j;\xi,\vartheta,m}$ in a similar manner as we did in Lemma 6.9, whose only nonzero column is equal to one of the vectors $v_{j,1}, \dots, v_{j,k_j}$ described above. We then define linear mappings $\mathscr{E}_j : \mathscr{W}_j \to \mathscr{R}(\Psi(0))$ as in (6.11), and it follows that $\mathscr{W}_j$ is isomorphic to $\hat{\mathscr{W}}_j = \mathscr{E}_j(\mathscr{W}_j)$.

Now, any $Z \in \mathrm{Mat}_{c\times c}$ can be partitioned as in (6.4), i.e.

$$Z = \begin{pmatrix} \mathscr{Z}_{1,1} & \mathscr{Z}_{1,2} & \cdots & \cdots & \mathscr{Z}_{1,r} \\ \mathscr{Z}_{2,1} & \mathscr{Z}_{2,2} & \cdots & \cdots & \mathscr{Z}_{2,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathscr{Z}_{r,1} & \mathscr{Z}_{r,2} & \cdots & \cdots & \mathscr{Z}_{r,r} \end{pmatrix}, \tag{6.17}$$

where $\mathscr{Z}_{p,q}$ is a $(n_{p,1} + \cdots + n_{p,k_p}) \times (n_{q,1} + \cdots + n_{q,k_q})$ matrix. Rewrite Z as

$$Z = \begin{pmatrix} \mathscr{Z}_{1,1} & 0 & \cdots & \cdots & 0 \\ 0 & \mathscr{Z}_{2,2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \mathscr{Z}_{r,r} \end{pmatrix} + \begin{pmatrix} 0 & \mathscr{Z}_{1,2} & \cdots & \cdots & \mathscr{Z}_{1,r} \\ \mathscr{Z}_{2,1} & 0 & \cdots & \cdots & \mathscr{Z}_{2,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathscr{Z}_{r,1} & \mathscr{Z}_{r,2} & \cdots & \cdots & 0 \end{pmatrix}, \tag{6.18}$$

where the second summand on the right-hand side of (6.18) belongs to range$(\mathscr{T})$ (Lemma 6.4). Furthermore, following the same method as in Lemma 6.9, for each $j \in \{1, \dots, r\}$ we write $\mathscr{Z}_{j,j}$ as a sum of matrices as is illustrated in Fig. 4, where the second summand belongs to range$(\mathscr{T})$. Thus, for Z as in (6.17), we define $\Gamma(Z)$ as the first summand on the right-hand side of (6.18), where the blocks $\mathscr{Z}_{j,j}$ are of the form of the first summand in Fig. 4. It follows that for any $Z \in \mathrm{Mat}_{c\times c}$, we have $Z - \Gamma(Z) \in \mathrm{range}(\mathscr{T})$.

Finally, we use an induction argument similar to that used in the proof of Lemma 6.9 to show that for each $j \in \{1, \dots, r\}$, we have $\mathscr{W}_j \subset \mathrm{range}(\mathscr{T}) + \hat{\mathscr{W}}_j$. We thus define $\hat{\mathscr{W}} = \hat{\mathscr{W}}_1 \oplus \cdots \oplus \hat{\mathscr{W}}_r$. This completes the proof. $\square$

7. Construction of a Λ-versal unfolding

In this section, we will solve the problem which was posed at the beginning of Section 6; that is, given a linear homogeneous RFDE such as (2.1) and a set Λ of solutions to the characteristic equation (2.2), find a Λ-versal unfolding (3.1) for (2.1).

7.1. *Main result*

The following useful result is proven in [8,12].

Lemma 7.1. *Let I be an interval in $\mathbb{R}$ and suppose that $\varphi_1, \ldots, \varphi_c \in C(I, \mathbb{C}^n)$ are linearly independent functions. For each $\theta \in I$, define $\Phi_c(\theta) = (\varphi_1(\theta), \ldots, \varphi_c(\theta))$. For a fixed $\tau_0 \in I$, denote by q the rank of the $n \times c$ matrix $\Phi_c(\tau_0)$. Then there exist $c - q$ distinct points $\tau_1, \ldots, \tau_{c-q} \in I \backslash \{\tau_0\}$ such that the $n(c-q+1) \times c$ matrix $\mathrm{col}(\Phi_c(\tau_0), \ldots, \Phi_c(\tau_{c-q}))$ has rank c.*

Now, suppose $\varphi_1, \ldots, \varphi_c$ are the basis elements of the space P in (2.3), and let $\Phi(\theta) = (\varphi_1(\theta), \ldots, \varphi_c(\theta))$, and $I = [-\tau, 0]$. Since $\dot{\Phi} = \Phi B$, it follows that the rows of Φ are solutions to the vector ordinary differential equation $\dot{\xi} = \xi B$, and so the rank of the matrix $\Phi(\theta)$ is independent of θ [14]. We now have the following

Proposition 7.2. *Let $q = \mathrm{rank}(\Phi(0))$, and let $\tau_0, \ldots, \tau_{c-q}$ be as in Lemma 7.1 with $I = [-\tau, 0]$. Then for any $n \times c$ matrix R, there exist matrices $A_0, \ldots, A_{c-q} \in \mathrm{Mat}_{n \times n}$ such that*

$$R = \sum_{j=0}^{c-q} A_j \Phi(\tau_j). \tag{7.1}$$

Proof. Define the linear mapping $\mathscr{K} : \mathrm{Mat}_{n \times (n(c-q+1))} \to \mathrm{Mat}_{n \times c}$ by

$$\mathscr{K}(\mathscr{A}) = \mathscr{A} \cdot \mathrm{col}(\Phi(\tau_0), \ldots, \Phi(\tau_{c-q})).$$

If we associate $\mathrm{Mat}_{n \times (n(c-q+1))} \cong \mathbb{C}^{n(n(c-q+1))}$ and $\mathrm{Mat}_{n \times c} \cong \mathbb{C}^{nc}$ in the standard way, then the $nc \times n(n(c-q+1))$ matrix representation of $\mathscr{K}$ is given by

$$\mathscr{K} \sim I_n \otimes (\mathrm{col}(\Phi(\tau_0), \ldots, \Phi(\tau_{c-q})))^T,$$

whose rank is nc. Thus, $\mathscr{K}$ is onto $\mathrm{Mat}_{n \times c}$. For a given $R \in \mathrm{Mat}_{n \times c}$, let $\mathscr{A} \in \mathrm{Mat}_{n \times (n(c-q+1))}$ be such that $\mathscr{K}(\mathscr{A}) = R$. The conclusion of the proposition follows by partitioning the matrix $\mathscr{A}$ as $\mathscr{A} = (A_0, \ldots, A_{c-q})$, where the A_j are $n \times n$ matrices, $j = 0, \ldots, c-q$. □

Remark 7.3. The crucial element in the proof of Proposition 7.2 is the fact that the rank of the matrix $\mathrm{col}(\Phi(\tau_0), \ldots, \Phi(\tau_{c-q}))$ is equal to c. Lemma 7.1 assures us that we can always achieve this if we use $q = \mathrm{rank}(\Phi(0))$ and if the delay times $\tau_0, \ldots, \tau_{c-q}$ are chosen appropriately. However, as is noted in [8], in certain cases it is possible to achieve $\mathrm{rank}(\mathrm{col}(\Phi(\tau_0), \ldots, \Phi(\tau_{c-q}))) = c$ with a value of q larger than $\mathrm{rank}(\Phi(0))$. The following example was given in [8]. Consider the linear RFDE (2.1) on $\mathbb{C}^2$ such that P is four dimensional with basis matrix

$$\Phi(\theta) = \begin{pmatrix} e^{i\omega_1\theta} & e^{-i\omega_1\theta} & 0 & 0 \\ 0 & 0 & e^{i\omega_2\theta} & e^{-i\omega_2\theta} \end{pmatrix},$$

where ω_1 and ω_2 are distinct real numbers. Thus, we have $c = 4$ and $\mathrm{rank}(\Phi(0)) = 2$. However, for any τ_0 and τ_1 such that $\tau_0 \neq \tau_1$, we have

$$\mathrm{rank}(\mathrm{col}(\Phi(\tau_0), \Phi(\tau_1))) = \mathrm{rank}\begin{pmatrix} e^{i\omega_1\tau_0} & e^{-i\omega_1\tau_0} & 0 & 0 \\ 0 & 0 & e^{i\omega_2\tau_0} & e^{-i\omega_2\tau_0} \\ e^{i\omega_1\tau_1} & e^{-i\omega_1\tau_1} & 0 & 0 \\ 0 & 0 & e^{i\omega_2\tau_1} & e^{-i\omega_2\tau_1} \end{pmatrix} = 4.$$

From Proposition 6.2, we know that

$$\mathrm{Mat}_{c\times c} = \mathrm{range}(\mathscr{T}) \oplus \hat{\mathscr{W}},$$

where $\hat{\mathscr{W}}$ is isomorphic to $\mathscr{W}$ (see (6.7)), and the isomorphism $\mathscr{E}$ is described in the proof of Lemma 6.9 and Proposition 6.2. Let $\{\Omega_1, \ldots \Omega_\delta\}$ be a basis for $\mathscr{W}$. From Proposition 7.2, there exist matrices $A_0^m, \ldots, A_{c-q}^m \in \mathrm{Mat}_{n\times n}$ such that

$$\mathscr{E}(\Omega_m) = \sum_{j=0}^{c-q} \Psi(0) A_j^m \Phi(\tau_j), \quad m = 1, \ldots, \delta, \tag{7.2}$$

where the set $\{\mathscr{E}(\Omega_m)\}$ spans a complement to $\mathrm{range}(\mathscr{T})$ in $\mathrm{Mat}_{c\times c}$. Clearly the previous statement still holds if instead of being as previously defined, $\mathscr{E}$ is any injective linear mapping from $\mathscr{W}$ into $\mathscr{R}(\Psi(0))$ whose range is a complement to $\mathrm{range}(\mathscr{T})$ in $\mathrm{Mat}_{c\times c}$.

The following is our main result of this section, and follows from Theorem 5.1, Propositions 6.2 and 7.2, and the previous discussion.

Theorem 7.4. *Let* $\{\Omega_1, \ldots, \Omega_\delta\}$ *be a basis for* $\mathscr{W}$ (*see* (6.7)), *and let* $\tau_0, \ldots, \tau_{c-q} \in [-\tau, 0]$ *be such that*

$$\mathrm{rank}(\mathrm{col}(\Phi(\tau_0), \ldots, \Phi(\tau_{c-q}))) = c.$$

Let $\mathscr{E}$ *be an injective linear mapping from* $\mathscr{W}$ *into* $\mathscr{R}(\Psi(0))$, *such that*

$$\mathrm{Mat}_{c\times c} = \mathrm{range}(\mathscr{T}) \oplus \mathscr{E}(\mathscr{W})$$

(*there exists at least one such* $\mathscr{E}$), *and let* A_j^m ($m \in \{1, \ldots, \delta\}, j \in \{0, \ldots, c-q\}$) *be* $n \times n$ *matrices which solve* (7.2). *For each* $m \in \{1, \ldots, \delta\}$, *let* L_m *be the bounded linear operator from* $C([-\tau, 0], \mathbb{C}^n)$ *into* $\mathbb{C}^n$ *defined by*

$$L_m(z) = \sum_{j=0}^{c-q} A_j^m z(\tau_j). \tag{7.3}$$

Let $\mathscr{L}(\alpha)$ *be the* δ*-parameter family of bounded linear operators from* $C([-\tau, 0], \mathbb{C}^n)$ *into* $\mathbb{C}^n$ *defined by*

$$\mathscr{L}(\alpha) = \mathscr{L}_0 + \sum_{m=1}^{\delta} \alpha_m L_m, \tag{7.4}$$

where the α_m *are complex parameters, and* $\mathscr{L}_0$ *is as in* (2.1). *Then* (3.1) *is a* Λ*-miniversal unfolding of* (2.1).

7.2. First-order scalar equations

In the case of first-order scalar linear RFDEs, that is $z_t \in C_1$ in (2.1), we are interested in the following question. Rewrite (7.4) as follows:

$$\mathscr{L}(\alpha)z = \mathscr{L}_0 z + \sum_{j=0}^{c-q} \left(\sum_{m=1}^{\delta} \alpha_m A_j^m \right) z(\tau_j). \tag{7.5}$$

We wish to show that it is possible to find a change of coordinates which simplifies (7.5) to

$$\mathscr{L}(\beta)z = \mathscr{L}_0 z + \sum_{j=0}^{c-q} \beta_j z(\tau_j), \tag{7.6}$$

where $\beta_j \in \mathbb{C}$ for all $j = 0, \ldots, c-q$. Recall that B is a $c \times c$ matrix and $\delta = \dim(\mathscr{W})$. We have the following result.

Proposition 7.5. *If* (2.1) *is a scalar equation, then* $\delta = c$.

Proof. By Theorem 5.1 of [8], since $n = 1$, the number of Jordan blocks for each eigenvalue of B is 1. Thus $c = n_{1,1} + \cdots + n_{r,1}$. Now, δ is given by (6.6) with $k_j = 1$ for all $j = 1, \ldots, r$. Hence the equality holds. □

Theorem 7.6. *Consider a* Λ*-versal unfolding of a scalar equation* (2.1) *given by* (7.4). *Then there exists a change of coordinates in parameter space* $\mathbb{C}^\delta$ *which brings* (7.5) *to* (7.6) *where* $q = 1$, $\beta = (\beta_0, \ldots, \beta_{c-1})$ *with* $\beta_j \in \mathbb{C}$ *for* $j = 0, \ldots, c-1$.

Proof. Since $n = 1$, $q = \mathrm{rank}(\Phi(0)) = 1$ and therefore c delays $\tau_0, \ldots, \tau_{c-1}$ are necessary to solve Eq. (7.1). Let $R_1, \ldots, R_c$ be the $1 \times c$ matrices such that $\mathscr{E}(\Omega_m) = \Psi(0) R_m$. By Proposition 7.2,

$$R_m = \sum_{j=0}^{c-1} A_j^m \Phi(\tau_j)$$

with $A_j^m \in \mathbb{C}$. We can rewrite this equation as a matrix equation

$$R_m^T = (\Phi(\tau_0)^T, \ldots, \Phi(\tau_{c-1})^T) A^m \equiv \mathscr{P} A^m,$$

where $A^m = (A_0^m, \ldots, A_{c-1}^m)^T$ and ^{T} is transposition. By choice of $\tau_0, \ldots, \tau_{c-1}$, the determinant of $\mathscr{P}$ is nonzero so that $A^m = \mathscr{P}^{-1} R_m^T$.

Let $\beta = (\beta_0, \dots, \beta_{c-1})^T$ and $\alpha = (\alpha_1, \dots, \alpha_c)^T$. Since $\delta = c$, set

$$\beta = (A^1, \dots, A^c)\alpha.$$

Now $(A^1, \dots, A^c) = \mathscr{P}^{-1}(R_1^T, \dots, R_c^T)$ is nonsingular since $\mathscr{P}^{-1}$ is a nonsingular $c \times c$ matrix and we claim that $(R_1^T, \dots, R_c^T)$ is also a nonsingular $c \times c$ matrix. Hence, this change of coordinates yields the result.

We now prove the claim. Again by Theorem 5.1 of [8], the number of Jordan blocks for each eigenvalue of B is 1. The set of $1 \times c$ vectors $\{R_1, \dots, R_c\}$ is partitioned into subsets $\{R_{j;1,1,m} \mid m \in \mathscr{Q}(1,1)\}$, for each $j \in \{1, \dots, r\}$ and defined as in the proof of Lemma 6.9 and Proposition 6.2. That is, the only nonzero element in $R_{j;1,1,m}$ is in the $(n_{1,1} + n_{2,1} + \cdots + n_{j,1} - m + 1)$th column. Hence the vectors $R_1, \dots, R_c$ are linearly independent since the unique nonzero element for each vector lies in a different column. $\square$

7.3. Independence of unfolding on choice of basis

Suppose we perform changes of bases in the spaces P and P^* (see (2.3)), $\Psi^\# = U^{-1}\Psi$ and $\Phi^\# = \Phi U$, where U is an invertible $c \times c$ matrix. Then obviously we have $(\Psi^\#, \Phi^\#)_n = I_c$, and Eq. (4.9) transforms into

$$\dot{x} = B^\# x + \Psi^\#(0)[\mathscr{L}(\alpha) - \mathscr{L}_0](\Phi^\# + h(\alpha)U)x, \tag{7.7}$$

where $B^\# = U^{-1}BU$. Let T_U be the linear invertible transformation of $\mathrm{Mat}_{c\times c}$ defined by $T_U(M) = U^{-1}MU$, let $\mathscr{T}^\# : \mathrm{Mat}_{c\times c} \to \mathrm{Mat}_{c\times c}$ be defined by $\mathscr{T}^\#(M) = [B^\#, M]$, and let $\mathscr{R}(\Psi^\#(0))$ be the set of all $c \times c$ matrices whose columns are in the range of the matrix $\Psi^\#(0)$. Then it is easy to show that

$$\mathrm{range}(\mathscr{T}^\#) = T_U(\mathrm{range}(\mathscr{T}))$$

and

$$\mathscr{R}(\Psi^\#(0)) = T_U(\mathscr{R}(\Psi(0))).$$

Consequently, from (6.1), we have that

$$\mathrm{Mat}_{c\times c} = \mathrm{range}(\mathscr{T}^\#) \oplus \hat{\mathscr{W}}^\#,$$

where $\hat{\mathscr{W}}^\# = T_U(\hat{\mathscr{W}})$. Let $\{\Omega_1, \dots, \Omega_\delta\}$ be a basis for $\mathscr{W}$ (see (6.7)), and $\mathscr{E}$ as in Theorem 7.4. Let $A_0^m, \dots, A_{c-q}^m$ be $n \times n$ matrices such that

$$\mathscr{E}(\Omega_m) = \sum_{j=0}^{c-q} \Psi(0) A_j^m \Phi(\tau_j), \quad m = 1, \dots, \delta. \tag{7.8}$$

Then

$$T_U(\mathscr{E}(\Omega_m)) = U^{-1}\mathscr{E}(\Omega_m)U = \sum_{j=0}^{c-q} U^{-1}\Psi(0)A_j^m\Phi(\tau_j)U$$
$$= \sum_{j=0}^{c-q} \Psi^{\#}(0)A_j^m\Phi^{\#}(\tau_j), \quad m = 1, \dots, \delta,$$

i.e. given the delay times $\tau_0, \dots, \tau_{c-q}$ which are such that Lemma 7.1 holds, the matrices $A_0^m, \dots, A_{c-q}^m$ which solve (7.8) are such that the parametrized family (7.4) generates a Λ-mini-versal unfolding of (2.1) independently of the choice of bases matrices Φ and Ψ for P and P^* respectively, provided that $(\Psi, \Phi)_n = I_c$.

7.4. Decomplexification

In applications, it is usually the case that $\mathscr{L}_0$ in (2.1) and $\mathscr{L}(\alpha)$ in (3.1) are real, i.e. they are bounded linear operators from $C([-\tau, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$. Although the previous theory has been carried out in complex spaces, it is straightforward to construct real versal unfoldings by a simple process of decomplexification of (7.4). We need to assume that the set Λ defined in Section 2 is invariant under complex conjugation (one important example where this is always the case is center manifold reduction, in which case Λ is the set of all roots of (2.2) with zero real parts).

Theorem 7.7. *Suppose that $\Lambda = \{\Lambda_0, \Lambda_h, \overline{\Lambda_h}\}$ where Λ_0 is a subset of real eigenvalues and Λ_h a subset of nonreal eigenvalues. Then, a real Λ-mini-versal unfolding of* (2.1) *is given by*

$$\mathscr{L}(\alpha) = \mathscr{L}_0 + \sum_{p=1}^{\delta_0} \alpha_p L_p + \sum_{s=\delta_0+1}^{\delta_0+\delta_h} (\alpha_s \operatorname{Re}(L_s) + \alpha_{s+\delta_h} \operatorname{Im}(L_s)), \tag{7.9}$$

where $\alpha_p \in \mathbb{R}$ for $p = 1, \dots, \delta_0$, $\alpha_s, \alpha_{s+\delta_h} \in \mathbb{R}$ for $s = \delta_0 + 1, \dots, \delta_0 + \delta_h$, L_p is a bounded linear operator from $C([-\tau, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$ for $p = 1, \dots, \delta_0$, and L_s is a bounded linear operator from $C([-\tau, 0], \mathbb{R}^n)$ into $\mathbb{C}^n$, for $s = \delta_0 + 1, \dots, \delta_0 + \delta_h$.

Proof. The $c \times c$ matrix B can be decomposed as $B = \operatorname{diag}(B^0, B^h, \overline{B^h})$ with $c = c_0 + 2c_h$, where B^0 is the $c_0 \times c_0$ diagonal block of real eigenvalues Λ_0 in real Jordan canonical form and B^h is the $c_h \times c_h$ diagonal block of nonreal eigenvalues Λ_h in complex Jordan canonical form.

We establish the following notation for the remainder of the proof. We let $\tilde{\varphi}_1, \dots, \tilde{\varphi}_{c_0}$ and $\varphi_1, \dots, \varphi_{c_h}$ be bases for the generalized eigenspace corresponding to the eigenvalues of Λ_0 and Λ_h, respectively, chosen so that the matrices $\Phi^0 = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_{c_0})$ and $\Phi^h = (\varphi_1, \dots, \varphi_{c_h})$ satisfy, respectively, $\dot{\Phi}^0 = \Phi^0 B^0$ and $\dot{\Phi}^h = \Phi^h B^h$. Consequently, if we set $\Phi = (\Phi^0, \Phi^h, \overline{\Phi^h})$, then the columns of Φ form a basis for P, and we have $\dot{\Phi} = \Phi B$. Now, let $\tilde{\psi}_1^*, \ldots, \tilde{\psi}_{c_0}^*$ be real linearly independent functions in P^* (corresponding to Λ_0) and $\psi_1^*, \ldots, \psi_{c_h}^*$ be nonreal linearly independent functions in P^* (corresponding to Λ_h). If we denote $\Psi^{0*} = \mathrm{col}(\tilde{\psi}_1^*, \ldots, \tilde{\psi}_{c_0}^*)$ and $\Psi^{h*} = \mathrm{col}(\psi_1^*, \ldots, \psi_{c_h}^*)$, then $\Psi^* = \mathrm{col}(\Psi^{0*}, \Psi^{h*}, \overline{\Psi^{h*}})$ is a basis for P^*. Define $\Psi = (\Psi^*, \Phi)_n^{-1} \Psi^*$, then $(\Psi, \Phi)_n = I_c$. Moreover, a simple computation shows that $\Psi = \mathrm{col}(\Psi^0, \Psi^h, \overline{\Psi^h})$, where Ψ^0 is a $c_0 \times n$ real matrix corresponding to Λ_0 whose rows are linearly independent, and Ψ^h is a $c_h \times n$ nonreal matrix corresponding to Λ_h whose rows are linearly independent.

We define three projections Π_0, Π_1, Π_2 as in Remark 6.7 where Π_0 corresponds to Ψ^0 and has all real components while Π_1 and Π_2 correspond to Ψ^h and $\overline{\Psi^h}$. Consequently, the mappings Π_1 and Π_2 are such that

$$\overline{\Pi_1} = \Pi_2. \tag{7.10}$$

Now let $\{\Omega_1, \ldots, \Omega_\delta\}$ be the elements of the basis of $\mathscr{W}$ used in the proof of Lemma 6.9 and Proposition 6.2. Assume that these basis elements are ordered so that the set $\{\Omega_1, \ldots, \Omega_{\delta_0}\}$ corresponds to the block B^0, $\{\Omega_{\delta_0+1}, \ldots, \Omega_{\delta_0+\delta_h}\}$ corresponds to the block B^h, $\{\Omega_{\delta_0+1+\delta_h}, \ldots, \Omega_{\delta_0+2\delta_h}\}$ corresponds to the block $\overline{B^h}$, with

$$\Omega_p = \begin{pmatrix} \Omega_p^0 & 0_{c_0 \times c_h} & 0_{c_0 \times c_h} \\ 0_{c_h \times c_0} & 0_{c_h \times c_h} & 0_{c_h \times c_h} \\ 0_{c_h \times c_0} & 0_{c_h \times c_h} & 0_{c_h \times c_h} \end{pmatrix}, \quad p = 1, \ldots, \delta_0,$$

$$\Omega_s = \begin{pmatrix} 0_{c_0 \times c_0} & 0_{c_0 \times c_h} & 0_{c_0 \times c_h} \\ 0_{c_h \times c_0} & \Omega_s^h & 0_{c_h \times c_h} \\ 0_{c_h \times c_0} & 0_{c_h \times c_h} & 0_{c_h \times c_h} \end{pmatrix}, \quad \Omega_{s+\delta_h} = \begin{pmatrix} 0_{c_0 \times c_0} & 0_{c_0 \times c_h} & 0_{c_0 \times c_h} \\ 0_{c_h \times c_0} & 0_{c_h \times c_h} & 0_{c_h \times c_h} \\ 0_{c_h \times c_0} & 0_{c_h \times c_h} & \Omega_s^h \end{pmatrix},$$
$$s = \delta_0 + 1, \ldots, \delta_0 + \delta_h,$$

where $0_{k \times \ell}$ is the $k \times \ell$ zero matrix, Ω_p^0 is a $c_0 \times c_0$ matrix with only one nonzero element, and Ω_s^h is a $c_h \times c_h$ matrix with only one nonzero element. Define $R_p = (R_p^0, 0, 0)$ for $p = 1, \ldots, \delta_0$ where 0 is the $n \times c_h$ zero matrix and R_p^0 is a $n \times c_0$ matrix with only one nonzero column, corresponding to a vector $v_l \in \mathbb{R}^n$ chosen as in Proposition 6.2, such that $\Psi(0) R_p$ has a 1 at the same position as the 1 in Ω_p. Note that v_l can be chosen in $\mathbb{R}^n$ since Π_0 is real. Now, for all $p = 1, \ldots, \delta_0$, by Proposition 7.2 we can find matrices $A_1^p, \ldots, A_{c-q}^p$ such that

$$(R_p^0, 0, 0) = \sum_{j=0}^{c-q} A_j^p (\Phi_0(\tau_j), \Phi^h(\tau_j), \overline{\Phi^h(\tau_j)}),$$

where we have written $\Phi(\tau_j) \equiv (\Phi^0(\tau_j), \Phi^h(\tau_j), \overline{\Phi^h(\tau_j)})$. The matrices A_j^p can be chosen to be real matrices for all $p = 1, \ldots, \delta_0$. We then let L_p be the bounded linear operator from $C([-\tau, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$ defined by $L_p(z) = \sum_{j=0}^{c-q} A_j^p z(\tau_j)$ for $p = 1, \ldots, \delta_0$.

Now, define $R_s = (0, R_s^h, 0)$ and $R_{s+\delta_h} = (0, 0, \overline{R_s^h})$ for $s = \delta_0 + 1, \ldots, \delta_0 + \delta_h$, where the first 0 is the $n \times c_0$ zero matrix and the second 0 designates the $n \times c_h$ zero matrix. The $n \times c_h$ matrix R_s^h has only one nonzero column, corresponding to a vector $v_l \in \mathbb{C}^n$ chosen as in Proposition 6.2, such that $\Psi(0)R_s$ has a one at the same position as the 1 in Ω_s. It then follows (from (7.10)) that $\Psi(0)\overline{R_s}$ has a one at the same position as the 1 in $\Omega_{s+\delta_h}$. Then, we set $\mathscr{E}(\Omega_s) = \Psi(0)R_s$ and $\mathscr{E}(\Omega_{s+\delta_h}) = \Psi(0)R_{s+\delta_h}$. From Proposition 7.2, there exist matrices $A_0^s, \ldots, A_{c-q}^s$ such that

$$(0, R_s^h, 0) = \sum_{j=0}^{c-q} A_j^s(\Phi^0(\tau_j), \Phi^h(\tau_j), \overline{\Phi^h(\tau_j)}),$$

which we rewrite as the system

$$\sum_{j=0}^{c-q} A_j^s \Phi^0(\tau_j) = 0, \quad \sum_{j=0}^{c-q} A_j^s \Phi^h(\tau_j) = R_s^h, \quad \sum_{j=0}^{c-q} A_j^s \overline{\Phi^h(\tau_j)} = 0.$$

If we take complex conjugates of the above system, we get

$$\sum_{j=0}^{c-q} \overline{A_j^s} \Phi^0(\tau_j) = 0, \quad \sum_{j=0}^{c-q} \overline{A_j^s}\, \overline{\Phi^h(\tau_j)} = \overline{R_s^h}, \quad \sum_{j=0}^{c-q} \overline{A_j^s} \Phi^h(\tau_j) = 0,$$

which is equivalent to

$$(0, 0, \overline{R_s^h}) = \sum_{j=0}^{c-q} \overline{A_j^s}(\Phi^0(\tau_j), \Phi^h(\tau_j), \overline{\Phi^h(\tau_j)}).$$

Thus,

$$R_{s+\delta_h} = \sum_{j=0}^{c-q} \overline{A_j^s} \Phi(\tau_j).$$

From this, we conclude that the matrices $A_0^{s+\delta_h}, \ldots, A_{c-q}^{s+\delta_h}$ of the decomposition

$$R_{s+\delta_h} = \sum_{j=0}^{c-q} A_j^{s+\delta_h} \Phi(\tau_j)$$

can be chosen so that $A_j^{s+\delta_h} = \overline{A_j^s}$, $s = \delta_0 + 1, \ldots, \delta_0 + \delta_h$, $j = 0, \ldots, c-q$. Now, for all $s \in \{\delta_0 + 1, \ldots, \delta_0 + \delta_h\}$, we let L_s and $L_{s+\delta_h}$ be the bounded linear operators from $C([-\tau, 0], \mathbb{R}^n)$ into $\mathbb{C}^n$ defined by $L_s(z) = \sum_{j=0}^{c-q} A_j^s z(\tau_j)$ and $L_{s+\delta_h}(z) = \sum_{j=0}^{c-q} \overline{A_j^s} z(\tau_j)$.

Taking real and imaginary parts of L_s as in (7.9) yields a real Λ-mini-versal unfolding. □

Corollary 7.8. *If $\Lambda = \Lambda_0$, then a real Λ-versal unfolding of* (2.1) *is given by* (7.9) *with* $\alpha_s = \alpha_{s+\delta_h} = 0$ *for all* $s = \delta_0 + 1, \dots, \delta_0 + \delta_h$.

8. Examples

In this section, we illustrate our theory with several examples.

Example 8.1. Consider the scalar delay differential equation

$$\dot{x}(t) = \mathscr{L}_0(x_t) = x(t) - x(t-1), \tag{8.1}$$

which has been studied extensively [6,17]. The characteristic equation has a double zero eigenvalue, therefore the center eigenspace P is two dimensional, i.e. $c = 2$. A basis for P is given by $\Phi = (1, \theta)$, and we notice that

$$\operatorname{rank}(\operatorname{col}(\Phi(0), \Phi(-1))) = \operatorname{rank}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = 2 = c.$$

By (6.6), $\delta = \dim(\mathscr{W}) = 2$, and if follows by Theorem 7.6 and Corollary 7.8 that a real Λ-mini-versal unfolding of (8.1) is given by

$$\dot{x}(t) = \mathscr{L}(\alpha)(x_t) = x(t) - x(t-1) + \alpha_1 x(t) + \alpha_2 x(t-1),$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$.

Example 8.2. Consider now the first-order scalar equation

$$\dot{x} = \mathscr{L}_0 x_t = A_1 x(t - \tau_1) + A_2 x(t - \tau_2), \tag{8.2}$$

where A_1, A_2 are real parameters and the delays τ_1, τ_2 are positive. Bélair and Campbell [2] have shown that (8.2) has points of nonresonant double Hopf bifurcation in parameter space $(A_1, A_2, \tau_1, \tau_2)$ with eigenvalues $\pm\omega_1 i$ and $\pm\omega_2 i$. The center eigenspace P is four dimensional, thus $c = 4$.

Suppose that (8.2) has a double Hopf bifurcation at $(A_1^*, A_2^*, \tau_1^*, \tau_2^*)$, with $\tau_1^* \neq \tau_2^*$. By Eq. (6.6) we have that $\delta = 4$, and Theorem 7.6 implies that a complex Λ-mini-versal unfolding of (8.2) has the form

$$\dot{x} = \mathscr{L}(\alpha)x_t = (A_1^* + \alpha_1)x(t - \tau_1^*) + (A_2^* + \alpha_2)x(t - \tau_2^*) + \alpha_3 x(t - \tau_3) + \alpha_4 x(t - \tau_4)$$

for a suitable choice of τ_3 and τ_4, where $\alpha_j \in \mathbb{C}$ for $j = 1, 2, 3, 4$.

Example 8.3. The second-order scalar delay equation

$$\ddot{x}(t) + \alpha\dot{x}(t) + \beta x(t) = f(x(t-\tau)),$$

where f is a smooth function such that $f(0) = 0$, was studied in [16] as a model for the pupil light reflex. In [5], it was shown that the linearization of this equation about the trivial equilibrium, which we write in first-order form as

$$\dot{x}_1(t) = x_2(t),$$

$$\dot{x}_2(t) = -\beta x_1(t) - \alpha x_2(t) + A x_1(t-\tau), \tag{8.3}$$

where $A = f'(0)$, has a double Hopf point with $1:2$ resonance (eigenvalues $\pm i$ and $\pm 2i$) at parameter value

$$(\alpha, \beta, \tau, A) = (0, \tfrac{5}{2}, \pi, \tfrac{-3}{2}). \tag{8.4}$$

Using Theorem 5.1, it is straightforward to show that if we treat α, β, τ and A as unfolding parameters which vary in a neighborhood of point (8.4), then (8.3) generates a complex Λ-mini-versal unfolding for the singularity at parameter value (8.4). Proving this amounts to constructing the 20×16 matrix S in Theorem 5.1 and showing that its rank is 16.

However, the goal of this example is to compute the real Λ-mini-versal unfolding for the singularity (8.3) (at parameter value (8.4)), which results from using the decomplexification procedure described in Section 7.4.

We have that a complex basis for the center subspace is given by

$$\Phi(\theta) = \begin{pmatrix} e^{i\theta} & e^{2i\theta} & e^{-i\theta} & e^{-2i\theta} \\ ie^{i\theta} & 2ie^{2i\theta} & -ie^{-i\theta} & -2ie^{-2i\theta} \end{pmatrix}.$$

A basis for the adjoint problem is given by

$$\Psi^*(s) = \begin{pmatrix} e^{-is} & ie^{-is} \\ e^{-2is} & 2ie^{-2is} \\ e^{is} & -ie^{is} \\ e^{2is} & -2ie^{2is} \end{pmatrix},$$

which we renormalize by defining

$$\Psi = (\Psi^*, \Phi)^{-1}\Psi^*.$$

The result is such that

$$\Psi(0) = \kappa \begin{pmatrix} -\frac{2}{9}i(3\pi+4i)(-3\pi+4i)^2 & -32\pi+128i+6\pi^3-8i\pi^2 \\ -\frac{i}{9}(-3\pi+8i)(3\pi+8i)^2 & 64\pi-16i\pi^2+64i-6\pi^3 \\ \frac{2}{9}i(3\pi-4i)(-3\pi-4i)^2 & -32\pi-128i+6\pi^3+8i\pi^2 \\ \frac{i}{9}(-3\pi-8i)(3\pi-8i)^2 & 64\pi+16i\pi^2-64i-6\pi^3 \end{pmatrix}, \tag{8.5}$$

where $\kappa = (9\pi^4 - 32\pi^2 - 256)^{-1}$. The matrix B is given by

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}.$$

A basis for $\mathscr{W}$ is given by the following four matrices:

$$\Omega_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For any τ_0 and τ_1 such that $\tau_0 \neq \tau_1$, the matrix $\mathrm{col}(\Phi(\tau_0), \Phi(\tau_1))$ has rank 4. Therefore we choose, for example, $\tau_0 = 0$ and $\tau_1 = -\pi$. Following Remark 6.7, we construct $\Pi_j : \mathbb{C}^2 \to \mathbb{C}$, $j = 1, \dots, 4$:

$$\Pi_j(v) = \psi_{j1}(0)v_1 + \psi_{j2}(0)v_2, \quad j = 1, \dots, 4,$$

where $\psi_{j\ell}(0)$ are the elements in the matrix $\Psi(0)$ in (8.5), and $v = (v_1, v_2)^T \in \mathbb{C}^2$. Therefore, we define

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \psi_{12}^{-1}(0) & 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \psi_{22}^{-1}(0) & 0 & 0 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{\psi_{12}^{-1}(0)} & 0 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\psi_{22}^{-1}(0)} \end{pmatrix}.$$

A simple computation shows that

$$R_1 = \frac{\psi_{12}^{-1}(0)}{4}\begin{pmatrix} 0 & 0 \\ 1 & -i \end{pmatrix}\Phi(0) + \frac{\psi_{12}^{-1}(0)}{4}\begin{pmatrix} 0 & 0 \\ -1 & i \end{pmatrix}\Phi(-\pi)$$

and

$$R_2 = \frac{\psi_{22}^{-1}(0)}{4}\begin{pmatrix} 0 & 0 \\ 1 & \frac{-i}{2} \end{pmatrix}\Phi(0) + \frac{\psi_{22}^{-1}(0)}{4}\begin{pmatrix} 0 & 0 \\ 1 & \frac{-i}{2} \end{pmatrix}\Phi(-\pi).$$

Thus, if we set

$$\beta_1 + i\gamma_1 = \psi_{12}^{-1}(0), \quad \beta_2 + i\gamma_2 = \psi_{22}^{-1}(0),$$

then the operators L_m in (7.3) are

$$L_1(z) = \frac{\beta_1 + i\gamma_1}{4}\left[\begin{pmatrix} 0 & 0 \\ 1 & -i \end{pmatrix} z(0) + \begin{pmatrix} 0 & 0 \\ -1 & i \end{pmatrix} z(-\pi)\right],$$

$$L_2(z) = \frac{\beta_2 + i\gamma_2}{4}\left[\begin{pmatrix} 0 & 0 \\ 1 & \frac{-i}{2} \end{pmatrix} z(0) + \begin{pmatrix} 0 & 0 \\ 1 & \frac{-i}{2} \end{pmatrix} z(-\pi)\right],$$

$$L_3 = \overline{L_1}, \quad L_4 = \overline{L_2}.$$

Thus, the decomplexification procedure yields the following real Λ-mini-versal unfolding of singularity (8.3) (at parameter value (8.4))

$$\begin{aligned} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} &= \begin{pmatrix} x_2(t) \\ -\frac{5}{2}x_1(t) - \frac{3}{2}x_1(t-\pi) \end{pmatrix} \\ &\quad + \left(\alpha_1\begin{pmatrix} 0 & 0 \\ \beta_1 & \gamma_1 \end{pmatrix} + \alpha_2\begin{pmatrix} 0 & 0 \\ \gamma_1 & -\beta_1 \end{pmatrix}\right)\begin{pmatrix} x_1(t) - x_1(t-\pi) \\ x_2(t) - x_2(t-\pi) \end{pmatrix} \\ &\quad + \left(\alpha_3\begin{pmatrix} 0 & 0 \\ 2\beta_2 & \gamma_2 \end{pmatrix} + \alpha_4\begin{pmatrix} 0 & 0 \\ 2\gamma_2 & \beta_2 \end{pmatrix}\right)\begin{pmatrix} x_1(t) + x_1(t-\pi) \\ x_2(t) + x_2(t-\pi) \end{pmatrix}. \end{aligned} \tag{8.6}$$

Now, since $\beta_1^2+\gamma_1^2\neq 0$ and $\beta_2^2-\gamma_2^2\neq 0$, we can perform a linear change of coordinates in the parameters bringing (8.6) to

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -(\frac{5}{2}+b_1)x_1(t)-(\frac{3}{2}+b_2)x_1(t-\pi)+b_3x_2(t)+b_4x_2(t-\pi) \end{pmatrix}.$$

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Appendix. Proof of Lemma 6.4

We use the following inner product on the space $\mathrm{Mat}_{c\times c}$:

$$\langle M_1, M_2 \rangle = \mathrm{trace}(M_1 M_2^*). \tag{A.1}$$

A simple computation shows that with respect to (A.1), the conjugate transpose of $\mathcal{T}$ is given by $\mathcal{T}^*(M)=[B^*,M]$. From the Fredholm alternative, we have $\mathrm{range}(\mathcal{T})=\ker(\mathcal{T}^*)^\perp$. Recall that $\ker(\mathcal{T}^*)$ has been characterized in Lemma 6.3. So $Y\in\ker(\mathcal{T}^*)^\perp$, if and only if Y is orthogonal to all elements in a basis of $\ker(\mathcal{T}^*)$. We construct a basis for $\ker(\mathcal{T}^*)$ as follows: for $j\in\{1,\dots,r\}$, let

$$n_o(j) \equiv \sum_{\ell=1}^{k_j} (2\ell-1)n_{j,\ell}$$

which is equal to the number of distinct oblique segments in the block $\mathcal{M}_j$, as in Fig. 1, and give some ordering to these oblique segments, numbering them from 1 to $n_o(j)$. Then, for $j\in\{1,\dots,r\}$ and $\ell\in\{1,\dots,n_o(j)\}$, we define $M_{j,\ell}$ to be the matrix structured as in (6.3), and such that all diagonal blocks are zero except the block j; and in this block j, the only oblique segment which is nonzero is the ℓth segment, whose elements all have value 1. Then $\{M_{j,\ell}\}$ forms a basis for $\ker(\mathcal{T}^*)$.

Now consider $Y\in\mathrm{Mat}_{c\times c}$, which we partition as in (6.4). Multiply Y on the right by $M_{j,\ell}^*$, for some $j\in\{1,\dots,r\}$, $\ell\in\{1,\dots,n_o(j)\}$. Then $YM_{j,\ell}^*$ is partitioned as in (6.4), and is such that

$$YM^*_{j,\ell} = \begin{array}{c} \begin{pmatrix} 0 & \cdots & 0 & \mathscr{Y}_{1,j}\mathscr{M}^*_{j,\ell} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \mathscr{Y}_{2,j}\mathscr{M}^*_{j,\ell} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \mathscr{Y}_{r,j}\mathscr{M}^*_{j,\ell} & 0 & \cdots & 0 \end{pmatrix}, \\ \uparrow \\ j\text{th vertical block} \\ \text{in partitioning (6.4)} \end{array}$$

where $\mathscr{M}_{j,\ell}$ is the jth diagonal block in decomposition (6.3) of $M_{j,\ell}$. Thus, $\langle Y, M_{j,\ell} \rangle = \operatorname{trace}(YM^*_{j,\ell}) = \operatorname{trace}(\mathscr{Y}_{j,j}\mathscr{M}^*_{j,\ell})$. It follows that $Y \in \ker(\mathscr{T}^*)^\perp$ if and only if for all $j \in \{1, \dots, r\}$, the block $\mathscr{Y}_{j,j}$ in partition (6.4) is such that $\operatorname{trace}(\mathscr{Y}_{j,j}\mathscr{M}^*_{j,\ell}) = 0$ for all $\ell \in \{1, \dots, n_o(j)\}$. Using the partition illustrated in Fig. 1, we write

$$\mathscr{Y}_{j,j} = \begin{pmatrix} \mathscr{Y}^{1,1}_{j,j} & \mathscr{Y}^{1,2}_{j,j} & \cdots & \cdots & \mathscr{Y}^{1,k_j}_{j,j} \\ \mathscr{Y}^{2,1}_{j,j} & \mathscr{Y}^{2,2}_{j,j} & \cdots & \cdots & \mathscr{Y}^{2,k_j}_{j,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathscr{Y}^{k_j,1}_{j,j} & \mathscr{Y}^{k_j,2}_{j,j} & \cdots & \cdots & \mathscr{Y}^{k_j,k_j}_{j,j} \end{pmatrix} \tag{A.2}$$

and

$$\mathscr{M}_{j,\ell} = \begin{pmatrix} \mathscr{M}^{1,1}_{j,\ell} & \mathscr{M}^{1,2}_{j,\ell} & \cdots & \cdots & \mathscr{M}^{1,k_j}_{j,\ell} \\ \mathscr{M}^{2,1}_{j,\ell} & \mathscr{M}^{2,2}_{j,\ell} & \cdots & \cdots & \mathscr{M}^{2,k_j}_{j,\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathscr{M}^{k_j,1}_{j,\ell} & \mathscr{M}^{k_j,2}_{j,\ell} & \cdots & \cdots & \mathscr{M}^{k_j,k_j}_{j,\ell} \end{pmatrix}, \tag{A.3}$$

where $\mathscr{Y}^{p,q}_{j,j}$ and $\mathscr{M}^{p,q}_{j,\ell}$ are $n_{j,p} \times n_{j,q}$, for $p, q \in \{1, \dots, k_j\}$. Let $\tilde{p}$ and $\tilde{q}$ be such that $\mathscr{M}^{\tilde{p},\tilde{q}}_{j,\ell}$ is the unique block in (A.3) which has the nonzero oblique segment. Then $\mathscr{Y}_{j,j}\mathscr{M}^*_{j,\ell}$ is partitioned as in (A.2) and has the form

$$\mathscr{Y}_{j,j}\mathscr{M}_{j,\ell}^{*} \quad = \quad \begin{array}{c} \begin{pmatrix} 0 & \cdots & 0 & \mathscr{Y}_{j,j}^{1,\tilde{q}}\cdot(\mathscr{M}_{j,\ell}^{\tilde{p},\tilde{q}})^{*} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \mathscr{Y}_{j,j}^{2,\tilde{q}}\cdot(\mathscr{M}_{j,\ell}^{\tilde{p},\tilde{q}})^{*} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \mathscr{Y}_{j,j}^{k_j,\tilde{q}}\cdot(\mathscr{M}_{j,\ell}^{\tilde{p},\tilde{q}})^{*} & 0 & \cdots & 0 \end{pmatrix} \\ \uparrow \\ \tilde{p}\text{th vertical block in} \\ \text{partitioning } (A.2) \end{array}.$$

It follows that $Y\in\ker(\mathscr{T}^{*})^{\perp}$ if and only if Y is of the form (6.4), and for all $j\in\{1,\dots,r\}$ and for all $\ell\in\{1,\dots,n_o(j)\}$, we have $\operatorname{trace}(\mathscr{Y}_{j,j}^{\tilde{p},\tilde{q}}\cdot(\mathscr{M}_{j,\ell}^{\tilde{p},\tilde{q}})^{*})=0$, where $\tilde{p}$ and $\tilde{q}$ are such that $\mathscr{M}_{j,\ell}^{\tilde{p},\tilde{q}}$ is the unique block in (A.3) which has the nonzero oblique segment. Now, suppose the block $\mathscr{M}_{j,\ell}^{\tilde{p},\tilde{q}}$ is such that the oblique segment whose elements are all 1's is the one which passes through the vth leftmost element in the bottom row of the block, where $v\in\{1,\dots,n_{j,\tilde{q}}\}$. Then a simple computation shows that $\mathscr{Y}_{j,j}^{\tilde{p},\tilde{q}}\cdot(\mathscr{M}_{j,\ell}^{\tilde{p},\tilde{q}})^{*}$ is an $n_{j,\tilde{p}}\times n_{j,\tilde{p}}$ matrix whose first $n_{j,\tilde{p}}-v$ rows are zero, and the remaining rows are the rows $1,\dots,v$ of $\mathscr{Y}_{j,j}^{\tilde{p},\tilde{q}}$. It follows that $\operatorname{trace}(\mathscr{Y}_{j,j}^{\tilde{p},\tilde{q}}\cdot(\mathscr{M}_{j,\ell}^{\tilde{p},\tilde{q}})^{*})$ is the sum of the elements in the oblique segment of the block $\mathscr{Y}_{j,j}^{\tilde{p},\tilde{q}}$ which is the same oblique segment as the nonzero segment of $\mathscr{M}_{j,\ell}^{\tilde{p},\tilde{q}}$, from which we get the conclusion. □

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