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An inversion formula for finding technology distribution of production functions

Joel M. Cohen [a,*], Mauro Pagliacci [b]

[a] *University of Maryland, College Park, MD, USA*
[b] *Dipartimento di Economia, Università degli Studi di Perugia, Perugia, Italy*

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Abstract

We solve the problem of finding the technology distribution for profit functions (equivalently production functions) in a discrete setting. This is done by finding an inversion formula for the profit function, making use of a sequence of recursively defined polynomials whose behavior is studied.

1. Introduction

Production functions are widely used to describe the capabilities of an economic unit. A production function is a mathematical expression relating the total production of a product to the inputs, typically, labor, capital, land, or other components necessary for its production. The function describes the maximum output obtainable from given amounts of the inputs.

There is an equivalence between the production function and the profit function. From each we can obtain the other (cf. [2, p. 217]). In this paper we shall work directly with the profit function.

Consider an industry with a homogeneous production which utilizes n factors of production, e.g., hours of work, dollars of capital investment, square feet of work space, etc. Let $x = (x_1, \dots, x_n)$ be the vector which represents the technology using the "recipe" x_1 hours of work, x_2 dollars of investment, etc., for each unit produced, and let $p = (p_1, \dots, p_n)$ be the vector of prices: p_1 is the price of an hour of labor, p_2 the price of

* Corresponding author.
E-mail addresses: jcohen@umd.edu (J.M. Cohen), pagliacci@unipg.it (M. Pagliacci).

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one dollar invested, etc. Let us suppose that the unit selling price is p_0. Then the profit for one unit using the technology x where the production costs are p is

$$p_0 - p \cdot x,$$

where $p \cdot x = p_1 x_1 + \cdots + p_n x_n$ is the scalar product.

We may assume that there are many technologies being used simultaneously. For example, if a new factory is opened, older factories are not necessarily closed immediately.

We let $\mu(x)$ represent the distribution of technologies with respect to the vectors x in $\mathbb{R}_+^n$, where $\mathbb{R}_+ = [0, \infty)$. That is, if $A \subset \mathbb{R}_+^n$, then $\mu(A)$ is the amount of the industry which uses technologies $x \in A$. The total profit is given by the *profit function*

$$\Pi(p, p_0) = \int_{\mathbb{R}_+^n} (p_0 - p \cdot x)_+ \, \mathrm{d}\mu(x),$$

where by y_+ we mean y if it is positive, and 0 otherwise. That is, we assume that production is halted where a loss is certain.

In [2], Henkin and Shananin studied the inverse problem: given the values of Π, calculate the distribution μ. They assumed that this distribution was continuous—indeed differentiable. In this work, we consider the discrete case, which could be easily applied to concrete situations.

The context of the economic problem will dictate whether it makes sense to talk about a differentiable context (in particular smoothly distributed means of production) or whether to pass to the discrete setting described here.

By normalizing the size of the units used, we may assume that the support of the distribution lies in the set $\mathbb{Z}_+^n$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. For a vector $t \in \mathbb{Z}_+^n$, we use a_t for $\mu(\{t\})$. Thus the discrete function of profit becomes

$$\Pi(p, p_0) = \sum_{t \in \mathbb{Z}_+^n} (p_0 - p \cdot t)_+ a_t.$$

The problem we study is to find the measure a given the function Π.

We notice that Π satisfies the property that $\Pi(p, p_0) = p_0 \Pi(p/p_0, 1)$. Thus, without loss of generality, we may normalize the problem and assume that $p_0 = 1$. We write $\Pi(p) = \Pi(p, 1)$, where $p \in I = [0, 1]$.

We now have the function $\Pi : I^n \to \mathbb{R}$ given by

$$\Pi(p) = \sum_{t \in \mathbb{Z}_+^n,\ p \cdot t < 1} (1 - p \cdot t) a_t,$$

and we want to describe each a_t in terms of Π.

We shall find an inverse to the operator $\{a_t\}_{t \in \mathbb{Z}_+^n} \to \Pi$ as follows: First in Section 2 we solve the case $n = 1$, with the inversion formula given by Theorem 1. In Section 3 we reduce the two-dimensional case to the one-dimensional case by means of a sequence of polynomials $B_n(z,w)$, which act as operators turning functions of two variables into functions of one variable. This resolves the two-dimensional case, which is perhaps the most important case for economists. In Section 4 we show how to use the operators defined by the polynomials in order to reduce the $(n+1)$-dimensional case to the n-dimensional case. The proof is virtually identical to that given in Section 3. In Section 5 we describe a second sequence of polynomials $A_n(z)$, which can be used to calculate the $B_n(z,w)$. These polynomials, although given recursively, are very difficult to describe in general. In Section 6 we show how to get some general information about the polynomials $A_n(z)$ by finding a generating function for them.

2. The one-dimensional case

Let us look at the case $n=1$. Then $\Pi(p)=\sum_{t=0,\ pt<1}^{\infty}(1-pt)a_t$, for $p\in I$. For $m\geqslant 1$, set

$$p_m=\Pi(1/m)=\sum_{k=0}^{m-1}(1-k/m)a_k,$$

and set $p_{-1}=p_0=0$. We claim that

$$a_m=(m+1)p_{m+1}-2mp_m+(m-1)p_{m-1}.$$

The first three claimed values are $a_0=p_1$, $a_1=2p_2-2p_1$, $a_2=3p_3-4p_2+p_1$. Notice that they all follow easily from the definitions:

$$p_1=a_0,\qquad p_2=a_0+\frac{1}{2}a_1,\qquad p_3=a_0+\frac{2}{3}a_1+\frac{1}{3}a_2.$$

We prove the full result by induction.

Theorem 1. *Assume that* $\Pi(p)=\sum_{t=0,\ pt<1}^{\infty}(1-pt)a_t$, *for* $p\in I$. *Then*

$$a_m=(m+1)\Pi\bigl(1/(m+1)\bigr)-2m\Pi(1/m)+(m-1)\Pi\bigl(1/(m-1)\bigr).$$

Proof. To prove the inductive step, assume that for some $m\geqslant 3$ the formula holds for $a_0,a_1,\ldots,a_{m-1}$. Multiplying the definition of p_{m+1} by $m+1$ yields

$$(m+1)p_{m+1}=\sum_{k=0}^{m}(m+1-k)a_k=\sum_{k=0}^{m-1}(m+1-k)a_k+a_m.$$

Using the inductive hypothesis, we have

$$\begin{aligned}\sum_{k=0}^{m-1}(m+1-k)a_k &= \sum_{k=0}^{m-1}(m+1-k)\big((k+1)p_{k+1}-2kp_k+(k-1)p_{k-1}\big)\\ &= \sum_{k=0}^{m-1}(m+1-k)(k+1)p_{k+1}-\sum_{k=0}^{m-2}(m-k)2(k+1)p_{k+1}\\ &\quad+\sum_{k=0}^{m-3}(m-1-k)(k+1)p_{k+1}.\end{aligned}$$

For each $k=0,\dots,m-3$ the coefficient of p_{k+1} is identically zero. The coefficient of p_{m-1} is $-(m-1)$, and the coefficient of p_m is $2m$. Thus we have just shown that $\sum_{k=0}^{m-1}(m+1-k)a_k=-(m-1)p_{m-1}+2mp_m$. This shows that

$$a_m=(m+1)p_{m+1}-2mp_m+(m-1)p_{m-1}. \qquad \square$$

3. The two-dimensional case

The case of two variables, typically the cost of labor and the cost of capital, is sufficiently important (cf. [3]) that we describe the results separately.

We shall use a sequence of polynomials, $\{B_n(z,w)\}$, given by

$$B_1(z,w)=z^2w^2, \qquad B_n(z,w)=z^{n+1}w^{n+1}-\sum_{j=1}^{n-1}B_{n-j+1}\big(z^j,w\big).$$

The method we will use to solve this case is to reduce the two-dimensional case to the one-dimensional by means of the polynomials B_m. Given Π a function on I^2, we shall define $B_m\Pi$ as a function on I. Note that $B_m(z,w)$ is a sum of monomials of the form z^kw^m, where $k,m\geqslant 1$. The action of $B_m(z,w)$ on $\Pi(p_1,p_2)$ will be given by linearizing the following formula:

$$z^mw^n\Pi(p)=m\big\{\Pi(p/m,1/n)-\Pi(p/m,1)\big\}.$$

We now describe the reduction of the two-dimensional case to the one-dimensional case solved in Theorem 1.

Theorem 2. *For each value of $m\geqslant 1$,*

$$\sum_{t=0,\ pt<1}^{\infty}(1-pt)a_{t,m}=B_m\Pi(p) \quad \textit{and} \quad \sum_{t\in\mathbb{Z}_+,\ pt<1}(1-pt)a_{t,0}=\Pi(p,1).$$

Remark 1. For $m=0$, this together with Theorem 1 imply that

$$a_{t,0} = (t+1)\Pi\bigl(1/(t+1),1\bigr) - 2t\Pi(1/t,1) + (t-1)\Pi\bigl(1/(t-1),1\bigr).$$

For $m = 1$, we have

$$\sum_{t=0,\ pt<1}^{\infty} (1-pt)a_{t,1} = B_1\Pi(p) = z^2w^2\Pi(p) = 2\bigl\{\Pi(p/2,1/2) - \Pi(p/2,1)\bigr\}.$$

Thus Theorem 1 yields

$$\begin{aligned} a_{t,1} = {} & 2(t+1)\Pi\bigl(1/2(t+1),1/2\bigr) - 4t\Pi(1/2t,1/2) + 2(t-1)\Pi\bigl(1/2(t-1),1/2\bigr) \\ & - 2(t+1)\Pi\bigl(1/2(t+1),1\bigr) + 4t\Pi(1/2t,1) - 2(t-1)\Pi\bigl(1/2(t-1),1\bigr). \end{aligned}$$

For $m = 2$, the situation becomes more complicated, since

$$\begin{aligned} \sum_{t=0,\ pt<1}^{\infty} (1-pt)a_{t,2} &= B_2\Pi(p) = \bigl(z^3w^3 - z^2w^2\bigr)\Pi(p) \\ &= 3\bigl\{\Pi(p/3,1/3) - \Pi(p/3,1)\bigr\} - 2\bigl\{\Pi(p/2,1/2) - \Pi(p/2,1)\bigr\}. \end{aligned}$$

So using Theorem 1, we get that the formula for $a_{t,2}$ contains 12 terms!

Proof. Note that $\Pi(p_1,1) = \sum_{(t_1,t_2)\in\mathbb{Z}_+^2,\ p_1t_1+t_2<1}(1 - p_1t_1 + p_2t_2)a_t$, so that the only value of t_2 in the sum is 0. Thus $\Pi(p,1) = \sum_{t\in\mathbb{Z}_+,\ pt<1}(1-pt)a_{t,0}$.

We start the induction with $m = 1$. Look at

$$\begin{aligned} \Pi(p/2,1/2) &= \sum_{s=0,1}\ \sum_{t\geqslant 0,\ (p/2)t+s/2<1} \bigl(1-(p/2)t - s/2\bigr)a_{t,s} \\ &= \sum_{t\geqslant 0,\ (p/2)t<1} \bigl(1-(p/2)t\bigr)a_{t,0} + \sum_{t\geqslant 0,\ (p/2)t+1/2<1} \bigl(1-(p/2)t-1/2\bigr)a_{t,1} \\ &= \Pi(p/2,1) + (1/2)\sum_{t\geqslant 0,\ p<1}(1-p)a_{t,1}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{t\geqslant 0,\ p<1}(1-p)a_{t,1} &= 2\bigl\{\Pi(p/2,1/2) - \Pi(p/2,1)\bigr\} \\ &= z^2w^2\Pi(p,1) = B_1(z,w)\Pi(p,1), \end{aligned}$$

completing the first step of the induction.

A direct calculation shows that $kz^m w^n\Pi(p/k) = z^{mk}w^n\Pi(p)$. In particular, if $Q(z,w)$ is any polynomial divisible by zw, then $kQ(z,w)\Pi(p/k) = Q(z^k,w)\Pi(p)$. Assume by induction that $\sum_{t=0,\ pt<1}^{\infty}(1-pt)a_{t,s} = B_s\Pi(p)$ for $s < m$.

In analogy to the case of $\Pi(p,1)$, we have

$$\begin{aligned}\Pi\left(p,\frac{1}{m+1}\right) &= \sum_{s=0}^{m}\sum_{t\in\mathbb{Z}_+,\ pt+\frac{s}{m+1}<1}\left(1-pt-\frac{s}{m+1}\right)a_{t,s}\\ &= \sum_{s=0}^{m}\sum_{t\in\mathbb{Z}_+,\ pt<\frac{m+1-s}{m+1}}\left(\frac{m+1-s}{m+1}-pt\right)a_{t,s}\\ &= \sum_{s=0}^{m}\frac{m+1-s}{m+1}\sum_{t\in\mathbb{Z}_+,\ \frac{m+1}{m+1-s}pt<1}\left(1-\frac{m+1}{m+1-s}pt\right)a_{t,s}.\end{aligned}$$

Thus

$$\Pi\left(\frac{p}{m+1},\frac{1}{m+1}\right) = \sum_{s=0}^{m}\frac{m+1-s}{m+1}\sum_{t\in\mathbb{Z}_+,\ \frac{p}{m+1-s}t<1}\left(1-\frac{p}{m+1-s}t\right)a_{t,s}.$$

Using the inductive hypothesis, we get

$$\begin{aligned}&(m+1)\Pi\left(\frac{p}{m+1},\frac{1}{m+1}\right)\\ &\quad= \sum_{s=0}^{m-1}(m+1-s)B_s(z,w)\Pi\left(\frac{p}{m+s-1}\right)+\sum_{t\in\mathbb{Z}_+,\ pt<1}(1-pt)a_{t,m}\\ &\quad= (m+1)\Pi\left(\frac{p}{m+1},1\right)+\sum_{s=1}^{m-1}B_s\left(z^{m+s-1},w\right)\Pi(p)+\sum_{t\in\mathbb{Z}_+,\ pt<1}(1-pt)a_{t,m}.\end{aligned}$$

This yields

$$\begin{aligned}\sum_{t\in\mathbb{Z}_+,\ pt<1}(1-pt)a_{t,m} &= (m+1)\Pi\left(\frac{p}{m+1},\frac{1}{m+1}\right)-(m+1)\Pi\left(\frac{p}{m+1},1\right)\\ &\quad-\sum_{s=1}^{m-1}B_s\left(z^{m+s-1},w\right)\Pi(p)\\ &= \left(z^{m+1}w^{m+1}-\sum_{s=1}^{m-1}B_s\left(z^{m+s-1},w\right)\right)\Pi(p)=B_m(z,w)\Pi(p).\end{aligned}$$

This completes the proof. □

4. The general case

We can now generalize the ideas to higher dimensions. In the above proof, Π is a function of two variables, whereas $B_m(z,w)\Pi$ is a function of one variable. In the general case, if Π is a function of $n+1$ variables, then $B_m(z,w)\Pi$ will be a function of n variables as follows.

Let $p \in I^n$ and $p' \in I$. If $\Pi(p, p')$ is given, let

$$z^k w^m \Pi(p) = k\{\Pi(p/k, 1/m) - \Pi(p/k, 1)\},$$

and extend linearly to get the definition of $B_m(z,w)\Pi(p)$.

We then get the generalization of Theorem 2, writing $B_m\Pi(p) = B_m(z,w)\Pi(p)$:

Theorem 3. *Assume that*

$$\Pi(p, p') = \sum_{t\in\mathbb{Z}_+^n,\, t'\in\mathbb{Z}_+,\, p\cdot t+p't'<1} (1 - p\cdot t - p't')a_{t,t'}.$$

Then for each value of $m \geqslant 1$,

$$\sum_{t=0,\, p\cdot t<1}^{\infty} (1 - p\cdot t)a_{t,m} = B_m\Pi(p) \quad \textit{and} \quad \sum_{t\in\mathbb{Z}_+,\, p\cdot t<1} (1 - p\cdot t)a_{t,0} = \Pi(p, 1).$$

The proof of Theorem 4 is identical to that of Theorem 3, except that p and t should now be considered as vectors, and their product is the inner product.

We can thus reduce everything to a dimension one, where we can solve the problem by Theorem 1.

Example. To calculate $a_{m,1,2}$: First we calculate $B_3\Pi(p_1, p_2)$. This is

$$\begin{aligned} &3\big(\Pi(p_1/3, p_2/3, 1/3)\big) - \big(\Pi(p_1/3, p_2/3, 1)\big) - 4\big(\Pi(p_1/4, p_2/4, 1/2)\big) \\ &\quad - \big(\Pi(p_1/4, p_2/4, 1)\big). \end{aligned}$$

Now applying B_2 to this we get

$$\begin{aligned} &6\big(\Pi(p/6, 1/6, 1/3) - \Pi(p/6, 1/3, 1/3) - \Pi(p/6, 1/6, 1) - \Pi(p/6, 1/3, 1)\big) \\ &\quad - 8\big(\Pi(p/8, 1/8, 1/2) - \Pi(p/8, 1/4, 1/2) - \Pi(p/8, 1/8, 1) - \Pi(p/8, 1/4, 1)\big). \end{aligned}$$

Finally, we can get $a_{n,1,2}$ by simply applying Theorem 1 to this value of Π.

5. Calculating $B_n(z, w)$

Let us define another sequence of polynomials, $\{A_n(z)\}$, where

$$A_1(z) = z, \qquad A_{n+1}(z) = -\sum_{j=1}^{n} A_j\big(z^{n-j+2}\big).$$

We use the $A_n(z)$ to calculate the $B_n(z, w)$ by means of the following result:

Theorem 4. $B_n(z, w) = \sum_{j=1}^{n} A_j(z^{n-j+2})w^{n-j+2}$.

Proof. We show this by induction. For $n = 1$ we get $A_1(z^2)w^2 = z^2w^2 = B_1(z, w)$, as required.

Assume now that $B_m(z, w) = \sum_{j=1}^{m} A_j(z^{m-j+1})w^{m-j+1}$, for all $m < n$. Then we have

$$\begin{aligned} B_n(z, w) &= z^{n+1}w^{n+1} - \sum_{s=2}^{n}\sum_{j=1}^{n-s+1} A_j\big(z^{s(n-s-j+2)}\big)w^{n-s-j+2} \\ &= z^{n+1}w^{n+1} - \sum_{s=2}^{n}\sum_{k=s}^{n} A_{k-s+1}\big(z^{s(n-k+2)}\big)w^{n-k+2} \\ &= A_1\big(z^{n+1}\big)w^{n+1} - \sum_{k=2}^{n}\sum_{s=2}^{k} A_{k-s+1}\big(z^{s(n-k+2)}\big)w^{n-k+2}. \end{aligned}$$

Since $\sum_{s=2}^{k} A_{k-s+1}(z^s) = -A_k(z)$, we get

$$B_n(z, w) = \sum_{j=1}^{n} A_j\big(z^{n-j+2}\big)w^{n-j+2},$$

which completes the proof of the inductive step. □

6. A generating function for the A_n

Although we are not able to find a useful generating function for the polynomials A_n, we will construct a generating function that does yield some interesting results about the polynomials. At the end of this section, however, we shall present a closed form which is elegant, but not practical.

Define a generating function $G(z, u)$ by the formula

$$G(z, u) = \sum_{m=0}^{\infty} A_{m+1}(z)u^m.$$

Then we get the following result.

Theorem 5.

$$\sum_{m=0}^{\infty} G\bigl(z^{m+1}, u\bigr)u^m = z. \tag{1}$$

Proof. Notice that from the definition of the polynomials A_n, we have that

$$\sum_{j=0}^{k} A_{k+1-j}\bigl(z^{j+1}\bigr) = \begin{cases} z & \text{for } k = 0, \\ 0 & \text{for } k \geqslant 0. \end{cases}$$

From this we get the following directly:

$$\begin{aligned}\sum_{m=0}^{\infty} G\bigl(z^{m+1}, u\bigr)u^m &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} A_{n+1}\bigl(z^{m+1}\bigr)u^n\right)u^m = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{n+1}\bigl(z^{m+1}\bigr)u^{n+m} \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{k} A_{n+1}\bigl(z^{k-n+1}\bigr)u^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k} A_{k+1-j}\bigl(z^{j+1}\bigr)u^k = z. \qquad \square\end{aligned}$$

An example of the use of this is the following. Set $z = 1$. Then we get $\sum_{m=0}^{\infty} G(1, u)u^m = 1$. Since $\sum_{m=0}^{\infty} u^m = 1/(1-u)$, this yields $G(1, u) = 1 - u$, so

$$A_n(1) = \begin{cases} 1, & n = 1, \\ -1, & n = 2, \\ 0, & n \geqslant 2. \end{cases}$$

Thus we have that $z - 1$ is a factor of $A_n(z)$ for all $n \geqslant 2$. Taking the partial derivative of $G(z, u)$ with respect to z and evaluating at the point $z = 1$, we also get that $A_n'(1) = 0$ for $n \geqslant 4$. Thus $(z-1)^2$ is a factor of $A_n(z)$ for all $n \geqslant 4$.

Next we look at $A_n(-1)$. Setting $z = -1$ in (1) yields:

$$\sum_m G(-1, u)u^{2m} + \sum_m G(1, u)u^{2m+1} = -1,$$

which gives us $G(-1) = 2u^2 - u - 1$, since we already know the formula for $G(1, u)$.

Thus

$$A_n(-1) = \begin{cases} -1, & n = 1, 2, \\ 2, & n = 3, \\ 0, & n \geqslant 4. \end{cases}$$

Thus $(z+1)(z-1)^2$ is a factor of $A_n(z)$ for $n \geqslant 4$.

Now let us look at the case of $z = \lambda = e^{2\pi i/3}$. Using the same ideas as above, we get

$$G(\lambda, u) + G\big(\bar{\lambda}, u\big)u + G(1, u)u^2 = \lambda\big(1 - u^3\big).$$

Since all of the coefficients of $G(z, u)$ are real (indeed they are integers), $G(\bar{\lambda}, u) = \overline{G(\lambda, u)}$. Thus we get

$$G(\lambda, u) = \lambda + \lambda u - \bar{\lambda}u^2 + \frac{u^3}{1 + u},$$

a rational function, but not a polynomial.

This yields

$$A_n(\lambda) = \begin{cases} \lambda, & n = 1, 2, \\ -\bar{\lambda}, & n = 3, \\ 1, & n \text{ even} \geqslant 4, \\ -1, & n \text{ odd} \geqslant 5. \end{cases}$$

This does not yield new factors, but still gives us information about every $A_n(z)$. Specifically, we know that $z - \lambda$ is a factor of $A_n(z) - (-1)^n$.

Similarly for $z = i$, we get

$$G(i, u) = i - u + (1 + i)u^2 + \frac{4u^3 + 2u^5}{1 - u^4},$$

which among other things yields the fact that for $n \geqslant 5$ odd, we have $A_n(i) = 0$. This given us that $A_n(\bar{i}) = 0$, so $z^2 + 1$ is a factor of $A_n(z)$ for $n \geqslant 5$ odd.

Now we show the following in general:

Theorem 6. *$G(\lambda, u)$ is a rational function of u if λ is a root of unity.*

Proof. We have already shown this result for λ where $\lambda^m = 1$ and $m = 1, 2, 3, 4$. Assume the result holds for $m < n$, where $n > 4$. Since $\lambda^m = \lambda^t$ for $m \equiv t \bmod n$, (1) yields:

$$\sum_{k=1}^{n} G\big(\lambda^k, u\big)u^{k-1} = \lambda\big(1 - u^n\big). \tag{2}$$

Let $\mu = r^{2\pi i/n}$, so that $\lambda = \mu^t$, for some $t = 1, \dots, n$. We shall refer to (2) as $(2)_t$ in this case. If t and n are not relatively prime, then by induction we already know that $G(\lambda^k, u)$ is a rational function. So we shall consider the $\varphi(n)$ equations $(2)_t$ for $\gcd(t, n) = 1$. In addition, for k not relatively prime to n, we already know that $G(\lambda^k, u)$ is a rational function. Thus each equation $(2)_t$ is linear in $\varphi(n)$ unknowns, which we may think of as $G(\mu^j)$ for $\gcd(j, n) = 1$.

Thus the solution is a rational function of rational functions—and hence a rational function—as long as we can show that the determinant of the coefficients of the $G(\mu^j)$, which is a polynomial in u, is nonzero. But the coefficient of $G(\mu^j)$ in $(2)_t$ is u^{k-1}, where $\mu^{tk} = \lambda^k = \mu^j$, i.e., there $tk \equiv j \bmod n$, where t, k, j are relatively prime to n. For each t the coefficient of $G(\mu^t)$ is 1, and all the other coefficients are higher powers of u. Thus the constant term in the determinant of the coefficients is 1, so the determinant is not zero, and the result is proved. □

Finally we end this with a closed form for the polynomials $A_n(z)$ which appears in [1] along with many other facts concerning the polynomials. The restrictions on the sum are such that calculating the value is almost as complicated as using the recursion formula

$$A_{n+1}(z) = \sum (-1)^a \frac{a!}{a_1! a_2! \cdots a_r!} z^{x_1^{a_1} \cdots x_r^{a_r}}$$

where the sum is taken for $1 < x_1 < \cdots < x_r \leqslant n+1$, $\sum_{j=1}^r a_j x_j = n + a$, and $a = \sum_{j=1}^r a_j \leqslant n$.

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