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# A Finite-Dimensional Tool-Theorem in Monotone Operator Theory*

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## 1. INTRODUCTION

Historically, this note stems from Kirszbraun's Theorem [2], sometimes called the Kirszbraun–Valentine Theorem, useful in Geometric Measure Theory and which was the original tool-theorem for Monotone Operator Theory. In 1962, the writer proved a variant on this theorem [3] as an attempt to construct a tool-theorem specifically for the latter theory, but it was unsuccessful because it lacked any useful feature making it applicable to analysis problems in infinite-dimensions. The theorem was improved by Debrunner and Flor [1] and is now a standard tool in the theory of "variational inequalities." Here, we improve the theorem still further by providing an additional feature which is closer to the coercivity conditions needed for analysis problems. At the same time, we penetrate more deeply into the logical structure of the theorem and generalize it so that a large number of useful corollaries flow out of it, which are directly applicable to analysis problems currently under attack by means of awkward applications of the Debrunner–Flor Lemma and Kirszbraun's Theorem.

The present work is "finite-dimensional" in the sense that no hypotheses are made on the dimensions of the linear spaces introduced, no topologies are hypothesized, and in fact it would be quite sufficient to prove these results in finite-dimensions. The methods are those of "convexity theory."

## 2. RESULTS

LEMMA 1. *Let $Z$ be a linear space over the reals $R$; let $A$, $B$, $C$ be three bilinear forms mapping $Z \times Z$ into $R$ (without loss of generality $A$*

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*and $C$ may be assumed symmetric) such that the two quadratic forms $\mathbf{Q}_1(z) = (-1)\,C(z, z)$ and $\mathbf{Q}_2(z) = A(z, z) + B(z, z) + C(z, z)$ are both positive semidefinite. Let a sequence $\{z_i: i = 1,..., n\}$ in $Z$ be given such that $A(z_i - z_j, z_i - z_j) \geqslant 0$ for $i, j = 1,..., n$. Then there exist real nonnegative $\lambda_1,..., \lambda_n$ such that*

$$A(z_i, z_i) + B(z_i, z) + C(z, z) \geqslant 0 \qquad (i = 1,..., n) \tag{1}$$

*where $z = \sum_i \lambda_i z_i$, and also* (Lemma 1A) $\sum_i \lambda_i \leqslant 1$ *and*

$$\sum_i \lambda_i A(z_i, z_i) - C(z, z) \leqslant 0 \tag{2}$$

*or alternatively* (Lemma 1B) $\sum_i \lambda_i = 1$.

*Proof.* Consider the function $\Phi: \mathbf{D} \times \mathbf{D} \to R$ defined by

$$\Phi(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \sum_i \mu_i[A(z_i, z_i) + B(z_i, z) + C(z, z)]$$

$$+ (\mu - 1)\left[\sum_i \lambda_i A(z_i, z_i) - C(z, z)\right]$$

where $\boldsymbol{\mu} = (\mu_1,..., \mu_n)$, $\boldsymbol{\lambda} = (\lambda_1,..., \lambda_n)$, $z = \sum_i \lambda_i z_i$, $\mu = \sum_i \mu_i$, $\mathbf{D} = \{\boldsymbol{\lambda}: \lambda_i \geqslant 0, \sum_i \lambda_i \leqslant 1\}$. The term in $\Phi$ which is quadratic in $\boldsymbol{\lambda}$ is $C(z, z) = (-1)\,\mathbf{Q}_1(z)$, which is a negative semidefinite form considered as a function of $\boldsymbol{\lambda}$. Thus $\Phi$ is a linear function of $\boldsymbol{\mu}$ and a concave function of $\boldsymbol{\lambda}$, and is continuous in both variables for $\boldsymbol{\mu}, \boldsymbol{\lambda} \in R^n \times R^n$. Apply the Minimax Theorem of von Neumann to see that there exist $\boldsymbol{\mu}^0, \boldsymbol{\lambda}^0 \in \mathbf{D}$ such that for all $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathbf{D}$ we have

$$\Phi(\boldsymbol{\mu}, \boldsymbol{\lambda}^0) \geqslant \Phi(\boldsymbol{\mu}^0, \boldsymbol{\lambda}^0) \geqslant \Phi(\boldsymbol{\mu}^0, \boldsymbol{\lambda})$$

and hence, putting $\boldsymbol{\lambda} = \boldsymbol{\mu}^0$, we have: for any $\boldsymbol{\mu} \in \mathbf{D}$,

$$\Phi(\boldsymbol{\mu}, \boldsymbol{\lambda}^0) \geqslant \Phi(\boldsymbol{\mu}^0, \boldsymbol{\mu}^0).$$

But

$$\Phi(\boldsymbol{\mu}^0, \boldsymbol{\mu}^0) = \mathbf{Q}_2\left(\sum_i \mu_i^0 z_i\right) + \tfrac{1}{2}\sum_{i,j} \mu_i^0 \mu_j^0 A(z_i - z_j, z_i - z_j) \geqslant 0.$$

Now, since $\boldsymbol{\mu}$ is still at our disposal, we put it equal to a Kronecker delta to obtain Eq. (1), and then set all $\mu_i$ equal to zero to obtain Eq. (2). Part A of the Lemma is proved; to prove part B, we repeat the argument, but with $\mathbf{D} = \{\boldsymbol{\lambda}: \lambda_i \geqslant 0, \sum_i \lambda_i = 1\}$, $\mu = 1$, and omission of the step "set all $\mu_i = 0$."

We now state a corollary which would suffice for most of the applications to "monotone operator theory." The interest lies in the insight it yields into the meaning of "monotonicity."

COROLLARY. *Let $Z$ be a linear space over $R$, with a distinguished quadratic form $\mathbf{Q}(z)$; let $P: Z \to Z$ be a linear mapping such that $(-1)\mathbf{Q}(Pz)$ and $\mathbf{Q}(z - Pz)$ are both positive semidefinite quadratic forms. Let $\{z_i: i = 1,..., n\}$ be a sequence in $Z$ such that $\mathbf{Q}(z_i - z_j) \geqslant 0$, $(i, j = 1,..., n)$. Then there exist $\lambda_1,..., \lambda_n \geqslant 0$ such that $\mathbf{Q}(z_i - Pz) \geqslant 0$ $(i = 1,..., n)$, where $z = \sum_i \lambda_i z_i$, and also* (Part A) $\sum_i \lambda_i \leqslant 1$ *and* $\sum_i \lambda_i \mathbf{Q}(z_i) - \mathbf{Q}(Pz) \leqslant 0$, *or* (Part B) $\sum_i \lambda_i = 1$.

*Proof.* Let $A$, $B$, $C$ be three bilinear forms mapping $Z \times Z$ into $R$ such that $A(z, z) = \mathbf{Q}(z)$, $B(z, w) = -A(Pz, w) - A(w, Pz)$, $C(z, w) = A(Pz, Pw)$, and apply Lemma 1.

However, in order to include some special cases of interest which do not fit the above corollary, and in order to be closer to applications, we introduce a more complex structure.

THEOREM 1. *Let $Z$, $X$, $Y$ be linear spaces over $R$; let $\langle\ ,\ \rangle$ be a bilinear form mapping $X \times Y$ into $R$. Let $L$, $M$, $P$, $Q$ be four linear mappings of $Z$ into $X$, $Y$, $X$, $Y$, respectively, such that the two quadratic forms on $Z$ given by*

$$\mathbf{Q}_1(z) = -\langle P(z), Q(z)\rangle$$

$$\mathbf{Q}_2(z) = \langle L(z) - P(z), M(z) - Q(z)\rangle$$

*are both positive semidefinite. Now let $\{z_i: i = 1,..., n\}$ be a sequence in $Z$ which is monotone with respect to the quadratic form $\langle L(z), M(z)\rangle$; that is, $\langle L(z_i - z_j), M(z_i - z_j)\rangle \geqslant 0$ $(i, j = 1,..., n)$. Then there exist $\lambda_1,..., \lambda_n \geqslant 0$ such that*

$$\langle L(z_i) - P(z), M(z_i) - Q(z)\rangle \geqslant 0 \qquad (i = 1,..., n) \tag{3}$$

*and also* (Theorem 1A) $\sum_i \lambda_i \leqslant 1$ *and*

$$\sum_i \lambda_i \langle L(z_i), M(z_i)\rangle + \mathbf{Q}_1(z) \leqslant 0 \tag{4}$$

(*throughout*, $z = \sum_i \lambda_i z_i$), *or alternatively* (Theorem 1B) $\sum_i \lambda_i = 1$.

*Proof.* Let $A$, $B$, $C$ be bilinear forms mapping $Z \times Z$ into $R$, with $A(z, z) = \langle L(z), M(z)\rangle$, $C(z, z) = \langle P(z), Q(z)\rangle$, and $B(z, w) = -\langle P(w), M(z)\rangle - \langle L(z), Q(w)\rangle$; apply Lemma 1.

In the rest of this paper, *we shall always take* $Z = X \dot{+} Y$ (the product-space $X \times Y$ endowed with the usual linear structure), so we write $P(x, y)$, $Q(x, y)$, $L(x, y)$, $M(x, y)$.

## 3. Corollaries

By taking $Z = X \dot{+} Y$ (as prescribed earlier), $X = Y$, $L(x, y) = x + y$, $M(x, y) = x - y$, $P(x, y) = y$, $Q(x, y) = -y$, and assuming $\langle\ ,\ \rangle$ is a symmetric, positive definite bilinear form, we obtain by Theorem 1B:

Corollary 1. *Let $X$ be an inner product space under the bilinear form $\langle\ ,\ \rangle$, and let the sequence $(x_i, y_i)$ $(i = 1,\ldots, n)$ satisfy*

$$\| y_i - y_j \| \leqslant \| x_i - x_j \| \qquad (i, j = 1,\ldots, n).$$

*Then there exists a $y$ in the convex hull of $\{ y_i\}$ such that*

$$\| y_i - y \| \leqslant \| x_i \| \qquad (i = 1,\ldots, n).$$

This is the Kirszbraun (–Valentine) Theorem referred to earlier. This theorem has been generalized in a quite different direction by the writer in [5].

*In all our remaining Corollaries, we shall take* $L(x, y) = x$, $M(x, y) = y$, as well as $Z = X \dot{+} Y$.

Corollary 2. *Let $K: Y \to X$ be a linear mapping whose associated quadratic form $\langle Ky, y\rangle$ is positive semidefinite: $\langle Ky, y\rangle \geqslant 0$ for all $y \in Y$. Let $\{(x_i, y_i): i = 1, n\}$ be a sequence in $X \times Y$ such that $\langle x_i - x_j, y_i - y_j\rangle \geqslant 0$. Then there exist $\lambda_1,\ldots, \lambda_n \geqslant 0$ with $\sum_i \lambda_i \leqslant 1$ such that*

$$\langle x_i + Ky, y_i - y\rangle \geqslant 0 \qquad (i = 1,\ldots, n)$$

*where $y = \sum_i \lambda_i y_i$, and furthermore*

$$\sum_i \lambda_i\langle x_i, y_i\rangle + \langle Ky, y\rangle \leqslant 0.$$

*Proof.* In addition to earlier mentioned specializations, take $P(x, y) = -Ky$, $Q(x, y) = y$, and apply Theorem 1A. This corollary appears to be useful for the proof of an existence theorem for Hammerstein's nonlinear integral equation, with a weak- or weak*-compactness argument in the space $Y$.

In case $K$ is identically zero, we obtain the Debrunner–Flor Lemma; or to be more precise, we would have obtained this Lemma if we had applied Theorem 1B. By relaxing the requirement $\sum_i \lambda_i = 1$ to $\sum_i \lambda_i \leqslant 1$, we obtain in addition the extremely useful condition $\sum_i \lambda_i\langle x_i , y_i\rangle \leqslant 0$, which is related to "coercivity conditions" in the theory of "monotone operators and variational inequalities."

In the remaining applications, we take $X = Y$ and $\langle\ ,\ \rangle$ as a form whose associated quadratic form $\langle x, y\rangle$ is positive semidefinite. (For concreteness, one may assume it also symmetric, though we shall not need this hypothesis.)

COROLLARY 3. *Let $X$, $\langle\ ,\ \rangle$ be as above; let $U$ be a linear subspace of $X \dotplus X$ such that $\langle p, q\rangle \geqslant 0$ for all $(p, q) \in U$, and enjoying the "unique decomposition property": any vector $v \in X$ can be represented as a sum $v = p + q$, where $(p, q) \in U$. Define the mappings $P$ and $Q$ by $-P(x, y) + Q(x, y) = -x + y$, $(-P(x, y), Q(x, y)) \in U$. Let $\{(x_i , y_i): i = 1,..., n\}$ be such that $\langle x_i - x_j , y_i - y_j\rangle \geqslant 0$ for $i, j = 1,..., n$. Then there exist $\lambda_1 ,..., \lambda_n \geqslant 0$ with $\sum_i \lambda_i \leqslant 1$ such that*

$$\langle x_i - P(x, y), y_i - Q(x, y)\rangle \geqslant 0 \qquad (i = 1,..., n)$$

*and also*

$$\sum_i \lambda_i\langle x_i , y_i\rangle - \langle P(x, y), Q(x, y)\rangle \leqslant 0$$

*where*

$$x = \sum_i \lambda_i x_i , \qquad y = \sum_i \lambda_i y_i .$$

*Proof.* The specializations needed are obvious. It is a routine matter to prove that the mappings $P$ and $Q$ are linear mappings. Observe that $\mathbf{Q}_2(x, y) = \langle x - P(x, y), y - Q(x, y)\rangle = \langle x - P(x, y), x - P(x, y)\rangle$ and thus $\mathbf{Q}_2$ is positive semidefinite.

The most interesting special case is when $X$ is a Hilbert space, and $U$ is a "maximal monotone" linear subspace. A subset $U$ of $X \times X$ is called "monotone" provided: for any $(p_1 , q_1)$, $(p_2 , q_2) \in U$, we have $\langle p_1 - p_2 , q_1 - q_2\rangle \geqslant 0$, and "maximal monotone" if it is not properly contained in another monotone set. The "unique decomposition property" for maximal monotone sets in $X \times X$ is proved in [4].

The product of two orthogonal complementary subspaces of a Hilbert space is a maximal monotone subspace of $X \dotplus X$. Thus we have the following.

COROLLARY 4. *Let $X$ be a Hilbert space with inner product $\langle\ ,\ \rangle$, and let $A$ and $B$ be the orthogonal projection operators onto a closed linear subspace and its orthogonal complement, respectively. Let $(x_i, y_i)$ be a monotone sequence in $X \times X$. Then there exist $\lambda_i \geqslant 0$ with $\sum_i \lambda_i \leqslant 1$ such that*

$$\langle x_i - Ax, y_i - By\rangle \geqslant 0 \qquad (i = 1,..., n)$$

*where $x = \sum_i \lambda_i x_i$ and $y = \sum_i \lambda_i y_i$, and furthermore*

$$\sum_i \lambda_i \langle x_i, y_i\rangle \leqslant 0.$$

This corollary has applications to the theory of differential equations.

## 3. Projected Applications

We are not really interested in the solution of a *finite* system of inequalities; this Theorem and Corollaries are designed as tools for existence-theorems for solutions of *infinite* systems. The methods are (weak- or weak*-) compactness arguments: one chooses a large "ball" in a topological vector space which is compact, and solves the inequalities "finitely many at a time" in the ball. The conditions like $\sum_i \lambda_i \langle x_i, y_i\rangle \leqslant 0$ are inputs to a coercivity argument to show the existence of a suitably large ball. Under suitable hypotheses, each inequality cuts out a closed subset from the ball (the inequalities we are solving describe convex sets). In order to keep this paper uncluttered with such topological considerations, we postpone these arguments to another paper.

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