Discrete Mathematics 62 (1986) 299–306
North-Holland

ENUMERATION OF PROJECTIVE-PLANAR EMBEDDINGS OF GRAPHS

Seiya NEGAMI
Department of Information Science, Tokyo Institute of Technology, Tokyo 152, Japan

Received 15 February 1984
Revised 17 December 1985

It will be shown that the number of equivalence classes of embeddings of a 3-connected nonplanar graph into a projective plane coincides with the number of isomorphism classes of planar double coverings of the graph and a combinatorial method to determine the number will be developed.

1. Introduction

Two embeddings of a graph G into a surface F^2, say $f_1, f_2 : G \to F^2$, are said to be *equivalent* if there exist an automorphism $\sigma : G \to G$ and a homeomorphism $h : F^2 \to F^2$ for which $h \circ f_1 = f_2 \circ \sigma$. If one would like to work in the category of labeled graphs, the automorphism σ should be chosen to be the identity map of G at any time. Our purpose is to develop a method for enumeration of the equivalence classes of embeddings of a graph into a projective plane. A 2-sphere and a projective plane will be denoted by S^2 and P^2, respectively, throughout this paper. Our idea to do the enumeration is closely related to the fact that S^2 is *the* double universal covering space of P^2. We shall count the number of the equivalence classes of projective-planar embeddings of a graph by making each class correspond to a planar graph which covers the graph 2-fold.

A graph $\tilde{G}$ is called an *n-fold covering* of a graph G if there is an n-to-one correspondence $q : V(\tilde{G}) \to V(G)$ between their vertex sets, called the *covering projection,* which sends bijectively neighbors of each vertex $v \in V(\tilde{G})$ to neighbors of $q(v) \in V(G)$. Topologically, the covering projection $q : V(\tilde{G}) \to V(G)$ is realized by a local homeomorphism $p : \tilde{G} \to G$ with $p|_{V(\tilde{G})} = q$, and p itself is called the covering projection. In particular, a 2-fold covering of a graph G is called a *double covering* of G. Our main result states that:

Theorem 1. *There is bijection between the equivalence classes of embeddings of a 3-connected nonplanar graph G into a projective plane and the isomorphism classes of planar double coverings of G.*

Let $q : S^2 \to P^2$ denote the canonical covering projection which projects each pair of antipodal points of S^2 to the same point of P^2. For an embedding

$f: G \to P^2$, the pull-back $\tilde{G} = q^{-1}(f(G))$ of the image $f(G)$ is clearly a planar double covering of G. The bijective correspondence in the above theorem sends the equivalence class of f to the isomorphism class of $\tilde{G}$. Note that the latter class means only the isomorphism class of $\tilde{G}$ as just a graph, not referring to how $\tilde{G}$ is embedded in S^2 and how it covers G. A single planar graph $\tilde{G}$ might cover a nonplanar graph G in two or more ways. Our theorem however asserts that such a phenomenon does not occur if G is 3-connected.

As an immediate consequence of Theorem 1, we have a characterization of *projective-planar* graphs, that is, those graphs which are embeddable in P^2:

Corollary 2. *A connected nonplanar graph is projective-planar if and only if it has a planar double covering.*

In Section 2, we show how the combinatorics of double coverings of graphs can be used to enumerate the double coverings of a graph, thus yielding by Theorem 1 a method to enumerate the projective-planar embedding of a graph. The proof of Theorem 1 is given in Section 3.

Our graph G is a finite, undirected and simple one with the canonical topology as a 1-complex and its vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. Our terminology is quite standard and can be found in [2] for graph theory and in [6] for topology.

2. Double coverings of a graph

The concept of covering space can be found in standard texts of topology, say [6]. However, they present it in a topological fashion completely. The best combinatorial approach to classifying regular coverings of graphs is that of Gross and Tucker [1]. In this section, we shall confine our objects to double coverings of graphs and show their standard construction and a special phenomenon we need later.

Let $\tilde{G}$ be an n-fold covering of a graph G with projection $p: \tilde{G} \to G$. Then an automorphism $\tau: \tilde{G} \to \tilde{G}$ is called a *covering transformation* if $p = p \circ \tau$. Such an automorphism of $\tilde{G}$ does not always exist except for the identity map of $\tilde{G}$. However if $\tilde{G}$ is a double covering of G, then $\tilde{G}$ admits a unique non-trivial covering transformation $\tau: \tilde{G} \to \tilde{G}$ which acts freely on $\tilde{G}$ and has period 2. That is, τ fixes no vertex and no edge of $\tilde{G}$ and τ^2 is the identity map of $\tilde{G}$. Hereafter, we shall say that τ is a covering transformation, meaning that it is not the identity map.

In general, two n-fold covering $\tilde{G}_i$ of a graph G with projection $p_i: \tilde{G}_i \to G$ are *equivalent* if there exists ismorphisms $\sigma: G \to G$ and $\tilde{\sigma}: \tilde{G}_1 \to \tilde{G}_2$ such that $\sigma \circ p_1 = p_2 \circ \tilde{\sigma}$. When $n = 2$, $\tilde{G}_1$ and $\tilde{G}_2$ are equivalent if and only if for their covering transformation $\tau_i: \tilde{G}_i \to \tilde{G}_i$, there is an isomorphism $\tilde{\sigma}: \tilde{G}_1 \to \tilde{G}_2$ such that $\tilde{\sigma} \circ \tau_1 = \tau_2 \circ \tilde{\sigma}$.

Let E be a subset of $E(G)$ and let G_1, G_2 be two copies of $G - E$. The copy of each vertex $v \in V(G)$ in G_i is denoted by $v_i \in V(G_i)$ with subscript i $(i = 1, 2)$. Join u_1 to v_2 and u_2 to v_1, respectively, by an edge when and only when $uv \in E$. Then the resulting graph, denoted by $\tilde{G}[E]$, is a double covering of G with projection $p:\tilde{G}[E] \to G$ such that $p(v_1) = p(v_2) = v \in V(G)$ for v_1, $v_2 \in V(\tilde{G}[E]) = V(G_1) \cup V(G_2)$ and its covering transformation $\tau:\tilde{G}[E] \to \tilde{G}[E]$ interchanges G_1 and G_2. In particular, $\tilde{G}[\emptyset]$ is a disjoint union of two copies of G.

Theorem 3. *Every double covering of a graph G is equivalent to $\tilde{G}[E]$ for a subset E of $E(G)$ as coverings. In particular, the set E can be chosen so as to be contained in a fixed cotree T^* if G is connected.*

Proof. Let $\tilde{G}$ be a double covering of G with transformation $\tau:\tilde{G} \to \tilde{G}$ and projection $p:\tilde{G} \to G$. Choose a vertex-coloring $\theta:V(\tilde{G}) \to \{1, 2\}$ such that $\theta(v) \neq \theta(\tau(v))$ for $v \in V(\tilde{G})$, which clearly exists. Then the subset $\tilde{E} = \{uv \in E(\tilde{G})$: $\theta(u) \neq \theta(v)\}$ of $E(\tilde{G})$ is a cutset of $\tilde{G}$ such that $\tau(uv) \in \tilde{E}$ for $uv \in \tilde{E}$, and $\tilde{G} - \tilde{E}$ consists of two disjoint subgraphs one of which is colored by 1 and the other by 2. Take $p(\tilde{E})$ as E, then these subgraphs are nothing but G_1 and G_2 in the definition of $\tilde{G}[E]$. Thus $\tilde{G}$ and $\tilde{G}[E]$ are equivalent.

Suppose that G is connected and let T be a fixed spanning tree of G. By well-known properties of covering projections, $p^{-1}(T)$ consists of two disjoint copies T_1 and T_2 of T. In the above, we can take a vertex-coloring θ especially so that all vertices on one copy T_i $(i = 1, 2)$ have the same color. Therefore, a subset E can be restricted within $E(T^*)$ for the cotree T^* corresponding to T. □

When a connected graph G has p vertices and q edges, then its Betti number, denoted by $\beta(G)$, is equal to $q - p + 1$. Since any cotree T^* has precisely $\beta(G)$ edges, G admits at most $2^{\beta(G)}$ double coverings, up to equivalence, by Theorem 3.

The following fact will play an important role in our proof of Theorem 1.

Lemma 4. *Every planar double covering of a 3-connected nonplanar graph is 3-connected.*

Proof. Let G be a 3-connected nonplanar graph and $\tilde{G}$ a double covering of G with projection $p:\tilde{G} \to G$ and transformation $\tau:\tilde{G} \to \tilde{G}$. Suppose that $\tilde{G}$ has a vertex-cut $\{v, u\}$. Let $H = G - \{p(v), p(u)\}$. Since G is 3-connected, H is connected. Let $\tilde{H} = p^{-1}(H)$. Then the restriction $p|_{\tilde{H}}:\tilde{H} \to H$ is a double covering projection. Since $\tilde{G} - \{v, u\}$ is disconnected, $\tilde{H}$ must consist of two disjoint copies H_1, H_2 of H mapped isomorphically by p onto H. Let G_1 and G_2 be the subgraph of $\tilde{G}$ induced by $H_1 \cup \{v, u\}$ and $H_2 \cup \{\tau(v), \tau(u)\}$, respectively. Then $\tau(G_1) = G_2$.

Suppose that $\tau(v) \neq u$. Since $\tilde{G}$ is connected and $\{v, u\}$ is its vertex-cut, there is a path joining H_1 to H_2 and any such path has to pass through at least one of v

and u. If there were a path xvy (or $xvuy$) in $\tilde{G}$ with $x \in V(H_1)$ and $y \in V(H_2)$, then the path $\tau(x)\tau(v)\tau(y)$ (or $\tau(x)\tau(v)\tau(u)\tau(y)$) would join H_2 to H_1 but would contain neither v nor u. To exclude these paths, $\tilde{G}$ must consist of the disjoint union $G_1 \cup G_2$ with precisely two edges $v\tau(u)$ and $u\tau(v)$. (If v and u were adjacent, then $p(v)$ and $p(u)$ would be joined by two multiple edges, contrary to G being simple.) It follows that G is isomorphic to $G_1 + uv$. However, $G_1 + uv$ is planar since its subdivision is contained in the planar graph $\tilde{G}$, a contradiction.

Now suppose that $\tau(v) = u$ and equivalently that $p(v) = p(u)$. Every edge e incident to $p(v)$ is lifted to two edges e_1 and e_2 each of which joins H_1 and H_2, respectively, to only one of v and u. Thus, $G_i + uv$ $(i = 1, 2)$ can be regarded as the result of vertex-splitting at $p(v)$ in G. In other words, G can be obtained from $G_i + uv$ by contracting the edge uv. Since $\tilde{G}$ contains a subdivision of $G_i + uv$, $G_i + uv$ is planar and so its contraction $G = (G_i + uv)/uv$ is, contrary to G being non-planar.

In either case, we have got a contradiction. Thus, $\tilde{G}$ is 3-connected. □

Remark that the planarity of a double covering is essential in Lemma 4. For example, the complete bipartite graph $K_{3,3}$ admits precisely 2 inequivalent double

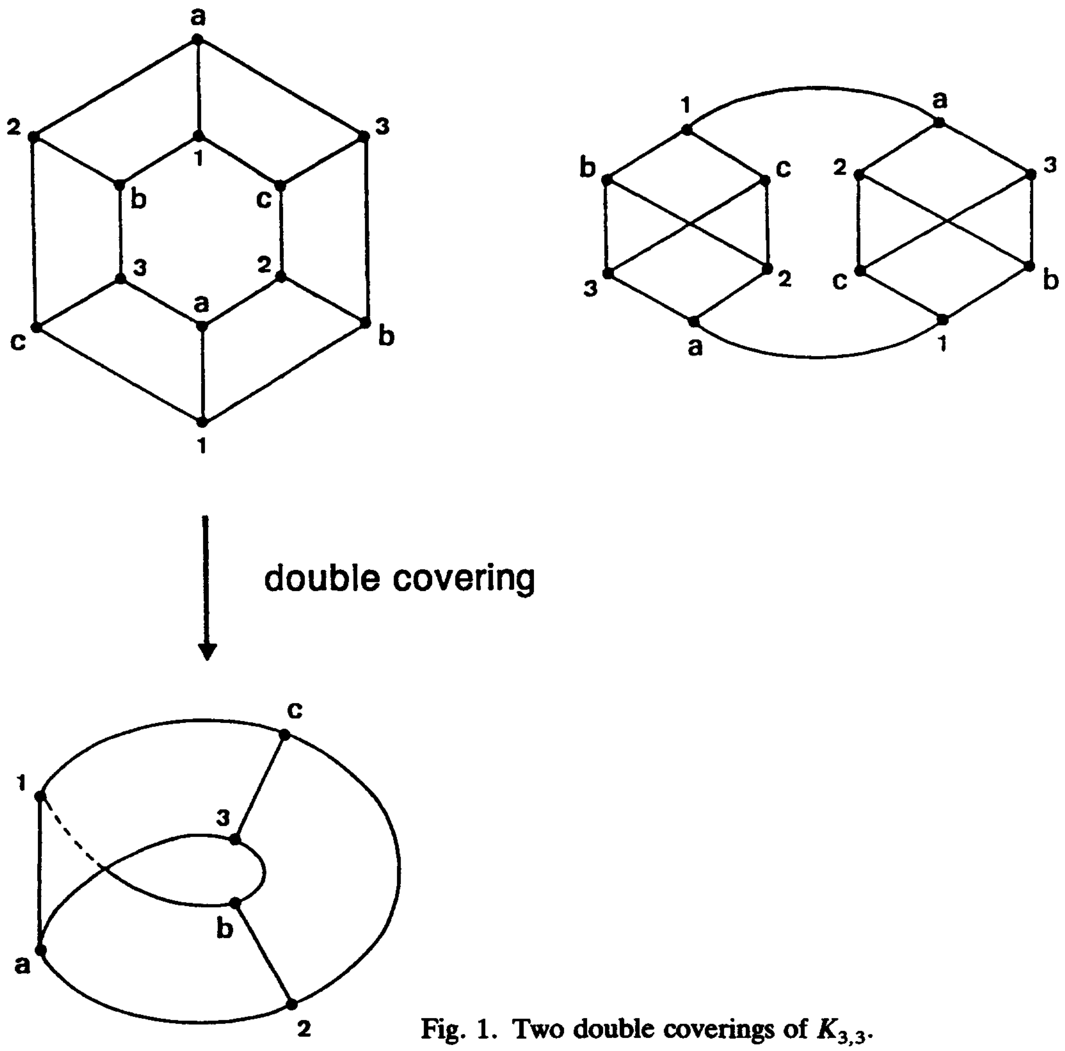

Fig. 1. Two double coverings of $K_{3,3}$.

coverings, shown in Fig. 1. (The labels on their vertices suggest how they cover $K_{3,3}$.) The right hand one is a 3-connected planar graph with the form $C_6 \times K_2$, but the left hand one is neither 3-connected nor planar.

Let G be a 3-connected nonplanar graph and consider the double covering $p : \tilde{G}[\{e\}] \to G$ for an edge $e \in E(G)$. Then $p^{-1}(e)$ is a 2-edge-cut of $\tilde{G}[\{e\}]$ and hence $\tilde{G}[\{e\}]$ is not 3-connected. Moreover it is not planar by Lemma 4. Therefore, if $\tilde{G}[E]$ is planar, then E consists of two or more edges. This implies that a 3-connected graph G admits at most $(2^{\beta(G)} - \beta(G))$ planar double coverings and also projective-planar embeddings by Theorem 1.

3. Graphs in a projective plane

Theorem 1 is closely related to the uniqueness and faithfulness of embedding of planar graphs into a 2-sphere S^2. Classically, Whitney proved in [7] that every 3-connected planar graph has a unique dual. In [3], the author has pointed out that the uniqueness of a dual for a 3-connected planar graph splits into the two concepts: uniqueness and faithfulness of the spherical embeddings of such a graph. A graph is *uniquely embeddable* in a surface F^2 if there is only one equivalence class of its embeddings into F^2. An embedding $f : G \to F^2$ is *faithful* if for any automorphism $\sigma : G \to G$, there is a homeomorphism $h : F^2 \to F^2$ such that $h \circ f = f \circ \sigma$. A graph G is *faithfully embeddable* in a surface F^2 if G has a faithful embedding into F^2. The uniqueness and faithfulness of embedding of graphs into a projective plane have been discussed in [4] and [5].

An *involution* on a surface F^2 is a homeomorphism $h : F^2 \to F^2$ of period 2, that is, h^2 is the identity map of F^2. An involution on a surface is said to be *free* if it has no fixed point.

Lemma 5. *Let K be a triangulation of a sphere S^2, and let $h_1, h_2 : S^2 \to S^2$ be two free involutions on S^2 which are simplicial with respect to K. Then $h_1 = h_2$.*

Proof. Since both h_1 and h_2 are simplicial with respect to K, the composition $h_1 \circ h_2$ has finite period, say n. Then h_1 and h_2 generate together a dihedral group D_n of order $2n$, and $h_1 \circ h_2$ generates the normal cyclic subgroup H of order n and of index 2 in D_n. A free involution on a sphere is orientation-reversing, so both h_1 and h_2 are orientation-reversing while $h_1 \circ h_2$ is orientation-preserving. By Smith theory, $h_1 \circ h_2$ is a rotation with two fixed points v and u, and the quotient S^2/H of S^2 by H is a sphere. Since $h_1 \circ h_2(v) = v$ and h_1 is an involution, we have $h_1(v) = h_2(v)$. Then

$$h_1 \circ h_2(h_1(v)) = h_1 \circ h_2(h_2(v)) = h_1(v) \neq v.$$

This implies that $h_1(v)$ is one of the two fixed points of $h_1 \circ h_2$ and hence $u = h_1(v)$.

Let $p:S^2 \to S^2/H$ be the canonical projection. Since H is normal in D_n, $D_n/H = \{H, h_1H\}$ is a cyclic group of order 2 and acts on S^2/H, reversing the orientation of S^2/H. By Smith theory again, D_n/H is realized by either a reflection or an antipodal map of S^2/H. Then there is a cycle C' in $p(K)$ which is invariant with respect to the action of D_n/H and which separates $p(v)$ and $p(u)$ on S^2/H. Consider $p(v)$ and $p(u)$ as the north and south pole, respectively, then C' may be called the equator. If h_1H is a reflection, then C' is its fixed point set. In either case, $C = p^{-1}(C')$ is a cycle in K such that $h_1(C) = C$ and $h_2(C) = C$.

Clearly, $h_1|_C$ and $h_2|_C$ coincide. Adding 2-simplices of K to C in order, we shall observe that the part where h_1 and h_2 coincide extends to the whole of S^2. □

Proof of Theorem 1. Let G be a 3-connected nonplanar graph. Define the correspondence Φ in question as follows; Let $q:S^2 \to P^2$ be the canonical double covering projection. Choose a representative $f:G \to P^2$ out of each equivalence class $[f]$ of embeddings of G into a projective plane P^2. Then $\tilde{G} = q^{-1}(f(G))$ is a double covering of G which is embedded in the sphere S^2 so as to cover $f(G)$ 2-fold via q. Set $\Phi([f]) = [\tilde{G}]$, where $[\tilde{G}]$ indicates the isomorphism class which $\tilde{G}$ belongs to. Clearly Φ is well-defined. Now $\tilde{G}$ is a 3-connected planar graph by Lemma 4, so it is uniquely and faithfully embeddable in S^2 by Whitney's result [7]. The uniqueness and faithfulness will imply that Φ is injective and surjective, respectively.

(1) *Φ is injective.* Let $f_1, f_2:G \to P^2$ be two embeddings of G into P^2 and $\tilde{G}$ a double covering of G. Suppose that $\Phi([f_1]) = \Phi([f_2]) = \tilde{G}$. Notice that $\tilde{G}$ might cover G in two ways. Then let $p_i:\tilde{G} \to G$ and $\tau_i:\tilde{G} \to \tilde{G}$ be the covering projection and transformation of the double covering of G corresponding to f_i $(i = 1, 2)$. There are two embeddings $\tilde{f}_1, \tilde{f}_2:\tilde{G} \to S^2$, as lifts of f_1 and f_2, such that $f_i \circ p_i = q \circ \tilde{f}_i$ $(i = 1, 2)$. Since $\tilde{G}$ is uniquely embeddable in S^2, there exist an automorphism $\tilde{\sigma}:\tilde{G} \to \tilde{G}$ and a self-homeomorphism $\tilde{h}:S^2 \to S^2$ such that $\tilde{h} \circ \tilde{f}_1 = \tilde{f}_2 \circ \tilde{\sigma}$.

Let $\tau:S^2 \to S^2$ be the covering transformation of the double covering $q:S^2 \to P^2$, namely the canonical antipodal map, then $\tau \circ \tilde{f}_i = \tilde{f}_i \circ \tau_i$ $(i = 1, 2)$. Place a vertex on each face of $\tilde{f}_1(\tilde{G})$ in S^2 and join it to all vertices lying on the boundary of the face, then a triangulation K of S^2 which contains $\tilde{f}_1(\tilde{G})$ as a subcomplex will be obtained. We may assume that both τ and $\tilde{h}^{-1} \circ \tau \circ \tilde{h}$ are simplicial with respect to K. By Lemma 5, these involutions are equal and hence $\tilde{h} \circ \tau = \tau \circ \tilde{h}$. Thus there is a self-homeomorphism $h:P^2 \to P^2$ such that $h \circ q = q \circ \tilde{h}$. From $\tilde{\sigma} = \tilde{f}_2^{-1} \circ \tilde{h} \circ \tilde{f}_1$ and $\tau_i = \tilde{f}_i^{-1} \circ \tau \circ \tilde{f}_i$ $(i = 1, 2)$, we conclude that $\tilde{\sigma} \circ \tau_1 = \tau_2 \circ \tilde{\sigma}$, so there is an automorphism $\sigma:G \to G$ such that $\sigma \circ p_1 = p_2 \circ \tilde{\sigma}$. Since $\tilde{h} \circ \tilde{f}_1 = \tilde{f}_2 \circ \tilde{\sigma}$, it holds that $h \circ f_1 = f_2 \circ \sigma$. Therefore f_1 and f_2 are equivalent, that is, $[f_1] = [f_2]$.

(2) *Φ is surjective.* Choose a representative $\tilde{G}$ out of each isomorphism class $[\tilde{G}]$ of planar double coverings of G. We can assume that $\tilde{G}$ is faithfully embedded in S^2. Then for the covering transformation $\tau:\tilde{G} \to \tilde{G}$, there is an involution $h:S^2 \to S^2$ such that $h|_{\tilde{G}} = \tau$. Since τ is free, each fixed point of h must

be contained in $S^2 - \tilde{G}$, and hence the fixed set of h is empty or consists of isolated points. By Smith theory, this implies that there is a homeomorphism $g: S^2 \to S^2$ such that $g \circ h \circ g^{-1}$ is either the canonical rotation around the axis or the canonical antipodal map. After $\tilde{G}$ is re-embedded by g, we can assume that h itself is such a canonical one.

In the former case, G could be embedded in the quotient S^2/h which is homeomorphic to a sphere, and G would be planar, a contradiction. Thus the latter is the case and S^2/h is nothing but the projective plane P^2, in turn, covered by S^2 via the canonical projection $q: S^2 \to P^2$. The embeddings of $\tilde{G}$ in S^2 induces naturally an embedding $f: G \to P^2$ and $\Phi([f]) = [\tilde{G}]$. □

It should be pointed out that the isomorphism classes in Theorem 1 do not refer to the structure of coverings of a graph. They are just isomorphism classes of graphs as graphs. Using a combinatorial method developed in Section 2, we can list all of the double coverings of a given graph. Pick out only the planar ones and count their number up to isomorphism. Theorem 1 translates the topological problem of counting the number of projective-planar embeddings of a graph into a graph-theoretical problem. We need not take account of a projective plane, an object in topology. For example, $K_{3,3}$ has a unique planar double covering $C_6 \times K_2$, as is mentioned in the last section, and hence it is uniquely embeddable in a projective plane.

Now we shall prove our characterization of projective-planar graphs, Corollary 2. Theorem 1 has already covered the case of 3-connected graphs. It remains to consider the case that the graph is not 3-connected.

Proof of Corollary 2. The necessity is clear, since the pre-image of G under the canonical double covering $q: S^2 \to P^2$ provides a planar double covering of G.

Now suppose that the sufficiency did not hold. Let G be one of the smallest counter examples to the corollary. By Theorem 1, G is not 3-connected and decomposes into $G = G_1 \cup G_2$ such that both G_1 and G_2 are connected and they have at most two vertices u, v, (possibly equal) in common. Observe that G has not vertex of degree 2. Put $H_i = G_i$ or $G_i + uv$ $(i = 1, 2)$, according to whether or not u and v are adjacent in G_i. Naturally, a planar double covering $\tilde{H}_i$ of each H_i can be derived from a planar covering $\tilde{G}$ of G. By the minimality of G, if H_i is nonplanar, then H_i is projective-planar. It is easy to see that if both or one of H_1 and H_2 were planar, then G would be planar or projective-planar, respectively, contrary to the hypothesis of G. Thus, both H_1 and H_2 are projective-planar, and hence both $\tilde{H}_1$ and $\tilde{H}_2$ are connected.

We choose G_1 to be small as possible. If H_1 splits into two subgraphs H' and H'' which meet in at most two vertices then one of them, say H', contains the edge uv and the other H'' can be regarded as a subgraph of G_1. If H' contained another vertex different from u and v, then H'' would be smaller than G_1, contrary to our choice of G_1. Thus, H_1 is 3-connected.

Embed $\tilde{G}$ into a sphere S^2. Then there is a simple closed curve Γ which separates the lifts $\tilde{G}_1$ and $\tilde{G}_2$ of G_1 and G_2, passing through the four (or two) vertices u_1, u_2 and v_1, v_2, the lifts of u and v, respectively. Join u_1 to v_1 and u_2 to v_2 with edges along Γ. Then an embedding of $\tilde{H}_1$ will be obtained from the embedding of $\tilde{G}_1$ in S^2. Let $\tau:\tilde{H}_1 \to \tilde{H}_1$ be the covering transformation. Since H_1 is 3-connected, so is $\tilde{H}_1$ by Lemma 4, and it is faithfully embedded in S^2. Then τ extends to a free involution $h:S^2 \to S^2$ which carries the face A containing $\tilde{G}_2$ onto itself. Since h is free, h would induce a free involution on the closure of A which is a disk. This contradicts the fact that any continuous map from a disk to itself has a fixed point, as is well-known as Brouwer's fixed point theorem. Therefore, there is no counter example to the corollary $\square$

References

[1] J.L. Gross and T.W. Tucker, Fast computations in voltage graph theory, Ann N.Y. Acad. Sci. 319 (1979) 247–253.

[2] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).

[3] S. Negami, Uniqueness and faithfulness of embedding of toroidal graphs, Discrete Math. 44 (1983) 161–180.

[4] S. Negami, Uniquely and faithfully embeddable projective-planar triangulations, J. Combin. Theory Ser. B 36 (1984) 189–193.

[5] S. Negami, Unique and faithful embeddings of projective-planar graphs, J. Graph Theory 9 (1985) 235–243.

[6] E. Spanier, Algebraic Topology, (McGraw-Hill, New York, 1966).

[7] H. Whitney, Congruent graphs and the connectivity of graphs, Amer. J. Math. 54 (1932) 150–168.