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On Paul Turán's Work in Number Theory

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1

Number theory has always played a central role in Paul Turán's wide and important work in mathematics. His very first publication, as well as 7 of the next 8 papers (1933–1936) are devoted to problems of analytical and elementary number theory. He published about one hundred papers dealing with number theoretical problems. His main achievement in mathematics, the power sum method—now called universally Turán's method—can be viewed, according to Turán's description, as a chapter in the theory of diophantine approximation. The method was created by Turán himself for an attack on the Riemann hypothesis. First successes of the method and its most fruitful applications were connected with the theory of the zeta-function and the distribution of prime numbers. In what follows we shall try to give a short account of some of his achievements which perhaps form the most important part of this outstanding work in number theory.

2

Turán was always interested in finding new pathways in mathematics. His first international success, his thesis [3] initiated probabilistic number theory—now a large chapter of number theory. In [3] he gave a new and very simple elementary proof of the theorem of Hardy and Ramanujan according to which if $U(n)$ denotes the number of prime factors of n and $H(x) \to \infty$ then

$$|U(n) - \log \log N| \leqslant H(N) \sqrt{\log \log N} \tag{2.1}$$

for almost all $n \leqslant N$, i.e., apart from $o(N)$ numbers $n \leqslant N$. He proves by an easy calculation that

$$D(N) \stackrel{\text{def}}{=} \sum_{n \leqslant N} (U(n) - \log \log N)^2 = O(N \log \log N) \tag{2.2}$$

of which (2.1) is an immediate consequence. His method—unlike the Hardy–Ramanujan method—opened the way for investigating general

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additive number theoretical functions. This first general theorem in this direction, the generalization of (2.1) for strongly additive ψ's with $\psi(p) = O(1)$ is contained in [6].

3

Turán was very much interested in the theory of the zeta-function. A characterisation of the zeta-function can be very important in reaching results, e.g., in the direction of the Riemann conjecture. If a proof for a zero-free region does not use all characteristic properties of the zeta-function then the method is probably too weak for proving the Riemann conjecture. The first characterisation by Hamburger and all others prior to Turán's work [112] hinged on a functional equation. Therefore the general impression was left that the best method in proving zero-free regions—the method of trigonometric sums, which doesn't use any functional equation—cannot lead to the proof of the Riemann conjecture. Turán discovered in 1959 [112] the very interesting and surprising fact that a Dirichlet series with positive monotonically decreasing coefficients and Euler product is identical with the zeta-function apart from a translation. The theorem is the multiplicative form of a theorem of Erdös according to which any monotone additive number theoretical function has the form $c \cdot \log n$. In [160] he gave a similar characterisation of Dirichlet's $\mathscr{L}$-functions. The theorem of Turán gives a good example that not always those theorems are important and beautiful where the proof is difficult and that in many cases not the proof of a theorem but the discovery of a new phenomenon is most fruitful in mathematics. He was also the first who at the same time as Linnik and Siegel (but independently) realised the importance of formulating and proving a so called density theorem for the zeros of Dirichlet's $\mathscr{L}$-functions—which has become now a very important area of analytic number theory. He proved in 1943 [26] that almost all $\mathscr{L}$-functions mod k have no zeros in the domain

$$\frac{1}{2} + \frac{5}{\sqrt{\log \log k}} \leqslant \sigma \leqslant 1, \qquad |t| \leqslant 5. \tag{3.1}$$

His most important results concerning the zeta and $\mathscr{L}$-functions were proved by the power sum method which we shall discuss later.

4

Hardy and Littlewood in their famous paper in 1922 introduced a new trailblazing method for the investigation of the Goldbach and twin-prime problem. They proved a heuristic formula for the number of solutions of the equations

$$p+p'=n \text{ and } p-p'=2, \qquad p\leqslant x$$

resp. and showed that the problem is intimately connected with the distribution of zeros of $\mathscr{L}$-functions.

Turán created a new method for the investigation of these problems—called by him the function theoretical sieve method which can be considered as the continuation of Hardy and Littlewood's heuristic method. In his publications [154, 158, 167, 170, 173, 175] he obtained explicit formulae for the number of solutions which contained the heuristic main term (which he gained in a new and far more simple way) and the error terms depending in a relatively easy way on zeros of $\mathscr{L}$-functions. The formulae can be considered as analogs of the exact prime number formula of Riemann and von Mangoldt. As consequences of those formulae he proved the very surprising fact that the solution of the twin-prime problem, e.g., depends only on $\mathscr{L}$-zeros in the domain

$$\tfrac{1}{3}\leqslant\sigma\leqslant\tfrac{3}{4}, \qquad |t|\leqslant\tfrac{7}{4} \tag{4.1}$$

and some analogous phenomenon occurs in case of the Goldbach problem too. Of course it is not yet known whether his completely new ideas can lead or are helpful to the solution of the Goldbach and the twin-prime problems.

5

It is a next to impossible task to sketch the power sum method and its applications, even in number theory. More than a hundred publications of Turán himself and many of the other authors from four continents, 3 books with the same title but with increasingly richer material [66–67, 92, 244], edited in Hungarian, German, Chinese and English deal with the method. The completely rewritten and enlarged new edition [244] which shall appear in the near future also can not cover the full spectrum of that epoch-making theory. The theory gained applications in various branches of mathematics as, e.g., number theory, theory of complex functions, differential equations, numerical analysis.

Turán succeeded in deriving from relatively simple inequalities—concerning oscillation of power sums of finitely many complex numbers—deep theorems in analysis. He himself considered this as the partial execution of a program which was conjectured to be possible by Weierstrass: to derive analysis from algebra.

From the many possible inequalities which have important applications we shall quote 3 main theorems here. They give lower bounds for the oscillation of the power sum

$$g(\nu)=\sum_{j=1}^{n} b_j z_j^{\nu} \tag{5.1}$$

on successive ν-values, where in general the b_j's and z_j's are arbitrary complex numbers, but for simplicity's sake we use some normalization for the z_j's.

The first main theorem asserts

$$\max_{m+1\leqslant\nu\leqslant m+n} |g(\nu)| \geqslant \left(\frac{n}{2e(m+n)}\right)^n \left|\sum_{j=1}^{n} b_j\right| \tag{5.2}$$

if ($m\geqslant 0$ is arbitrary and)

$$\min_{1\leqslant j\leqslant n} |z_j| = 1. \tag{5.3}$$

This had important consequences in the theory of complex functions, differential equations and (indirectly) in transcendental number theory.

The second main theorem applied in analytical number theory and numerical analysis states in terms of the deeper normalization

$$1=|z_1|\geqslant|z_2|\geqslant\cdots\geqslant|z_n| \tag{5.4}$$

the inequality

$$\max_{m+1\leqslant\nu\leqslant m+n} |g(\nu)| \geqslant \left(\frac{n}{8e(m+n)}\right)^n \min_{l=1,\ldots,n} \left|\sum_{j=1}^{l} b_j\right|. \tag{5.5}$$

Turán worked for a long time until he could get what he called one-sided power sum inequalities, where one can guarantee oscillation in both directions for Re $g(\nu)$. Those inequalities were needed for the application in analytic number theory, in the comparative prime number theory created and named by himself and S. Knapowski. A characteristic inequality of this type operates with the normalization (5.4); assuming further that

$$0<\kappa<|\arg z_j|\leqslant\pi, \qquad (j=1,2,\ldots,n) \tag{5.6}$$

it asserts that

$$\max_{m+1\leqslant\nu\leqslant m+(20n/\kappa)} \operatorname{Re} g(\nu)$$

$$\geqslant \left(\frac{n}{8e(m+(20n/\kappa))}\right)^{2n} \min_{l=1,\ldots,n} \operatorname{Re}\left|\sum_{j=1}^{l} b_j\right| \tag{5.7}$$

which gives "large" negative values for Re $g(\nu)$ too if applied for $-b_j$ instead of b_j.

It is an interesting phenomenon in connection with the previously mentioned program of Weierstrass that though all the power sum theorems have purely algebraic features their proof is analytic in most cases.

6

Many of Turán's results in the theory of the Riemann zeta-function were helpful in elucidating the mysterious connection between zeta-function and primes. The starting point for one part of his investigations was a remark in Landau's Handbuch:

> "Die Tatsache, dass $\sum_\rho x^\rho/\rho$ gerade in der Nähe der Primzahlen und höheren Primzahlpotenzen und sonst in der Nähe keiner Stelle ungleichmässig konvergiert, deutet auf einen arithmetischen Zusammenhang zwischen komplexen Wurzeln ρ der Zetafunktion und den Primzahlen hin. Ich habe keine Ahnung, worin derselbe besteht."

In 1947 he proved [31] that if for the constants α, β and for a given $t_0 > c(\alpha, \beta)$ the inequality

$$\left| \sum_{N_1 \leqslant p \leqslant N_2} e^{-it_0 \log p} \right| \leqslant \frac{N \log^{10} N}{t_0^{\beta}} \tag{6.1}$$

holds for all N_1, N_2 with

$$t_0^{\alpha} \leqslant N \leqslant N_1 < N_2 \leqslant 2N \leqslant \exp(t_0^{\beta/10}) \tag{6.2}$$

then $\zeta(s) \neq 0$ on the segment

$$\sigma > 1 - \frac{c^{-10}\beta^3}{\alpha^2}, \qquad t = t_0. \tag{6.3}$$

One can formulate the global form of this theorem if one requires (6.1) for all $t > c(\alpha, \beta)$. Then we get as consequence the existence of a non-trivial zero-free half plane, i.e., the truth of the so called quasi-Riemann conjecture. It is not known at present whether better estimates for exponential sums can lead in the future to a proof of the quasi-Riemann conjecture through (6.1). He also showed [209, 245] that it is enough to assume (6.1) for the far shorter prime range

$$t_0^{\alpha(1-\epsilon)} \leqslant N \leqslant N_1 < N_2 \leqslant 2N < t_0^{\alpha(1+\epsilon)} \tag{6.4}$$

which comes closer to the problem of localized connection between primes and zeta-roots. In [245] he got results which assert that at least under the assumption of the Lindelöf conjecture roughly speaking primes in certain ranges are dependent only on zeta-roots in certain corresponding ranges, and conversely.

7

Another area of the theory of the zeta-function is the problem of the density hypothesis

$$N(\sigma, T) = \sum_{\substack{\zeta(\rho)=0 \\ \operatorname{Re}\rho \geqslant \sigma, 0 < \operatorname{Im}\rho \leqslant T}} 1 = O(T^{2(1-\sigma)+\varepsilon}) \tag{7.1}$$

and related questions were the power sum theory has its most important application in all of mathematics. The significance of (7.1) is that from it follows easily that every interval $[x, x + x^{(1/2)+\varepsilon}]$ for $x > x_0(\varepsilon)$ contains prime numbers. Even assuming the Riemann conjecture we get only slightly more. Turán proved approximately (7.1) near 1 already in 1951 [54], more precisely he gave the estimate:

$$N(\sigma, T) = O(T^{2(1-\sigma)+600(1-\sigma)^{1.01}\log^6 T}), \tag{7.2}$$

which was one of the first great successes of his method. In 1969 Turán and G. Halász [181] could first prove the density hypothesis for a definite range $c_0 \leqslant \sigma \leqslant 1$ (with $c_0 < 1$). They even proved the stronger estimate

$$N(\sigma, T) = O(T^{(1-\sigma)^{3/2}\log^3(1/1-\sigma)}). \tag{7.3}$$

Another interesting problem is the connection of the Lindelöf conjecture

$$\zeta(\tfrac{1}{2} + it) = O(|t|^{\varepsilon}) \tag{7.4}$$

with possible density type theorems. The first surprising result proved by Ingham in 1937 was that it implies the truth of the density hypothesis. Turán gave a new proof for this phenomenon in 1954. In [77] he stated the corollary that it implies even the far stronger inequality

$$N(\sigma, T) = O(T^{\varepsilon}) \tag{7.5}$$

for $\frac{1}{2} < \sigma \leqslant 1$. This surprising fact was proved at least for $\frac{3}{4} < \sigma$ by himself and G. Halász [181]. The method was the same as applied for the proof of (7.3)—showing that unconditional theorems and conditional theorems being perhaps "only of theoretical interest" are much closer to one another than one should think.

8

The power sum method was successful also in demonstrating a connection between zero-free region and distribution of primes. In [48]—solving a problem of Littlewood— he gave the first explicit Ω-type estimate for the remainder term $\Delta(x) = \psi(x) - x = \sum_{n \leqslant x} \Lambda(n) - x$ of the prime number formula in dependence of one ζ-zero $\rho_0 = \beta_0 + i\gamma_0$. His estimate is

$$\max_{x \leqslant T} |\Delta(x)| > \frac{T^{\beta_0}}{|\rho_0|^{10 \log T/\log\log T}} \exp\left(-c_1 \frac{\log T \log\log\log T}{\log\log T}\right) \tag{8.1}$$

for $T > \max(c_2, \exp(|\rho_0|^{60}))$ with explicit absolute constants c_1 and c_2.

Another important problem of Heilbronn and Ingham whether non-trivial bounds for the error term $\Delta(x)$ imply zero-free region for $\zeta(s)$. The other direction is well known and the classical theorem of Ingham states, e.g., that from

$$\zeta(s) \neq 0 \qquad \text{for} \quad \sigma > 1 - \frac{c_3}{\log^{\beta} |t|}, \qquad |t| \geqslant c_4 \tag{8.2}$$

follows

$$\Delta(x) = O(x \exp(-c_5 \log^{(1/1+\beta)} x)) \tag{8.3}$$

but nearly nothing was known in the other direction. Turán's method led to a positive answer for this problem in 1950. Namely, he could show [50] that conversely (8.3) implies (8.2), proving again a conditional theorem of high theoretical interest and giving a new contribution to the task of discovering the mysterious connection between primes and zeta-roots.

9

The problem of finding "one sided" power sum theorems (as given in (5.6)–(5.7)) was stated in the first edition of Turán's book ([66] = [67]) as one of the main open problems of the theory. These theorems were investigated originally for the purpose of attacking an old problem of Riemann and Littlewood. Riemann stated in his famous memoir without proof the assertion that

$$\pi(x) < li\, x = \int_0^x \frac{dr}{\log r} \qquad \text{for} \quad x > 2. \tag{9.1}$$

Riemann's assertion though checked by D. N. Lehmer for all $x \leqslant 10^7$ in 1914 was disproved by Littlewood in the same year, who showed that

$\pi(x) - li\, x$ changes sign infinitely often as $x \to \infty$. Curiously enough his proof was incapable of exhibiting any explicit upper bound for the first sign change and the problem was solved only in 1955 by Skewes.

But Knapowski was the first who using one-sided power sum theorems could prove in 1961–1962 weaker effective ($c \log_4 Y$) and stronger ineffective ($c \log_2 Y$) lower bounds for the number $V(Y)$ of sign changes of $\pi(x) - li\, x$ in the interval $[2, Y]$.[1] It is again of theoretical and historical interest that later Turán and Knapowski could investigate this "one-sided" problem with two-sided power sum theorems—which were developed earlier, nearly ten years before Skewes's proof—with even greater success. Namely, they proved the effective inequality [226]

$$V(Y) > c_6 \log_3 Y \qquad \text{for} \quad Y > c_7 \tag{9.2}$$

and the stronger ineffective one [223]

$$V(Y) > c_8 \frac{\log^{1/4} Y}{(\log_2 Y)^4} \qquad \text{for} \quad Y > Y_0, \tag{9.3}$$

where c_6, c_7, c_8 are explicitly calculable but Y_0 is an ineffective absolute constant.

10

Though one-sided power sum theorems could be eliminated from the problem of investigation of sign changes of the remainder term of the prime number formula they were useful and also now necessary in another closely related theory, in the so called comparative prime number theory. This theory which was founded and developed in nearly 20 papers of Turán and Knapowski had as its aim finding irregularities in the distribution of primes in arithmetic progressions.

The starting point for such a theory was the mysterious assertion of Chebyshev in 1853, stating

$$\lim_{x \to \infty} \sum_{p > 2} (-1)^{(p-1)/2} e^{-p/x} = -\infty \tag{10.1}$$

without any proof or even background. The only results in 100 years after it were 3 papers of Hardy and Littlewood and Landau in which they proved that (10.1) is equivalent to the very deep analog of the Riemann conjecture

$$\mathscr{L}(s, \chi_1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^s} \neq 0 \qquad \text{for} \quad \sigma > \frac{1}{2}, \tag{10.2}$$

[1] $\log_\nu x$ stands for the ν-times iterated logarithm function.

further the result of Littlewood that

$$\pi_1(x) - \pi_3(x) = \sum_{2 < p \leqslant x} (-1)^{(p-1)/2} \tag{10.3}$$

changes sign infinitely often. The small number of results was probably not due to lack of interest—Chebyshev's assertion seemed to be sufficiently attractive especially in view of the equivalence with (10.2)—but due to the fact that these problems were indeed as characterized by O. Toeplitz in a lecture in the 1930s as "fast unangreifbar bei dem heutigen Stand der Wissenschaft." The main problem is what can one prove on possible sign changes of the functions $\psi(x, k, l_1) - \psi(x, k, l_2)$ and $\pi(x, k, l_1) - \pi(x, k, l_2)$, where

$$\psi(x, k, l) = \sum_{\substack{n \leqslant x \\ n \equiv l(k)}} \Lambda(n), \qquad \pi(x, k, l) = \sum_{\substack{p \leqslant x \\ p \equiv l(k)}} 1 \tag{10.4}$$

and $(l_1, k) = (l_2, k) = 1$, $l_1 \not\equiv l_2 \pmod{k}$. A necessary disadvantage of the theory is that for general modulus k one has to assume that the $\mathscr{L}$-functions mod k have no real zeros or more explicitly they don't vanish in the region

$$0 \leqslant \sigma \leqslant 1, \qquad |t| \leqslant A(k). \tag{10.5}$$

This is verified for $k < 25$ by calculations of Hazelgrove and Spira but it is very deep and was unknown generally until now. The necessity of this assumption can be seen easily from the formula

$$\psi(x, k, l) = \frac{x}{\varphi(k)} - \sum_{\chi} \bar{\chi}(l) \sum_{\substack{\rho = \rho_\chi \\ |\mathrm{Im}\, \rho| \leqslant x}} \frac{x^\rho}{\rho} + O(\log^2 kx) \tag{10.6}$$

where in case of existence of a real zero being right from all complex zeros $\psi(x, k, l_1) > \psi(x, k, l_2)$ is possible for all $x > x_0$.

Now Turán and Knapowski could prove under the assumption (10.5) that all the functions $\psi(x, k, l_1) - \psi(x, k, l_2)$ change their sign in every interval of the form $[\omega, \exp(2\sqrt{\omega})]$ if

$$\omega \geqslant \max(e^{k^c}, e^{(2/A(k))^3}). \tag{10.7}$$

The problem of finding sign changes for the functions $\pi(x, k, l_1) - \pi(x, k, l_2)$ turned to be far deeper. Here they had to assume beside the natural (10.5) also a finite form of the Riemann–Piltz conjecture ($\mathscr{L}(s, \chi) \neq 0$ for $\sigma > \frac{1}{2}$, $|t| \leqslant ck^{10}$). Under these assumptions they could prove the occurrence of infinitely many sign changes in the case were both l_1 and l_2 are quadratic residues mod k or both are non-residues. However, in the important special case $l_1 = 1$ they were able to show sign changes in every interval $[\log_3 T, T]$ for

$$T \geqslant \max\ (c, \exp\exp\exp\exp\exp k, \exp(1/A(k))) \tag{10.8}$$

assuming only (10.5) independently from the quadratic nature of l_2. An interesting feature of the problems discussed in Sections 8–10 is that no other known method can lead to the results mentioned there.

The comparative prime number theory like other areas where Turán worked is now also full of deep unsolved problems which will be solved perhaps with other methods in the future. However, his pioneering contribution to many branches of mathematics can never be forgotten. We hope that this relatively short paper also can give an adequate idea of the main characteristics of Paul Turán's number theoretical work, of the richness and importance of his results, of the new and exciting art of proposing the questions and of working out methods for solving the new problems, of his endeavor of always searching for new paths in mathematics.