Serial Rings and Finitely Presented Modules*

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Communicated by I. N. Herstein

Received May 10, 1973

INTRODUCTION

A module is serial if its submodules are linearly ordered with respect to inclusion. A ring $R$ is called left serial if $R R$ is a direct sum of serial modules, and $R$ is called serial if it is both left and right serial. Serial rings provide a natural generalization of both commutative valuation rings and generalized uniserial algebras. In this paper, it is shown that a finitely presented module over a serial ring is a direct sum of serial modules. Applications are given to the structure of semiperfect semihereditary rings and "locally serial" algebras over commutative Noetherian rings. A fairly complete structure theory is given for Noetherian serial rings.

In the first section we review the basic properties of semiperfect rings. If $R$ is semiperfect, $M$ is a finitely presented (FP) $R$-module, and $S$ is a simple $R$-module, we introduce two numerical invariants, $\text{Gen}(M; S)$ and $\text{Rel}(M; S)$, which we can use to refine statements usually made in terms of generators and relations. We use this to give the structure theory of semiperfect rings for which every finitely generated left ideal is principal (1.14). We also show (1.4 and 1.5) that every stable isomorphism class of finitely generated modules over a semiperfect ring contains a unique (up to isomorphism) minimal element. (Recall that $A$ and $B$ are stably isomorphic if there are projective $P$ and $Q$ such that $A \oplus P \cong B \oplus Q$.) These results are used in Section 2 to give a refined form of the duality theory of Auslander and Bridger [1], between FP right modules and FP left modules.

In Section 3 we prove that every finitely presented module over a serial ring is a direct sum of serial modules. We actually show more. Say that a module is local if it has a unique maximal proper submodule, and $M$ is locally presented (LP) if there is an exact sequence $P \to Q \to M \to 0$ in which $P$ and $Q$ are local projectives. We show, then (2.6 and 3.4), that the following

* Research supported in part by National Science Foundation grant GP-28946.

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properties are equivalent for a semiperfect ring $R$: (i) Every FP module (right or left) is a direct sum of local modules; (ii) every FP left module is a direct sum of LP modules; (iii) every FP left module is a direct sum of serial modules; (iv) $R$ is serial.

There are many special cases of this theorem in the literature. In 1949, Kaplansky proved [24] that a finitely presented module over a local commutative serial ring is a direct sum of cyclic modules. (There is no gain in generality in considering semiperfect rings in the commutative case.) He also proved this for a local serial ring $R$ such that if $r \in R$, then $Rr = rR$ (i.e., all left and right ideals are two-sided). Roux [36] proved the same result for local serial rings in general in an ingenious paper that also contains a partial converse. Converses to Kaplansky's result in the commutative case seem to have been first proved by the author [39] and Lafon [27]. Kaplansky also proved uniqueness results for his decompositions. No such result has been proved in the noncommutative case, though the uniqueness is obvious in certain cases (e.g., for Noetherian serial rings).

If a serial ring is nonsingular, it is semihereditary. In Section 4 we investigate what conditions are needed for a semiperfect semihereditary ring to be serial. In particular (4.7), a semiperfect, semiprime, left and right Goldie ring is semihereditary if and only if it is serial. The nonsingular local serial rings are characterized (4.9) in a way that points up their relation to commutative valuation rings, and it is shown (4.10) that an indecomposable serial semihereditary ring is an order in a block upper triangular matrix ring over a division ring.

In Section 5, we give a fairly complete structure theory for Noetherian serial rings. Such a ring is the product of an Artinian serial ring and a finite number of prime rings (necessarily hereditary, by the results of Section 4). If $R$ is a prime Noetherian serial ring that is not semisimple, a theorem of Michler's [31] says that there is a discrete valuation ring $D$ (not necessarily commutative) with Jacobson radical $J$, such that $R$ is a $(D; J)$ block upper triangular matrix ring. (For this terminology, see 5.12 or Robson [34]. Any such ring is Morita equivalent to a matrix ring over $D$ in which the entries below the diagonal are restricted to lie in $J$.) We give a new, short proof of Michler's theorem, using the methods of this paper. Section 5 was considerably influenced by the recent papers on Artinian serial rings by Eisenbud and Griffith [10, 11].

In the last section of the paper, we apply our previous results to algebras over commutative Noetherian rings. We assume that $R$ is a commutative Noetherian ring and $A$ is an $R$-algebra that is finitely generated as an $R$-module. If for every maximal ideal $m$ of $R$, we let $R_m^*$ be the $m$-adic completion of $R$, then the $m$-adic completion of $A$ is just $A \otimes R_m^* = A_m^*$. $A_m^*$ is a semiperfect ring. The main theorem (6.6) of Section 6 is that under these
circumstances, the following properties of $A$ are equivalent: (i) For every ideal $I$ such that $A/I$ is Artinian, $A/I$ is serial; (ii) every finitely generated module is the direct sum of a projective module and a finite number of Artinian serial modules; (iii) $A$ is the product of an Artinian serial ring and a finite number of hereditary orders over Dedekind domains; and (iv) for every maximal ideal $m$ of $R$, $A_m \ast$ is a serial ring. This yields, as an easy corollary, a structure theorem for a class of hereditary algebras (6.7). We also show (6.8) that an algebra $A$ has the property that all of its finitely generated modules are balanced if and only if it is the product of an Artinian uniserial ring (in the sense of Nakayama) and a finite number of maximal orders over Dedekind domains. (For the background and literature on this problem, we refer to Section 6 below.)

The use of the complete localization in Section 6 is based on its use in the classical theory of orders over Dedekind rings (as in [35]). If one regards semiperfect rings as the right noncommutative generalization of local rings (as we do), and if one wants to study some class of rings in terms of its "localizations", one must have an analogue of the complete localization for algebras over commutative Noetherian rings. (This is not a localization, in the usual modern sense, since the homomorphism $A \to A_m \ast$ is not a ring epimorphism.) We do not plan to develop such a theory here, but we note that in at least one recent study [18, Chap. 2], the complete localization seems to be a more natural object than the localization.

We close this introduction by mentioning some open questions. Perhaps the outstanding one is the uniqueness question for decompositions of an FP module into serial summands (proved in the commutative case and in one noncommutative case by Kaplansky [24]). Another, very general, problem is to extend the results of Section 6, getting global results in a more general setting. Presumably this would require an analogue of the complete localization theory used in Section 6.

At the end of Section 2, we discuss some non-Artinian variants of the problem of Köthe rings. Another problem related to those in Section 2 is that of finding the results analogous to ours for finitely generated modules. For the commutative case of this, see [15, 24, 25, 27]. Finally, in connection with Theorem 6.10, there are a large number of open questions concerning rings for which certain modules are balanced. (For example, what semiperfect rings have the property that all FP modules are balanced?)

All rings in this paper are associative with identity. If $R$ is a ring, we denote by $R\mathcal{R}$ and $R_\mathcal{R}$, respectively, the modules obtained by regarding $R$ as a left or right module over itself. In a discussion of modules, if it does not matter whether the module in question is left or right, we will usually assume that the module is a left module. However, when we consider a property of a ring, such as "all $R$-modules have property $X$," then this will be intended to cover
both left and right modules, unless otherwise specified. Similarly, terms such as "hereditary," "serial," and "Noethrian," when applied to a ring, will apply on both sides, unless otherwise specified.

1. Modules over Semiperfect Rings

In this section we review the basic properties of semiperfect rings. We show that if $R$ is a semiperfect ring, then every stable isomorphism class of finitely generated modules contains a unique (up to isomorphism) minimal element (1.4 and 1.5). A refinement of the usual notions of the number of generators and relations of a finitely presented module (1.6–1.11) is given and we use this to give a characterization of left serial rings (1.13) and to give the structure of semiperfect rings for which every finitely generated left ideal is principal (1.14).

**Definition 1.1.** A module $M$ is local if it has a unique maximal proper submodule.

**Lemma 1.2.** If $R$ is a ring with Jacobson radical $J$, and $M$ a finitely generated left $R$-module, then: (i) if $N$ is a submodule of $M$ and $M = N + JM$, then $M = N$, (ii) if $R/J$ is Artinian, then $M/JM$ is a direct sum of a finite number of simple modules; and (iii) if $R/J$ is Artinian, then $M$ is local if and only if $M/JM$ is simple.

Part (i) is merely a form of "Nakayama's lemma," and (ii) and (iii) are trivial. We now review quickly the various equivalent conditions that characterize semiperfect rings.

**Lemma-Definition 1.3.** If $R$ is a ring with Jacobson radical $J$ and $R/J$ is Artinian, the following conditions on $R$ are equivalent, and the rings satisfying them are called semiperfect rings:

(i) $R$ is a direct sum of local modules;  
(ii) $R$ is a direct sum of local modules;  
(iii) if $e \in R/J$ is an idempotent, there is an idempotent $f \in R$ such that $e = f + J$.  
(iv) All projective modules are direct sums of local cyclic projectives.

All of 1.3 is quite standard [3, 32].

**Theorem 1.4.** Let $M$ be a finitely generated module over a semiperfect ring
R. Then there is a decomposition \( M = N \oplus P \), where \( P \) is projective and \( N \) has no projective summands. Further, if \( M = N' \oplus P' \) is another such decomposition, then \( N \cong N' \) and \( P \cong P' \).

Proof. The existence of the indicated decomposition is a triviality—we just start separating off projective summands, if they exist, and the process will eventually stop by the chain condition on \( M/JM \) and 1.2. Suppose, then, that we have two such decompositions: \( M = N \oplus P = N' \oplus P' \). We recall that the endomorphism ring of an indecomposable projective module over a semiperfect ring is a local ring, from which it follows that any finitely generated projective module has the exchange property [38, 40]. Applying the exchange property for \( P \), we obtain submodules \( P'_0 \subseteq P' \) and \( N'_0 \subseteq N' \), such that \( M = P \oplus P'_0 \oplus N'_0 \) (that, incidentally, is the definition of the exchange property). This clearly implies that \( N \cong P'_0 \oplus N'_0 \), so that \( P'_0 = 0 \) (since \( N \) has no nonzero projective summands). We can decompose \( N' = N'_0 \oplus (N' \cap P) \), and comparing the two complements of \( N'_0 \), we obtain \( (N' \cap P) \oplus P' \cong P \). Since by hypothesis, \( N' \) has no nonzero projective summands, it follows that \( N' \cap P = 0 \), \( N' = N'_0 \). The above formulas now imply that \( N \cong N' \) and \( P \cong P' \), as desired.

We recall that modules \( X \) and \( Y \) are stably isomorphic if there are projectives \( P \) and \( Q \) such that \( X \oplus P \cong Y \oplus Q \). (If \( X \) and \( Y \) are finitely generated, we may assume \( P \) and \( Q \) are also). The previous theorem therefore yields the following sharpening of the notion of stable isomorphism for modules over semiperfect rings.

**Corollary 1.5.** If \( R \) is a semiperfect ring then: (i) Every finitely generated module is stably isomorphic to a module with no nonzero projective summands; and (ii) two modules with no nonzero projective summands are stably isomorphic if and only if they are isomorphic.

This result will allow considerable precision in the next section in applying the Auslander-Bridger duality theory, which initially is only a relation between stable isomorphism classes of modules. To use that duality theory, we will also need some fairly precise notions concerning generators and relations for modules over semiperfect rings.

**Lemma 1.6.** If \( R \) is a semiperfect ring with Jacobson radical \( J \) and \( M \) is a finitely generated module, then there is a projective module \( P \) and an epimorphism \( f : P \to M \) such that the induced homomorphism \( P/JP \to M/JM \) is an isomorphism. If \( g : Q \to M \) is another such epimorphism, (with \( Q \) projective) then there is an isomorphism \( \phi : P \to Q \), of \( P \) onto \( Q \) such that \( g\phi = f \).

Lemma 1.6 is just a description of the projective cover [3] of \( M \). The
uniqueness of the projective cover makes it possible to give a precise and refined form to certain statements concerning generators and relations.

**Definition 1.7.** Let $M$ be a finitely generated module over a semiperfect ring $R$, and let $S$ be a simple module. We define $\text{Gen}(M)$ to be the number of summands in a decomposition of $M/\text{J}M$ as a direct sum of simple modules, and $\text{Gen}(M; S)$ to be the number of such summands isomorphic to $S$.

**Lemma 1.8.** A finitely generated module $M$ over a semiperfect ring $R$ is cyclic if and only if for every simple module $S$, $\text{Gen}(M; S) \leq \text{Gen}(\text{K}R; S)$.

**Definition 1.9.** Let $M$ be a finitely presented module over a semiperfect ring $R$, and $f: P \to M$ a projective covering (as in 1.4), and $K = \ker(f)$. Then we define $\text{Rel}(M)$ and $\text{Rel}(M; S)$, for all simple modules $S$, by $\text{Rel}(M) = \text{Gen}(K)$, $\text{Rel}(M; S) = \text{Gen}(K; S)$. These are well-defined by 1.6.

**Lemma 1.10.** If $M$ is a finitely generated module over a semiperfect ring, $S$ is a simple module, and $M = A \oplus B$, then $\text{Gen}(M; S) = \text{Gen}(A; S) + \text{Gen}(B; S)$. If $M$ is finitely presented then $\text{Rel}(M; S) = \text{Rel}(A; S) + \text{Rel}(B; S)$.

This is easily shown by taking projective covers (1.6) for $A$ and $B$ and combining them to get one for $M$.

It is clear from 1.8 that $\text{Gen}(M)$ is only indirectly related to the number of generators required to generate $M$. However, if we restrict ourselves to certain kinds of elements, the connection becomes closer. If $M$ is an $R$-module and $x \in M$, we say $x$ is a local element if $Rx$ is a local module. The fact that any projective module is generated by local elements (1.3) implies that any module is generated by a set of local elements.

**Lemma 1.11.** If $M$ is a finitely generated module over a semiperfect ring and $X$ is a set of local elements of $M$, then $X$ is a minimal set of local generators for $M$ if and only if the natural map $M \to M/\text{J}M$ takes $X$ bijectively onto a minimal set of local generators of $M/\text{J}M$. The number of elements in any minimal set of local generators is exactly $\text{Gen}(M)$.

**Definition 1.12.** A module is serial if its submodules are linearly ordered with respect to inclusion. A ring $R$ is left serial if $\text{K}R$ is a direct sum of serial modules. $R$ is serial if it is both left and right serial.

Artinian serial rings are traditionally called “generalized uniserial rings,”
and were introduced by Nakayama [33] and studied more recently by Eisenbud and Griffith [10, 11].

**Lemma 1.13.** If $M$ is a finitely generated module over a left serial ring and $N$ a finitely generated submodule, then

$$\text{Gen}(N) \leq \text{Gen}(M).$$

**Proof.** Let $L$ be a local submodule of $M$ such that $L$ is not contained in $JM$. [You can find such a submodule by taking a projective cover of $M$ (1.6) and looking at the image in $M$ of one of the indecomposable summands of the projective module.] Look at the short exact sequence

$$0 \to N \cap L \to N \to N/N \cap L \to 0.$$

Given any short exact sequence $0 \to X \to Y \to Z \to 0$, there is an induced exact sequence $X/JX \to Y/JY \to Z/JZ \to 0$. Note that if $X$ is serial, then either $X = JX$ or $X/JX$ is simple. Applying this to the sequence above, and using the fact that $N \cap L$ is serial, we obtain $\text{Gen}(N) \leq 1 + \text{Gen}(N/N \cap L)$. By our choice of $L$, $\text{Gen}(M/L) + 1 = \text{Gen}(M)$, and since $N/N \cap L$ is a finitely generated submodule of $M/L$, our result follows by induction on $\text{Gen}(M)$.

Lemma 1.13 gives a characterization of left serial rings similar to the well-known characterizations of rings whose finitely generated left ideals are principal. (If $R$ is such a ring, $M$ is a module that can be generated by $n$ elements, and $N$ is a finitely generated submodule, then $N$ can be generated by $n$ elements.) It is fairly easy to characterize semiperfect rings in which every finitely generated left ideal is principal, and we will do this now as an application of the previous methods.

If $M$ is a serial module, we say $M$ is **homogeneously serial** if for all pairs of nonzero finitely generated submodules $A$ and $B$, $A/JA \cong B/JB$. If $M$ satisfies both the ascending and descending chain conditions, this means that all the simple composition factors of $M$ are isomorphic.

**Theorem 1.14.** The following properties are equivalent for a semiperfect ring $R$:

(i) Every finitely generated left ideal is principal;

(ii) $R$ is left serial and the indecomposable summands of $_RR$ are homogeneously serial;

(iii) $R$ is the product of a finite number of full matrix rings over local, left serial rings.
Remark. The artinian case of this theorem is well-known [11, 20]. For related results see [23].

Proof. To show that (i) implies (ii), first show that $R$ must be left serial. If not, $\mathfrak{r}R = P \oplus Q$, where $P$ is an indecomposable projective that is not serial. If $A$ is a finitely generated submodule of $P$ such that $\text{Gen}(A) \geq 2$, then $\text{Gen}(A \oplus Q) = \text{Gen}(A) - \text{Gen}(Q) > \text{Gen}(\mathfrak{r}R)$ so that $A \oplus Q$ is certainly not a cyclic module. We now suppose that $R$ is left serial and has an indecomposable summand $P$ that is not homogeneously serial. Again, we will find a left ideal that is not principal. Write $\mathfrak{r}R = P \oplus Q$ and let $A$ be a cyclic submodule of $P$ such that $A/IA = S$, where $S$ is a simple module that is not isomorphic to $P/JP$. In this case, $\text{Gen}(Q; S) = \text{Gen}(\mathfrak{r}R; S)$, so $\text{Gen}(A \oplus Q; S) = \text{Gen}(\mathfrak{r}R; S) + 1$. By 1.8, $A \oplus Q$ is not a principal left ideal.

We now show that (ii) implies (iii). Condition (ii) implies that if $P$ and $Q$ are nonisomorphic indecomposable projectives then $\text{hom}(P, Q) = 0$, since if $A$ were the image of such a homomorphism, $A \neq 0$, then we would have $A/\text{J}A \cong P/\text{J}P \ncong Q/\text{J}Q$. $R$ is therefore a product of a finite number of rings satisfying (ii), with the additional property that all of their indecomposable projectives are isomorphic. It is standard that such a ring is a full matrix ring over a local ring [29, p. 79]. The property of being a left serial ring is preserved under Morita equivalence (since it has a categorical description: projective modules are direct sums of serial modules); thus, the local rings that arise again must be left serial rings. This completes the proof of (iii).

To show that (iii) implies (i), we may restrict ourselves to indecomposable rings [since the statement of (i) is preserved by products of rings]. An indecomposable ring satisfying (iii) has only one simple module up to isomorphism. Combining 1.8 and 1.13 we see that if $R$ is a left serial ring with only one simple module up to isomorphism, then every submodule of a cyclic module is cyclic, which implies (i).

2. Auslander–Bridger Duality

In this section, we give our promised refinement of the Auslander–Bridger duality theory and use it to study rings for which every finitely presented module is a direct sum of local modules. In particular, we show that this property implies that the ring is serial. Some related questions concerning Köthe rings and rings of bounded representation type are discussed. We conclude by proving a theorem that gives a restriction on the structure of a ring $R$ for which there is an upper bound on the number of generators required for indecomposable FP $R$-modules.
If \( R \) is any ring with identity and \( M \) is a finitely presented (FP) left \( R \)-module, we define the Auslander–Bridger dual \( D(M) \) as follows. Choose an exact sequence \( Q \to P \to M \to 0 \) in which \( P \) and \( Q \) are finitely generated projective modules. Define \( D(M) \) to be the cokernel of the homomorphism \( \phi^*: P^* \to Q^* \) (where \( X^* = \text{hom}(X, \_R) \), and if \( X \) is a left module, \( X^* \) is a right module). \( D(M) \) is well-defined up to stable isomorphism (defined in the discussion prior to 1.5), and this defines a category anti-isomorphism between the categories of stable isomorphism classes of FP left modules and stable isomorphism classes of FP right modules [1].

Note that if \( M \) is given by \( n \) generators and \( K \) relations, by a sequence \( R^n \to R^k \to M \to 0 \), then \( D(M) \) is given by \( k \) generators and \( n \) relations. For modules over semiperfect rings, we can make this more precise using the invariants defined in the previous section (1.7 and 1.9) and the fact that over such a ring, each stable isomorphism class of finitely generated modules contains, in some sense, a canonical minimal element (1.4 and 1.5).

**Lemma 2.1.** If \( R \) is a semiperfect ring and \( P \) is an indecomposable projective left module, then the dual \( P^* \) is an indecomposable projective right module. Further, if \( P/JP \cong S \), then \( P^*/P^*J \cong S' \), where \( S' \) is the dual of the module \( S \) with respect to the ring \( R/J \).

**Lemma 2.2 [36, Proposition 1.3].** If \( R \) is a semiperfect ring, \( P \) is a finitely generated projective module, and \( N \) is a submodule not contained in \( JP \), then \( N \) contains a nonzero summand of \( P \).

If \( M \) is a finitely presented module over a semiperfect ring and \( Q \to P \to M \to 0 \) is an exact sequence, then we call this sequence a **minimal presentation** of \( M \) if: (i) the induced homomorphism \( P/JP \to M/JM \) is an isomorphism, and (ii) if \( K \) is the kernel of the homomorphism \( P \to M \), then the induced homomorphism \( Q/JQ \to K/JK \) is an isomorphism. In this case, the isomorphism types of \( P/JP \) and \( Q/JQ \) are invariants of \( M \) (1.6, 1.7, and 1.9).

**Lemma 2.3.** If \( R \) is a semiperfect ring, \( M \) is a finitely presented left \( R \)-module with no nonzero projective summands, and \( Q \to P \to M \to 0 \) is a minimal presentation for \( M \), then the induced sequence \( P^* \to Q^* \to D \to 0 \) is a minimal presentation for \( D \), and \( D \) has no nonzero projective summands.

**Remark.** Strictly speaking, the dual \( D(M) \) is a stable isomorphism class, of which the above module \( D \) is a representative. The point of this lemma is that by choosing a suitable resolution, one can assure that the resulting representative is the canonical member of the stable isomorphism class, whose existence is guaranteed by 1.5.
Proof. If $\phi: Q \to P$ is the homomorphism appearing in the presentation of $M$, and $\phi^*: P^* \to Q^*$ is its dual, then we must first show that $\phi^*(P^*) \subseteq Q^* J$. If not, by 2.2 we could decompose $P^* = A \oplus B$, where $\phi^*$ maps $B$ isomorphically onto a nonzero summand of $Q^*$. Dualizing again, and identifying $P^{**} = P$, we would get a nonzero summand $B^*$ of $P$ in the kernel of the homomorphism from $P$ to $M$, contradicting the fact that the presentation is minimal.

Now let $L$ be the kernel of the homomorphism $Q^* \to D$, and by our previous argument we know that $L \subseteq Q^* J$. We wish to show that the homomorphism $\phi^*$ induces an isomorphism $P^*/P^* J \to L/L J$. If not, by 2.2 we can decompose $P^* = C \oplus E$, where $C \neq 0$ and $C$ is in the kernel of $\phi^*$. Dualizing again, we obtain $P = C^* \oplus E^*$ and $\phi(Q) \subseteq E^*$. Hence, $M$ would have a projective summand, which it does not.

The final point is that $D$ has no nonzero projective summands. One argues, just as before that if it had such a summand, then $\phi^*(P^*)$ would lie in a proper summand of $Q^*$, and dualizing, one would obtain a nonzero summand of $Q$ in the kernel of $\phi$, contradicting the minimality of the presentation.

**Theorem 2.4.** Let $R$ be a semiperfect ring, $\mathcal{X}$ be the class of finitely presented left modules with no nonzero projective summands, and $\mathcal{Y}$ be the class of finitely presented right modules with no projective summands. To each $M \in \mathcal{X}$ there is an element $D(M) \in \mathcal{Y}$, uniquely determined up to isomorphism, such that: (i) $M \cong N$ if and only if $D(M) \cong D(N)$; and (ii) $D(M \oplus N) \cong D(M) \oplus D(N)$. Furthermore, if to each simple left module $S$ we associate a corresponding simple right module $S'$ as in (2.1), then the following equations hold: $\text{Gen}(M; S) = \text{Rel}(D(M); S')$, and $\text{Rel}(M; S) = \text{Gen}(D(M); S')$.

This theorem is an immediate consequence of the original Auslander-Bridger duality [1], outlined at the beginning of this section and 2.3, 1.5, 1.7, and 1.8. $D(M)$ can be computed explicitly from 2.3. If we start from right modules instead of left modules, we again get such a function $D$, and, of course, it is the inverse of the one discussed here.

**Definition 2.5.** A module $M$ is locally presented (LP) if there is an exact sequence $Q \to P \to M \to 0$ in which $P$ and $Q$ are both local projectives. Equivalently, $M$ is finitely presented and $\text{Gen}(M) = 1$, $\text{Rel}(M) \leq 1$.

**Theorem 2.6.** The following properties of a semiperfect ring are equivalent:

(i) Every FP left module is a direct sum of LP modules;

(ii) every FP left module is a direct sum of serial modules;

(iii) every FP right module is a direct sum of LP modules;
(iv) every FP left module and every FP right module is a direct sum of local modules.

Remark. These conditions clearly imply that $R$ is serial (i.e., left and right serial). We will show in the next section, that these rings are exactly the serial rings.

Proof. Condition (i) says that if $M$ is an indecomposable finitely presented left module which is not projective, then $\text{Gen}(M) = \text{Rel}(M) = 1$. This implies that if $P$ is an indecomposable projective and $A$ a finitely generated submodule ($A \neq 0$) then $\text{Gen}(A) = 1$ since $\text{Gen}(A) = \text{Rel}(P/A)$ and $P/A$ is indecomposable. Therefore, (i) implies that $R$ is left serial, and for a left serial ring, (i) and (ii) are trivially equivalent.

Again, since (i) says that if $M$ is an indecomposable FP left module which is not projective, then $\text{Gen}(M) = \text{Rel}(M) = 1$, it follows by 2.4 that this is equivalent to the corresponding condition for right modules, whence (i) and (iii) are equivalent.

Finally, it is clear that (i) and (iii) imply (iv). Conversely, if $R$ satisfies (iv) and $M$ is an indecomposable, nonprojective FP module, then $\text{Gen}(M) = 1$ by (iv) and $\text{Gen}(D(M)) = 1$, by (iv) and the fact that $D(M)$ is indecomposable (2.4). Since $\text{Gen}(D(M)) = \text{Rel}(M)$ (by 2.4), it follows that $M$ is LP, and thus, (iv) implies (i).

In the rest of this section, we point out the connections between the results of this and the next section to the general problem of Köthe rings and some related problems. Faith [12] calls a ring $R$ a Köthe ring if every left module and every right module is a direct sum of cyclic modules. (One could clearly talk about left Köthe rings and right Köthe rings if one wished.) Chase [6] shows that such a ring is Artinian. If we want to look at rings that are not Artinian, it is reasonable to restrict attention to finitely presented modules. One obtains two obvious generalizations of the original question (which coincide in the Artinian case because of the Krull–Schmidt theorem).

Question 1. What rings have the property that every FP module is a direct sum of cyclic modules?

Question 2. What rings have the property that every FP module is a summand of a direct sum of FP cyclic modules?

An answer to Question 2 in the commutative case is in [39, Theorem 3]. Descriptions of the rings answering Question 1 have been given [24, 28], but whether these are “answers” depends on one’s taste. In particular, whether a Bezout domain necessarily has the property described in Theorem 1 is still an outstanding problem.

If a problem cannot be solved, it is always tempting to change it. For example, if we replace the word “cyclic” in question 1 with the word “local,” then
the answer is that the rings in question are exactly the serial rings (Theorems 2.6 and 3.4). One virtue of this question is that the answer is clearly invariant under Morita equivalence, which is not the case for the original question. If one likes Morita equivalence, there are two other natural ways to make Questions 1 and 2 "Morita invariant."

**Question 3.** What rings $R$ have the property that every finitely presented module over a ring Morita equivalent to $R$ is a direct sum of cyclic modules?

Obviously, one can make a similar variant of Question 2. In a different direction, one might ask what rings $R$ have the property that some ring Morita equivalent to $R$ has the required property for its FP modules. If we restrict ourselves to semiperfect rings (so that every FP module is a direct sum of indecomposable modules), then it is easy to see that $R$ is Morita equivalent to a ring over which every FP module is a direct sum of cyclic modules if and only if $R$ is of *bounded representation type*, i.e., there is an upper bound on the number of generators required for indecomposable FP modules.

**Question 4.** For what semiperfect rings is there an upper bound on the number of generators required for the indecomposable finitely presented modules?

We do not have an answer for Question 4, but we will show that there is a serious restriction on the structure of such a ring. In particular, if a ring is semiprimary and of bounded representation type (in this sense), then it is Artinian. (This special case is essentially what was proved by Eisenbud and Griffith in [11], also using Auslander–Bridger duality. Their theorem provided the inspiration for this one.)

We recall that a module $M$ has *finite Goldie dimension* (or rank) if it satisfies one of the following two equivalent conditions: (i) The injective hull $E(M)$ is a finite direct sum of indecomposable modules; or (ii) $M$ does not contain any infinite direct sums of nonzero submodules [18, pp. 214–217]. The number of summands appearing in a decomposition of $E(M)$ is an invariant of $M$—the Goldie dimension of $M$, which we usually write rank($M$).

**Theorem 2.7.** Let $R$ be a semiperfect ring and suppose that there is an upper bound on the number of generators required for indecomposable FP right $R$-modules. Then if $I$ is any left ideal of $R$, $R/I$ has finite Goldie dimension, and there is an upper bound for the numbers rank($R/I$). In particular, if $R$ is semiprimary, then $R$ is left Artinian.

**Proof.** We first establish an upper bound for rank($P/A$), for indecomposable projective modules $P$ and submodules $A$. If rank($P/A$) $\geq n$, then $P/A$ has a finitely generated submodule $B$ such that Gen($B$) = $n$. (We may
assume $A$ is a direct sum of local modules if we wish.) Let $B' = \{x \in P: x + A \in B \}$. Clearly, $B'/JB'$ has rank at least $n$ (where $J$ is the Jacobson radical of $R$), and thus, $B'$ contains a finitely generated submodule $B''$ such that $\text{Gen}(B'') = n$. $P/B''$ is an indecomposable FP module, with $\text{Gen}(P/B'') - 1$ and $\text{Rel}(P/B'') - n$. If $N$ is the Auslander–Bridger dual of $P/B''$ (in the strong sense of 2.4) then $N$ is an indecomposable FP right module and $\text{Gen}(N) = n$. Since, by hypothesis, there is an upper bound on the numbers $\text{Gen}(N)$, it follows that there is an upper bound on the numbers $\text{rank}(P/A)$, independent of the indecomposable projective $P$ or the submodule $A$.

We now let $I$ be any left ideal of $R$, and write $\_R R = P_1 \oplus \cdots \oplus P_k$ as a direct sum of indecomposable projectives. $R/I$ is the sum (not direct) of the submodules $P_i/I \cap P_i$, $1 \leq i \leq k$, from which it follows easily that $\text{rank}(R/I) \leq \text{rank}(P_1/I \cap P_1) + \cdots + \text{rank}(P_k/I \cap P_k)$. The existence of a bound on the numbers $\text{rank}(P_i/I \cap P_i)$, therefore implies the existence of a bound on the numbers $\text{rank}(R/I)$ (since $k$ is fixed).

Finally, if $R$ is semiprimary (by which we mean that $R/J$ is Artinian, and for some $n$, $J^n = 0$), then $R$ is left Artinian unless for some integer $m$, $J^m/J^{m+1}$ is an infinite direct sum of simple modules. Clearly, this would imply that $R/J^{n+1}$ was not of finite Goldie dimension, which proves the last statement of the theorem.

The referee remarks that "semiprimary" can be replaced by "right perfect" in this last result, using a result of B. Osofsky's [48, p. 382].

3. The Decomposition Theorem

In this section we prove that a finitely presented module over a serial ring is a direct sum of LP modules. The proof was inspired by the proof given by Roux [36] for local rings.

**Lemma 3.1.** If $M$ is a finitely presented module over a serial ring, then $\text{Gen}(M) \geq \text{Rel}(M)$. $M$ has no nonzero projective summands if and only if $\text{Gen}(M) = \text{Rel}(M)$.

**Proof.** The first statement follows by applying 1.13 to a projective cover for $M$ (1.6) and using the definition of $\text{Rel}(M)$ (1.9). If $M$ has no projective summands, then we apply 2.4 to obtain the reverse inequality, $\text{Rel}(M) \geq \text{Gen}(M)$, from the original inequality applied to the dual $D(M)$. Conversely, if $M = N \oplus P$ where $P$ is projective and $\text{Gen}(P) > 0$, then $\text{Rel}(M) = \text{Rel}(N) \leq \text{Gen}(N) < \text{Gen}(M)$, so $\text{Rel}(M) < \text{Gen}(M)$ as desired.
Lemma 3.2. Let $P$ be a finitely generated projective module over a serial ring, and $x \in P$ a local element. Then there is an indecomposable summand $Q$ of $P$ such that $x \in Q$.

Proof. If $x \notin JP$ then $Rx$ is a local summand of $P$ by 2.2. If $x \in JP$ and $M = P/Rx$, then $Gen(M) = Gen(P)$ and $Rel(M) = 1$. If we write $M = A \oplus B$, where $B$ is projective and $A$ has no nonzero projective summands (1.4), then by 1.10, $Rel(A) = 1$. By 3.1, this implies $Gen(A) = 1$. If $Q = \{y \in P : y + Rx \in A\}$, then $Q$ is clearly a summand of $P$ whose complement is isomorphic to $B$, and $Q$ is clearly local.

Theorem 3.3. Let $R$ be a serial ring, $P$ be a finitely generated projective module, and $M$ be a finitely generated submodule of $P$. Then there is a decomposition $P = P_1 \oplus \cdots \oplus P_n$ of $P$ into indecomposable projectives such that if $M_i = M \cap P_i$, then $M = M_1 \oplus \cdots \oplus M_n$.

This theorem is essentially stated in classical elementary divisor form [24, 36], but it may be valuable to point out a rephrasing that is even closer to the classical form. The theorem says that there is a minimal set $\{y_1, \ldots, y_n\}$ of local generators of $P$, a minimal set $\{x_1, \ldots, x_k\}$ of local generators of $M$, and elements $\{r_1, \ldots, r_k\}$ of $R$, such that $k \leq n$ and $r_i y_i = x_i$, $1 \leq i \leq k$. We will state an immediate corollary before proceeding to the proof.

Corollary 3.4. A finitely presented module over a serial ring is a direct sum of local cyclic modules.

This shows that a ring is serial if and only if it satisfies the equivalent conditions of Theorem 2.6.

Proof. We prove the result by induction on $Gen(M)$ (defined in 1.7). The case $Gen(M) = 0$ is trivial, and $Gen(M) = 1$ has actually already been done in 3.2. We will assume, therefore, that the result is known if $Gen(M) = n$ and prove it under the hypothesis that $Gen(M) = n + 1$. By 3.1, there is a decomposition $P = A \oplus B$ such that $Gen(A) = n + 1$, and $M \subseteq A$. Therefore, we may assume without loss of generality that $Gen(P) = n + 1$. Assuming this, we remark that to prove the theorem it will suffice to find a decomposition $P = Q \oplus L$, where $Q$ is an indecomposable projective and $M = M \cap Q \oplus M \cap L$. We will refer to this fact a number of times in the following.

We now recall the basic facts about decompositions of projective modules over a semiperfect ring that we will need. Let $P$ be a finitely generated projective module, $Q$ be an indecomposable summand of $P$, and $P = \bigoplus_{i=1}^n P_i$.
be a decomposition of $P$ into indecomposable summands. For any $j \in I$, we let $I(j) = I - \{j\}$, and we say $Q$ replaces $P_j$ if

$$P = Q \oplus \left( \bigoplus_{i \in I(j)} P_i \right). \quad (*)$$

We let $\pi_i$ be the projection onto $P_i$ and $\phi_i$ the restriction of $\pi_i$ to $Q$. In this setting, we draw three conclusions: (i) $Q$ replaces $P_j$ if and only if $\phi_j$ is an isomorphism (of $Q$ onto $P_j$); (ii) if $Q$ replaces $P_j$, then in the decomposition (*), the projection onto $Q$ is $\phi_j^{-1}\pi_j$; and (iii) there is at least one index $j$ such that $Q$ replaces $P_j$. The first two of these are essentially trivialities, and the third is just the fact that $Q$ has the exchange property [38], which is an easy consequence of the fact that the endomorphism ring of $Q$ is a local ring. (Briefly, if $\pi$ is the original projection onto $Q$, then in $\text{End}(Q)$, $1 = \sum \pi\phi_i$, so for some $j$, $\pi\phi_j$ is an automorphism, and this $j$ will work.)

We now turn to the proof of the theorem. We may suppose that $P = P_1 \oplus \cdots \oplus P_{n-1}$, where each $P_i$ is indecomposable, and that $M$ has a minimal set of local generators $\{x_1, \ldots, x_{n+1}\}$. By the induction hypothesis, applied to the submodule generated by $\{x_1, \ldots, x_n\}$, we may assume that the $P_i$ and $x_i$ have been chosen so that $x_i \in P_i$, $1 \leq i \leq n$. By 3.2, there is an indecomposable projective summand $Q$ of $P$ containing $x_{n+1}$. In the notation of the previous remarks, we let $\pi_i$ be the projection onto $P_i$ and $\phi_i$ the restriction of $\pi_i$ to $Q$. If $\phi_{n+1}$ is an isomorphism, then by (i), $Q$ replaces $P_{n+1}$ and we are done. Otherwise, by (iii), $\phi_i$ is an isomorphism for some $j$, $1 \leq j \leq n$. Without loss of generality, we may assume that $\phi_1$ is an isomorphism. Since $P_1$ is a serial module, either $\phi_1(x_{n+1}) \in Rx_1$ or $x_1 \in R\phi_1(x_{n+1})$. In the first case, $\phi_1(x_{n+1}) \in Rx_1$ implies that $\pi_1(M) \subseteq Rx_1 = M \cap P_1$; thus, $M = (M \cap P_1) \oplus (M \cap (P_2 \oplus \cdots \oplus P_{n+1}))$. This proves the result by induction. In the second case, we write $P = Q \oplus P_2 \oplus \cdots \oplus P_{n+1}$ [as we may by (i)], and in this decomposition, the projection onto $Q$ is $f = \phi_1^{-1}\pi_1$. By construction, $f(x_1) \in Rx_{n+1}$, from which we conclude that $f(M) \subseteq Rx_{n+1} = M \cap Q$. As before, this implies that $M = M \cap Q \oplus M \cap (P_2 \oplus \cdots \oplus P_{n+1})$, and the result is again proved by induction.

4. Semihereditary Serial Rings

A serial ring is semihereditary if and only if it is nonsingular (4.1). The first point of this section is to show that, in some sense, “good” semiperfect semihereditary rings are serial (4.6, 4.7, and 4.8). For example, if $R$ is semi-perfect, semiprime, and a left and right Goldie ring, then $R$ is semihereditary if and only if it is serial (4.7). We regard these rings as, in some sense, the
local representatives of a class of good semihereditary rings, isolated in Theorem 4.4, that are in some sense noncommutative analogues of Prüfer domains. We then give a characterization (4.9) of nonsingular local serial rings, similar to the standard characterization of commutative valuation rings. Using this, we prove, finally (4.10), the main result about serial semihereditary rings: that an indecomposable serial semihereditary ring is an order in a block upper triangular matrix ring.

If $M$ is an $R$-module, an element $x \in M$ is a singular element if its annihilator ideal is an essential left ideal of $R$. A module is singular if all of its elements are singular, and nonsingular if no nonzero elements are singular. In particular, $R$ is left (or right) nonsingular if the module $_RR$ (or $RR$) is nonsingular. The set of singular elements of a module $M$ form a submodule $Z(M)$. If $R$ is left nonsingular, then $M/Z(M)$ is nonsingular for any left $R$-module $M$. A good survey of these notions is in the first chapter of [19] (see also [18, pp. 214–227; 14, pp. 416–421]).

**Theorem 4.1.** If $R$ is left serial, then $R$ is left nonsingular if and only if $R$ is left semhereditary.

**Proof.** If $M$ is a module and $N$ is an essential submodule, then the $M/N$ clearly cannot be projective. Hence, a left semhereditary ring is left nonsingular. Conversely, if $Q$ is an indecomposable projective and $B$ is a finitely generated submodule, then $B$ is local (since $R$ is left serial), and hence of the form $P/A$ for some indecomposable projective $P$ and submodule $A$. Since $R$ is left serial, any nonzero submodule of $P$ is essential, so if $A \neq 0$, $B \cong P/A$ is singular. If $R$ is left nonsingular, therefore, we must have $A = 0$, so $B \cong P$, which shows that $R$ is left semhereditary.

A module $M$ has finite Goldie dimension if its injective hull $E(M)$ is a finite direct sum of indecomposable injective modules (see the remarks prior to Theorem 2.7). The number of summands appearing in a decomposition of $E(M)$ is an invariant of $M$ the Goldie dimension (or rank) of $M$. We usually write this dimension as $\text{rank}(M)$. If $\text{rank}(M) = 1$, $M$ is called a uniform module. Clearly, $M$ is uniform if and only if $M \neq 0$ and $M$ and all its submodules are indecomposable (or equivalently, every nonzero submodule of $M$ is essential in $M$). If $\text{rank}(M) = n$ then $M$ has an essential submodule that is a direct sum of $n$ uniform submodules.

We say $R$ is left finite dimensional if the module $_RR$ is finite dimensional. $R$ is a left Goldie ring if it is left finite dimensional and has the ascending chain condition on left annihilators [20, p. 62]. We remark that if $R$ is left nonsingular, then $R$ is a left Goldie ring if and only if it is left finite dimensional.

If $R$ is a left nonsingular ring and $E$ is the injective envelope of $_RR$, then the ring structure of $R$ can be extended in a unique way to a ring structure on $E$. 
This new ring, usually written \( Q \), is the maximal left quotient ring of \( R \), and is left-self-injective and von Neumann regular [13, Chap. 8]. \( Q \) is semisimple Artinian if and only if \( R \) is left finite dimensional [13, Chap. 8 or 14, p. 418].

Recall that a ring \( R \) is semiprime if it has no nonzero nilpotent ideals.

**Lemma 4.2** [16, Theorem 11]. If \( R \) is left and right nonsingular, left and right finite dimensional, and semiprime, and \( Q \) is the maximal left quotient ring of \( R \), then \( QR \) is the injective hull of \( RR \), so that \( Q \) is also the maximal right quotient ring of \( R \).

In this case, of course, \( Q \) is the classical quotient ring of \( R \), or, equivalently, \( R \) is a (left and right) order in \( Q \). (Note that \( R \) is a left order in \( Q \) if every element of \( Q \) is of the form \( a^{-1}b \) for suitable elements \( a \) and \( b \) of \( R \), with \( a \) regular.)

**Lemma 4.3.** If \( R \) is a ring that is either semiperfect or left or right finite dimensional, then \( R \) is left semihereditary if and only if \( R \) is right semihereditary.

**Proof.** Small proves in [37] that if for all positive integers \( n \), the ring \( R_n \) has no infinite families of orthogonal idempotents, then \( R \) is left semihereditary if and only if it is right semihereditary. Semiperfect rings and rings that are finite dimensional on either side clearly have no such infinite families, and these properties are inherited by matrix rings, so Small’s result trivially implies 4.3.

We will now arrive at a good class of semihereditary rings to work with, and by 4.2, this will include those semihereditary rings that are left and right finite dimensional and semiprime.

**Theorem 4.4.** Let \( R \) be a left semihereditary, left finite dimensional ring, and let \( Q \) be the maximal left quotient ring of \( R \). The following are equivalent:

(i) Every finitely generated nonsingular left \( R \)-module is projective,
(ii) \( RQ \) is flat,
(iii) \( RR \) is essential in \( QR \),
(iv) \( Q \) is the maximal right quotient ring of \( R \).

Under these circumstances, \( R \) is right semihereditary, right finite dimensional, and left and right nonsingular. Also, \( RQ \) and \( RR \) are direct sums of uniform left and right ideals respectively.

**Proof.** By 4.3, \( R \) is right semihereditary and by the proof of 4.1, \( R \) is left and right nonsingular. The equivalence of (i), (ii), (iii), and (iv) is now trivial.
from [19; 2.5, 1.16, and 1.25]. Clearly (iv) implies that $R$ is right finite dimensional.

For the statement on the decomposition of $\mathbb{R}R$, it suffices to show that an indecomposable finitely generated projective module is uniform. If $P$ is the module and $N$ a nonzero submodule, we let $M = \{x \in P: x + N \in Z(P/N)\}$. $P/M$ is nonsingular (by our original remarks or [19, 1.6]) and hence projective by (i). Hence, since $P$ is indecomposable, $P = M$, so that $N$ is essential in $P$. This holds for any nonzero submodule $N$, and thus, $P$ is uniform.

**Corollary 4.5.** If $R$ is an indecomposable semihereditary ring satisfying the conditions of 4.4, then the maximal quotient ring of $R$ is a full matrix ring over a division ring.

**Remark.** If, in particular, $R$ is a left and right finite dimensional, semi-prime, semihereditary ring, then this show shows that $R$ is a product of prime rings—a result previously obtained by Levy [30, Theorem 4.3].

**Proof.** Write $\mathbb{R}R$ as a direct sum $\mathbb{R}R = \bigoplus_{i=1}^{n} L_i$, where the $L_i$ are uniform left ideals (as we may by 4.4). (This is the only fact about $R$ that we will use.) We conclude that $E(\mathbb{R}R) = \bigoplus_{i=1}^{n} E(L_i)$, and the $E(L_i)$ are indecomposable nonsingular injective modules. We recall that the maximal left quotient ring $Q$ is the endomorphism ring of $E(\mathbb{R}R)$, and we can identify $\mathcal{O}Q$ with $E(\mathbb{R}R)$ (the $Q$-module structure being entirely determined by the $R$-module structure). The decomposition $\mathcal{O}Q = \bigoplus_{i=1}^{n} E(L_i)$ is a decomposition of $\mathcal{O}Q$ into a direct sum of simple left modules. If we group the summands $E(L_i)$ by isomorphism types, we obtain a decomposition $Q = Q_1 \oplus \cdots \oplus Q_k$ of $Q$ into a direct sum of simple rings. If $R_i = R \cap Q_i$, then $R_i$ is clearly a two-sided ideal of $R$. By construction, $R = \bigoplus_{i=1}^{k} R_i$ (this is where the fact that $\mathbb{R}R$ is a direct sum of uniform left ideals is used). Hence, in particular, $R$ is indecomposable (as a ring) if and only if $Q$ is indecomposable, which is the statement to be proved.

**Theorem 4.6.** Let $R$ be a semiperfect ring. The following conditions are equivalent:

(i) $R$ is left nonsingular and left and right serial.

(ii) $R$ is left nonsingular and left finite dimensional, and all finitely generated nonsingular left modules are projective.

(iii) $R$ is left semihereditary, left finite dimensional, and if $Q$ is the maximal left quotient ring of $R$, then $\mathbb{R}Q$ is flat.

(iv) $R$ is left and right serial and left and right semihereditary.

**Proof.** (i) $\Rightarrow$ (ii). Let $M$ be a finitely generated module with no nonzero
projective summands, and let $M = P/A$, where $P$ is a finitely generated projective and $A \subseteq JP$ (as in 1.6, where $J$ is the Jacobson radical of $R$). $A$ must contain a nonzero local element $x$, and by 3.2, $P$ has a local summand $S$ such that $x \in S$. Since an indecomposable projective module over a left serial ring is uniform. $S/A \cap S$ must be a singular module (since $A \cap S \neq 0$), and $S/A \cap S \neq 0$ since $A \subseteq JP$. It follows that $M$ cannot be a nonsingular module. This implies (by 1.4) that any nonsingular finitely generated module is projective.

(ii) $\Leftrightarrow$ (iii). (ii) trivially implies that $R$ is left semihereditary, and the equivalence then follows from 4.4.

(ii) $\Rightarrow$ (iv). By 4.4, local projective (left or right) $R$-modules are uniform. If $P$ is a local projective module and $A$ is a finitely generated submodule, then $A$ is projective (since $R$ is left and right semihereditary by 4.4) and indecomposable (since $P$ is uniform), and hence, $A$ is local (since $R$ is semiperfect). This implies that $P$ is a serial module and, therefore, that $R$ is left and right serial.

(iv) $\Rightarrow$ (i). This follows from the fact that semihereditary rings are nonsingular.

**Corollary 4.7.** If $R$ is a semiperfect, semiprime, left and right Goldie ring, then the following are equivalent:

(i) $R$ is left semihereditary.

(ii) $R$ is left serial.

(iii) $R$ is right serial.

**Proof.** The previous theorems apply, since a semiprime left Goldie ring is left nonsingular ([20, p. 84] shows that an essential left ideal contains a regular element, and therefore, cannot annihilate any element.) If (i) holds, then by 4.2, condition (iii) of 4.6 holds, which (by 4.6) implies that $R$ is left and right serial. By 4.3, (i) is equivalent to the statement that $R$ is right semihereditary, so it will suffice to show that (ii) implies (i), which is a trivial consequence of 4.1.

**Corollary 4.8.** If $R$ is a semiperfect, semiprime, semihereditary ring, then the following properties are equivalent:

(i) $R$ is serial.

(ii) $R$ is left and right finite dimensional.

(iii) There is an upper bound on the number of generators required for (left or right) indecomposable FP $R$-modules.
**Proof.** (i) implies (iii) by the decomposition theorem 3.5. (iii) implies (ii) by 2.7. (ii) implies (i) by 4.7 (assisted by 4.2 and 4.6).

**THEOREM 4.9.** If \( R \) is a local ring, the following properties are equivalent:

(i) \( R \) is serial and semihereditary.

(ii) \( R \) is a subring of a division ring \( D \) in such a way that if \( x \in D \) and \( x \notin R \) then \( x^{-1} \in R \).

**Proof.** It is clear that if \( R \) satisfies (i) then \( R \) is a prime Goldie ring, and (i) clearly implies that \( R \) has no left or right zero divisors, so \( R \) is an order in a division ring \( D \), and \( D \) is a two-sided quotient ring of \( R \). If \( x \in D \), \( x = a^{-1}b \), where \( a \in R \), \( b \in R \). (All nonzero elements of \( R \) are regular). Using the fact that \( R \) is right serial as a module over itself, either \( b = ar \) for some \( r \in R \) (in which case \( x = r \)) or \( a = bs \) for some \( s \in R \) (in which case \( x = s^{-1} \)).

To show that (ii) implies (i), we first show that \( R \) is serial. If \( a \) and \( b \) are elements of \( R \) and \( x \) is chosen in \( D \) such that \( ax = b \), then if \( x \in R \), \( b \in aR \), while otherwise, \( x^{-1} \in R \), and thus, \( a \in bR \). This shows that \( R \) is right serial as a module over itself, and the proof of left seriality is the same. Clearly, \( R \) has no zero divisors, which implies that every principal left or right ideal is free.

**THEOREM 4.10.** Let \( R \) be an indecomposable nonsingular serial ring. Then \( R \) is a two-sided order in a ring of block upper triangular matrices over a division ring.

**Proof.** First note that if \( M \) is a nonsingular left \( R \)-module, then by 4.6 (ii) and the structure theorem of Section 3 any finitely generated submodule of \( M \) is a direct sum of serial modules. It follows that a uniform nonsingular left \( R \)-module is serial. Now let \( A \) and \( B \) be two indecomposable summands of \( rR \). Since \( E(A) \cong E(B) \) (by 4.5), it follows that \( A \) is isomorphic to a submodule \( A' \) of the uniform module \( E(B) \). Since \( E(B) \) is a serial module, either \( A' \subseteq B \) or \( B \subseteq A' \). It follows that either \( \text{Hom}(A, B) \neq 0 \) or \( \text{Hom}(B, A) \neq 0 \). We will say that \( A \) and \( B \) are equivalent if \( \text{Hom}(A, B) \neq 0 \) and \( \text{Hom}(B, A) \neq 0 \). An easy iterative argument shows that we can write \( rR = \bigoplus_{i=1}^{n} L_i \) in such a way that the \( L_i \) are indecomposable, and \( j > i \) implies that \( \text{Hom}(L_i, L_j) \neq 0 \). We now collect the summands \( L_i \) into equivalence classes (with respect to the above equivalence relation) where the equivalence classes are \( C_1, \ldots, C_k \), ordered so that if \( j > i \), \( A \in C_i \), and \( B \in C_j \) then \( \text{Hom}(A, B) \neq 0 \) and \( \text{Hom}(B, A) = 0 \). If \( C_i \) contains \( m_i \) members, then \( \sum_{i=1}^{C} m_i = n \), \( C_1 \) contains the summands \( L_j \) (\( 1 \leq j \leq m_1 \)), \( C_2 \) contains the summands \( L_j \) (\( m_1 < j \leq m_1 + m_2 \)), etc.

Now let \( E_i = E(L_i) \), and let \( S \) be the subring of \( Q = \text{End}(E(rR)) \) generated
by the elements of \( \text{Hom}(E_i, E_j) \) where either: (a) \( j > i \); or (b) \( L_i \) and \( L_j \) are equivalent. [Here, we regard \( \text{Hom}(E_i, E_j) \) as a subset of \( Q \) consisting of those elements \( f \in Q \) such that \( f(E_p) = 0 \) if \( p \neq i \), and \( f(E_i) \subseteq E_j \).] \( S \) is clearly a block upper triangular matrix ring over a division ring, and the sizes of the blocks are the integers \( m_1, \ldots, m_k \). [The division ring \( D \) is the endomorphism ring of one (any) of the indecomposable summands \( E_i \) of \( E(\mathcal{R}) \).]

If \( f \) is a nonzero endomorphism of \( E_i \), then either \( f(L_i) \subseteq L_i \), or \( L_i \subseteq f(L_i) \). It follows that either \( f \) or \( f^{-1} \) is in \( \text{End}(L_i) \). \( \text{End}(L_i) \) is therefore (by 4.9) a serial ring and an order in the division ring \( \text{End}(E_i) \).

Now let \( f_{ij} \) be an element of \( S \) that kills all summands except \( E_i \) and takes \( E_i \) into \( E_j \). To obtain the natural ring structure on \( \text{End}(E(\mathcal{R})) = Q \), write endomorphisms on the right. If \( (L_i) f_{ij} \subseteq L_j \) then \( f_{ij} \in R \). Otherwise, by the conditions on \( i \) and \( j \), there is an endomorphism \( g \) of \( E_j \) such that \( (L_i) f_{ij} g \subseteq L_j \). If \( g \notin \text{End}(L_j) \), then \( g^{-1} \notin \text{End}(L_j) \). However, this is impossible, since \( (L_j) g^{-1} \) properly contains \( L_j \). Hence, \( g \in \text{End}(L_j) \), and thus, in particular, \( g \in R \).

Now let \( f \) be an element of \( S \), and write it as \( \sum f_{ij} \), where \( f_{ij} \) kills \( E_p \) unless \( p = i \), and \( E_i f_{ij} \subseteq E_j \). By the previous argument, there are elements \( g_{ij} \in \text{End}(L_j) \), chosen by induction on \( i \), such that \( g_{ij} \not= 0 \) and \( (f_{ij} g_{ij} g_{ij} \cdots g_{j(-1)i}) g_{ij} \in \text{Hom}(L_i, L_i) \). Let \( g_j = \prod_{i=1}^n g_{ij} \). (Note that if \( f_{ij} = 0 \), we may choose \( g_{ij} = 1 \).) The element \( g = \sum_{i=1}^n g_j \) is a regular element of \( R \), and \( f g \in R \). This proves that \( R \) is a right order in \( S \).

Returning to an earlier point, if \( f_{ij} \in \text{Hom}(E_i, E_j) \) and \( f_{ij} \in S \), then either \( L_i f_{ij} \subseteq L_j \) (so \( f_{ij} \in R \)) or \( L_i f_{ij} \supseteq L_j \). In this case, \( f_{ij} \) is an isomorphism of \( E_i \) onto \( E_j \), so we may look at \( L_j f_{ij}^{-1} \) in \( E_i \). For some \( g \in \text{End}(E_i) \), \( L_i g \subseteq L_j f_{ij}^{-1} \), and as before, \( g \in R \). Clearly \( L_j (g f_{ij}) \subseteq L_j \) so \( g f_{ij} \in R \). Now, following the argument used above, we see that if \( f \in S \) there is a regular element \( g \in R \) such that \( g f \in R \), so \( R \) is a left order in \( S \).

Corollary 4.11. A semihereditary serial ring is prime if and only if for every pair \( P, Q \) of indecomposable projectives, \( \text{Hom}(P, Q) \neq 0 \).

5. Noetherian Serial Rings

In this section we prove that a Noetherian serial ring is a product of an Artinian ring and a finite number of rings, each of which is Morita equivalent to an \((R: J)\)-upper triangular matrix ring over a discrete valuation ring \( R \) with Jacobson radical \( J \). This section was influenced by the already well developed theory of Artinian serial rings, as presented by Eisenbud and Griffith [10, 11], and descending in part from the important paper of Kupisch [47]. We also refer to [11; 17; 19, Chap. 2; 43] for results in the Artinian case. This section
was also influenced by Michler’s paper on semiperfect hereditary Noetherian rings [31], which, in turn, generalized a result of Harada’s [46] on hereditary orders over a complete discrete valuation ring.

**Lemma 5.1.** A uniform module over a left and right serial, left Noetherian ring $R$ is serial. In particular, an indecomposable injective $R$-module is serial.

**Proof.** Since $M$ is uniform, any submodule of $M$ is indecomposable. It follows that any finitely generated submodule is serial, which implies 5.1.

**Definition 5.2.** If $S$ and $T$ are simple modules we say $T$ is a successor of $S$ if $\text{Ext}(S, T) \neq 0$. Under the same conditions, $S$ is called a predecessor of $T$.

**Lemma 5.3.** If $R$ is a left Noetherian, left and right serial ring, and $S$ is a simple module, then $S$ has at most one successor and one predecessor up to isomorphism. Further, $S$ has a successor unless $S$ is projective.

**Proof.** Let $M$ be a module with a submodule $N$ such that $N \cong T$, and $M/N \cong S$. If $M$ corresponds to a nonzero element of $\text{Ext}(S, T)$, then $JM = N$. Let $P$ be an indecomposable projective module such that $P/JP = S$. Clearly there is a surjective homomorphism $P \twoheadrightarrow M$ that induces a surjective homomorphism $JP/J^2P \twoheadrightarrow N$. Since $JP$ is serial, we conclude that $T \cong JP/J^2P$. This proves the uniqueness of $T$. Further, the module $M = P/J^2(P)$ provides a nonzero element of $\text{Ext}(S, T)$ for a suitable choice of $T$, thus showing that $S$ has a successor unless $JP = J^2P$. This last can happen only if $JP = 0$, since $R$ is left Noetherian.

For the statement concerning predecessors, we let $E$ be the injective hull of $S$, $\text{Soc}(E)$ the usual socle (the join of the simple submodules, which, in this case is the module $S$ itself) and we let $\text{Soc}^2(E)$ be the set of elements in $E$ which are mapped into the socle of $E/\text{Soc}(E)$ under the natural homomorphism. By 5.1, $\text{Soc}^2(E)$ is serial, which implies that $\text{Soc}^2(E)/\text{Soc}(E)$ is simple or zero. If $M$ is any module that is local and such that $JM \cong S$, then $M \cong \text{Soc}^2(E)$, and $M/JM \cong \text{Soc}^2(E)/\text{Soc}(E)$. This shows that the predecessor of $S$, if it exists, is unique up to isomorphism.

We denote the successor of a simple module $S$ by $\sigma(S)$. More generally, we say $S$ has a simple module $T$ as a successor of degree $k$ if there is a serial module $M$ with $M/JM \cong S$ and $J^kM \cong T$. In this case, we write $T = \sigma_k(S)$. $\sigma_k(S)$ is clearly well-defined, if at all, by an iteration of the above argument. However, we should remark that if $\sigma(S) = T$ and $\sigma(T) = U$, it may not follow that $\sigma_3(S) = U$, since there may not be a corresponding serial module of length 3. On the other hand, if $P$ is the indecomposable projective
module associated to $S$, and if $J^nP \neq 0$ for all positive integers $n$, then $\sigma_k(S)$ is clearly well defined for all $k$.

If $S$ is a simple left module we denote by $S'$ the corresponding simple right module (as in 2.1).

**Lemma 5.4.** If $R$ is a serial ring and $S = \sigma_k(T)$, then $T' = \sigma_k(S')$.

**Proof.** Let $e$ and $f$ be minimal idempotent elements of $R$ such that $Re$ and $Rf$ are indecomposable projective modules associated to $S$ and $T$, respectively. If $J$ is the Jacobson radical of $R$, and $S = \sigma_k(T)$, then there is a homomorphism $\psi: Re \to Rf$ such that $\psi(e) \in J^k - J^{k+1}$. Since such a homomorphism is given by right multiplication, we conclude that there are ring elements $r$ and $s$ such that $rf = es$, and $rf \in J^k - J^{k+1}$. This equation shows that left multiplication by $r$ gives a homomorphism $\psi: fR \to eR$ such that $\psi(f) \in J^k - J^{k+1}$. This immediately implies that $\sigma_k(S') = T'$.

**Lemma 5.5.** If $R$ is a left and right serial, left and right Noetherian ring, $S$ is a simple module with no predecessor, and $P$ is an indecomposable projective module associated to $S$, then $P$ is Artinian.

**Proof.** If $P$ is not Artinian, then $\sigma_n(S)$ is well-defined for all positive integers $n$. Since there are only a finite number of isomorphism classes of simple modules, there must be integers $n$ and $k$ such that $\sigma_n(S) = \sigma_{n+k}(S)$. By the uniqueness of predecessors (5.3), it follows that $S = \sigma_k(S)$, which implies that $S$ itself has a predecessor, thus proving the contrapositive of the indicated lemma.

**Lemma 5.6.** If $R$ is a left and right serial, left Noetherian ring, with Jacobson radical $J$, and $P$ is an indecomposable projective module such that $\bigcap_{n>0} J^nP \neq 0$, then $R$ has a simple module $S$ with no predecessor, and if $Q$ is the indecomposable projective associated with $S$, then $\text{Hom}(Q, P) \neq 0$.

**Proof.** Let $N = \bigcap_{n>0} J^nP$, let $S = N/JN$, and let $E = E(P/JN)$. $E$ is the injective hull of $S$. Since $P/N$ has no simple submodules, $\text{Soc}^2(E) \cap P/JN = S$. Since $E$ is serial (by 5.1) we must have $\text{Soc}^2(E) = S$, which, by the proof of 5.3, implies that $S$ has no predecessor.

**Definition 5.7.** If $R$ is a left and right Noetherian ring, $M$ an $R$-module, and $x \in M$, we say that $x$ is $\delta$-torsion if $Rx$ is Artinian, and we let $\delta(M)$ be the set of $\delta$-torsion elements.

It is well-known that $M/\delta(M)$ has no nonzero $\delta$-torsion elements [7]. If $R$ is a left and right Noetherian ring, then we can regard $R$ as a left or
right module over itself, getting in this way two torsion submodules that are both clearly two-sided ideals.

**Lemma 5.8.** If $R$ is a left and right Noetherian, left and right serial ring, then $\delta(R_R) = \delta(R_R)$.

**Proof.** $R/\delta(R_R)$ is again a left and right Noetherian, left and right serial ring. If we can show that $R/\delta(R_R)$ has no elements $x \neq 0$ such that $xR$ is $\delta$-torsion, then we will have shown that $\delta(R_R) \subseteq \delta(R_R)$, which will prove the result. It will suffice to show, therefore, that if $\delta(R_R) = 0$, then $\delta(R_R) = 0$.

By 5.6, 5.5, and the fact that $\delta(R_R) = 0$, we already know that $\bigcap_{n>0} J^n = 0$, so that if $P$ is an indecomposable right projective, the only submodules are $PJ^n$. If $PJ^n$ were Artinian, $P$ would be also, so we can have $\delta(R_R) \neq 0$ only if there is an Artinian projective summand. If $S$ is a simple left $R$-module, then, since $\delta(R_R) = 0$, $\sigma_n(S)$ is well-defined for all positive integers $n$. As in the proof of 4.7, this implies that there is a positive integer $k$ such that for all positive integers $n$, $S \cong \sigma_{kn}(S)$. By 4.6, this implies that $S' \cong \sigma_{kn}(S')$ for all positive integers $n$, which implies that an indecomposable projective module associated to $S'$ is not Artinian. This shows that $R_R$ has no Artinian projective summands, which, by our earlier remarks, implies that $\delta(R_R) = 0$.

**Corollary 5.9.** If $R$ is a Noetherian serial ring and $e$ a minimal idempotent, then $Re$ is Artinian if and only if $eR$ is Artinian.

**Lemma 5.10.** If $R$ is a Noetherian serial ring, $P$ is an indecomposable projective that is Artinian, and $Q$ is an indecomposable projective that is not Artinian, then $\text{Hom}(Q, P) = \text{Hom}(P, Q) = 0$.

**Proof.** If $S$ and $T$ are the simple modules associated to $P$ and $Q$, respectively, and if $\text{Hom}(Q, P) \neq 0$, then we would have $T = \sigma_k(S)$ for some $k$. In this case, $S' = \sigma_k(T')$ by 5.4. Since the indecomposable projective $(P^*)$ associated to $S'$ is Artinian by 5.9, this would imply that the indecomposable projective $Q^*$ associated to $T'$ was Artinian, which (by 5.9 again) would imply that $Q$ was Artinian, which it is not. Similarly, if $\text{Hom}(P, Q) \neq 0$ then $\text{Hom}(Q^*, P^*) \neq 0$, which is impossible by the previous argument, applied on the right instead of on the left.

**Theorem 5.11.** If $R$ is a Noetherian serial ring, with Jacobson radical $J$, then $\bigcap_{n>0} J^n = 0$, and $R$ is the product of an Artinian serial ring and a hereditary, serial ring with no simple left or right ideals.

**Proof.** If there were an indecomposable projective $P$ with $\bigcap_{n>0} J^n P \neq 0$,
then by 5.5 and 5.6 there would be an Artinian projective $Q$ with $\text{Hom}(Q, P) \neq 0$, which is impossible by (5.10). Hence, $\bigcap_{n>0} J^n = 0$.

If we write $R = A \oplus B$, where $A$ is a direct sum of Artinian projective left ideals, and $B$ is a direct sum of non-Artinian indecomposable projective summands, then 5.10 implies that this is a ring decomposition. Since $B$ has no nonzero Artinian projectives, and $\bigcap_{n>0} J^n B = 0$, it is clear that $\delta_B = 0$, so (by 5.8) $B$ has no simple left or right ideals. If $P$ is an indecomposable projective of $B$, then the only nonzero submodules of $P$ are $J^n P$, $n = 0, 1, \ldots$, and thus, any proper factor of $P$ is Artinian. Since $B$ has no nonzero Artinian submodules, it follows that any local submodule of $B$ is projective, and (since $B$ is serial) this implies that $B$ is hereditary.

**Definition 5.12.** A discrete valuation ring is a Noetherian non-Artinian, local, serial ring. Equivalently, it is a ring $R$ with Jacobson radical $J$ such that: (i) $R/J$ is a division ring; (ii) $\cap_{n>0} J^n = 0$; (iii) $J^n \neq 0$ for all $n > 0$; and $J^n/J^{n+1}$ is simple, as a left and a right $R$-module.

**Definition 5.13.** If $R$ is a ring and $J$ is an ideal in $R$, a ring $A$ is an $(R: J)$-upper triangular matrix ring if $A$ is the subring of the ring of all $m \times n$ matrices with entries in $R$, such that the entries below the diagonal ($a_{ij}$ such that $j > i$) are all in $J$.

The rings Morita equivalent to $(R: J)$-upper triangular matrix rings are the $(R: J)$-block upper triangular matrix rings, for which we refer to [34, p. 60; 31].

**Theorem 5.14.** If $A$ is an indecomposable, Noetherian, serial ring, that is not Artinian, then $A$ is Morita equivalent to an $(R: J)$-upper triangular matrix ring over a discrete valuation ring $R$ with Jacobson radical $J$.

**Proof.** We let $P_1, \ldots, P_n$ be a list of the distinct (nonisomorphic) indecomposable projective $A$-modules. If $P = P_1 \oplus \cdots \oplus P_n$, then $A$ is Morita equivalent to the endomorphism ring of $P$, and we will show that this endomorphism ring has the desired form.

We let $S_i$ be the simple module associated with $P_i$, $S_i = P_i/\text{LP}_i$, where $L$ is the Jacobson radical of $A$. By 5.11 and 5.3, each $S_i$ has a well-defined successor. We may assume that there is an index $k$ such that for $i < k$, $\sigma(S_i) = S_{i+1}$, while $\sigma(S_k)$ is one of the simple modules $S_i$, $1 \leq i \leq k$. (We do not exclude the possibility $k = 1$). We claim that in this case $\sigma(S_k) = S_1$. Note first that since there are no Artinian submodules of $A$, if $\sigma(S_k) = S_i$, $i > 1$, then $\sigma(S_i) = S_1 = \sigma_{i-1}(S_i)$. It follows that $\sigma_{k-i+1}(S_i) = S_1$, from which, by the uniqueness of predecessors (5.3) we conclude that $\sigma_{k-i+1}(S_i) = S_1$. This is only possible if $k - i + 1 = k$, and $i = 1$ as desired.
We next claim that \( k = n \). What we have shown implies that all iterated predecessors or successors of simple modules in the list \( S_1, \ldots, S_k \) are again in this list, from which it follows that if \( k < m \leq n \), then \( \text{Hom}(P_m, P_i) = \text{Hom}(P_i, P_m) = 0 \) if \( 1 \leq i \leq k \). For this to be consistent with the indecomposability of \( A \), we must have \( n = k \).

Note that by 4.10, we have shown that \( A \) is a prime ring, so that our result follows by Michler's theorem [31]. However, we have done most of the work to prove this theorem from our point of view, so we will finish the proof.

We let \( Q_i = L^{k-1}P_i \), where \( L \) is the Jacobson radical of \( A \). Clearly \( Q_i \cong P_k \).

If we let \( Q = Q_1 \oplus \cdots \oplus Q_n \), and choose identifications of the \( Q_i \), then \( \text{End}(Q) \) is the ring of all \( n \times n \) matrices over a ring \( R \), where \( R \) is the endomorphism ring of \( P_k \). It is clear that any endomorphism of the original module \( P \) takes \( Q \) into itself, so that \( \text{End}(P) \) is a subring of the \( n \times n \) matrix ring over \( R \). It is also clear that \( R \) is a discrete valuation ring and that if \( J \) is its Jacobson radical, then \( JP_k = L^{k}P_k \). (We recall that \( L \) is the Jacobson radical of \( A \) and that \( L^{k}P_k \) is the largest proper submodule of \( P_k \) that is isomorphic to \( P_k \).) Therefore we have embedded \( \text{End}(P) \) into an \( n \times n \) matrix ring over a discrete valuation ring, and we need only identify it as the subring claimed in the theorem.

To establish the form of the subring \( \text{End}(P) \) of the matrix ring, we must show that if \( i < j \), the restriction map \( \text{Hom}(P_i, P_j) \rightarrow \text{Hom}(Q_i, Q_j) \) is an isomorphism, while if \( i > j \), the restriction map is one-to-one and its image is \( J \text{Hom}(Q_i, Q_j) \).

The restriction map is trivially one-to-one, since if \( f \) and \( g \) are in \( \text{Hom}(P_i, P_j) \) and their restrictions to \( Q_i \) coincide, then the image of \( f - g \) is Artinian (since \( P_i/Q_i \) is Artinian), which implies that \( f = g \) by 5.11. We can identify \( P_i \) as a submodule of the injective hull \( E(Q_i) \) by \( P_i = \{ x \in E(Q_i) : L^{n-1}x \subseteq Q_i \} \). An element \( f \in \text{Hom}(Q_i, Q_j) \) extends uniquely to an element \( f' \in \text{Hom}(E(Q_i), E(Q_j)) \), and \( f \) extends to an element of \( \text{Hom}(P_i, P_j) \) if and only if \( f'(P_i) \subseteq P_j \) (using the above identifications). By inspection, this is always the case if \( i \leq j \). Otherwise it is not necessarily the case, but if \( f \in J \text{Hom}(Q_i, Q_j) \), then \( f(Q_i) \subseteq JQ_j = L^nQ_j \), so that \( f'(P_i) \subseteq L^{n-1}Q_j \subseteq L^{n-1}P_j = P_j \). This shows that \( \text{End}(P) \) is exactly the \( (R: J) \)-upper triangular subring of the \( n \times n \) matrix ring over \( R \), which completes the proof of the theorem.

**Theorem 5.15.** Let \( R \) be a semiperfect, Noetherian, hereditary ring, and let \( Q \) be its maximal left quotient ring. The following properties of \( R \) are then equivalent:

(i) \( RQ \) is flat.

(ii) \( R \) is a serial ring.
(iii) \( R \) is a product of rings, each of which is Morita equivalent to either
(a) an upper triangular matrix ring over a division ring, or
(b) a \((D; J)\)-upper triangular matrix ring over a discrete valuation ring \( D \) with Jacobson radical \( J \).

Proof. This is an immediate consequence of 5.11, 5.14, 4.7, and Goldie's theorem on hereditary serial rings ([17, Theorem 8.11] or [11, Theorem 4.1]).

6. Algebras over Commutative Noetherian Rings

Throughout this section, \( R \) is a commutative Noetherian ring and \( A \) is an \( R \)-algebra that is finitely generated as an \( R \)-module. Such an algebra is called a finite \( R \)-algebra. Note that by Eisenbud's theorem [44], if \( A \) is any left and right Noetherian ring that is finitely generated over its center, then the center is Noetherian, so \( A \) fits into this context. For any maximal ideal \( m \) of \( R \), we let \( R_m^* \) be the completion of \( R \) in the \( m \)-adic topology. We let \( A_m^* = A \otimes R_m^* \), which we regard as a finite \( R_m^* \)-algebra. It turns out that \( A_m^* \) is semiperfect. In this section, we study algebras \( A \) such that for all maximal ideals \( m \) of \( R \), \( A_m^* \) is a serial ring.

We begin with a preliminary theorem (6.1) giving the structure of such algebras when \( R \) is a complete local ring. We then pass from the local case to the global case, using the lemma of Auslander and Goldman (6.2) as our main tool. The main result is 6.6, which shows that an algebra \( A \) has the property that all Artinian factor rings are serial if and only if it is the product of an Artinian serial ring and a finite number of hereditary orders over Dedekind rings. This yields as a corollary (6.7) the structure of hereditary algebras whose maximal left and right quotient rings coincide. (Harada [45] proves a related result, showing that a hereditary order over a Noetherian domain is a product of hereditary orders over Dedekind domains. Other theorems relating the structure of a ring and its Artinian factors are given by Faith [12] and Zaks [41].)

The section ends with a discussion of algebras over which all finitely generated modules are balanced. (A module \( M \) is balanced if the natural map from \( A \) to the bicommutator of \( M \) is surjective.) It is shown (6.8) that an algebra has this property if and only if it is the product of an Artinian ring that is both a left and right principal ideal ring and a finite number of maximal orders over Dedekind rings. This is essentially a corollary of 6.6 and the theorems proved for Artinian rings by Camillo and Fuller [5] and Dlab and Ringel [9].

At the suggestion of the referee, we have included (Lemma 6.2) a fairly complete catalogue of the facts from commutative algebra needed for this section, to make the material more accessible.
Theorem 6.1. If \( R \) is a Noetherian complete local ring, and \( A \) a finite \( R \)-algebra, then \( A \) is semiperfect, and the following properties of \( A \) are equivalent:

(i) For every ideal \( I \) such that \( A/I \) is Artinian, \( A/I \) is serial.

(ii) \( A \) is serial.

(iii) Every finitely generated \( A \)-module is a direct sum of a projective module with no simple submodules and a finite number of Artinian serial modules.

(iv) \( A \) is the product of an Artinian serial ring and a finite number of hereditary orders over complete commutative discrete valuation rings.

Proof. First note that \( A \) is complete in the topology defined by taking the ideals \( m^n A = (mA)^n \) as neighborhoods of zero [4, Chap. III, Sect. 3, No. 4, Theorem 3]. It follows that \( mA \) is in the Jacobson radical of \( A \), since an element \( 1 + x (x \in mA) \) has the inverse \( 1 - x + x^2 - x^3 \cdots \). If \( J \) is the radical of \( A \), then \( J + mA \) is the radical of \( A/mA \), which is a finite dimensional algebra with nilpotent radical. It follows that for some integer \( n \), \( J^n \subseteq mA \), so that \( A \) is also complete and Hausdorff in the topology defined by taking powers of \( J \) as neighborhoods of 0. To show that \( A \) is semiperfect, we must show that idempotents modulo \( J \) can be lifted. Referring to the proof of [29, p. 72, Proposition 1], we see that if \( u \in A \), and \( n = u^2 - u \in J \), then one can obtain a formal power series (in powers of \( n \)) for an element \( x \) such that \( e - u + x(1 - 2n) \) is an idempotent in \( A \) and such that \( e - u \in J \). In [29], this series converges because the ideal containing \( n \) is a nil ideal. In our present situation, it converges because of the completeness of \( A \) in the \( J \)-adic topology. This completes the proof that \( A \) is a semiperfect ring, and also, in passing, that \( \cap_{k>0} J^k = 0 \).

To prove that (i) implies (ii), we recall that for any positive integer \( n \), there is an integer \( k \) such that \( J^k \subseteq m^n A \). If \( P \) is an indecomposable projective summand of \( A \) and \( L \) is a nonlocal submodule of \( P \), then \( L/JL \) is not simple. The above remark plus the Artin–Rees Lemma [4, Chap. III, Sect. 3, No. 1, Corollary 1] implies that for some integer \( k \), \( J^k \cap L \subseteq JL \), so \( P/J^k P \) would not be a serial module, contradicting the seriality of \( A/J^k \).

(i) implies (iii) by 3.5, and (iii) trivially implies (i).

(ii), 5.11, and 5.14 imply that \( P \) is the product of an Artinian serial ring and a finite number of hereditary, prime, serial rings. Without loss of generality, we may assume that \( A \) is an hereditary, prime serial ring that is an \((S: I)\) block upper triangular matrix ring over a (possibly noncommutative) discrete valuation ring \( S \) with radical \( I \). An examination of the structure theorem (5.14) shows that \( A \) is free as an \( S \)-module, and that the \( J \)-adic, \( m \)-adic, and \( I \)-adic topologies on \( A \) all agree. From this it follows that \( S \) is complete in its \( I \)-adic topology. \( S \) is an order in a division ring \( D \) (its quotient division ring), and \( A \) is a subring of the full matrix ring \( Q \) over \( D \). Standard arguments show
that the center, $C$, of $Q$ can be identified with the center of $D$ (where $D$ is identified with the subring of scalar matrices). Since $C$ is a subfield of $D$, $S \cap C$ is still a discrete valuation ring (by 4.9, for example). The hypotheses imply that $A$ and $S$ are both finitely generated as $S \cap C$-modules, and hence are free, from which it follows (by similar arguments to those used previously) that $S \cap C$ is complete discrete valuation ring (with quotient field $C$). $A$ is therefore an hereditary order over $S \cap C$, in the classical sense ($A$ is a finite $S \cap C$ subalgebra of $Q$, where $Q$ is a finite dimensional, central simple $C$-algebra, $C$ is the quotient field of $S \cap C$, and $A$ spans $Q$ over $C$.) This shows that (ii) implies (iv).

That (iv) implies (ii) follows from 5.15, the fact that an order is semiprime, and 4.2.

For convenience of reference, we collect in the next lemma a variety of facts about completions of commutative Noetherian rings that will be needed in the rest of this section. In all cases, $R$ will be a commutative Noetherian ring, $m$ will be a maximal ideal of $R$, and $R_m^*$ will be the completion of $R$ in the topology defined by taking the ideals $m^k$, $k \geq 0$, as neighborhoods of zero.

**Lemma 6.2.** (i) (Chinese remainder theorem). Let $M$ be an $R$-module, $m_i$ ($i \in I$) be a finite family of maximal ideals in $R$, $k(i)$ be a positive integer for each $i$, and $J = \bigcap_{i \in I} m_i^{k(i)}$. Then the natural map $M \rightarrow \bigoplus_{i \in I} M/m_i^{k(i)}M$ is surjective, with kernel $J$ [4, Chap. II, p. 71, Proposition 3 and p. 73, Proposition 6].

(ii) $R_m^*$ is a flat $R$-module [4, Chap. III, p. 68, Theorem 3].

(iii) If $M$ is a finitely generated $R$-module, the natural map $M \otimes R_m^* \rightarrow M_m^*$ (where $M_m^*$ is the $m$-adic completion of $M$) is an isomorphism [4, Chap. III, p. 68, Theorem 3].

(iv) If $M$ and $N$ are finitely generated $R$-modules, and $f : M \rightarrow N$ is a homomorphism, then for each of the following properties, $f$ has the property if and only if, for all maximal ideals $m$, the induced map $f_m^*: M_m^* \rightarrow N_m^*$ does: epimorphism, monomorphism, zero, isomorphism [4, Chap. III, p. 73, Corollary 5].

(v) $R_m^*$ is a Noetherian, local ring, with maximal ideal $mR_m^*$, $R_m^*$ is complete in its $mR_m^*$-adic topology, and the natural map $R/m^k \rightarrow R_m^*/m^kR_m^*$ is an isomorphism [4, Chap. III, p. 70, Proposition 8].

(vi) If $M$ is any finitely generated $R$-module, the natural map $M/m^kM \rightarrow M_m^*/m^kM_m^*$ is an isomorphism. (This follows from (iii) and (v), since this map is essentially the map $M \otimes R/m^k \rightarrow M_m^* \otimes R_m^*/m^kR_m^* \rightarrow M \otimes R_m^*/m^kR_m^*$.)

(vii) If $M$ is an $R$-module, and $M_m^* = 0$ for all maximal ideals $m$ of $R$, then $M = 0$ (by (iv) above.) Further, if $M$ is a finitely generated $R$-module and
$M/mM = 0$, then $M_m^* = 0$ (the Nakayama lemma [4, Chap. II, p. 105, Proposition 4]), and if $M$ is finitely generated and $M/mM = 0$ for all $M$, then $M = 0$.

(viii) (Auslander–Goldman). If $A$ is a left Noetherian $R$-algebra, and $M$ and $N$ are left $A$-modules such that $M$ is finitely generated, then there is a natural isomorphism

$$R_m^* \otimes \operatorname{Ext}^n_A(M, N) \to \operatorname{Ext}^n_{A_m^*}(M_m^*, N_m^*), \quad n \geq 0.$$ 

When $n = 0$, this isomorphism takes $r \otimes m$ to $r'r \otimes f(m)$ ([2], or [35, Chap. III, Theorem 1.2].)

We recall from 5.7 that if $M$ is a module, $\delta(M)$ is the submodule consisting of all elements $x$ such that $Ax$ is Artinian. (Over a non-Noetherian ring we would insist that $Ax$ satisfy both chain conditions.) We next show, in effect, that $\delta(M)$ is the same if $M$ is regarded as an $A$-module or as an $R$-module.

We let $\Omega$ be the set of maximal ideals of $R$ and $\Omega^*$ be the set of finite products of powers of these ideals. If $S$ is a simple $A$-module, then there is a unique $m \in \Omega$ such that $mS = 0$. [Nakayama's lemma (6.2, vii) implies that there is at least one such $m$, and the Chinese remainder theorem (6.2(i)), implies that there is at most one.] We call this $m$ the maximal ideal of $R$ associated to $S$. For any $R$-module $M$, we let $\delta_m(M)$ be the submodule of $M$ consisting of those elements $x$ such that $Ax$ is Artinian and all composition factors of $Ax$ have $m$ as their associated maximal ideal. The following statement is an immediate consequence of the Chinese remainder theorem (6.2(i)):

**Lemma 6.3.** If $T$ is a torsion $A$-module (i.e., $\delta(T) = T$) then $T = \bigoplus_{m \in \Omega^*} \delta_m(T)$.

As before, for any $A$-module $M$, we let $M_m^* = M \otimes R_m^*$, which we regard as an $A_m^*$-module.

**Lemma 6.4.** If $M$ is a finitely generated $A$-module, then the natural homomorphism $\delta_m(M) \to \delta(M_m^*)$ is an isomorphism. In particular, $M$ is $\delta$-torsion-free if and only if $M_m^*$ is $\delta$-torsion-free for all $m \in \Omega$.

**Proof.** $M_m^*$ is finitely generated over $R_m^*$, and since $R_m^*$ is Noetherian (6.2(v)), $\delta(M_m^*)$ is finitely generated. This implies that for some integer $k$, $m^k\delta(M_m^*) = 0$. $\delta(M_m^*)$ is therefore a finitely generated module over the ring $R_m^*/m^kR_m^*$, which, by 6.2(v) implies that $\delta(M_m^*)$ is finitely generated over $R$. 6.2(viii) now gives us a natural isomorphism

$$R_m^* \otimes \operatorname{Hom}(\delta(M_m^*), M) \to \operatorname{Hom}(\delta(M_m^*), M_m^*).$$
where we use the fact (6.2(vi)) that $\delta(M_m^*) \otimes R_m^* = \delta(M_m^*)$. If $\phi: M \to M_m^*$ is the natural map, and $g: \delta(M_m^*) \to M_m^*$ is the natural embedding, then this isomorphism says there are homomorphisms $f_1, \ldots, f_n: \delta(M_m^*) \to M$ and elements $r_1, \ldots, r_n$ in $R_m^*$ such that $g = r_1 \phi f_1 + \cdots + r_n \phi f_n$. Since the image of each $f_i$ is in $\delta_m(M)$, which is an $R$-module annihilated by some power of $m$, multiplication of $f_i$ by $r_i$ is already defined in $\text{Hom}(\delta(M_m^*), M)$, so there is a homomorphism $f = \sum f_i + \cdots + r_n f_n$, $f: \delta(M_m^*) \to M$, such that $g = \phi f$. If we let $\phi'$ be the restriction of $\phi$ to $\delta_m(M)$, then $\phi'$ is injective, by 6.2(ii) and (iii). Since $g = \phi' f$, and $g$ is an isomorphism and $\phi'$ is injective, it follows that $\phi'$ is also surjective, which is what we wished to prove.

**Lemma 6.5.** A finitely generated $A$-module $M$ is projective if and only if $M_m^*$ is projective as an $A_m^*$-module for each $m \in \Omega$.

**Proof.** Apply 6.2(viii) with $n = 1$. This shows immediately that if $M$ is projective then so is $M_m^*$ for all $m \in \Omega$. Conversely, if $M_m^*$ is projective for all $m \in \Omega$, then

$$R_m^* \otimes \text{Ext}_A^1(M, N) = 0$$

which, by 6.2(vii) implies that $\text{Ext}_A^1(M, N) = 0$, for all $N$, as desired.

**Theorem 6.6.** Let $R$ be a commutative Noetherian ring and $A$ an $R$-algebra that is finitely generated as an $R$-module. Then the following properties of $A$ are equivalent:

(i) For every ideal $I$ such that $A/I$ is Artinian, $A/I$ is serial.

(ii) Every finitely generated module is a direct sum of a projective module with no simple submodules and a finite number of Artinian serial modules.

(iii) $A$ is the product of an Artinian serial ring and a finite number of hereditary orders over Dedekind domains.

(iv) For every maximal ideal $m$ of $R$, $A_m^*$ (the complete localization of $A$) is a serial ring.

**Proof.** We first prove the equivalence of (i) and (iv), if $\phi: A \to A_m^*$ is the natural map, and $L$ is an ideal of $A_m^*$ such that $A_m^*/L$ is Artinian, then the induced homomorphism $A \to A_m^*/L$ is onto. [To see this, note that $A_m^*/L$ is a finitely generated Artinian $R_m^*$-module, so that $m^k(A_m^*/L) = 0$ for some $k$ (by the argument preceding 6.3). Hence, $m^k A_m^* \subseteq L$, and since the natural map $A/m^k A \to A_m^*/m^k A_m^*$ is bijective (6.2(vi)), the result follows.] Hence, (i) implies that the equivalent conditions of 6.1 hold for $A_m^*$.
(using 6.2(v)), form which (iv) follows. Conversely, if \( A/I \) is Artinian, then \( A/I \) has a ring decomposition, \( A/I = \bigoplus_{m \in \Omega} \delta_m(A/I) \) (where all but a finite number of the summands are necessarily zero) by 6.3. \( \delta_m(A/I) \) is an Artinian factor of \( A_m^* \), and hence serial by (iv), so (iv) implies (i).

We now show that (i) implies (ii). If \( M \) is a finitely generated \( A \)-module, then \( \delta(M) \) is also finitely generated (since \( A \) is Noetherian), and by 6.3, \( \delta(M) = \bigoplus_{m \in \Omega} \delta_m(M) \), where all but a finite number of the summands are zero. The natural map \( M \rightarrow M_m^* \) carries \( \delta_m(M) \) isomorphically onto a summand of \( M_m^* \) (by 6.1 and 6.4). This gives us a projection of \( M \) onto \( \delta_m(M) \), making it a summand of \( M \). It is clear that the kernel of this projection contains the complement of \( \delta_m(M) \) in the natural decomposition of \( \delta(M) \) [since this complement is in the kernel of the map \( M \rightarrow M_m^* \) by 6.2(i) and (vii)], so that \( \delta(M) \) is a summand of \( M \). It is clear from (i) that \( \delta(M) \) is a direct sum of a finite number of Artinian, serial modules. \( M/\delta(M) \) has no simple submodules (5.7) from which we know that \((M/\delta(M))_m^* \) is projective over \( A_m^* \) [by (iv), 6.1, and 6.4]. By 6.5, this shows that \( M/\delta(M) \) is projective, which proves (ii).

It is trivial that (ii), specialized to Artinian modules, implies (i).

We now show that (iii) follows from (i), (ii), and (iv). If we use the procedure used above to construct a projection \( A \rightarrow \delta(A) \), then from the structure theory of serial rings (5.11) we know that locally this projection can be chosen to be a ring homomorphism, from which it follows that \( \delta(A) \) is a summand of \( A \) as a ring. It follows (again from the local structure theory, 6.4 and 6.5) that \( A \) is the product of an Artinian serial ring and an hereditary \( R \)-algebra with no simple left or right submodules.

We now assume that \( A \) is an hereditary \( R \)-algebra with no simple left or right submodules, satisfying (i), (ii), and (iv), and show that \( A \) is a product of a finite number of hereditary orders over Dedekind domains. By Chatters's theorem [42] \( A \) is a product of hereditary prime rings, so we may assume \( A \) is prime. Next note that (i) and (ii) are independent of the ring \( R \) (if, for example, \( A \) can be regarded as an algebra over several different rings \( R \)).

In particular, \( A \) is clearly finite over its center, and this center is Noetherian (since it is an \( R \)-submodule of \( A \)), so we may assume that \( R \) is the center of \( A \).

With this adjustment, we will show that \( R \) is a Dedekind domain, which will complete the proof of (iv).

Either by direct arguments, or by [34, Theorem 6.1], it is clear that \( A_m^* \) is a prime ring [to which 6.1 applies, since \( R_m^* \) is Noetherian by 6.2(v)], for each \( m \in \Omega \). Since \( R_m^* \) is flat over \( R \), the embedding \( R \rightarrow A \) induces an embedding \( R_m^* \rightarrow A_m^* \), so we may think of \( R_m^* \) as a subring of \( A_m^* \). As in the proof of 6.1, if \( Q \) is the classical quotient ring of \( A_m^* \), \( C \) its center, \( C_0 \) the subfield of \( C \) that is the quotient field of \( R_m^* \), and \( S_0 = C_0 \cap A_m^* \), then \( S_0 \) is a complete discrete valuation ring, which is finite over \( R_m^* \) and has the
same quotient field. By routine commutative ring theory [26, Theorems 17 and 44], every nonzero prime in $R_m^*$ is maximal, since this is true of $S_0$. It follows that $S_0/R_m^*$ is Artinian as an $R_m^*$-module (since it is finitely generated and torsion in the usual sense for domains, over a ring of Krull dimension one). By our original construction, we know that $A/R$ is torsion-free (in the usual sense for domains) as an $R$-module, and in particular has no simple submodules. By 6.4, this implies that $A_m^*/R_m^*$ has no simple submodules, which means, in particular, that $S_0/R_m^*$ has no simple submodules. Since $S_0/R_m^*$ is Artinian, this implies that $S_0 = R_m^*$. Hence, $R_m^*$ is a discrete valuation ring, for all $m \in \Omega$, so $R$ is a Dedekind domain.

Finally, we must show that (iii) implies (i). This is an immediate consequence of [10, Theorem 3.1]. (A direct proof is easy from the results of the previous sections.)

**Theorem 6.7.** Let $R$ be a commutative Noetherian ring and $A$ an $R$-algebra that is finitely generated as an $R$-module. Suppose that $A$ is hereditary. Then the following conditions are equivalent:

(i) $A$ is the product of a finite number of block triangular matrix rings over division rings and a finite number of hereditary orders over Dedekind domains.

(ii) For every ideal $I$ such that $A/I$ is Artinian, $A/I$ is serial.

(iii) If $Q$ is the maximal left quotient ring of $A$, then $AQ$ is flat.

**Proof.** The equivalence of (i) and (ii) is a consequence of 6.6 and the structure theory of nonsingular Artinian serial rings, due to Goldie [11, Theorem 4.1; 17, Theorem 8.11]. The structure theory of 6.6 clearly implies that a ring satisfying (i) and (ii) has the property that every finitely generated nonsingular module is projective, so the equivalent conditions of 4.4 hold. By [10, Theorem 3.1], (iii) implies (i). (With a little additional effort, one can prove the necessary cases of the results cited from other papers directly from the earlier results of this paper.)

A special class of rings covered by 6.6 have an interesting property with respect to bicommutators (or double centralizers). Recall that if $R$ is any ring and $M$ a left $R$-module, and $E$ is the $R$-endomorphism ring of $M$, acting on the right, then there is a natural ring homomorphism

$$R \rightarrow \text{End}(M_E) = \text{Bi}(M).$$

$M$ is said to be *balanced* if the map $R \rightarrow \text{Bi}(M)$ of $R$ to the bicommutator is onto. (Older papers sometimes require that it be one-to-one, but we do not.) The result of the papers of Dickson and Fuller [8], Jans [21], Camillo and Fuller [5], and Dlab and Ringel [9], is that if $R$ is an Artinian ring that is
finitely generated (as a module) over its center, then every $R$-module is balanced if and only if $R$ is a principal left and right ideal ring (classically, if $R$ is uniserial, using the terminology of Nakayama). A module theoretic description of these rings can be obtained from 1.14, [22], or [11]. Dlab and Ringel remark [9, p. 126] that a commutative Noetherian ring has the property that every finitely generated module is balanced if and only if it is the product of an Artinian principal ideal ring and a finite number of Dedekind domains. It is this result that we generalize here.

**Theorem 6.8.** Let $R$ be a commutative Noetherian ring and $A$ an $R$-algebra that is finitely generated as an $R$-module. The following properties of $A$ are equivalent:

(i) For every ideal $I$ such that $A/I$ is Artinian, $A/I$ is a principal left and right ideal ring.

(ii) Every finitely generated module is a direct sum of a projective module with no simple submodules and a finite number of Artinian, homogeneously serial modules (see 1.14).

(iii) $A$ is the product of an Artinian ring that is a principal left and right ideal ring, and a finite number of maximal orders over Dedekind domains.

(iv) For every maximal ideal $\mathfrak{m}$ of $R$, $A_{\mathfrak{m}}^*$ (the complete localization) is a principal left and right ideal ring.

(v) Every finitely generated module is balanced.

**Proof.** (i) implies (ii) immediately from 1.14 and 6.6.

(ii) implies (iii) by 6.6 and [10, Theorem 3.3].

(iii) implies (i) also by [10, Theorem 3.3].

(i) implies (iv) since by 1.14 and 6.6 it is enough to show that the indecomposable projectives of $A_{\mathfrak{m}}^*$ are homogeneously serial, which clearly follows from the Noetherian structure theory (5.11 and 5.14), and the fact that this holds for the Artinian factor rings $A/I$.

(iv) implies (i), since by the beginning of the proof of 6.6, any Artinian factor of $A$ is a product of Artinian factors of the various $A_{\mathfrak{m}}^*$.

To show that (i) and (iii) imply (v), first note that since any module over an Artinian left and right principal ideal ring is balanced, it suffices to prove the result for a module over a maximal order. Any finitely generated module over a maximal order (more generally, over a Dedekind prime ring) is the direct sum of a projective generator and a module annihilated by an ideal $I$ such that
SERIAL RINGS

\( A/I \) is Artinian. The result follows, since any \( A/I \)-module is balanced and any finitely generated projective generator (over any ring) is balanced (by [5, Lemma 1.2.3] for example).

(v) implies (i) by the previously mentioned result of Camillo and Fuller [5] and Dlab and Ringel [9].

REFERENCES


