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The enumeration of four-dimensional polytopes

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Abstract

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An algorithm to enumerate the combinatorial types of three-spheres is described. The respective numbers are given for three-spheres with up to 9 facets. Simple geometrical arguments are used to find most of the nonpolytopal three-spheres.

1. Introduction

The morphology of polytopes has a long tradition in combinatorial geometry. Its beginnings go back to Euler, but since then little progress has been made. An efficient algorithm for enumerating the combinatorial types of 3-polyhedra was described by the author in [9]. Brückner [8] investigated the morphology of 4-polytopes without, however, realizing that some of his constructions do not correspond to 4-polytopes. Only after Grünbaum and coworkers [11–12] revealed the nonpolytopality of one of Brückners construction has a renewed interest in polytope morphology emerged.

Whereas the characterization of 3-polytopes is completely solved by Steinitz' theorem [17], no corresponding theorem is known to hold for d-polytopes, $d \geqslant 4$. Reformulated in the language of graph theory, Steinitz' fundamental theorem states that every 3-connected planar graph is isomorphous to the edge graph of a 3-polytope. However, according to Bokowski and Sturmfels [7] it seems likely that in higher dimensions there is no local solution to the Steinitz problem, i.e., to find intrinsic characterizations for boundary complexes of d-polytopes.

Under a (combinatorial) $(d-1)$-sphere we understand a face-to-face tiling with convex tiles of the $(d-1)$-sphere in $\mathbb{R}^d$, where each k-face has at least $k+1$ subordinated $(k-1)$-faces, and where the intersection of any two k-faces is either empty or an h-face of each. By Steinitz' theroem each 2-sphere is polytopal that is, it is isomorphic to a 3-polytope. However, in higher dimensions there exist

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$(d-1)$-spheres, $d \geq 4$, that are nonpolytopal. Only for d-polytopes with few vertices, $N_0 \leq d+3$, the Steinitz theorem still holds by results of Mani [14] and Kleinschmidt [13]. Perles [15] enumerated the combinatorial types of d-polytopes with $N_0 = d+3$, for $d = 4, 5, 6$. The simplicial 4-polytopes with 8 vertices were completely determined by Grünbaum and Sreedharan [12], correcting earlier results of Brückner [8]. Altshuler and coworkers [1–4] enumerated the 3-spheres with 8 vertices and the simplicial ones with 9 vertices and decided their polytopality.

2. Derivation of the combinatorial types of 3-spheres

Following Brückner [8], we consider a bounded, convex 4-polytope $P_N \subset \mathbb{R}^4$ with N facets as the intersection of N closed halfspaces H_i,

$$P_N := \bigcap_{i=1}^{N} H_i.$$

For computation of the d-polytopes we use the halfspace-intersection algorithm described by the author in [10]. Obviously, given all polytopes P_{N-1} with $N-1$ facets we obtain the polytopes with N facets by intersecting the P_{N-1} with a further halfspace H_N in all possible ways subject to the requirement that H_N does not cut off a complete facet of P_{N-1}. It follows that H_N does cut off certain vertices v_j of a particular facet F_k of P_{N-1}. We denote by v^+, v^0, or v^- a vertex which lies inside, on the boundary, or outside of H_N respectively. Given a polytope P_{N-1} and a halfspace H_N, the combinatorial type of the resulting polytope P_N is uniquely determined through the identification of the vertices of P_{N-1} with v^+, v^0, or v^-. Using the halfspace-intersection algorithm of [10] we can transform the face-lattice of P_{N-1} into the face-lattice of P_N. We call such a transformation a cut. We note that not every possible cut can be realized as a halfspace-intersection. Moreover, the present algorithm checks that the resulting face-lattice corresponds to a 3-sphere and eliminates impossible cuts.

The present algorithm to derive the combinatorial types of 3-spheres starts with the face-lattice $L(P_5)$ of the 4-simplex in $\mathbb{R}^4$. By appropriate cuts we can obtain the face-lattices of all 3-spheres with N facets. This algorithm proved to be very efficient for deriving the face-lattices of the simple 3-spheres, that is, those whose vertices are all 4-valent. In this case the vertices of $L(P_{N-1})$ have to be identified with v^+, or v^- only. It was already stated by Grünbaum and Sreedharan [12] that the list of possible cuts given by Brückner [8] is not complete. Indeed the complexity of the cuts depends on the number N of facets. Clearly, it is possible to get 4-spheres of the same combinatorial type with cuts from different starting 4-spheres.

In order to derive the nonsimple 3-spheres in a systematic way we use the method of edge-reduction described by the author in [9]. An edge E_{ij} of P_N is shrunk until both adjacent vertices v_i and v_j coincide. The corresponding transformation performed on the face-lattice $L(P_N)$ results in the derived face-lattice $L(P'_N)$. An edge-reduction may result in a bridge, that is, the intersection of two k-faces of the derived face-lattice is not a h-face. The algorithm eliminates such edge-reductions. If an edge E_{ij} belongs to a k-simplex T^k, $k \geq 2$ then the reduction of E_{ij} results in a $(k-1)$-simplex T^{k-1}. The corresponding transformation may generate a bridge. This is always true if P_N is simple. In this case a further edge E_{ik} of T^{k-1} is reduced. This step is possibly repeated until the bridge disappears.

3. Characterization of the combinatorial types of 3-spheres

The algorithm to enumerate the combinatorial types of 3-spheres requires a unique characterization of the face-lattice of P_N. In what follows, we propose a unified polytope-scheme. We call a subseries of mutually subordinated k-faces F^k of P_N, $F^0 \subset F^1 \subset \cdots \subset F^d \equiv P_N$, a d-flag. Given an arbitrary d-flag we can number all the faces of P_N in a unique way. For each level k we use a separate numbering scheme. The numbering starts at level 0 with the vertex $v_1 := F^0$. Next, we go up along the d-flag to F^1 and find the vertex v_2 adjacent to v_1 with respect to F^1. Thereby the d-flag is transformed to $F^0 := v_2$. If all faces subordinated to F^k are numbered through, we also assign a successive number to F^k and ascent along the d-flag to F^{k+1}. Among all unnumbered faces F^k subordinated to F^{k+1} we find a face $F^{k'}$ which itself has subordinated a face F_m^{k-1} with relative minimal number m. The d-flag is transformed to $F^k := F^{k'}$. In a similar way we descend within the face-lattice until we reach F^0, or a face F^h, all subordinated faces of which are already numbered. We number this face with a successive number according to the corresponding level and go up again along the d-flag to the next higher level. Thus, ascending and descending within the face-lattice, we can number through all faces. Having completed the numbering, all the faces of P_N appear in a determined order. Next, we write down for each facet the numbers of its subordinated vertices in increasing order. With a slash we indicate the completion of a facet. The resulting polytope-scheme consists of a sequence of numbers and slashes. A subsequence of numbers which are successive numbers from n_1 to n_2 is written, by abbreviation, as n_1–n_2.

This polytope-scheme depends only on the arbitrarily chosen initial d-flag. We set up the polytope-scheme for each possible d-flag and retain those polytope-schemes wherein a smaller number first occurs. We can reduce the number of possible initial d-flags if we require that each k-face F^k within the initial d-flag has subordinated a relative maximal number of $(k-1)$-faces. The minimal polytope-scheme we call the unified polytope-scheme. It gives a unique characterization of the face-lattice of P_N, and hence, of its combinatorial type. Vice versa, given a polytope-scheme we can regenerate again the complete face-lattice. The number of identical unified polytope-schemes obtained in the above process corresponds to the order of the combinatorial automorphism group of the face-lattice. The unified polytope-scheme is particularly useful for computer applications.

4. The determination of nonpolytopal 3-spheres

We note that some of the cuts and edge-reductions may result in a 3-sphere which cannot be realized as a polytope through halfspace-intersections. Such 3-spheres are called nonpolytopal. Given any 3-sphere, it seems to be extremely difficult to decide whether it is polytopal. Representing the polytopes through halfspace-intersections, we are able to give a simple geometrical condition for a given cut to be nonrealizable through a halfspace-intersection.

It is easy to see that we can always realize through a halfspace-intersection the cutting off of any part of a 2-face, of a k-simplex, or of a complete k-face, $k = 0-(d-1)$. These cuts generate most of the polytopes P_N from the P_{N-1} which we call the free generated polytopes. The more complicated cuts which involve two adjacent 2-faces have to be checked in each case individually in order to decide whether they can be realized through a halfspace-intersection. In what follows, we show under which conditions such a cut cannot be realized through a halfspace-intersection.

Consider a facet F^3 of a 4-polytope P_N. We say that three 2-faces of F^3 which share no common vertex form a trihedron if there exists a trigonal section through the three 2-faces. (Thus a trigonal pyramid forming a vertex of F^3 is not considered as a trihedron.) It is an elementary geometrical fact that the carrier planes of the three 2-faces which build a trihedron intersect in a common point (which may be at infinity).

Let F^3_h be a facet of P_N and consider an edge E_{ij} of F^3_h. Suppose E_{ij} is common to at least two different trihedra which do not belong to F^3_h. We shall prove that some of the cuts involving the edge E_{ij} and both adjacent 2-faces cannot be realized through halfspace-intersections. As shown in Fig. 1 we define the

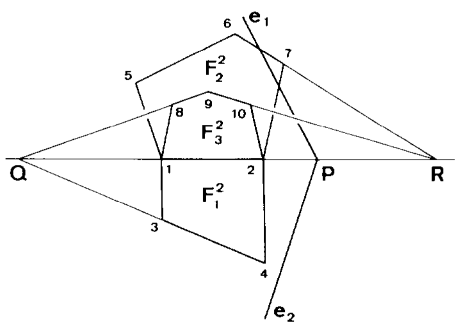

Fig. 1. A nonrealizable halfspace-intersection.

following 2-faces,

$$F_1^2 := \{1, 2, 3, 4\}, \qquad F_2^2 := \{1, 2, 5, 6, 7\}, \qquad F_3^2 := \{1, 2, 8, 9, 10\}.$$

Suppose that F_1^2 and F_3^2 belong to a trihedron having edges $E_{1,2}$, $E_{3,4}$, and $E_{8,9}$. The straight lines defined by these edges intersect in a common point Q. Similarly, suppose that F_2^2 and F_3^2 belong to a second trihedron with edges $E_{1,2}$, $E_{6,7}$, and $E_{9,10}$ with the corresponding straight lines intersecting in a common point R (one or both of the points Q and R may be at infinity). Because of the convexity of F_3^2 the point Q lies outside of the closed interval $[v_1, R]$, either beyond v_1 or else beyond R. Now assume that we want to cut off the following sequence of vertices: v_6^-, v_5^-, v_1^-, v_2^-, and v_4^-. As a shorthand we write for this cut $6^- - 5^- - 1^- = 2^- - 4^-$. The double line indicates the edge $E_{1,2}$ which is common to both trihedra. This implies that the trace e_1 of the intersection of H_N with the carrier plane of F_2^2 intersects the edge $E_{6,7}$ and intersects the straight line through $E_{1,2}$ in P. Thus the point P is within the open interval (R, v_2). Now consider the trace e_2 of the intersection of H_N with the carrier plane of F_1^2. Necessarily, e_2 meets P but avoilds v_4^-. Since the point Q is outside of the interval $[R, v_1]$ it follows that the halfspace H_N necessarily cuts off the vertex v_3. Hence, the cut $6^- - 5^- - 1^- = 2^- - 4^-$ cannot be realized through a halfspace-intersection.

From this we may not conclude that the cut $5^- - 1^- = 2^- - 4^-$ may be realized. It still depends on the shape of F_h^3 which may not be preassigned as was shown by Barnette [6].

Let M be the set of vertices of a facet F^3 and let Σ be a sequence of vertices of M which define a cut. We denote by Σ' the complement of Σ in M. We observed that for simple 3-spheres the two cuts defined by Σ and by the complement Σ' result in the same combinatorial type. If the sequence Σ defines a nonrealizable halfspace-intersection then does also the complement Σ' and vice versa.

The nonpolytopality of a given 3-sphere can be established by indicating the cut which leads to a nonrealizable halfspace-intersection. The individual proof may be a slight variation of the above model case. For example, it may happen that the edge E_{ij} is common to more than two trihedra. In this case additional restrictions occur which are easily recognized.

We will show that it is sufficient to prove the nonpolytopality only for a few special 3-spheres which we call initial ones. Given a nonsimple 3-sphere S we can cut off a complete facet which contains at least one vertex with valence higher than 4. The derived 3-sphere S' has the same number of facets and, in general, is of a different combinatorial type than S. Now, if there exists a derived type S' which is nonpolytopal then necessarily S also is nonpolytopal. This is an important construction in finding large classes of nonpolytopal 3-spheres. There exist nonpolytopal 3-spheres S which have not any nonpolytopal derived type S'. These non-polytopal 3-spheres are exactly the initial nonpolytopal 3-spheres introduced above. Clearly, all simple nonpolytopal 3-spheres are initial. But there

Table 1
Unified polytope-schemes of the combinatorial types of 4-polytopes with 5, 6, 7 facets

```
5.5-1       S    5 10 10       AUT=120
  1-4/1-3,5/1,2,4,5/1,3-5/2-5/


6.6-1       A    6 13 13       AUT= 16
  1-5/1-4,6/1,2,5,6/1,3,5,6/2,4-6/3-6/
6.7-1       H    7 15 14       AUT= 12
  1-6/1-4,7/1,2,5-7/1,3,5,7/2,4,6,7/3-7/
6.8-1       S    8 16 14       AUT= 48
  1-6/1-4,7,8/1,2,5-8/1,3,5,7/2,4,6,8/3-8/
6.9-1       S    9 18 15       AUT= 72
  1-6/1-4,7,8/1,2,5-7,9/1,3,5,7-9/2,4,6-9/3-6,8,9/


7.6-1       A    6 14 15       AUT= 12
  1-5/1-3,6/1,2,4,6/1,3,4,6/2,3,5,6/2,4-6/3-6/
7.7-1       A    7 16 16       AUT= 20
  1-6/1-5,7/1,2,6,7/1,3,6,7/2,4,6,7/3,5-7/4-7/
7.7-2       A    7 16 16       AUT=  2
  1-6/1-4,7/1,2,5-7/1,3,5,7/2,4,6,7/3-5,7/4-7/
7.7-3       A    7 17 17       AUT=  4
  1-5/1-3,6,7/1,2,4,6,7/1,3,4,6/2,3,5,7/2,4,5,7/3-7/
7.8-1       A    8 18 17       AUT=  2
  1-7/1-5,8/1,2,6-8/1,3,6,8/2,4,7,8/3,5-8/4,5,7,8/
7.8-2       A    8 18 17       AUT=  4
  1-7/1-4,8/1,2,5,6,8/1,3,5,7,8/2,4,6-8/3,4,7,8/5-8/
7.8-3       A    8 18 17       AUT=  2
  1-6/1-4,7,8/1,2,5-8/1,3,5,7/2,4,6,8/3-5,7,8/4-6,8/
7.8-4       A    8 18 17       AUT=  8
  1-6/1-4,7,8/1,2,5-7/1,3,5,7,8/2,4,6-8/3-6,8/5-8/
7.8-5       A    8 19 18       AUT=  2
  1-6/1-4,7,8/1,2,5-7/1,3,5,7,8/2,4,6,7/3-5,8/4-8/
7.9-1       H    9 20 18       AUT=  4
  1-8/1-5,9/1,2,6-9/1,3,6,9/2,4,7,9/3,5,6,8,9/4,5,7-9/
7.9-2       H    9 20 18       AUT= 48
  1-8/1-4,9/1,2,5,6,9/1,3,5,7,9/2,4,6,8,9/3,4,7-9/5-9/
7.9-3       A    9 19 17       AUT=  4
  1-7/1-5,8,9/1,2,6-9/1,3,6,8/2,4,7,9/3,5-9/4,5,7,9/
7.9-4       A    9 20 18       AUT=  2
  1-7/1-5,8,9/1,2,6-9/1,3,6,8/2,4,7,9/3,5-8/4,5,7-9/
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```
7.9-5      A    9 20 18      AUT=  4
  1-7/1-4,8,9/1,2,5,6,8,9/1,3,5,7,8/2,4,6,7,9/3,4,7-9/5-9/
7.9-6      A    9 20 18      AUT=  2
  1-7/1-4,8,9/1,2,5,6,8,9/1,3,5,7,8/2,4,6,9/3,4,6-9/5-8/
7.9-7      A    9 20 18      AUT=  2
  1-6/1-4,7,8/1,2,5-7,9/1,3,5,7-9/2,4,6-9/3-5,8,9/4-6,9/
7.10-1     H   10 21 18      AUT=  2
  1-8/1-5,9,10/1,2,6-10/1,3,6,9/2,4,7,10/3,5,6,8-10/4,5,7,8,10/
7.10-2     A   10 21 18      AUT= 16
  1-8/1-4,9,10/1,2,5,6,9,10/1,3,5,7,9/2,4,6,8,10/3,4,7-10/5-10/
7.10-3     A   10 22 19      AUT=  2
  1-7/1-5,8,9/1,2,6-8,10/1,3,6,8-10/2,4,7-10/3,5-7,9,10/4,5,7,9/
7.10-4     A   10 21 18      AUT=  4
  1-7/1-4,8-10/1,2,5,6,8,9/1,3,5,7,8,10/2,4,6,7,9,10/3,4,7,10/5-10/
7.10-5     A   10 22 19      AUT=  2
  1-7/1-4,8-10/1,2,5,6,8,9/1,3,5,7,8,10/2,4,6,7,9/3,4,7,9,10/5-10/
7.10-6     H   10 21 18      AUT= 12
  1-7/1-4,8-10/1,2,5,6,8,9/1,3,5,7,8,10/2,4,6,9/3,4,7,10/4-10/
7.11-1     S   11 22 18      AUT= 12
  1-8/1-5,9-11/1,2,6-11/1,3,6,9/2,4,7,10/3,5,6,8,9,11/4,5,7,8,10,11/
7.11-2     A   11 23 19      AUT=  4
  1-8/1-5,9,10/1,2,6-9,11/1,3,6,9-11/2,4,7,9-11/3,5,6,8,10,11/4,5,7,8,10,11/
7.11-3     H   11 23 19      AUT=  2
  1-8/1-5,9,10/1,2,6-9,11/1,3,6,9-11/2,4,7,9/3,5,6,8,10,11/4,5,7-11/
7.11-4     H   11 23 19      AUT=  4
  1-8/1-4,9-11/1,2,5,6,9,10/1,3,5,7,9,11/2,4,6,8,10,11/3,4,7,8,11/5-11/
7.12-1     S   12 24 19      AUT=  8
  1-8/1-5,9-11/1,2,6-10,12/1,3,6,9,11,12/2,4,7,10/3,5,6,8,11,12/4,5,7-12/
7.12-2     H   12 25 20      AUT=  2
  1-8/1-5,9-11/1,2,6-9,12/1,3,6,9-12/2,4,7,9,11,12/3,5,6,8,10,12/4,5,7,8,10-12/
7.12-3     S   12 24 19      AUT= 48
  1-8/1-4,9-12/1,2,5,6,9,10/1,3,5,7,9,11/2,4,6,8,10,12/3,4,7,8,11,12/5-12/
7.13-1     S   13 26 20      AUT=  8
  1-8/1-5,9-11/1,2,6-9,12,13/1,3,6,9-13/2,4,7,9,11,13/3,5,6,8,10,12/4,5,7,8,10-13/
7.14-1     S   14 28 21      AUT= 14
  1-8/1-5,9-11/1,2,6-9,12,13/1,3,6,9-12,14/2,4,7,9,11-14/3,5,6,8,10,12-14/4,5,7,8,10,11,13,14/
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Table 2
Unified polytope-schemes of the combinatorial types of nonpolytopal 3-spheres with 8 facets

```
N8.10-1     A   10 24 22      AUT=  1     A1,S10             N8.12-2
  1-7/1-4,8,9/1,2,5,6,8,9/1,3,5,7,8,10/2,4,6,9,10/3,4,6,7,10/5-10/3,4,8-10/
N8.11-1     A   11 26 23      AUT=  2     A5                 N8.16-1
  1-8/1-4,9,10/1,2,5,6,9,10/1,3,5,7,9,11/2,4,6,8,10,11/3,4,7,9-11/5,6,8-11/4,5,7,8,11/
N8.11-2     A   11 26 23      AUT=  1     A3,S12             N8.13-2
  1-7/1-4,8-10/1,2,5,6,8,9,11/1,3,5,7,8,10/2,4,6,7,9,11/3,4,7,9-11/5-7,10,11/5,8-11/
N8.11-3   I A   11 25 22      AUT=  2     A2,S1              P7.11-4: 1°-3°-7¯=5¯-6°-8°
  1-7/1-4,8,9/1,2,5,6,8,9/1,3,5,7,8,10/2,4,6,9,11/3,4,6,7,10,11/5-11/3,4,8-11/
N8.11-4     A   11 26 23      AUT=  2     A4                 N8.15-1
  1-7/1-4,8,9/1,2,5,6,8,10/1,3,5,8-11/2,4,7-11/3,4,6,7,9,11/2,6,7,10,11/3,5,6,10,11/
N8.12-1     A   12 27 23      AUT=  2     A8,S11             N8.14-6
  1-8/1-5,9-11/1,2,6-9,12/1,3,6,9-12/2,4,7,9,11,12/3,5,6,8,10,12/4,5,7,8,10,11/7,8,10-12/
N8.12-2   I A   12 27 23      AUT=  1     A9,S19             P7.12-2: 10°-5°-4¯=11¯-9°-2°
  1-8/1-5,9,10/1,2,6,7,9,11,12/1,3,6,8,9,11/2,4,7,9,10,12/3,5,8-12/4-6,8,10,11/4,6,7,10-12/
N8.12-3     A   12 28 24      AUT=  3     A125               N8.14-1
  1-8/1-4,9-11/1,2,5,6,9,10,12/1,3,5,7,9,11,12/2,4,6,8,10,11/3,4,7,8,11,12/5-8,10,12/8-12/
N8.12-4     A   12 28 24      AUT=  1     A137               N8.16-2
  1-8/1-4,9-11/1,2,5,6,9,10,12/1,3,5,7,9,11/2,4,6,10-12/3,4,7,8,11,12/4-6,8,12/5,7-12/
N8.12-4     A   12 27 23      AUT=  2     A7                 N8.16-1
  1-8/1-4,9,10/1,2,5,6,9,10/1,3,5,7,9,11/2,4,6,8,10,12/3,4,7,9-12/5,6,8-12/4,5,7,8,11,12/
N8.12-6   I A   12 27 23      AUT=  2     A6,S3              P7.11-4: 3°-7¯=5¯-6°-8°
  1-7/1-4,8-10/1,2,5,6,8,9,11/1,3,5,7,8,10,12/2,4,6,7,9,11,12/3,4,7,9,10,12/5-8,11,12/8-12/
N8.13-1     A   13 29 24      AUT=  1     A12,S13            N8.15-4
  1-9/1-5,10-12/1,2,6-8,10,13/1,3,6,9-11,13/2,4,7,10-13/3,5,8,9,11,12/4,5,7,8,12,13/6,8,9,11-13/
N8.13-2   I A   13 29 24      AUT=  1     A13,S21            P7.12-2: 10°-5¯-4¯=11¯-9°-2°
  1-9/1-5,10,11/1,2,6-8,10,12/1,3,6,9-11,13/2,4,7,10-13/3,5,8,9,11-13/4,5,7,8,11,12/6,8-10,12,13/
N8.13-3   I A   13 28 23      AUT=  4     A10,K4,S2          P7.13-1: 7°-8°-5¯=4¯-11°-10°
  1-8/1-5,9-11/1,2,6-9,12,13/1,3,6,9-13/2,4,7,9,11,13/3,5,6,8,10,12/4,5,7,8,10,11/7,8,10-13/
N8.13-4     A   13 29 24      AUT=  2     A14                N8.17-1
  1-8/1-5,9-11/1,2,6-9,12,13/1,3,6,9,10,12/2,4,7,9,10,12/3,5,6,10-13/4-6,8,11,13/4,7,8,10-13/
N8.13-5     A   13 29 24      AUT=  1     A11                N8.15-1
  1-8/1-5,9-11/1,2,6,7,9,10,12/1,3,6,8,9,12,13/2,4,7,10-13/3,5,8-13/4-6,8,11,13/4,6,7,12,13/
N8.13-6     A   13 29 24      AUT=  2     A15                N8.17-1
  1-8/1-5,9,10/1,2,6,7,9,11-13/1,3,6,8,9,11/2,4,7,9,10,12/3,5,8-13/4-6,8,10,11,13/4,6,7,10,12,13/
N8.13-7   I A   13 29 24      AUT=  6     A194               P7.11-4: 3°-7¯=5¯-6¯-8°
  1-8/1-4,9-11/1,2,5,6,9,10,12/1,3,5,7,9,11,12/2,4,6,8,10,11,13/3,4,7,8,11-13/5-8,10,12,13/9-13/
N8.13-8     A   13 29 24      AUT=  2     A199               N8.17-2
  1-8/1-4,9-11/1,2,5,6,9,10,12/1,3,5,7,9,11/2,4,6,10-13/3,4,7,8,11,13/4-6,8,12,13/5,7-13/
N8.14-1   I A   14 31 25      AUT=  3     A349               P7.12-2: 10°-5¯-4¯=11¯-9°
  1-10/1-5,11,12/1,2,6-8,11,13/1,3,6,9,11,12,14/2,4,7,11-14/3,5,9,10,12-14/4,5,7,8,10,12,13/6,8-11,13,14/
N8.14-2     A   14 31 25      AUT=  2     A354               N8.17-2
  1-9/1-5,10-13/1,2,6-8,10,14/1,3,6,9-11,14/2,4,7,10-12,14/3,5,8,9,11,13/4,5,7,8,12-14/6,8,9,11-14/
N8.14-3     A   14 30 24      AUT=  1     A16                N8.16-1
  1-9/1-5,10-12/1,2,6-8,10,13,14/1,3,6,9-11,13/2,4,7,10-14/3,5,8,9,11,12/4,5,7,8,12,14/6,8,9,11-14/
N8.14-4     A   14 31 25      AUT=  1     A348               N8.16-2
  1-9/1-5,10-12/1,2,6-8,10,13/1,3,6,9-11,13,14/2,4,7,10-14/3,5,8,9,11,12,14/4,5,7,8,12,13/6,8,9,12-14/
N8.14-5     A   14 31 25      AUT=  1     A356               N8.18-1
  1-9/1-5,10-12/1,2,6-8,10,13/1,3,6,9,10,13,14/2,4,7,10-14/3,5,9,10,12,14/4,5,7,8,11,13/5,6,8,9,11-14/
N8.14-6   I A   14 30 24      AUT=  2     A17                P7.14-1: 7°-4°-5¯=8¯-6°-3°
  1-8/1-5,9-11/1,2,6,7,9,10,12,13/1,3,6,8,9,12-14/2,4,7,10-14/3,5,8-11,13,14/4-7,11,12/5,6,8,11,12,14/
N8.14-7     A   14 31 25      AUT=  2     A355               N8.16-2
  1-8/1-5,9-11/1,2,6,7,9,10,12/1,3,6,8,9,12-14/2,4,7,10-14/3,5,8-13/4-6,8,11,13,14/4,6,7,12,14/
N8.15-1     A   15 32 25      AUT=  2     A18                N8.16-1
  1-10/1-6,11-13/1,2,7-9,11,14,15/1,3,7,10-12,14/2,4,8,11,12,14,15/3,5,10,12-15/4,6,8,9,12,13,15/
  5-7,9,10,13-15/
```

```
N8.15-2     A   15 32 25      AUT=  1     A20             N8.17-1
  1-9/1-5,10-13/1,2,6-8,10,11,14/1,3,6,9,10,12,13,15/2,4,7,11-15/3,5,8,9,12,14,15/4,5,7,8,12,14/
  6,8-11,13-15/
N8.15-3     A   15 32 25      AUT=  2     A416            N8.17-2
  1-9/1-5,10-13/1,2,6-8,10,14,15/1,3,6,9-11,14/2,4,7,10-12,14,15/3,5,8,9,11,13/4,5,7,8,12,13,15/
  6,8,9,11-15/
N8.15-4     A   15 32 25      AUT=  1     A19             N8.17-1
  1-9/1-5,10-12/1,2,6-8,10,13,14/1,3,6,9-13,15/2,4,7,10,12-15/3,5,9,11,13-15/4,5,7,8,11,12,14,15/
  5,6,8,9,13,14/
N8.15-5     A   15 32 25      AUT=  2     A401            N8.17-2
  1-9/1-5,10-12/1,2,6-8,10,13,14/1,3,6,9-11,13,15/2,4,7,10-15/3,5,8,9,11,12,15/4,5,7,8,12,14/
  6,8,9,12-15/
N8.16-1   I A   16 33 25      AUT=  4     A21,K3,S5       P7.13-1: 8--5-=4--11°-10°
  1-10/1-6,11-14/1,2,7-9,11,15,16/1,3,7,10-12,15/2,4,8,11-13,15,16/3,5,10,12-16/4,6,8,9,13,14,16/
  5-7,9,10,14-16/
N8.16-2     A   16 34 26      AUT=  1     A488            N8.17-2
  1-10/1-6,11-13/1,2,7-9,11,14-16/1,3,7,10-12,14/2,4,8,11,12,14,15/3,5,10,12-16/4,6,8,9,12,13,15,16/
  5-7,9,10,13,14,16/
N8.16-3     A   16 34 26      AUT=  1     A487            N8.18-1
  1-10/1-5,11-14/1,2,6-8,11,12,15/1,3,6,9,11,13,14,16/2,4,7,12-16/3,5,9,10,13,15,16/4,5,7,8,10,13,15/
  6,8-12,14-16/
N8.16-4     A   16 34 26      AUT=  2     A490            N8.18-1
  1-9/1-5,10-13/1,2,6-8,10,11,14,15/1,3,6,9,10,12,13,16/2,4,7,11-16/3,5,8,9,12,14-16/4,5,7,8,12,14/
  6,8-11,13,15,16/
N8.16-5     A   16 34 26      AUT=  2     A489            N8.18-1
  1-9/1-5,10-13/1,2,6-8,10,14,15/1,3,6,9-11,14-16/2,4,7,10-12,15,16/3,5,8,9,11,13/4,5,7,8,12-16/
  6,8,9,11-14,16/
N8.17-1   I A   17 35 26      AUT=  2     A22,S23         P7.14-1: 4--5-=8--6°-3°
  1-10/1-6,11-14/1,2,7-9,11,12,15,16/1,3,7,10,11,15-17/2,4,8,12-14,16,17/3,5,10-12,14,16,17/4,6,8,9,13,15-17/
  5-7,9,10,13-15,17/
N8.17-2     H   17 35 26      AUT=  2     A506            N8.19-1
  1-10/1-6,11-14/1,2,7-9,11,15-17/1,3,7,10-12,15/2,4,8,11-13,15,16/3,5,10,12-17/4,6,8,9,13,14,16,17/
  5-7,9,10,14,15,17/
N8.18-1     H   18 37 27      AUT=  1     A519            N8.20-1
  1-10/1-6,11-14/1,2,7-9,11,12,15-17/1,3,7,10,11,15,16,18/2,4,8,12-14,17,18/3,5,10-12,14,16-18/
  4,6,8,9,13,15-18/5-7,9,10,13-15,18/
N8.18-2     H   18 37 27      AUT=  6     A520,K2         N8.19-1
  1-10/1-6,11-13/1,2,7-9,11,14-17/1,3,7,10-12,14,18/2,4,8,11,12,14-16,18/3,5,10,12,13,15-18/4,6,8,9,12,13,
  15,17/5-7,9,10,13,14,16-18/
N8.19-1   I S   19 38 27      AUT= 12     Barnette        P7.13-1: 8--5-=4--11-
  1-10/1-6,11-14/1,2,7-9,11,15-18/1,3,7,10-12,15,19/2,4,8,11-13,15-17,19/3,5,10,12-14,16-19/4,6,8,9,13,14,
  16,18/5-7,9,10,14,15,17-19/
N8.20-1   I S   20 40 28      AUT=  6     Brückner        P7.14-1: 4--5-=8--6-
  1-10/1-6,11-14/1,2,7-9,11,12,15-17/1,3,7,10,11,15,16,18-20/2,4,8,12-14,17-20/3,5,10-12,14,16-19/
  4,6,8,9,13,15-17,19,20/5-7,9,10,13-15,18,20/
```

Table 3
Unified polytope-schemes of the combinatorial types of simple 4-polytopes with 8 facets

```
8.14-1       S  14 28 22          AUT= 12
 1-10/1-6,11-14/1,2,7-14/1,3,7,11/2,4,8,12/3,5,7,9,11,13/4,6,8,10,12,14/5,6,9,10,13,14/
8.14-2       S  14 28 22          AUT=  4
 1-10/1-6,11-14/1,2,7-9,11-13/1,3,7,10,11,14/2,4,8,12/3,5,10,14/4,6,8,9,12,13/5-7,9-11,13,14/
8.14-3       S  14 28 22          AUT= 12
 1-10/1-6,11-14/1,2,7-9,11-13/1,3,7,11/2,4,8,12/3,5,7,9-11,13,14/4,6,8-10,12-14/5,6,10,14/
8.15-1       S  15 30 23          AUT=  2
 1-10/1-6,11-14/1,2,7-13,15/1,3,7,11,14,15/2,4,8,12/3,5,7,9,14,15/4,6,8,10,12,13/5,6,9-11,13-15/
8.15-2       S  15 30 23          AUT=  4
 1-10/1-6,11-14/1,2,7-13,15/1,3,7,11/2,4,8,12/3,5,7,9,11,13-15/4,6,8,10,12-15/5,6,9,10,14,15/
8.15-3       S  15 30 23          AUT=  4
 1-10/1-6,11-14/1,2,7-9,11,12,15/1,3,7,10,11,13-15/2,4,8,12/3,5,10,13/4,6,8,9,11,12,14,15/5-7,9,10,13-15/
8.15-4       S  15 30 23          AUT=  4
 1-10/1-5,11-15/1,2,6-8,11-13/1,3,6,9,11,14/2,4,7,12/3,5,9,10,14,15/4,5,7,8,10,12,13,15/6,8-11,13-15/
8.15-5       S  15 30 23          AUT= 60
 1-10/1-5,11-15/1,2,6,7,11,12/1,3,6,8,11,13/2,4,7,9,12,14/3,5,8,10,13,15/4,5,9,10,14,15/6-15/
8.16-1       S  16 32 24          AUT=  2
 1-10/1-6,11-14/1,2,7-12,15,16/1,3,7,11,13-16/2,4,8,12/3,5,7,9,13,15/4,6,8,10-12,14,16/5,6,9,10,13-16/
8.16-2       S  16 32 24          AUT= 16
 1-10/1-6,11-14/1,2,7-12,15,16/1,3,7,11,13,15/2,4,8,12,14,16/3,5,7,9,13,15/4,6,8,10,14,16/5,6,9-16/
8.16-3       S  16 32 24          AUT=  8
 1-10/1-6,11-14/1,2,7-12,15,16/1,3,7,11,13,15/2,4,8,12/3,5,7,9,13-16/4,6,8,10-16/5,6,9,10,14,16/
8.16-4       S  16 32 24          AUT=  8
 1-10/1-6,11-14/1,2,7-9,11-13,15,16/1,3,7,10,11,15/2,4,8,12,14,16/3,5,10,11,13,15/4,6,8,9,14,16/5-7,9,10,
 12-16/
8.16-5       S  16 32 24          AUT=  1
 1-10/1-6,11-14/1,2,7-9,11-13,15,16/1,3,7,10,11,15/2,4,8,12/3,5,10,11,13-16/4,6,8,9,12-14,16/5-7,9,10,14-16/
8.16-6       S  16 32 24          AUT=  4
 1-10/1-5,11-15/1,2,6-8,11,12,16/1,3,6,9,11,13/2,4,7,12,14,16/3,5,9,10,13,15/4,5,7,8,10,14-16/6,8-16/
8.16-7       S  16 32 24          AUT=  8
 1-10/1-5,11-13/1,2,6,7,11-16/1,3,6,8,11,14/2,4,7,9,12,15/3,5,8,10,11,13,14,16/4,5,9,10,12,13,15,16/6-10,
 14-16/
8.16-8       S  16 32 24          AUT=384
 1-8/1-4,9-12/1,2,5,6,9,10,13,14/1,3,5,7,9,11,13,15/2,4,6,8,10,12,14,16/3,4,7,8,11,12,15,16/5-8,13-16/9-16/
8.17-1       S  17 34 25          AUT=  2
 1-10/1-6,11-14/1,2,7-12,15,16/1,3,7,11,13-15,17/2,4,8,12/3,5,7,9,13,15-17/4,6,8,10-12,14-17/5,6,9,10,13,
 14,16,17/
8.17-2       S  17 34 25          AUT=  8
 1-10/1-6,11-14/1,2,7-11,15-17/1,3,7,11-17/2,4,8,11,13,16/3,5,7,9,12,15/4,6,8,10,13,14,16,17/5,6,9,10,12,
 14,15,17/
8.17-3       S  17 34 25          AUT=  4
 1-10/1-6,11-14/1,2,7-11,15-17/1,3,7,11-13,15,16/2,4,8,11,13,14,16,17/3,5,7,9,12,15/4,6,8,10,14,17/5,6,9,
 10,12-17/
8.17-4       S  17 34 25          AUT=  2
 1-10/1-6,11-14/1,2,7-11,15-17/1,3,7,11-13,15,16/2,4,8,11,13,16/3,5,7,9,12,14,15,17/4,6,8,10,12-17/5,6,9,
 10,14,17/
8.17-5       S  17 34 25          AUT=  2
 1-10/1-6,11-14/1,2,7-9,11-13,15,16/1,3,7,10,11,15-17/2,4,8,12/3,5,10,11,13,14,16,17/4,6,8,9,12-17/5-7,9,
 10,14,15,17/
8.17-6       S  17 34 25          AUT=  2
 1-10/1-6,11-14/1,2,7-9,11,12,15-17/1,3,7,10,11,15/2,4,8,12-14,16,17/3,5,10-12,14,15,17/4,6,8,9,13,16/5-7,
 9,10,13-17/
8.17-7       S  17 34 25          AUT=  2
 1-10/1-5,11-15/1,2,6-8,11,12,16/1,3,6,9,11,13/2,4,7,12,14-17/3,5,9,10,13-15,17/4,5,7,8,10,14,16,17/6,8-13,
 15-17/
8.17-8       S  17 34 25          AUT= 12
 1-10/1-5,11-13/1,2,6-8,11,14,15/1,3,6,9,11,12,14,16/2,4,7,11-17/3,5,9,10,12,13,16,17/4,5,7,8,10,13,15,17/
 6,8-10,14-17/
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```
8.18-1      S  18 36 26       AUT=  1
 1-10/1-6,11-14/1,2,7-11,15-17/1,3,7,11-16,18/2,4,8,11,13,16-18/3,5,7,9,12,15/4,6,8,10,13,14,17,18/5,6,9,
 10,12,14-18/
8.18-2      S  18 36 26       AUT=  2
 1-10/1-6,11-14/1,2,7-11,15-17/1,3,7,11-13,15,18/2,4,8,11,13,15,16,18/3,5,7,9,12,14-18/4,6,8,10,12-14,16-18/
 5,6,9,10,14,17/
8.18-3      S  18 36 26       AUT=  2
 1-10/1-6,11-14/1,2,7-9,11,12,15-17/1,3,7,10,11,15,16,18/2,4,8,12-18/3,5,10-12,14,16,18/4,6,8,9,13,17/5-7,
 9,10,13-15,17,18/
8.18-4      S  18 36 26       AUT=  2
 1-10/1-6,11-14/1,2,7-9,11,15-18/1,3,7,10-12,15,16/2,4,8,11-13,16,17/3,5,10,12-18/4,6,8,9,13,14,17,18/5-7,
 9,10,14,15,18/
8.18-5      S  18 36 26       AUT=  4
 1-10/1-6,11-14/1,2,7-9,11,15-18/1,3,7,11-13,15,16/2,4,8,11,13,14,16,17/3,5,7,9,10,12,15,18/4,6,8-10,14,17,
 18/5,6,10,12-18/
8.18-6      S  18 36 26       AUT=  4
 1-10/1-6,11-14/1,2,7-9,11,15,16/1,3,7,10-12,15-18/2,4,8,11-13,16,18/3,5,10,12-14,17,18/4,6,8,9,13-18/5-7,
 9,10,14,15,17/
8.19-1      S  19 38 27       AUT=  2
 1-10/1-6,11-14/1,2,7-11,15-17/1,3,7,11-15,18,19/2,4,8,11,13,15-19/3,5,7,9,12,15,17,18/4,6,8,10,13,14,16,
 19/5,6,9,10,12,14,16-19/
8.19-2      S  19 38 27       AUT=  1
 1-10/1-6,11-14/1,2,7-11,15-17/1,3,7,11-13,15,16,18,19/2,4,8,11,13,14,16,19/3,5,7,9,12,15,17,18/4,6,8,10,
 14-19/5,6,9,10,12-14,17-19/
8.19-3      S  19 38 27       AUT=  2
 1-10/1-6,11-14/1,2,7-9,11,12,15-17/1,3,7,10,11,15-19/2,4,8,12-14,16-19/3,5,10-12,14,16,18/4,6,8,9,13,15,
 17,19/5-7,9,10,13-15,18,19/
8.19-4      S  19 38 27       AUT=  4
 1-10/1-6,11-14/1,2,7-9,11,15-18/1,3,7,10-12,15,16/2,4,8,11-13,16-19/3,5,10,12-16,18,19/4,6,8,9,13,14,17,
 19/5-7,9,10,14,15,17-19/
8.20-1      S  20 40 28       AUT= 16
 1-10/1-6,11-14/1,2,7-11,15-17/1,3,7,11-15,18,19/2,4,8,11,13,15-17,19,20/3,5,7,9,12,15,17-20/4,6,8,10,13,
 14,16,18-20/5,6,9,10,12,14,16-18,20/
8.20-2      S  20 40 28       AUT=  4
 1-10/1-6,11-14/1,2,7-11,15-17/1,3,7,11-15,18,19/2,4,8,11,13,15,16,18-20/3,5,7,9,12,15-18,20/4,6,8,10,13,
 14,16,17,19,20/5,6,9,10,12,14,17-20/
8.20-3      S  20 40 28       AUT=  2
 1-10/1-6,11-14/1,2,7-11,15-17/1,3,7,11-13,15,16,18,19/2,4,8,11,13,14,16,18-20/3,5,7,9,12,15,17-20/4,6,8,
 10,14-17,19,20/5,6,9,10,12-14,17,18,20/
```

exist other initial nonpolytopal 3-spheres which are characterized by having at least one edge of valence higher than 3. In the example shown in Fig. 1 the following special cuts, in general, will lead to nonsimple, initial 3-spheres:

$$7^0 - 6^0 - 5^- - 1^- = 2^- - 4^-, \quad 6^- - 5^- - 1^- = 2^- - 4^0 - 3^0, \quad \text{and}$$

$$7^0 - 6^0 - 5^- - 1^- = 2^- - 4^0 - 3^0.$$

It is easily verified that these special cuts are nonrealizable halfspace-intersections.

5. Results of calculations

For applications of this algorithm a computer program POLYTOPE has been written in PL/I programming language. All calculations were performed using the IBM 3090 computer of BEDAG (Berne). The 3-spheres with up to 7 facets are readily obtained. They are all polytopal and are well known. In Table 1 we give their unified polytope-schemes; these are needed for the derivation of the 3-spheres with 8 facets.

From all polytopes P_7 we obtained the 37 simple polytopes P_8 by cutting off parts of the 2-faces. By construction, these are free generated polytopes. In order to complete the list of simple 3-spheres with 8 facets we used the process of edge-reduction followed by cutting off complete facets. By this we obtained two further simple 3-spheres, the Barnette sphere [5], and the Brückner sphere [8]. The complete list of all 3-spheres with 8 facets was derived through successive edge-reductions starting from the simple ones. For the initial nonpolytopal 3-spheres we had to find the corresponding cuts. The remaining nonpolytopal 3-spheres are then readily found. In Table 2 are given the unified polytope-schemes of the nonpolytopal 3-spheres with 8 facets. In Table 2 the cut is given for each initial 3-sphere, marked with the code I, which proves the non-polytopality. The references stated in Table 2 are as follows: A1–A22 refer to Altshuler [1], A125–A520 refer to Altshuler [2], S1–S23 refer to Schulz [16], K1–K4 refer to Kleinschmidt [13], Barnette to [5], and Brückner to [8]. In Table 3 we list the simple 4-polytopes with 8 facets which are needed to derive the 3-spheres with 9 facets.

From all polytopes P_8 we obtained 1135 simple polytopes P_9 by cutting off of any part of the 2-faces, or of 3-simplices. By construction these are free generated polytopes. The complete list of simple 3-spheres with 9 facets was obtained by edge-reduction followed by cutting off complete facets. By this process further 161 simple 3-spheres were found. Starting from the simple 3-spheres we calculated all 3-spheres with 9 facets through edge-reductions. The amount of computer time increases very rapidly and further progress seems to be beyond reasonable expense.

Table 4

Unified polytope-schemes of the combinatorial types of simple nonpolytopal 3-spheres with 9 facets

```
N9.22-1     I S   22 44 31        AUT=  4                    P8.16-3: 9--5-=3--13-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,15,18,22/2,4,9,14/3,5,12,15-17,19-22/4,6,9,11,13-15,17,18,20-22/
  5,7,8,10,12,16,18-20,22/6,7,10,11,16,17,19,21/
N9.22-2     I S   22 44 31        AUT=  1                    P8.16-1: 9--5-=6--14-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18-20,22/2,4,9,14/3,5,8,10,15,17,18,21/4,6,9,11-14,16,19,22/
  5,7,10-12,17,19-22/6,7,12,15-18,20-22/
N9.22-3     I S   22 44 31        AUT=  3                    P8.16-9: 9--16-=15--11-
  1-12/1-6,13-18/1,2,7-10,13-16,19,20/1,3,7,11,13,17,19,21/2,4,8,14/3,5,9-12,17,20-22/4,6,8,10,12,14,16,18,20,22/
  5,6,12,13,15,17-19,21,22/7,9,11,15,16,18-22/

N9.23-1     I S   23 46 32        AUT=  1                    P8.17-1: 7--15-=11--14-
  1-12/1-7,13-17/1,2,8-11,13-15,18-20/1,3,8,12,13,18,19,21-23/2,4,9,14/3,5,12,13,15,16,19-22/
  4,6,9,11,14-17,20-23/5,7,8,10,12,16-18,21,23/6,7,10,11,17-20,22,23/
N9.23-2     I S   23 46 32        AUT=  1                    P8.17-12: 1--7-=10--17-
  1-12/1-7,13-17/1,2,8-11,13-15,18-20/1,3,8,13,16-18,21-23/2,4,9,14/3,5,8,10,16,18-22/4,6,9,11,12,14,15,19-23/
  5,7,10-12,16,17,19,21,23/6,7,12,13,15,17,18,20,22,23/
N9.23-3     I S   23 46 32        AUT=  2                    P8.17-1: 5--9-=7--15-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,15,16,18,22,23/2,4,9,14/3,5,12,15,17,19-23/
  4,6,9,11,13,14,16,18,20-23/5,7,8,10,12,17-20,22/6,7,10,11,15-17,19,21,23/
N9.23-4     I S   23 46 32        AUT=  1                    P8.17-4: 9--5-=3--12-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,19,23/2,4,9,13,15,19/3,5,12,14,16,17,20-23/
  4,6,9,11,14,15,17-19,21-23/5,7,8,10,12,16,18,20,21,23/6,7,10,11,16,17,20,22/
N9.23-5     I S   23 46 32        AUT=  2                    P8.17-2: 5--9-=10--17-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23/2,4,9,13-16,18-21,23/3,5,12,14,16,17,20-23/
  4,6,9,11,15,19/5,7,8,10,12,17,18,20,22,23/6,7,10,11,15-17,19,21,22/
N9.23-6     I S   23 46 32        AUT=  2                    P8.16-41: 5--3--1-=2--12-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23/2,4,9,13-15,18-20,23/3,5,12,14-17,19-23/
  4,6,9,11,15,17,19,22/5,7,8,10,12,16,18,20,21,23/6,7,10,11,16,17,21,22/
N9.23-7     I S   23 46 32        AUT=  1                    P8.17-13: 2--12-=11--15-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23/2,4,9,13-15,18-20,23/3,5,12,14-16,19,20,22,23/
  4,6,9,11,15-17,19,21,22/5,7,8,10,12,16-18,20-23/6,7,10,11,17,21/
N9.23-8     I S   23 46 32        AUT=  3                    P8.17-12: 1--2--12-=13--14--4-
  1-12/1-6,13-18/1,2,7-10,13-16,19,20/1,3,7,11,13,17,19,21-23/2,4,8,14/3,5,9-12,17,20-22/4,6,8,10,12,14,16,18,20-23/
  5,6,12,13,15,17-19,22,23/7,9,11,15,16,18-21,23/
N9.23-9     I S   23 46 32        AUT=  2                    P8.17-12: 7--10-=17--14-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16-18,22,23/2,4,8,14/3,5,11,12,16,19-23/4,6,8,10,14,15,17,18,20-23/
  5,6,9,10,12,16,18,19,21,23/7,9,11-13,15,17,19,20,22/
N9.23-10    I S   23 46 32        AUT=  2                    P8.17-13: 7--1-=2--12-
  1-12/1-6,13-18/1,2,7-10,13,14,19-22/1,3,7,11,13,15/2,4,8,14,16,17,19-21,23/3,5,11,12,15-19,21-23/
  4,6,8,10,16,18,19,22/5,6,9,10,12,18,20-23/7,9,11-15,17,20,23/
N9.23-11    I S   23 46 32        AUT=  6                    P8.17-28: 10--17-=16--12-
  1-12/1-6,13-16/1,2,7-9,13,14,17-19/1,3,7,10,11,13,17,18,20-23/2,4,8,14-16,19-21,23/3,5,10,13,14,16,18-21/
  4,6,8,9,12,15,17-19,21-23/5,6,10-12,15,16,20,22,23/7,9,11,12,17,22/

N9.24-1     I S   24 48 33        AUT=  1                    P8.17-1: 5--6-=14--11--12-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-24/2,4,9,14-17,20-24/3,5,12-14,16,19,21-23/
  4,6,9,11,15,20/5,7,8,10,12,16-18,22,24/6,7,10,11,15,17-21,23,24/
N9.24-2     I S   24 48 33        AUT=  1                    P8.18-10: 2--1-=7--10-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-24/2,4,9,14-16,20,22-24/3,5,12-14,16,19,20,22,23/
  4,6,9,11,15,17-21,23,24/5,7,8,10,12,15-18,21,22,24/6,7,10,11,17,21/
N9.24-3     I S   24 48 33        AUT=  1                    P8.17-12: 1--2--12-=13--14-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-24/2,4,9,14,15,20/3,5,12-16,19-23/
  4,6,9,11,15-17,20-24/5,7,8,10,12,16-18,22,24/6,7,10,11,17-19,21,23,24/
N9.24-4     I S   24 48 33        AUT=  1                    P8.18-1: 10--6-=4--13-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,22-24/2,4,9,14,17,19/3,5,8,10,15,18,20-23/
  4,6,9,11,12,17,19-24/5,7,10-12,15,16,20,22,24/6,7,12-14,16-19,21,23,24/
N9.24-5     I S   24 48 33        AUT=  1                    P8.18-5: 5--9-=7--15-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,23,24/2,4,9,13,15,18-20,23,24/3,5,12,14,16,17,19-24/
  4,6,9,11,14,15,17,19,22,24/5,7,8,10,12,16,18,20,21,23/6,7,10,11,16,17,21,22/
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Table 4 (contd.)

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N9.24-6    I S   24 48 33      AUT=  1                  P8.17-13: 9--7--1-=2--12-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,23,24/2,4,9,13,15,18,19,24/3,5,12,14,16,17,20-23/
  4,6,9,11,14,15,17,19,21-24/5,7,8,10,12,16,18-21,23,24/6,7,10,11,16,17,20,22/
N9.24-7    I S   24 48 33      AUT=  1                  P8.18-5: 9--5-=3--12-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,19,23,24/2,4,9,13-15,19-21,24/3,5,12,14-17,20-24/
  4,6,9,11,15,17,20,22/5,7,8,10,12,16,18,23/6,7,10,11,16-19,21-24/
N9.24-8    I S   24 48 33      AUT=  1                  P8.17-13: 9--16--17-=15--11-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23/2,4,9,13-16,18-20,23,24/3,5,12,14,16,17,19-24/
  4,6,9,11,15,19,22,24/5,7,8,10,12,17,18,20,21,23/6,7,10,11,15-17,21,22,24/
N9.24-9    I S   24 48 33      AUT=  2                  P8.17-4: 10--6-=4--13--12-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23/2,4,9,13-15,18-21,23,24/3,5,12,14-17,19,21-24/
  4,6,9,11,15,17,19,22/5,7,8,10,12,16,18,20,23,24/6,7,10,11,16,17,20-22,24/
N9.24-10   I S   24 48 33      AUT=  1                  P8.17-3: 9--5-=6--14--13-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23/2,4,9,13-15,18-21,23,24/3,5,12,14-17,20-24/
  4,6,9,11,15,17,19,24/5,7,8,10,12,16,18,20,22,23/6,7,10,11,16,17,19,21,22,24/
N9.24-11   I S   24 48 33      AUT=  1                  P8.18-10: 5--14-=13--17-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23/2,4,9,13-15,18-20,23/3,5,12,14-16,19-24/
  4,6,9,11,15-17,19,22,24/5,7,8,10,12,16-18,20,21,23,24/6,7,10,11,17,21,22,24/
N9.24-12   I S   24 48 33      AUT=  1                  P8.18-10: 1--7--10--18-; N8.20-1: 2--4-
  1-12/1-7,13-17/1,2,8-10,13-15,18-21/1,3,8,11-13,18-20,22-24/2,4,9,14,16,17,21-24/3,5,11,13,15,19/
  4,6,9,10,16,18,20,21,23,24/5,7,11,12,14,15,17,19-22,24/6-8,10,12,16-18,22,23/
N9.24-13   I S   24 48 33      AUT=  1                  P8.18-11: 1--7-=9--18-
  1-12/1-7,13-17/1,2,8-10,13,14,18-20/1,3,8,11-13,18,19,21-24/2,4,9,14-16,20-22,24/3,5,11,13,14,16,19-22/
  4,6,9,10,15,17-20,22-24/5,7,11,12,15-17,21,23,24/6-8,10,12,17,18,23/
N9.24-14   I S   24 48 33      AUT=  2                  P8.18-11: 6--9--8--17-; N8.19-1: 1--2--4-
  1-12/1-7,13-17/1,2,8-10,13,14,18-20/1,3,8,11-13,18,21-23/2,4,9,14-16,19-24/3,5,11,13,14,16,18,20-22/
  4,6,9,10,15,17,19,24/5,7,11,12,15-17,21,23,24/6-8,10,12,17-20,22-24/
N9.24-15   I S   24 48 33      AUT=  1                  P8.18-10: 9--6-=5--14-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16,19,22/2,4,8,14,17,18,20,23/3,5,11,14-16,18,20-24/
  4,6,8,10,12,17,20,21,23,24/5-7,9,11,12,17-19,22-24/9,10,12,13,15,16,19,21,22,24/
N9.24-16   I S   24 48 33      AUT=  2                  P8.18-10: 7--10-=18--14-
  1-12/1-6,13-18/1,2,7-10,13,14,19-22/1,3,7,11,13,15-17,23,24/2,4,8,14,16-21,23,24/3,5,11,12,15,20-24/
  4,6,8,10,18,19/5,6,9,10,12,15,17-19,21,22,24/7,9,11-14,16,20,22,23/
N9.24-17   I S   24 48 33      AUT=  1                  P8.18-10: 1--2-=12--14-
  1-12/1-6,13-18/1,2,7-10,13,14,19-22/1,3,7,11,13,15/2,4,8,14,16,19-21,23,24/3,5,11,12,15,17-19,21-24/
  4,6,8,10,16-19,22,23/5,6,9,10,12,17,20-22,24/7,9,11-16,18,20,23,24/
N9.24-18   I S   24 48 33      AUT=  2                  P8.18-11: 9--6-=4--13-
  1-12/1-6,13-18/1,2,7-10,13,19-23/1,3,7,11,13-15,19,20,24/2,4,8,12,13,15,16,20/3,5,11,14,17,18,21-24/
  4,6,9,10,12,16-18,21,23/5-7,9,11,17,19,21,22,24/8,10,12,14-16,18-20,22-24/
N9.24-19   I S   24 48 33      AUT=  2                  P8.18-11: 7--1-=3--12-
  1-12/1-6,13-18/1,2,7-10,13,19-21/1,3,7,11,13-15,19,20,22-24/2,4,8,12,13,15,20,24/3,5,9-11,14,16,17,21,22/
  4,6,12,14,15,17,18,22-24/5,6,8,10,12,16,18-21,23,24/7,9,11,16-19,21-23/
N9.24-20   I S   24 48 33      AUT=  2                  P8.18-13: 7--1-=3--12-
  1-12/1-6,13-18/1,2,7-10,13,19-21/1,3,7,11,13-15,19,22,23/2,4,8,12,13,15,19,20,23,24/3,5,9-11,14,16,17,21,22/
  4,6,12,14,15,17,18,22-24/5,6,8,10,12,16,18,20,21,24/7,9,11,16-24/
N9.24-21   I S   24 48 33      AUT=  4                  P8.18-11: 9--18--15--16-; N8.19-1: 3--10--5--6--9-
  1-12/1-6,13-18/1,2,7-10,13,19-21/1,3,7,11,13-15,19,22,23/2,4,8,13,15-17,19,20,23/3,5,9-12,14,21,22,24/
  4,6,8,10,12,16,18,20,21,24/5,6,12,14,15,17,18,22-24/7,9,11,16-24/
N9.24-22   I S   24 48 33      AUT=  6                  P8.18-5: 3--1--2--8-
  1-12/1-6,13-16/1,2,7-9,13-15,17-20/1,3,7,10,11,13,17,18,21-24/2,4,8,14,16,19-22,24/3,5,10,13,15,18-22/
  4,6,8,9,12,16-19,22-24/5,6,10-12,14-16,20,21,23,24/7,9,11,12,17,23/

N9.25-1    I S   25 50 34      AUT=  1                  P8.19-1: 7--15-=11--13-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-17,21-25/3,5,12-14,16,19,21-23/
  4,6,9,11,15,19-21,23,25/5,7,8,10,12,16-18,22,24/6,7,10,11,15,17,18,20,24,25/
N9.25-2    I S   25 50 34      AUT=  1                  P8.19-3: 2--12-=14--18-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-16,21-23,25/3,5,12-14,16,19,21-23/
  4,6,9,11,15,17,19-21,23-25/5,7,8,10,12,15-18,22,24,25/6,7,10,11,17,18,20,24/
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N9.25-3     I S   25 50 34        AUT=  1                            P8.19-1: 5--9-=7--15-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-16,21,23-25/3,5,12-14,16,17,19,21-24/
  4,6,9,11,15,19-21,23,25/5,7,8,10,12,17,18,22/6,7,10,11,15-18,20,22,24,25/
N9.25-4     I S   25 50 34        AUT=  1                            P8.18-10: 1--2-=12--14--13-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-24/2,4,9,14-17,20,22-25/3,5,12-14,16,19-23,25/
  4,6,9,11,15,20,21,25/5,7,8,10,12,16-18,22,24/6,7,10,11,15,17-19,21,23-25/
N9.25-5     I S   25 50 34        AUT=  1                            P8.18-1: 4--6--10--9--7-; N8.20-1: 2--4--6-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3;8,12,13,18,19,22-24/2,4,9,14-16,20-25/3,5,12-14,16,19,21-23/
  4,6,9,11,15,17,20,25/5,7,8,10,12,15-18,22,24,25/6,7,10,11,17-21,23-25/
N9.25-6     I S   25 50 34        AUT=  1                            P8.18-5: 4--6--10--9--7-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-24/2,4,9,14-16,20,25/3,5,12-14,16,19-23,25/
  4,6,9,11,15,17,20-25/5,7,8,10,12,15-18,22,24,25/6,7,10,11,17-19,21,23,24/
N9.25-7     I S   25 50 34        AUT=  1                            P8.18-1: 10--9--15-=16--11-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-24/2,4,9,14,15,20,21,25/3,5,12-16,19,21-23,25/
  4,6,9,11,15-17,20,22-25/5,7,8,10,12,16-18,22,24/6,7,10,11,17-21,23-25/
N9.25-8     I S   25 50 34        AUT=  1                            P8.18-10: 10--7-=1--2--8-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,22-24/2,4,9,14,17,19-25/3,5,8,10,15,18,20-23/
  4,6,9,11,12,17,19,25/5,7,10-12,15,16,19,21,22,24,25/6,7,12-14,16-18,20,23-25/
N9.25-9     I S   25 50 34        AUT=  1                            P8.18-5: 11--15-=7--9--17-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,22-24/2,4,9,14,17,19,20,25/3,5,8,10,15,18-23,25/
  4,6,9,11,12,17,19,21-25/5,7,10-12,15,16,21,22,24/6,7,12-14,16-18,20,23-25/
N9.25-10    I S   25 50 34        AUT=  1                            P8.18-5: 14--17--9--7--15-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,19,23-25/2,4,9,13,15,19-21,24,25/3,5,12,14,16,17,20-25/
  4,6,9,11,14,15,17,20,22,24/5,7,8,10,12,16,18,23/6,7,10,11,16-19,21-23,25/
N9.25-11    I S   25 50 34        AUT=  2                            P8.18-11: 1--7--9--8--17-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,23,24/2,4,9,13,15,18-21,23-25/3,5,12,14,16,17,19,21-25/
  4,6,9,11,14,15,17,19,22,24/5,7,8,10,12,16,18,20,23,25/6,7,10,11,16,17,20-22,25/
N9.25-12    I S   25 50 34        AUT=  1                            P8.19-1: 9--5-=6--14-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,23,24/2,4,9,13,15,18-21,23-25/3,5,12,14,16,17,20-25/
  4,6,9,11,14,15,17,19,24,25/5,7,8,10,12,16,18,20,22,23/6,7,10,11,16,17,19,21,22,25/
N9.25-13      I S   25 50 34        AUT=  1                            P8.18-1: 6--10-=9--15--16-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,23,24/2,4,9,13,15,18-20,23,24/3,5,12,14,16,17,19,20,22-25/
  4,6,9,11,14,15,17,19,21,22,24,25/5,7,8,10,12,16,18,20-23,25/6,7,10,11,16,17,21,25/
N9.25-14    I S   25 50 34        AUT=  2                            P8.18-5: 5--9-=7--15--18-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,23,24/2,4,9,13,15,18-20,24,25/3,5,12,14,16,17,19-23,25/
  4,6,9,11,14,15,17,19,22-25/5,7,8,10,12,16,18,20,21,23-25/6,7,10,11,16,17,21,22/
N9.25-15    I S   25 50 34        AUT=  1                            P8.18-5: 5--9-=7--15--16-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,23,24/2,4,9,13,15,18,19,23-25/3,5,12,14,16,17,20-25/
  4,6,9,11,14,15,17,19,21,22,24,25/5,7,8,10,12,16,18-21,23,25/6,7,10,11,16,17,20,22/
N9.25-16    I S   25 50 34        AUT=  1                            P8.19-1: 6--14-=12--18-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,19,23,24/2,4,9,13-16,19,24,25/3,5,12,14,16,17,20-25/
  4,6,9,11,15,18,19,21-25/5,7,8,10,12,17,18,20,21,23/6,7,10,11,15-17,20,22,25/
N9.25-17    I S   25 50 34        AUT=  1                            P8.18-10: 7--9--6-=5--14-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,19,23,24/2,4,9,13-15,19-21,23-25/3,5,12,14-17,20-23,25/
  4,6,9,11,15,17,20,22/5,7,8,10,12,16,18,23-25/6,7,10,11,16-19,21,22,24,25/
N9.25-18    I S   25 50 34        AUT=  1                            P8.19-3: 13--19--18--16-; N8.19-1: 3--10--5--6--9--8-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,19,23,24/2,4,9,13-15,19-21,24/3,5,12,14-16,20-25/
  4,6,9,11,15-17,20,22,25/5,7,8,10,12,16-18,23,25/6,7,10,11,17-19,21-25/
N9.25-19    I S   25 50 34        AUT=  1                            P8.18-5: 9--5-=3--12--18-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,19,23,24/2,4,9,13-15,19,23-25/3,5,12,14-17,20-23,25/
  4,6,9,11,15,17-19,21,22,24,25/5,7,8,10,12,16,18,20,21,23-25/6,7,10,11,16,17,20,22/
N9.25-20    I S   25 50 34        AUT=  2                            P8.18-1: 5--9-=10--17--16-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23/2,4,9,13-16,18-20,23,24/3,5,12,14,16,17,19,20,22-25/
  4,6,9,11,15,19,21,22,24,25/5,7,8,10,12,17,18,20-23,25/6,7,10,11,15-17,21,24,25/
N9.25-21    I S   25 50 34        AUT=  2                            P8.18-1: 10--17--16-=15--12-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23/2,4,9,13-15,18-21,23,24/3,5,12,14-16,20-25/
  4,6,9,11,15-17,19,24,25/5,7,8,10,12,16-18,20,22,23,25/6,7,10,11,17,19,21,22,24,25/
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Table 4 (contd.)

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N9.25-22    I S   25 50 34       AUT=  2                           P8.18-10: 9--6-=5--14--10-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,13-15,18-20,23-25/2,4,9,13,15,16,19,21,23,24/3,5,8,10,14,17,18,22/
  4,6,9,11,12,16,21,23-25/5,7,10-12,17,19-22,24,25/6,7,12,14-18,20,22,23,25/
N9.25-23    I S   25 50 34       AUT=  1                           P8.18-10: 1--2-=12--14--4-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,13-15,18-20,23-25/2,4,9,13,15,19,21,24/3,5,8,10,14,16-18,22,23/
  4,6,9,11,12,14,15,17,21,23-25/5,7,10-12,16,19-22,24,25/6,7,12,16-18,20,22,23,25/
N9.25-24    I S   25 50 34       AUT=  2                           P8.19-1: 12--18-=19--16-
  1-12/1-7,13-17/1,2,8-11,13,18-20/1,3,8,12-15,18,21,22/2,4,9,13,15,18-20,22-25/3,5,12,14,16,17,21,23-25/
  4,6,9,11,14,15,17,19,21-24/5,7,8,10,12,16,18,20-22,24,25/6,7,10,11,16,17,19,20,23,25/
N9.25-25    I S   25 50 34       AUT=  1                           P8.18-13: 11--1--7-=10--17-
  1-12/1-7,13-17/1,2,8-11,13,18-20/1,3,8,12-14,18,19,21-24/2,4,9,13-15,19,20,22,25/3,5,12,14-16,21-23,25/
  4,6,9,11,15-17,20,21,23-25/5,7,8,10,12,16-18,21,24/6,7,10,11,17-20,22-25/
N9.25-26    I S   25 50 34       AUT=  2                           P8.18-13: 4--2--1--7--10-
  1-12/1-7,13-17/1,2,8-10,13-15,18-21/1,3,8,11-13,18,19,22-25/2,4,9,14,16,17,20-25/3,5,11,13,15,19-23/
  4,6,9,10,16,18-20,23,25/5,7,11,12,14,15,17,21,22,24/6-8,10,12,16-18,24,25/
N9.25-27    I S   25 50 34       AUT=  1                           P8.18-5: 2--1--3--12--14-; N8.20-1: 1--2--4-
  1-12/1-7,13-17/1,2,8-10,13-15,18-21/1,3,8,11-13,18,19,22-25/2,4,9,14,16,17,20,23-25/3,5,11,13,15,19,21,22/
  4,6,9,10,16,18-22,24,25/5,7,11,12,14,15,17,20-23,25/6-8,10,12,16-18,23,24/
N9.25-28    I S   25 50 34       AUT=  2                           P8.19-3: 6--9-=8--17-
  1-12/1-7,13-17/1,2,8-10,13-15,18-21/1,3,8,11-13,18,22-24/2,4,9,14,16,17,19,25/3,5,11,13,15,18,20-23/
  4,6,9,10,16,19-25/5,7,11,12,14,15,17,19,21,22,24,25/6-8,10,12,16-18,20,23-25/
N9.25-29    I S   25 50 34       AUT=  1                           P8.19-2: 6--5-=9--7-
  1-12/1-7,13-17/1,2,8-10,13,14,18-22/1,3,8,11-13,18-20,23-25/2,4,9,14-16,21,23-25/3,5,11,13,14,16,19,21,23,24/
  4,6,9,10,15,17,19-22,24,25/5,7,11,12,15-18,20,22,23,25/6-8,10,12,17,18,22/
N9.25-30    I S   25 50 34       AUT=  6                           P8.18-10: 6--9-=7--15--18-
  1-12/1-7,13-17/1,2,8-10,13,14,18-22/1,3,8,11-13,18,19,23,24/2,4,9,14-16,18-21,24,25/3,5,11,13,14,16,19,23-25/
  4,6,9,10,15,17,20,22/5,7,11,12,15-17,20-23,25/6-8,10,12,17,18,21-25/
N9.25-31    I S   25 50 34       AUT=  2                           P8.18-10: 9--6-=5--14--18-
  1-12/1-7,13-17/1,2,8-10,13,14,18-22/1,3,8,11-13,18,23-25/2,4,9,14-16,19-21,23-25/3,5,11,13,14,16,18,20,23,24/
  4,6,9,10,15,17,19,22/5,7,11,12,15-17,19,21-23,25/6-8,10,12,17,18,20-22,24,25/
N9.25-32    I S   25 50 34       AUT=  1                           P8.18-13: 7--1-=3--12--18-
  1-12/1-7,13-17/1,2,8-10,13,14,18-20/1,3,8,11-13,18,21-25/2,4,9,14-16,19-22,24,25/3,5,11,13,14,16,18,20-22/
  4,6,9,10,15,17,19,23-25/5,7,11,12,15-17,21,23,25/6-8,10,12,17-20,22-24/
N9.25-33    I S   25 50 34       AUT=  4                           P8.18-13: 1--7-=10--17--18-
  1-12/1-7,13-17/1,2,8-10,13,14,18-20/1,3,8,11-13,18,21-25/2,4,9,14-16,19,21,23,25/3,5,11,13,14,16,18-23/
  4,6,9,10,15,17,19,20,22-25/5,7,11,12,15-17,21,24,25/6-8,10,12,17,18,20,22,24/
N9.25-34    I S   25 50 34       AUT=  2                           P8.19-4: 7--1-=3--12-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11-13,16,22/2,4,8,14,17-19,23-25/3,5,11,14-16,18,19,21,22,24,25/
  4,6,8,10,17,19-21,23,24/5,6,9-12,17,18,20,22,23,25/7,9,12,13,15,16,20-25/
N9.25-35    I S   25 50 34       AUT=  1                           P8.19-3: 8--17-=16--18--12--14-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16-18,22-25/2,4,8,12,14,19,20,23-25/3,5,9-11,16,19-24/
  4,6,12,14,15,17,18,20-22,24,25/5,6,8,10,12,16,18,19,23,25/7,9,11,13,15,17,21,22/
N9.25-36    I S   25 50 34       AUT=  2                           P8.19-3: 2--4--8--1-=7--10-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16,17,19,22-25/2,4,8,12,14,18,20-25/3,5,9-11,16,20-23/
  4,6,12,13,15,17-19,24,25/5,6,8,10,12,16,17,20,23,25/7,9,11,14,15,18,19,21,22,24/
N9.25-37    I S   25 50 34       AUT=  1                           P8.19-3: 2--4--8--17-=16--18-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16,17,19,22,23/2,4,8,12,14,18,20,21,24,25/3,5,9-11,16,20-25/
  4,6,12,13,15,17-19,22-25/5,6,8,10,12,16,17,20,23,24/7,9,11,14,15,18,19,21,22,25/
N9.25-38    I S   25 50 34       AUT=  2                           P8.19-3: 10--18--14--12-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16,17,22-24/2,4,8,12,14,19,20,25/3,5,9-11,16,19-23,25/
  4,6,12,14,15,18,20-25/5,6,8,10,12,16-19,23-25/7,9,11,13,15,17,18,21,22,24/
N9.25-39    I S   25 50 34       AUT=  2                           N8.19-1: 7--1-=3--12-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16,19,22/2,4,8,14,17,18,20,23-25/3,5,11,14-16,18,20-22,24,25/
  4,6,8,10,12,17,20,21,23,24/5-7,9,11,12,17-19,22,23,25/9,10,12,13,15,16,19,21-25/
N9.25-40    I S   25 50 34       AUT=  1                           P8.19-3: 9--6-=5--14-
  1-12/1-6,13-18/1,2,7-10,13,14,19-22/1,3,7,11,13,15-17,23,24/2,4,8,14,16,17,19-21,23-25/3,5,11,12,15,18,19,21-25/
  4,6,8,10,15,17-19,22,24/5,6,9,10,12,18,20-22,25/7,9,11-14,16,20,23,25/
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N9.25-41    I S   25 50 34       AUT=  1                      P8.19-3: 2--1-=7--10-
  1-12/1-6,13-18/1,2,7-10,13,14,19-22/1,3,7,11,13,15-17,23,24/2,4,8,14,18-20,25/3,5,11,12,15,19-25/
  4,6,8,10,16-19,22-25/5,6,9,10,12,15,17,21,22,24/7,9,11-14,16,18,20,21,23,25/
N9.25-42    I S   25 50 34       AUT=  1                      P8.19-2: 12--18-=19--16-
  1-12/1-6,13-18/1,2,7-10,13,14,19,20/1,3,7,11-13,15,16,21,22/2,4,8,14-17,19,21-24/3,5,11,15,17,18,21,23-25/
  4,6,8,10,17-20,23,25/5,6,9-12,18,20-22,24,25/7,9,12-14,16,19,20,22-25/
N9.25-43    I S   25 50 34       AUT=  1                      P8.19-2: 5--6-=10--8-
  1-12/1-6,13-18/1,2,7-10,13,14,19,20/1,3,7,11-13,15,21-23/2,4,8,14,16,19-25/3,5,11,15,17,18,21,22,24,25/
  4,6,8,10,16-19,24,25/5,6,9-12,17,19-21,23,25/7,9,12-16,18,20,22-24/
N9.25-44    I S   25 50 34       AUT=  4   M963               P8.19-3: 5--6-=9--8-
  1-12/1-6,13-18/1,2,7-10,13,14,19,20/1,3,7,11,13,15-17,21,22/2,4,8,14,18-20,23-25/3,5,11,12,15,21-25/
  4,6,8,10,16-19,21-24/5,6,9,10,12,15,17,19,20,22,24,25/7,9,11-14,16,18,20,21,23,25/
N9.25-45    I S   25 50 34       AUT=  2                      P8.19-3: 6--5--14--12-
  1-12/1-6,13-18/1,2,7-10,13,14,19,20/1,3,7,11,13,15,16,21-23/2,4,8,12,14,17,19,20,24,25/3,5,9-11,15,19-22,24,25/
  4,6,12,17,18,21-25/5,6,8,10,12,15,16,18,19,22-24/7,9,11,13,14,16-18,20,21,23,25/
N9.25-46    I S   25 50 34       AUT=  2                      P8.19-2: 2--1--3--12-
  1-12/1-6,13-18/1,2,7-10,13,19-23/1,3,7,11,13-15,19,20,24/2,4,8,12,13,15,16,20,21,24/3,5,9-11,14,17,22,23,25/
  4,6,12,16,18-25/5,6,8,10,12,17,18,21,23,25/7,9,11,14-19,22,24,25/

N9.26-1     I S   26 52 35       AUT=  1                      P8.19-2: 1--11--13-=12--18-
  1-12/1-7,13-17/1,2,8-13,18-21/1,3,8,13-16,18,22-25/2,4,9,13,15,17-19,23,26/3,5,8,10,14,18-20,22-24,26/
  4,6,9,11,17,19-22,24-26/5,7,10,12,14,16,20-22,25/6,7,11,12,15-17,21,23-26/
N9.26-2     I S   26 52 35       AUT=  2                      P8.19-1: 10--6--14-=12--18-
  1-12/1-7,13-17/1,2,8-13,18-21/1,3,8,13-16,18,22-25/2,4,9,13,15,18-20,22-24,26/3,5,8,10,14,17,18,20,22,26/
  4,6,9,11,15,16,19,21,23,25/5,7,10,12,17,19-21,23-26/6,7,11,12,14,16,17,21,22,24-26/
N9.26-3     I S   26 52 35       AUT=  1                      P8.19-4: 5--3--1-=2--8-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-17,21-23,25,26/3,5,12-14,16,19,21-23/
  4,6,9,11,15,19-21,23-26/5,7,8,10,12,16-18,22,24-26/6,7,10,11,15,17,18,20,24,26/
N9.26-4     I S   26 52 35       AUT=  1                      P8.19-1: 7--15-=11--13--19-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-17,21,22,24-26/3,5,12-14,16,19,21-24,26/
  4,6,9,11,15,19-21,23,26/5,7,8,10,12,16-18,22,25/6,7,10,11,15,17,18,20,23-26/
N9.26-5     I S   26 52 35       AUT=  1                      P8.19-1: 7--15-=11--13--18-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-17,21,23-26/3,5,12-14,16,19,21-24,26/
  4,6,9,11,15,19-21,23,25/5,7,8,10,12,16-18,22,26/6,7,10,11,15,17,18,20,22,24-26/
N9.26-6     I S   26 52 35       AUT=  1                      P8.19-3: 4--8--17-=16--18-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-17,21,23-26/3,5,12-14,16,19,21,22,26/
  4,6,9,11,15,19-22,24-26/5,7,8,10,12,16-18,22-24,26/6,7,10,11,15,17,18,20,23,25/
N9.26-7     I S   26 52 35       AUT=  1                      P8.19-2: 1--7--15-=16--19-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-16,19,20,23,25/3,5,12-14,16,17,19,22-24,26/
  4,6,9,11,15,18,20-22,24-26/5,7,8,10,12,17,18,21,22,26/6,7,10,11,15-17,21,23-26/
N9.26-8     I S   26 52 35       AUT=  1                      P8.19-3: 9--2--1-=7--10-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-16,21-26/3,5,12-14,16,17,19,21-23,26/
  4,6,9,11,15,19-21,23,25/5,7,8,10,12,17,18,22,24,26/6,7,10,11,15-18,20,24-26/
N9.26-9     I S   26 52 35       AUT=  1                      P8.19-2: 4--6--5-=9--7-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-16,21-26/3,5,12-14,16,19,21-23/
  4,6,9,11,15,17,19-21,23,25,26/5,7,8,10,12,15-18,22,24,26/6,7,10,11,17,18,20,24-26/
N9.26-10    I S   26 52 35       AUT=  1                      P8.19-4: 1--7-=9--17--18-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14,15,19-21,23,26/3,5,12-16,19,22-24,26/
  4,6,9,11,15-17,21,22,24-26/5,7,8,10,12,16-18,22,25/6,7,10,11,17,18,20,21,23-26/
N9.26-11    I S   26 52 35       AUT=  1                      P8.19-2: 1--2--8-=10--17-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14-17,20,22,23,25,26/3,5,12-14,16,19,20,22,23/
  4,6,9,11,15,18-21,23,24,26/5,7,8,10,12,16-18,21,22,24,25/6,7,10,11 15,17,21,24-26/
N9.26-12    I S   26 52 35       AUT=  1                      P8.19-3: 1--2-=12--14--13-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14-17,20,22,24-26/3,5,12-14,16,19-24/
  4,6,9,11,15,20,21,23,24,26/5,7,8,10,12,16-18,22,25/6,7,10,11,15,17-19,21,23,25,26/
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Table 4 (contd.)

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N9.26-13    I S   26 52 35        AUT=  1                        P8.20-3: 8--16--11--13-; N8.20-1: 1--2--4--6-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14-16,20-23,25,26/3,5,12-14,16,19,21-23/
  4,6,9,11,15,17,20,24-26/5,7,8,10,12,15-18,22,24,25/6,7,10,11,17-21,23,24,26/
N9.26-14    I S   26 52 35        AUT=  2                        P8.19-1: 5--6--4-=13--11-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14-16,20,22,24,26/3,5,12-14,16,19-24/
  4,6,9,11,15,17,20,21,23-26/5,7,8,10,12,15-18,22,25,26/6,7,10,11,17-19,21,23,25/
N9.26-15    I S   26 52 35        AUT=  1                        P8.20-2: 5--6-=10--8-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14-16,20,23,24,26/3,5,12-14,16,17,19,20,22-24/
  4,6,9,11,15,18-21,23,25,26/5,7,8,10,12,17,18,21,22,25/6,7,10,11,15-17,21,22,24-26/
N9.26-16    I S   26 52 35        AUT=  1                        P8.19-2: 3--1--2--4--14-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14,15,20,21,23,24,26/3,5,12-16,19,21-24/
  4,6,9,11,15-17,20,22,24-26/5,7,8,10,12,16-18,22,25/6,7,10,11,17-21,23,25,26/
N9.26-17    I S   26 52 35        AUT=  2                        P8.19-3: 3--1--2--4--13-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-24/2,4,9,14-17,20,25,26/3,5,12-14,16,19-23,26/
  4,6,9,11,15,20-26/5,7,8,10,12,16-18,22,24-26/6,7,10,11,15,17-19,21,23-25/
N9.26-18    I S   26 52 35        AUT=  1                        P8.20-3: 2--4--14--20-; N8.20-1: 12--17--2--14-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-24/2,4,9,14-16,20,21,25,26/3,5,12-14,16,19,21-23,25,26/
  4,6,9,11,15,17,20,22-26/5,7,8,10,12,15-18,22,24,25/6,7,10,11,17-21,23,24,26/
N9.26-19    I S   26 52 35        AUT=  1                        P8.19-2: 3--5--6-=10--8-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-24/2,4,9,14-16,20,22-26/3,5,12-14,16,19-23,26/
  4,6,9,11,15,17,20,21,25,26/5,7,8,10,12,15-18,22,24,25/6,7,10,11,17-19,21,23-26/
N9.26-20    I S   26 52 35        AUT=  1                        P8.20-3: 2--1--3--12-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-24/2,4,9,14,15,20-26/3,5,12-16,19,21-23,25/
  4,6,9,11,15-17,20,25,26/5,7,8,10,12,16-18,22,24-26/6,7,10,11,17-21,23,24,26/
N9.26-21    I S   26 52 35        AUT=  1                        P8.19-2: 9--5-=6--14--13-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,19,22-25/2,4,9,14,17-20,23,26/3,5,8,10,15,18,20-24,26/
  4,6,9,11,12,17,20-22,24-26/5,7,10-12,15,16,21,22,25/6,7,12-14,16,17,19,23-26/
N9.26-22    I S   26 52 35        AUT=  1                        P8.19-2: 3--12--18-=19--16-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,19,22,23/2,4,9,14,17-20,22-25/3,5,8,10,15,18,20-22,24-26/
  4,6,9,11,12,17,20,21,24,26/5,7,10-12,15,16,21-23,25,26/6,7,12-14,16,17,19,23-26/
N9.26-23    I S   26 52 35        AUT=  1                        P8.19-3: 8--2--1-=7--10-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,22-26/2,4,9,14,17,19-23,25,26/3,5,8,10,15,18,20-23/
  4,6,9,11,12,17,19,24-26/5,7,10-12,15,16,19,21,22,24,26/6,7,12-14,16-18,20,23-25/
N9.26-24    I S   26 52 35        AUT=  1                        P8.20-3: 6--10-=8--16-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,22-24/2,4,9,14,17,19-21,25,26/3,5,8,10,15,18,20-23,25,26/
  4,6,9,11,12,17,19,22-26/5,7,10-12,15,16,19,21,22,24,26/6,7,12-14,16-18,20,23-25/
N9.26-25    I S   26 52 35        AUT=  1                        P8.19-2: 11--16-=8--10--17-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,22-24/2,4,9,14,17,19,20,22-26/3,5,8,10,15,18-23,26/
  4,6,9,11,12,17,19,21,25,26/5,7,10-12,15,16,21,22,24-26/6,7,12-14,16-18,20,23-25/
N9.26-26    I S   26 52 35        AUT=  1                        P8.19-3: 9--6--5--10--18-; N8.20-1: 6--5--14--3-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,19,23-25/2,4,9,13,15,19-21,23-26/3,5,12,14,16,17,20-24,26/
  4,6,9,11,14,15,17,20,22,24/5,7,8,10,12,16,18,23,25,26/6,7,10,11,16-19,21,22,25,26/
N9.26-27    I S   26 52 35        AUT=  3                        P8.19-2: 7--15-=16--19--13-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18-20,23-25/2,4,9,13-15,19,21,22,26/3,5,12,14-16,19,20,22-24,26/
  4,6,9,11,15-17,21,23-26/5,7,8,10,12,16-18,23,25/6,7,10,11,17,18,20-22,24-26/
N9.26-28    I S   26 52 35        AUT=  1                        P8.19-3: 4--2--12-=14--18-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,19,23-26/2,4,9,13-16,19,24,26/3,5,12,14,16,17,20-25/
  4,6,9,11,15,18,19,21-23,25,26/5,7,8,10,12,17,18,20,21,23/6,7,10,11,15-17,20,22,24-26/
N9.26-29    I S   26 52 35        AUT=  1                        P8.19-3: 8--17-=16--18--14-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,19,23-26/2,4,9,13-15,19-21,24/3,5,12,14-16,20-25/
  4,6,9,11,15-17,20,22,23,25,26/5,7,8,10,12,16-18,23,26/6,7,10,11,17-19,21,22,24-26/
N9.26-30    I S   26 52 35        AUT=  2                        N8.19-1: 3--1--2--8--9-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,19,23,24/2,4,9,13-16,19-21,24,25/3,5,12,14,16,20-26/
  4,6,9,11,15,17,20,22,25,26/5,7,8,10,12,15-18,23,25,26/6,7,10,11,17-19,21-24,26/
N9.26-31    I S   26 52 35        AUT=  1                        P8.19-3: 4--13-=14--12--19-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23-25/2,4,9,13-16,18,19,24-26/3,5,12,14,16,17,20-24,26/
  4,6,9,11,15,19,21-26/5,7,8,10,12,17-21,23,25/6,7,10,11,15-17,20,22,26/
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N9.26-32   I S   26 52 35        AUT=  2                        P8.19-3: 14--12--2--8--17-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23-25/2,4,9,13-15,18-21,24,25/3,5,12,14-16,19,21-24,26/
  4,6,9,11,15-17,19,22,26/5,7,8,10,12,16-18,20,23,25,26/6,7,10,11,17,20-26/
N9.33-26   I S   26 52 35        AUT=  2                        N8.19-1: 3--1--2--8--7-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,13-16,18-20,23,24/2,4,9,13,15,17,19,21,24,25/3,5,8,10,12,14,18,22,23,26/
  4,6,9,11,17,18,20-23,25,26/5,7,12,14,16,19-21,23-26/6,7,10-12,15-17,22,24-26/
N9.26-34   I S   26 52 35        AUT=  1                        P8.19-3: 8--17-=16--18--12-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,13-16,18,19,23-25/2,4,9,13,15,17,19-21,26/3,5,8,10,14,18,20-24,26/
  4,6,9,11,12,17,20,22-26/5,7,10-12,14,16,22,23,25/6,7,12,15-19,21,24-26/
N9.26-35   I S   26 52 35        AUT=  1                        P8.20-3: 4--2--1--7-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,13-16,18,19,23/2,4,9,13,15,17,19,20,23-26/3,5,8,10,12,14,18,21,22,24-26/
  4,6,9,11,17,20-22,24,25/5,7,12,14,16,18-20,22,23,25,26/6,7,10-12,15-17,21,23,24,26/
N9.26-36   I S   26 52 35        AUT=  1                        P8.19-3: 1--2--12--14--4-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,13-15,18,19,23-26/2,4,9,13,15,19-21,24,25/3,5,8,10,14,16-18,22,23/
  4,6,9,11,12,14,15,17,20,23,24,26/5,7,10-12,16,20-22,24-26/6,7,12,16-19,21-23,25,26/
N9.26-37   I S   26 52 35        AUT=  1                        P8.20-3: 5--6-=10--8-
  1-12/1-7,13-17/1,2,8-11,13,18-20/1,3,8,12-15,18,19,21-23/2,4,9,13,15,19,20,22-26/3,5,12,14,16,17,21,24-26/
  4,6,9,11,14,15,17,20,21,23-25/5,7,8,10,12,16,18,21-23,25,26/6,7,10,11,16-20,22,24,26/
N9.26-38   I S   26 52 35        AUT=  1                        P8.19-3: 6--5--14--12--3-
  1-12/1-7,13-17/1,2,8-11,13,18-20/1,3,8,13-15,18,21-25/2,4,9,13,15,18-20,22-24,26/3,5,8,10,14,16-18,20-22,26/
  4,6,9,11,12,14,15,17,19,21,23,25/5,7,10-12,16,19,20,23-26/6,7,12,16,17,21,22,24-26/
N9.26-39   I S   26 52 35        AUT=  1                        P8.19-3: 4--2--1-=7--10-
  1-12/1-7,13-17/1,2,8-10,13-15,18-21/1,3,8,11-13,18,22-26/2,4,9,14,16,17,19,24-26/3,5,11,13,15,18,20-23/
  4,6,9,10,16,19-23,25,26/5,7,11,12,14,15,17,19,21,22,24,26/6-8,10,12,16-18,20,23-25/
N9.26-40   I S   26 52 35        AUT=  1                        P8.20-3: 10--8-=16--11-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11-13,16,22-26/2,4,8,14,17-19,22,24,26/3,5,11,14-16,18,19,21-24/
  4,6,8,10,17,19-21,23-26/5,6,9-12,17,18,20,22,25,26/7,9,12,13,15,16,20,21,23,25/
N9.26-41   I S   26 52 35        AUT=  1                        P8.20-3: 9--10--17--8-=16--11-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11-13,16,22-24/2,4,8,14,17-19,22-26/3,5,11,14-16,18,19,21-23,26/
  4,6,8,10,17,19-21,25,26/5,6,9-12,17,18,20,22,24,25/7,9,12,13,15,16,20,21,23-26/
N9.26-42   I S   26 52 35        AUT=  1                        P8.20-3: 1--2--4--14-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16-18,22,23/2,4,8,12,14,19,20,24-26/3,5,9-11,16,19-22,25,26/
  4,6,12,14,15,17,18,20,21,23-25/5,6,8,10,12,16,18,19,22-24,26/7,9,11,13,15,17,21-26/
N9.26-43   I S   26 52 35        AUT=  1                        N8.20-1: 1--2--8--17-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16,17,19,22-25/2,4,8,12,14,18,20-23,25,26/3,5,9-11,16,20-23/
  4,6,12,13,15,17-19,24,26/5,6,8,10,12,16,17,20,23-26/7,9,11,14,15,18,19,21,22,24-26/
N9.26-44   I S   26 52 35        AUT=  1                        N8.20-1: 6--5--3--11-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16,17,22-26/2,4,8,12,14,19,20,23,24,26/3,5,9-11,16,19-24/
  4,6,12,14,15,18,20-22,24-26/5,6,8,10,12,16-19,23,25,26/7,9,11,13,15,17,18,21,22,25/
N9.26-45   I S   26 52 35        AUT=  1                        N8.20-1: 7--1-=2--12-
  1-12/1-6,13-18/1,2,7-10,13-15,19-21/1,3,7,11,13,16,19,22-26/2,4,8,14,17,18,20,22,24,26/3,5,11,14-16,18,20-24/
  4,6,8,10,12,17,20,21,23-26/5-7,9,11,12,17-19,22,25,26/9,10,12,13,15,16,19,21,23,25/

N9.27-1    I S   27 54 36        AUT=  1   A31                  P8.20-3: 3--5--6-=10--8-
  1-12/1-7,13-17/1,2,8-13,18-21/1,3,8,13-16,18,19,22-24/2,4,9,13,15,17,19,23-27/3,5,8,10,14,18,20-22,25-27/
  4,6,9,11,17-19,21,22,24-26/5,7,10,12,14,16,20,22-24,26,27/6,7,11,12,15-17,20,21,23,25,27/
N9.27-2    I S   27 54 36        AUT=  1   A26                  P8.20-1: 4--6--5-=9--7-
  1-12/1-7,13-17/1,2,8-13,18-21/1,3,8,13-16,18,19,22-24/2,4,9,13,15,19-21,24-27/3,5,8,10,14,17,18,22,23,25-27/
  4,6,9,11,15,16,20,22-26/5,7,10,12,17-19,21,23,24,26,27/6,7,11,12,14,16,17,20-22,25,27/
N9.27-3    I S   27 54 36        AUT=  1   A47                  P8.20-2: 4--6--5-=9--7-
  1-12/1-7,13-17/1,2,8-13,18-21/1,3,8,13-16,18,22-25/2,4,9,13,15,18,19,22-24,26,27/3,5,8,10,14,17-20,22,26,27/
  4,6,9,11,15,16,19-21,23,25,26/5,7,10,12,17,20,21,23-27/6,7,11,12,14,16,17,21,22,24,25,27/
N9.27-4    I S   27 54 36        AUT=  1   A43                  P8.20-2: 11--15-=7--9--17-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-17,19,20,23,25,26/3,5,12-14,16,19,22-24,26,27/
  4,6,9,11,15,18,20-22,24,25,27/5,7,8,10,12,16-18,21,22,26,27/6,7,10,11,15,17,21,23-27/
N9.27-5    I S   27 54 36        AUT=  2   A33                  P8.20-3: 3--5-=9--10--8-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-16,19-21,23,26,27/3,5,12-14,16,19,22-24,26,27/
  4,6,9,11,15,17,21,22,24-27/5,7,8,10,12,15-18,22,25,26/6,7,10,11,17,18,20,21,23-25,27/
N9.27-6    I S   27 54 36        AUT=  1   A27                  P8.20-3: 11--1--7-=15--17-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-16,21-23,25-27/3,5,12-14,16,17,19,21-23,27/
  4,6,9,11,15,19-21,23-26/5,7,8,10,12,17,18,22,24-27/6,7,10,11,15-18,20,24,26,27/
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Table 4 (contd.)

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N9.27-7    I S   27 54 36      AUT=  1   A34                  P8.20-3: 9--10--8-=16--11-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-16,21,23-27/3,5,12-14,16,17,19,21,22,26,27/
  4,6,9,11,15,19-22,24-26/5,7,8,10,12,17,18,22-24,26,27/6,7,10,11,15-18,20,23,25,27/
N9.28-7    I S   27 54 36      AUT=  1   A36                  P8.20-2: 3--5--6-=10--8-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18-20,22-25/2,4,9,14-16,21,23-27/3,5,12-14,16,19,21-24,27/
  4,6,9,11,15,17,19-21,23,25,26/5,7,8,10,12,15-18,22,26,27/6,7,10,11,17,18,20,22,24-27/
N9.27-9    I S   27 54 36      AUT=  2   A44                  P8.20-2: 10--8-=16--15--11-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14-17,20,22,24,26,27/3,5,12-14,16,19,20,22-24,27/
  4,6,9,11,15,18-21,23,25,27/5,7,8,10,12,16-18,21,22,25,26/6,7,10,11,15,17,21,23-27/
N9.27-10   I S   27 54 36      AUT=  1   A35                  P8.20-3: 9--7-=15--16--11-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14-17,20,23,24,26,27/3,5,12-14,16,19,20,22-24,27/
  4,6,9,11,15,18-21,23,25,26/5,7,8,10,12,16-18,21,22,25,27/6,7,10,11,15,17,21,22,24-27/
N9.27-11   I S   27 54 36      AUT=  2   A32                  N8.20-1: 3--1--2--4--13-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14-17,20,24-27/3,5,12-14,16,19-23,27/
  4,6,9,11,15,20-23,25-27/5,7,8,10,12,16-18,22,24,25,27/6,7,10,11,15,17-19,21,23,24,26/
N9.27-12   I S   27 54 36      AUT=  1   A40                  P8.20-3: 11--13--14--20--4-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14-16,18-20,23,25,26/3,5,12-14,16,17,19,22-24,27/
  4,6,9,11,15,20-22,24-27/5,7,8,10,12,17,18,20-22,25,27/6,7,10,11,15-17,21,23,24,26,27/
N9.27-13   I S   27 54 36      AUT=  1   A30                  P8.20-3: 11--16--8-=10--15-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14-16,20,22,23,25-27/3,5,12-14,16,17,19,
  20,22,23,27/4,6,9,11,15,18-21,23,24,26/5,7,8,10,12,17,18,21,22,24,25,27/6,7,10,11,15-17,21,24-27/
N9.27-14   I S   27 54 36      AUT=  1   A41                  P8.20-3: 6--5--12--13--11-; N8.20-1: 2--12--17--19--18-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14,15,20-23,25-27/3,5,12-16,19,21-23,27/
  4,6,9,11,15-17,20,24-27/5,7,8,10,12,16-18,22,24,25,27/6,7,10,11,17-21,23,24,26/
N9.27-15   I S   27 54 36      AUT=  2   A45                  P8.20-3: 5--6--14--13--11-; N8.20-1: 1--3--5--14--13-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,12,13,18,19,22-26/2,4,9,14,15,20,21,23-27/3,5,12-16,19,21-24/
  4,6,9,11,15-17,20,22,24,26,27/5,7,8,10,12,16-18,22,25-27/6,7,10,11,17-21,23,25,27/
N9.27-16   I S   27 54 36      AUT=  2   A29                  P8.20-3: 6--10-=8--16--19-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,22-26/2,4,9,14,17,19-22,24,26,27/3,5,8,10,15,18,20-24,27/
  4,6,9,11,12,17,19,23-27/5,7,10-12,15,16,19,21,22,25,26/6,7,12-14,16-18,20,23,25,27/
N9.27-17   I S   27 54 36      AUT=  2   A28                  P8.20-3: 6--10-=8--16--15-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,22-26/2,4,9,14,17,19-21,23,24,26,27/3,5,8,10,15,18,20-24,27/
  4,6,9,11,12,17,19,22,24-27/5,7,10-12,15,16,19,21,22,25,27/6,7,12-14,16-18,20,23,25,26/
N9.27-18   I S   27 54 36      AUT=  1   A38                  P8.20-3: 11--16-=8--10--17-
  1-12/1-7,13-17/1,2,8-11,13,14,18-21/1,3,8,13,15,16,18,22-26/2,4,9,14,17,19,20,22,23,25-27/3,5,8,10,15,18-23,27/
  4,6,9,11,12,17,19,21,24-27/5,7,10-12,15,16,21,22,24,26,27/6,7,12-14,16-18,20,23-25/
N9.27-19   I S   27 54 36      AUT=  1   A39                  N8.20-1: 7--1--2--8--17-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-15,18,19,23-25/2,4,9,13,15,19-21,24-27/3,5,12,14,16,17,20-23,26,27/
  4,6,9,11,14,15,17,20,22,23,25,26/5,7,8,10,12,16,18,23-27/6,7,10,11,16-19,21,22,24,27/
N9.27-20   I   S 27 54 36      AUT=  6   A24                  P8.20-3: 11--2--4--14--20-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18-20,23-25/2,4,9,13-16,19,21,22,26,27/3,5,12,14,16,19,20,22-24,26,27/
  4,6,9,11,15,17,21,23-27/5,7,8,10,12,15-18,23,25,26/6,7,10,11,17,18,20-22,24,25,27/
N9.27-21   I S   27 54 36      AUT=  2   A48                  N8.20-1: 1--2--4--13--14-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,19,23-26/2,4,9,13-16,19-21,24,27/3,5,12,14,16,20-25,27/
  4,6,9,11,15,17,20,22,23,25-27/5,7,8,10,12,15-18,23,26,27/6,7,10,11,17-19,21,22,24-26/
N9.27-22   I S   27 54 36      AUT=  1   A42                  N8.20-1: 9--7--1--2--12-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23-27/2,4,9,13-16,18,19,24,26,27/3,5,12,14,16,17,20-25/
  4,6,9,11,15,19,21-23,25-27/5,7,8,10,12,17-21,23,26/6,7,10,11,15-17,20,22,24,25,27/
N9.27-23   I S   27 54 36      AUT=  2   A49                  N8.20-1: 2--1--7--15--16-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,12-14,18,23-27/2,4,9,13-15,18-21,24,26/3,5,12,14-16,19,21-25/
  4,6,9,11,15-17,19,22,23,25,27/5,7,8,10,12,16-18,20,23,26,27/6,7,10,11,17,20-22,24-27/
N9.27-24   I S   27 54 36      AUT=  2   A37                  N8.20-1: 6--5--3--11--14-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,13-16,18-20,23,24/2,4,9,13,15,17,19,21,24-27/3,5,8,10,12,14,18,22,23,25-27/
  4,6,9,11,17,18,20-23,25,26/5,7,12,14,16,19-21,23,24,26,27/6,7,10-12,15-17,22,24,25,27/
N9.27-25   I S   27 54 36      AUT=  2   A25                  N8.20-1: 7--1--2--12--8-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,13-16,18,19,23-25/2,4,9,13,15,17,19-22,26,27/3,5,8,10,14,18,21-24,26,27/
  4,6,9,11,12,17,20,23-27/5,7,10-12,14,16,20,22,23,25,27/6,7,12,15-19,21,24-26/
N9.27-26   I S   27 54 36      AUT=  2   A46                  N8.20-1: 3--12--14--5--6-
  1-12/1-7,13-17/1,2,8-11,13,18-22/1,3,8,13-15,18,19,23-26/2,4,9,13,15,19-22,24,25,27/3,5,8,10,14,16-18,21-23,27/
  4,6,9,11,12,14,15,17,20,23,24,26/5,7,10-12,16,20,22,24-27/6,7,12,16-19,21,23,25-27/
N9.27-27   I S   27 54 36      AUT=  6   A50                  N8.20-1: 4--2--1--7--10-
  1-12/1-7,13-17/1,2,8-10,13-15,18-21/1,3,8,11-13,18,22-26/2,4,9,14,16,17,19,23-27/3,5,11,13,15,18,20-24,27/
  4,6,9,10,16,19-22,24,26,27/5,7,11,12,14,15,17,19,21,22,25,26/6-8,10,12,16-18,20,23,25,27/
```

For the 161 nonfree generated 3-spheres, we had to decide their polytopality. For each of them the relevant cut was found and for 123 the nonpolytopality could directly be proved. Among the remaining ones, 7 could be realized as polytopes by constructing them. There are left 31 3-spheres for which we could not prove nonpolytopality, but which were determined to be nonpolytopal by Altshuler and coworkers [3, 4], so they were added to the list of nonpolytopal 3-spheres. Nonsimple initial 3-spheres were found through special cuts. For the 3-spheres with 24 vertices we looked at all those which have an edge of valence 4 and thus could prove the completeness of the results. However, for those with less vertices the numbers of 3-spheres involved becomes so large that we were not able to prove completeness although, we are convinced, to have missed only a few ones. In Table 4 we give the unified polytope-schemes of the simple nonpolytopal 3-spheres with 9 facets. The references A24-A50 in Table 4 refer to Altshuler [4].

The above results suggest the following proposition.

Almost all d-polytopes in $\mathbb{R}^d$ are free generated.

In Table 5 we give the numbers of combinatorial types of simple 3-spheres with

Table 5
Number of combinatorial types of simple 3-spheres with N_3 facets and N_0 vertices. The numbers of nonpolytopal 3-spheres are given within parentheses

N_0 \ N_3	5	6	7	8	9
5	1				
6					
7					
8		1			
9		1			
10					
11			1		
12			2		
13			1		
14			1	3	
15				5	
16				8	
17				8	7
18				6	23
19				(1) 5	45
20				(1) 4	84
21					128
22					(3) 175
23					(11) 223
24					(22) 231
25					(46) 209
26					(45) 121
27					(27) 50
Total	1	2	5	(2) 37	(154) 1296

Table 6
Number of combinatorial types of 3-spheres with N_3 facets and N_0 vertices. The numbers of nonpolytopal 3-spheres are given within parentheses

N_0 \ N_3	5	6	7	8		9	
5	1						
6		1	1		1		1
7		1	3		5		7
8		1	5		27		76
9		1	7		76	(>1)	467
10			6	(1)	138	(>6)	1908
11			4	(4)	209	(>57)	5411
12			3	(6)	231	(>260)	11974
13			1	(8)	226	(>778)	21129
14			1	(7)	173	(>1706)	31234
15				(5)	122	(>3046)	39875
16				(5)	70	(>4488)	44461
17				(2)	33	(>5529)	43870
18				(2)	16	(>5836)	38493
19				(1)	5	(>5408)	30216
20				(1)	4	(>4313)	21089
21						(>3154)	13231
22						(>1872)	7181
23						(>1133)	3604
24						(444)	1390
25						(236)	567
26						(45)	121
27						(27)	50
Total	1	4	31	(42)	1336	(>38339)	316355

up to 9 facets. Within parenthesis are given the numbers of nonpolytopal 3-spheres. In Table 6 we give the numbers of combinatorial types of 3-spheres with up to 9 facets. The numbers of nonpolytopal 3-spheres with 9 facets and 9 to 23 vertices are lower bounds only.

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