Note

On the Evaluation at (3, 3) of the Tutte Polynomial of a Graph

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We give a combinatorial interpretation of the evaluation at (3, 3) of the Tutte polynomial of a planar graph. As a corollary we obtain a divisibility property.


1. INTRODUCTION

The dichromatic polynomial of a graph—now currently called the Tutte polynomial—was introduced by Tutte in 1954 as a generalization of chromatic polynomials studied by Birkhoff and Whitney. The generalization to matroids (combinatorial geometries) is due to Crapo in 1969. The Tutte polynomial of a matroid is relevant in a large number of problems involving numerical invariants of the matroid. We have introduced in [1, 3] a Tutte polynomial attached to the more general situation of a matroid perspective. Properties of this polynomial are particularly used in [2] to study Eulerian cycles of 4-valent regular graphs imbedded in surfaces.

In his thesis [5], Martin has introduced a polynomial in one variable related to the enumeration of Eulerian cycles for several classes of graphs (4-regular planar graphs, 4- and 6-regular undirected graphs, 4-regular directed graphs). In each case this polynomial is defined by means of inductive relations obtained from reduction operations: it is shown that all sequences of reductions yield the same polynomial. We have generalized in [4] this polynomial—the Martin polynomial—to all graphs by using a different (algebraic) technique.

The Tutte polynomial of a planar graph $G$ is related to the Martin

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polynomial of its medial graph \( H \) by the relation \( t(G; \zeta, \zeta) = m(H; \zeta) \). Our purpose in the present note is to derive new properties of the Martin polynomial of a 4-regular planar graph, which result in a combinatorial interpretation of \( t(G; 3, 3) \) and a divisibility property of \( t(G; 3, 3) \).

All definitions not given in this paper can be found in \([2–4]\).

2. A COMBINATORIAL INTERPRETATION OF \( t(G; 3, 3) \)

Let \( G \) be a plane graph (i.e., a planar graph imbedded in the plane). Let \( H \) be the medial graph of \( G \) and \( \mathbf{H} \) be an Eulerian orientation of \( H \): \( d_+^H(x) = d^-_H(x) = 2 \) for every vertex \( x \) of \( \mathbf{H} \). The vertices of \( \mathbf{H} \) can be put into two classes depending on the directions of the four incident edges (or half-edges in the case of loops) with respect to the embedding (see Fig. 1).

We say that a vertex corresponding to the situation in Fig. 1a is a saddle vertex of \( \mathbf{H} \).

**Theorem 2.1.** Let \( G \) be plane graph with medial graph \( H \). We have

\[
t(G; 3, 3) = \sum_{k \geq 0} 2^{k-1} e_k(H),
\]

where \( t(G; \zeta, \eta) \) is the Tutte polynomial of \( G \) and \( e_k(H) \) is the number of Eulerian orientations of \( H \) with exactly \( k \) saddle vertices.

**Proof.** Let \( f_k(H) \) be the number of Eulerian partitions without crossings of \( H \) \([2, 4]\) constituted of exactly \( k \) circuits (non-crossing Eulerian \( k \)-partitions for short). By \([2, \text{Proposition 4.1}]\) we have \( t(G; \zeta, \zeta) = \sum_{k \geq 0} f_{k+1}(H)(\zeta - 1)^k \). In particular

\[
t(G; 3, 3) = \sum_{k \geq 0} 2^k f_{k-1}(H).
\]

There is a natural \( 1 - 2^k \) correspondence between non-crossing Eulerian \( k \)-partitions of \( H \) and non-crossing directed Eulerian \( k \)-partitions of Eulerian orientations of \( H \). This correspondence is obtained by directing consistently all edges in each of the \( k \) circuits of a given non-crossing
Eulerian $k$-partition. Hence, denoting by $f(H)$ the number of non-crossing directed Eulerian partitions of an Eulerian orientation $H$ of $H$, we get

$$\sum_{k \geq 1} 2^k f_k(H) = \sum_{H} f(H),$$

where the right sum is over all Eulerian orientations $H$ of $H$. On the other hand we have

$$f(H) = 2^s(H),$$

where $s(H)$ denotes the number of saddle vertices of $H$, since there are two non-crossing directed transitions at a saddle vertex and only one otherwise.

### 3. A Divisibility Property

With notations of Section 2, we denote by $c(H)$ the number of crossing circuits of $H$.

**Proposition 3.1.** $e_0(H) = 2^{c(H)}$.

*Proof.* The $c(H)$ crossing circuits of $H$ partitions its edge-set. By directing consistently edges on each crossing circuit we get an Eulerian orientation $H$ of $H$ without saddle vertices. Furthermore each such orientation $H$ is obtained exactly once.

**Proposition 3.2.** For all $k \geq 1$, the number $2^{k - c(H) - 1} e_k(H)$ is an integer.

*Proof.* Let $\gamma_1, \gamma_2, \ldots, \gamma_c, c = c(H)$, be the crossing circuits of $H$. Let $H$ be an Eulerian orientation of $H$ with $k$ saddle vertices $x_1, x_2, \ldots, x_k$. Each $x_j$ is on 1 or 2 $\gamma_i$'s. We form an auxiliary graph $L$ representing the adjacencies between the $\gamma_i$'s corresponding to the $x_j$'s: let $\gamma_{i_1}, \gamma_{i_2}$ be joined by as many edges in $L$ as there are $x_j$'s belonging to both $\gamma_{i_1}$ and $\gamma_{i_2}$. We note that each $\gamma_i$ contains necessarily an even number of $x_j$'s since the edge directions on $\gamma_i$ are reversed at each $x_j$ and are consistent otherwise. Thus $L$ is a graph with even degrees containing $c(H)$ vertices and $k$ edges. It follows that $L$ has at least $c(H) - k + 1$ connected components. We observe that by reversing directions of all edges of $H$ on $\gamma_i$'s corresponding to a union of connected components of $L$ we get an Eulerian orientation of $H$ having $x_1, x_2, \ldots, x_k$ as saddle vertices. Hence the number of Eulerian orientations of $H$ with $x_1, x_2, \ldots, x_k$ as saddle vertices is an integer multiple of $2^p$, where $p$ is the number of connected components of $L$. Since $p \geq c(H) - k + 1$, this number is also an integer multiple of $2^{c(H) - k + 1}$. Proposition 3.2 follows.
As an immediate consequence of Propositions 3.1 and 3.2 and Theorem 2.1 we get

**Theorem 3.3.** Let $G$ be a plane graph with medial graph $H$. We have $t(G; 3, 3) = 2^{c(H)-1}q$, where $q$ is an odd integer.

**Remark 3.4.** We have $t(G; 3, 3) \equiv t(G; 1, 1) \pmod{2}$; hence $t(G; 3, 3)$ has the parity of the number of spanning forests of $G$. Therefore by Theorem 3.3, $c(H) = 1$ if and only if the number of spanning forests of $G$ is odd, a result due to Shank [7].

**Remark 3.5.** By a result of Martin–Rosenstiehl–Read [5, 6, 8] we have $t(G; -1, -1) = (-1)^{\nu(G)}(-2)^{c(H)-1}$. Hence Theorem 3.3 can be equivalently restated: (Theorem 3.3') If $G$ is a planar graph $|t(G; -1, -1)|$ divides $t(G, 3, 3)$ and the quotient is an odd integer.

### 4. Conjectures

Theorem 3.3' suggests the following generalization to graphs and binary matroids.

**Conjecture 4.1.** Let $M$ be a binary matroid; then $|t(M; -1, -1)|$ divides $t(M; 3, 3)$ and the quotient is an odd integer.

Note that by a result of Rosenstiehl–Read [7] we have $t(M; -1, -1) = (-1)^{\nu(M)}(-2)^d$, where $d = \dim V_2(M) \cap V_2^\perp(M)$ with $V_2(M)$ the vector space over $GF(2)$ generated by the circuits of $M$. Based on the evidence of several examples we state more generally

**Conjecture 4.2.** Let $M$ be a binary matroid. For all integer $p \in \mathbb{Z}$ the number $2^d$ (see above) divides $t(M; 4p-1, 4p-1)$ and the quotient is an odd integer.

**Conjecture 4.3.** Let $M$ be a binary matroid. For $k = 0, 1, \ldots, d$ the $k$th derivative of $t(M; \zeta, \zeta)$ evaluated at $\zeta = -1$ is an integer multiple of $2^{d-k}$.

**Proposition 4.4.** Conjecture 4.3 implies Conjecture 4.2.

**Proof.** Set $u(\zeta) = t(M; \zeta, \zeta)$. We have

$$u(\zeta) = \sum_{k \geq 0} \frac{u^{(k)}(-1)}{k!} (\zeta + 1)^k.$$

In particular

$$u(4p-1) = \sum_{k \geq 2} 2^{2k} \frac{u^{(k)}(-1)}{k!} p^k.$$
It can easily be established that the exponent of 2 in \( k! \) is \( \leq k - 1 \) for \( k \geq 1 \) (with equality if \( k \) is a power of 2). Hence if \( u^{(k)}(-1) \) is an integer multiple of \( 2^{d-k} \), we obtain that \( 2^{2k} u^{(k)}(-1)/k! \) is an integer multiple of \( 2^{2k + (d-k) - (k-1)} = 2^d + 1 \) for \( k \geq 1 \). Since for \( k = 0 \) we have \( |u(-1)| = 2^d \), Proposition 4.4 follows.

5. Generalizations to Surfaces

We have generalized in [2, Proposition 4.11] the relation
\[ t(G; \zeta, \xi) = m(H; \zeta) \]
for graphs imbedded in the projective plane and in the torus. Denoting by \( G^* \) the dual graph of \( G \) we have (projective plane)
\[ t(G, G^*; \zeta, \xi, 1) = m(H; \zeta) \]
(torus)
\[ t_2(G, G^*; \zeta, \xi) + (\zeta + 1) t_1(G, G^*; \zeta, \xi) + t_0(G, G^*; \zeta, \xi) = m(H; \zeta), \]
where \( t_i(G, G^*), \ i = 0, 1, 2, \) are defined by \( t(G, G^*; \zeta, \eta, \xi) = \sum_{i=0,1,2} \xi^i t_i(G, G^*; \zeta, \eta) \).

The proofs of Theorems 2.1 and 3.3 clearly yield analogous results in these situations. Also conjectures of Section 4 can be extended to binary matroid perspective \( M \to M' \) with \( r(M) - r(M') = 1 \) or 2.

Actually the proofs of Theorems 2.1 and 3.3 provide properties of the Martin polynomial of a 4-regular graph imbedded in any surface [2, 4]. We get

**Theorem 5.1.** Let \( H \) be a 4-regular graph imbedded in a surface \( S \). We have
\[ m(H; 3) = \sum_{k \geq 0} 2^{k-1} e_k(H), \]
where \( m(H; \zeta) \) is the Martin polynomial of \( H \) imbedded in \( S \), and \( e_k(H) \) is the number of Eulerian orientations of \( H \) with exactly \( k \) saddle vertices.

Furthermore \( m(H; 3) = 2^{c(H)} q \) where \( c(H) \) is the number of crossing circuits of \( H \) on \( S \) and \( q \) is an odd integer.

Note that \( m(H; -1) = (-1)^{\nu(H)} (-2)^{c(H)-1} \) [2, Proposition 5.2].

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Since the completion of the present work (1982) Conjecture 4.1 has been proven by A. Bouchet in the more general context of isotropic spaces. Interesting developments have been recently established by F. Jaeger.
REFERENCES


