On real Paley–Wiener theorems for certain integral transforms

Nils Byrial Andersen

School of Mathematics, University of New South Wales, Sydney NSW 2052, Australia

Received 11 November 2002
Submitted by B. Bongiorno

Abstract

We prove real Paley–Wiener theorems for the (inverse) Jacobi transform, characterising the space of $L^2$-functions whose image under the Jacobi transform are (smooth) functions with compact support.

Keywords: Paley–Wiener theorems; Jacobi transform; Chébli–Trimèche transform

1. Introduction

Recently there has been a great interest in real Paley–Wiener theorems for certain integral transform, see [7] for an overview, references and details, using real analysis techniques to give a description of those $L^2$-functions whose image are compactly supported functions.

The set-up is as follows: Let $\Delta$ be a differential operator with a continuous spectrum $\Omega_1$, let $\varphi_\lambda(x)$ be an eigenfunction of $\Delta$ with eigenvalue $(-\lambda)$, and suppose that the (inverse) integral transform $T^{-1}: L^2(\Omega_1, d\mu_1) \to L^2(\Omega_2, d\mu_2)$ defined by

$$f(x) = T^{-1}g(x) := \int_{\Omega_1} g(\lambda)\varphi_\lambda(x) d\mu_1(\lambda),$$

E-mail address: byrial@maths.unsw.edu.au.

1 Supported by a research grant from the Australian Research Council.

0022-247X/ - see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/S0022-247X(03)00585-7
is an isometry. Assume furthermore \((*)\) that \(\lambda^n g(\lambda) \in L^2(\Omega_1, d\mu_1)\); then
\[
\Delta^n f(x) = \int_{\Omega_1} (-\lambda)^n g(\lambda) \varphi_\lambda(x) d\mu_1(\lambda),
\]
and thus
\[
\int_{\Omega_2} |\Delta^n f(x)|^2 d\mu_2(x) = \int_{\Omega_1} |\lambda^n g(\lambda)|^2 d\mu_1(\lambda),
\]
which after a short computation yields
\[
\lim_{n \to \infty} \|\Delta^n f\|_{L^2}^{1/n} = \sup_{|\lambda| \in \text{supp} g} |\lambda|,
\]
where the support of \(g\) is the smallest closed set outside which the function vanishes almost everywhere.

It is obvious that (1) can be used to study the transform \(T^{-1}\) of functions \(g\) with compact support, whose image can be seen to be the set of functions \(f\) satisfying
\[
\lim_{n \to \infty} \|\Delta^n f\|_{L^2}^{1/n} < \infty,
\]
and some extra conditions arising from assumption \((*)\), again see [7] for details. However, these extra conditions lead to proofs and statements of a somewhat technical nature.

Now furthermore assume that we have a Parseval formula
\[
\int_{\Omega_2} f_1(x) f_2(x) d\mu_2(x) = \int_{\Omega_1} Tf_1(\lambda) T f_2(\lambda) d\mu_1(\lambda),
\]
for the transform \(T = (T^{-1})^{-1}\),
\[
T f(x) := \int_{\Omega_2} f(x) \varphi_\lambda(x) d\mu_2(x),
\]
and that \(\Delta\) is a self-adjoint differential operator with respect to the measure \(d\mu_2\). It then follows that \(T(\Delta f)(\lambda) = -\lambda T f(\lambda)\) for all smooth functions \(f\) such that \(\Delta f \in L^2(\Omega_2, d\mu_2)\), and we can show that the image of the compactly supported functions under \(T^{-1}\) are characterised as the \(L^2\)-functions satisfying (2) and the condition \(\Delta^n f \in L^2(\Omega_2, d\mu_2)\) for all \(n \in \mathbb{N} \cup \{0\}\).

In this paper, we consider the above approach for the inverse Jacobi transform \(I_{\alpha,\beta} := (F_{\alpha,\beta})^{-1}\). We also consider the functions whose image under the Jacobi transform are smooth functions with compact support. Besides the conditions already mentioned, the functions \(f\) also need to satisfy an extra polynomial decay condition, to ensure smoothness of the Jacobi transform.

Our approach can be used for many other transforms (and not only on \(\mathbb{R}\)) and we conclude the paper with some examples: the classical Fourier transform and the Chébli–Trimèche transform. Note that the latter transform is a generalisation of the Jacobi transform. Note also that our approach can be used in many cases where one already knows a version of the classical Paley–Wiener theorem for some transform and one would be interested in a (real) Paley–Wiener theorem for the inverse transform, see, e.g., [1].
2. Jacobi functions and the Jacobi transform

Let $\alpha, \beta, \lambda \in \mathbb{C}$ and $0 < t < \infty$. We consider the differential equation

$$\Delta^{\alpha,\beta} u(t) := (J^{\alpha,\beta}(t))^{-1} \frac{d}{dt} \left( J^{\alpha,\beta}(t) \frac{d u(t)}{dt} \right) = -(\lambda^2 + \rho^2) u(t),$$

(3)

where $\rho = \alpha + \beta + 1$ and $J^{\alpha,\beta}(t) = (2 \sinh(t))^{2\alpha+1}(2 \cosh(t))^{2\beta+1}$. Using the substitution $x = -\sinh^2(t)$, we can rewrite (3) as a hypergeometric differential equation with parameters $(\rho + i\lambda)/2, (\rho - i\lambda)/2$ and $\alpha + 1$ (see [3, 2.1.1]). Let $2F_1$ denote the Gauß hypergeometric function. The Jacobi function (of order $(\alpha, \beta)$),

$$\phi^{\alpha,\beta}_{\lambda}(t) := 2F_1 \left( \frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda), \alpha + 1; \cosh^{-2}(t) \right),$$

is for $\alpha \not\in \mathbb{N}$ the unique solution to (3) satisfying $\phi^{\alpha,\beta}_{\lambda}(0) = 1$ and $(d/dt)|_{t=0}\phi^{\alpha,\beta}_{\lambda}(t) = 0$.

We remark that $\phi^{\alpha,\beta}_{\lambda}(t)$ is even in both variables $\lambda$ and $t$. The Jacobi functions satisfy the following growth estimates.

Lemma 1. There exists for each $\alpha, \beta \in \mathbb{C}$, a positive constant $C$ such that

$$\left| \Gamma(\alpha + 1) \phi^{\alpha,\beta}_{\lambda}(t) \right| \leq C \left( 1 + |\lambda| \right)^k (1 + t) e^{(|\Im\lambda| - \Re\rho)t}$$

for all $\lambda \in \mathbb{C}$ and all $t \geq 0$, where $k = 0$ if $\Re\alpha > -1/2$ and $k = [1/2 - \Re\alpha]$ if $\Re\alpha \leq -1/2$.

Proof. See [5, Lemma 2.3]. □

Here $[\cdot]$ denotes integer part. We note that $\Gamma(\alpha + 1) \phi^{\alpha,\beta}_{\lambda}(t)$ is an entire function in the variables $\alpha, \beta$ and $\lambda \in \mathbb{C}$ (also for $\alpha \in \mathbb{N}$). The Jacobi transform (of order $(\alpha, \beta)$) is defined by

$$\mathcal{F}^{\alpha,\beta} f(\lambda) := \int_{\mathbb{R}} f(t) \phi^{\alpha,\beta}_{\lambda}(t) J^{\alpha,\beta}(t) dt$$

for all even functions $f$ and all complex numbers $\lambda$ for which the right-hand side is well defined. The (classical) Paley–Wiener theorem for the Jacobi transform [5, Theorem 3.4] states that the normalised application $f \mapsto \Gamma(\alpha + 1) \mathcal{F}^{\alpha,\beta} f$ is a bijection from $\mathcal{C}_c^\infty(\mathbb{R})^{\text{even}}$ onto $\mathcal{H}(\mathbb{C})^{\text{even}}$, the space of even entire rapidly decreasing functions of exponential type, for all $\alpha, \beta \in \mathbb{C}$.

We also note that

$$\mathcal{F}^{\alpha,\beta}(\Delta^{\alpha,\beta} f)(\lambda) = - (\lambda^2 + \rho^2) \mathcal{F}^{\alpha,\beta} f(\lambda) \quad (\lambda \in \mathbb{C}),$$

for all $f \in \mathcal{C}_c^\infty(\mathbb{R})^{\text{even}}$, by self-adjointness of $\Delta^{\alpha,\beta}$ with respect to the measure $J^{\alpha,\beta}(t) dt$.

The Jacobi functions of the second kind,

$$\phi^{\alpha,\beta}_{\lambda}(t) := (2 \cosh(t))^{i\lambda-\rho} 2F_1 \left( \frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho - \beta + 1 - i\lambda), 1 - i\lambda; \cosh^{-2}(t) \right)$$

defines for \( \lambda \notin -i\mathbb{N} \) another solution of (3), characterised by the property that \( \phi_{\lambda}^{\alpha,\beta}(t) \sim e^{i(\lambda - \rho)t} \) for \( t \to \infty \). Define the meromorphic Jacobi \( c \)-functions as

\[
\Phi_{\lambda}^{\alpha,\beta} := \frac{\Gamma(\alpha + 1) \Gamma(i\lambda)}{\Gamma((i\lambda + \rho)/2) \Gamma((i\lambda + \alpha - \beta + 1)/2)},
\]

then

\[
\phi_{\lambda}^{\alpha,\beta} = \Phi_{\lambda}^{\alpha,\beta} \phi_{\lambda}^{\alpha,\beta} + \Phi_{-\lambda}^{\alpha,\beta} \phi_{-\lambda}^{\alpha,\beta},
\]

as a meromorphic identity, see [6, (2.15–18)].

**Lemma 2.** Fix \( \alpha, \beta \in \mathbb{C} \) and let \( 0 \leq \eta < 1/2 \). There exists for \( \Im \lambda \geq -\eta \), a converging series such that

\[
\phi_{\lambda}^{\alpha,\beta}(t) = e^{i(\lambda - \rho)t} \sum_{n=0}^{\infty} \Gamma_{n}^{\alpha,\beta}(\lambda)e^{-nt} \quad (t > 0),
\]

where the \( \Gamma_{n}^{\alpha,\beta} \)'s are rational functions with \( \Gamma_{0}^{\alpha,\beta} \equiv 1 \). There furthermore exist positive constants \( C \) and \( d \) (depending on \( \alpha \) and \( \beta \)) such that

\[
\left| \Gamma_{n}^{\alpha,\beta}(\lambda) \right| < C(1 + n)^{d}
\]

for \( \Im \lambda \geq -\eta \) and all \( n \in \mathbb{N} \).

**Proof.** Follows by extending [4, Lemma 7] to complex \( \alpha, \beta \). □

**Lemma 3.** Let \( 0 \leq \eta < 1/2 \). Let \( r > 0 \). There exists a positive constant \( C \), such that

\[
\left| \Phi_{-\lambda}^{\alpha,\beta} \right| \leq C \left( 1 + |\lambda| \right)^{\alpha+1/2}
\]

for \( \Im \lambda \geq -\eta \) and \( \lambda \) at a distance larger than \( r \) from the poles of \( \Phi_{-\lambda}^{\alpha,\beta} \).

**Proof.** Follows from the definition of the \( c \)-functions. □

Assume for the remainder of this paper that \( \alpha, \beta \in \mathbb{R}, \alpha > -1/2, \) and that \( |\beta| < \alpha + 1 \). The inversion formula for the Jacobi transform can then be written as

\[
f(t) = \frac{1}{4\pi} \int_{\mathbb{R}} F_{\alpha,\beta} f(\lambda) \phi_{\lambda}^{\alpha,\beta}(t) \left| \Phi_{\lambda}^{\alpha,\beta}(\lambda) \right|^{-2} d\lambda \quad (t \in \mathbb{R})
\]

for \( f \in C_{\mathbb{R}}^{\infty}(\mathbb{R})^{\text{even}} \), see [6, Theorem 2.3].

Parseval’s formula for the Jacobi transform can be written as

\[
\langle f, f_{1} \rangle := \int_{\mathbb{R}^{+}} f(t) f_{1}(t) J_{\alpha,\beta}(t) \, dt
\]

\[
= \frac{1}{4\pi} \int_{\mathbb{R}} F_{\alpha,\beta} f(\lambda) \Phi_{\lambda}^{\alpha,\beta}(\lambda) \left| \Phi_{\lambda}^{\alpha,\beta}(\lambda) \right|^{-2} d\lambda =: \langle F_{\alpha,\beta} f, F_{\alpha,\beta} f_{1} \rangle
\]
for $f, f_1 \in C_c^\infty(\mathbb{R})$ even, see [6, Theorem 2.3], which for $f = f_1$ becomes $\|f\|_2 = \|\mathcal{F}f\|_2$.

A density argument then shows that the Jacobi transform extends to an isometry from $L^2(\mathbb{R}^+, J^{\alpha,\beta}(t) dt)^{\text{even}}$ onto $L^2(\mathbb{R}, (4\pi)^{-1}|c^{\alpha,\beta}(\lambda)|^{-2} d\lambda)^{\text{even}}$.

For sake of brevity, we use the notation $L^p(J^{\alpha,\beta}) := L^p(\mathbb{R}^+, J^{\alpha,\beta}(t) dt)^{\text{even}}$ and $L^p(c^{\alpha,\beta}) := L^p(\mathbb{R}, (4\pi)^{-1}|c^{\alpha,\beta}(\lambda)|^{-2} d\lambda)^{\text{even}}$ for $p = 1, 2$ in the following.

Let $f \in C^\infty(\mathbb{R})$ even such that $(\Delta^{\alpha,\beta} f)^n \in L^2(J^{\alpha,\beta})$ for all $n \in \mathbb{N} \cup \{0\}$, and let $f_1 \in C_c^\infty(\mathbb{R})$ even. The self-adjointness of $\Delta^{\alpha,\beta}$ yields

$$\langle \mathcal{F}^{\alpha,\beta}(\Delta^{\alpha,\beta} f), \mathcal{F}^{\alpha,\beta} f_1 \rangle = \langle \Delta^{\alpha,\beta} f, f_1 \rangle = \langle f_1, \Delta^{\alpha,\beta} f \rangle = \langle \mathcal{F}^{\alpha,\beta} f, \mathcal{F}^{\alpha,\beta}(\Delta^{\alpha,\beta} f_1) \rangle = \langle \mathcal{F}^{\alpha,\beta} f, -(\lambda^2 + \rho^2) \mathcal{F}^{\alpha,\beta} f_1 \rangle = \langle -(\lambda^2 + \rho^2) \mathcal{F}^{\alpha,\beta} f, \mathcal{F} f_1 \rangle$$

for $\lambda \in \mathbb{R}$, and we conclude that $\mathcal{F}^{\alpha,\beta}((\Delta^{\alpha,\beta} + \rho^2) f)(\lambda) = -\lambda^2 \mathcal{F} f(\lambda)$ a.e., by a density argument, whence $\mathcal{F}^{\alpha,\beta}((\Delta^{\alpha,\beta} + \rho^2)^n f)(\lambda) = (-1)^n \lambda^{2n} \mathcal{F} f(\lambda)$ a.e. (**) and

$$\int_{\mathbb{R}_+} |(\Delta^{\alpha,\beta} + \rho^2)^n f(t)|^2 J^{\alpha,\beta}(t) dt = \frac{1}{4\pi} \int_\mathbb{R} |\lambda|^{4n} |\mathcal{F}^{\alpha,\beta} f(\lambda)|^2 |c^{\alpha,\beta}(\lambda)|^{-2} d\lambda$$

for all $n \in \mathbb{N} \cup \{0\}$.

**Remark 4.** The two most important results in this section, for the purpose of this paper, are the intertwining property (**) and the extended Parseval’s formula (7). We note that to prove these identities, we only needed Parseval’s formula, density of $C_c^\infty(\mathbb{R})$ even and self-adjointness of $\Delta^{\alpha,\beta}$, not the full classical Paley–Wiener theorem.

### 3. A characterisation of the support of a function

We define the support $\text{supp} g$ of $g \in L^2(c^{\alpha,\beta})$ to be the smallest closed set, outside which the function $g$ vanishes almost everywhere, and $R_g := \sup_{\lambda \in \text{supp} g} |\lambda|$ to be the radius of the support of $g$; $R_g = R < \infty$ if, and only if, $g$ has support in the closed interval $[-R, R]$.

**Lemma 5.** Let $g \in L^2(c^{\alpha,\beta})$ such that $|\lambda|^{2n} g(\lambda) \in L^2(c^{\alpha,\beta})$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$R_g = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}} |\lambda|^{4n} |g(\lambda)|^2 |c^{\alpha,\beta}(\lambda)|^{-2} d\lambda \right\}^{1/4n}.$$

**Proof.** See also [7, Lemma 2]. Assume $g$ has compact support with $R_g > 0$. Then

$$\limsup_{n \to \infty} \left\{ \int_{\mathbb{R}} |\lambda|^{4n} |g(\lambda)|^2 |c^{\alpha,\beta}(\lambda)|^{-2} d\lambda \right\}^{1/4n} \leq R_g \limsup_{n \to \infty} \left\{ \int_{|\lambda| \leq R_g} |g(\lambda)|^2 |c^{\alpha,\beta}(\lambda)|^{-2} d\lambda \right\}^{1/4n} = R_g.$$
On the other hand,

$$\int_{R_g - \varepsilon \leq |\lambda| \leq R_g} |g(\lambda)|^2 |c_{\alpha,\beta}(\lambda)|^{-2} \, d\lambda > 0$$

for any $\varepsilon > 0$, hence

$$\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}} |\lambda|^{4n} |g(\lambda)|^2 |c_{\alpha,\beta}(\lambda)|^{-2} \, d\lambda \right\}^{1/4n} \geq \liminf_{n \to \infty} \left\{ \int_{R_g - \varepsilon \leq |\lambda| \leq R_g} |\lambda|^{4n} |g(\lambda)|^2 |c_{\alpha,\beta}(\lambda)|^{-2} \, d\lambda \right\}^{1/4n}$$

$$\geq (R_g - \varepsilon) \liminf_{n \to \infty} \left\{ \int_{|\lambda| \geq N} |g(\lambda)|^2 |c_{\alpha,\beta}(\lambda)|^{-2} \, d\lambda \right\}^{1/4n} = R_g - \varepsilon,$$

and thus

$$\lim_{n \to \infty} \left\{ \int_{\mathbb{R}_n} |\lambda|^{4n} |g(\lambda)|^2 |c_{\alpha,\beta}(\lambda)|^{-2} \, d\lambda \right\}^{1/4n} = R_g.$$

Now assume that $g$ has unbounded support. Then

$$\int_{|\lambda| \geq N} |g(\lambda)|^2 |c_{\alpha,\beta}(\lambda)|^{-2} \, d\lambda > 0$$

for any $N > 0$, so

$$\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}} |\lambda|^{4n} |g(\lambda)|^2 |c_{\alpha,\beta}(\lambda)|^{-2} \, d\lambda \right\}^{1/4n} \geq \liminf_{n \to \infty} \left\{ \int_{|\lambda| \geq N} |g(\lambda)|^2 |c_{\alpha,\beta}(\lambda)|^{-2} \, d\lambda \right\}^{1/4n}$$

$$\geq N \liminf_{n \to \infty} \left\{ \int_{|\lambda| \geq N} |g(\lambda)|^2 |c_{\alpha,\beta}(\lambda)|^{-2} \, d\lambda \right\}^{1/4n} = N$$

for arbitrary $N > 0$, and we conclude that

$$\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}} |\lambda|^{4n} |g(\lambda)|^2 |c_{\alpha,\beta}(\lambda)|^{-2} \, d\lambda \right\}^{1/4n} = \infty. \quad \square$$
4. The inverse Jacobi transform

We define the inverse Jacobi transform (of order \((\alpha, \beta)\)) of an even function \(g\) on \(\mathbb{R}\) via (6),

\[
I^{\alpha, \beta}g(t) := \frac{1}{4\pi} \int_{\mathbb{R}} g(\lambda) \phi_{\alpha, \beta}^{\alpha, \beta}(t) \left| \epsilon_{\alpha, \beta}(\lambda) \right|^{-2} d\lambda
\]  

(8)

for all \(t \in \mathbb{R}\) for which the integral exists.

Definition 6. The \(L^2\)-Paley–Wiener space \(PW^2(\mathbb{R})^{\alpha, \beta}\) is defined as the space of all even functions \(f \in C^\infty(\mathbb{R})\) even such that

1. \((\Delta^{\alpha, \beta})^n f \in L^2(J^{\alpha, \beta})\) for all \(n \in \mathbb{N} \cup \{0\}\),
2. \(R_f^\Delta := \liminf_{n \to \infty} \| (\Delta^{\alpha, \beta} + \rho^2)^n f \|_2^{1/2n} < \infty\),

and \(PW^2_R(\mathbb{R})^{\alpha, \beta} := \{ f \in PW^2(\mathbb{R})^{\alpha, \beta} | R_f^\Delta = R \} \) for \(R \geq 0\).

Let also \(L_c(\mathbb{R})^{\text{even}} \subset L^2(\mathbb{R})^{\alpha, \beta}\) denote the subspace of even \(L^2\)-functions with compact support and let \(L_R(\mathbb{R})^{\text{even}} := \{ g \in L_c(\mathbb{R})^{\text{even}} | R_g = R \} \).

The real \(L^2\)-Paley–Wiener theorem for the inverse Jacobi transform can be formulated as follows.

Theorem 7. The inverse Jacobi transform \(I^{\alpha, \beta}\) is a bijection of \(L_c(\mathbb{R})^{\text{even}}\) onto \(PW^2(\mathbb{R})^{\alpha, \beta}\), mapping \(L_R(\mathbb{R})^{\text{even}}\) onto \(PW^2_R(\mathbb{R})^{\alpha, \beta}\).

Proof. Let \(g \in L_R(\mathbb{R})^{\text{even}}\). Then \(|\lambda|^{4n} g(\lambda) \in L^1(\mathbb{R})^{\alpha, \beta} \cap L^2(\mathbb{R})^{\alpha, \beta}\) for all \(n \in \mathbb{N} \cup \{0\}\), which implies that \(I^{\alpha, \beta}g \in C^\infty(\mathbb{R})\). We also have \((\Delta^{\alpha, \beta} + \rho^2)^n (I^{\alpha, \beta}g) = I^{\alpha, \beta}((-1)^n \lambda^{2n} g) \in L^2(J^{\alpha, \beta})\) for all \(n \in \mathbb{N} \cup \{0\}\), by the definition of the inverse Jacobi transform (8). Identity (7) thus yields

\[
\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}} \left| (\Delta^{\alpha, \beta} + \rho^2)^n (I^{\alpha, \beta}g)(t) \right|^2 J^{\alpha, \beta}(t) dt \right\}^{1/4n} = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}} |\lambda|^{4n} |g(\lambda)|^2 \left| \epsilon_{\alpha, \beta}(\lambda) \right|^{-2} d\lambda \right\}^{1/4n} = R,
\]

whence \(I^{\alpha, \beta}g \in PW^2_R(\mathbb{R})^{\alpha, \beta}\).

Let now \(f \in PW^2_R(\mathbb{R})^{\alpha, \beta}\). It then follows that \(\mathcal{F}^{\alpha, \beta}((\Delta^{\alpha, \beta} + \rho^2)^n f)(\lambda) = (-1)^n \lambda^{2n} \times \mathcal{F}^{\alpha, \beta}f(\lambda) \in L^2(\mathbb{R})^{\alpha, \beta}\) for all \(n \in \mathbb{N}\), and another application of (7) shows that

\[
\lim_{n \to \infty} \left\{ \int_{\mathbb{R}} |\lambda|^{4n} |\mathcal{F}^{\alpha, \beta}f(\lambda)|^2 \left| \epsilon_{\alpha, \beta}(\lambda) \right|^{-2} d\lambda \right\}^{1/4n} = R.
\]
\( = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}} \left| (\Delta^{\alpha,\beta} + \rho^2)^n f(t) \right|^2 J^{\alpha,\beta}(t) \, dt \right\}^{1/4n} = R, \)

and we conclude that \( F^{\alpha,\beta} f \) has compact support with \( R^{\alpha,\beta} f = R. \) \( \square \)

We note that \( \lim \inf_{n \to \infty} \| (\Delta^{\alpha,\beta} + \rho^2)^n f \|_2^{1/2n} = \lim_{n \to \infty} \| (\Delta^{\alpha,\beta} + \rho^2)^n f \|_2^{1/2n} \) for any \( g \in PW_2^{2}(\mathbb{R})^{\alpha,\beta}. \) It furthermore follows that

**Corollary 8.** Let \( g \) be an even function on \( \mathbb{R}. \) Then \( |\lambda|^n g(\lambda) \in L^2(\mathbb{R}^{\alpha,\beta}) \) for all \( n \in \mathbb{N} \cup \{0\} \) if and only if \( (\Delta^{\alpha,\beta})^n \mathbb{F}^{\alpha,\beta} g \in L^2(J^{\alpha,\beta}) \) for all \( n \in \mathbb{N} \cup \{0\}. \)

**Corollary 9.** Let \( f \in C^\infty(\mathbb{R})^{\alpha,\beta} \) even such that \( (\Delta^{\alpha,\beta})^n f \in L^2(J^{\alpha,\beta}) \) for all \( n \in \mathbb{N} \cup \{0\}. \) Then \( \lim_{n \to \infty} \| (\Delta^{\alpha,\beta})^n f \|_2^{1/2n} < \infty \) if and only if \( \lim_{n \to \infty} \| (\Delta^{\alpha,\beta} + \rho^2)^n f \|_2^{1/2n} < \infty. \) Furthermore, \( \lim_{n \to \infty} \| (\Delta^{\alpha,\beta})^n f \|_2^{1/2n} = (R^2 + \rho^2)^{1/2} \) for \( 0 \neq f \in PW_2^{2}(\mathbb{R})^{\alpha,\beta}. \)

**Proof.** Let \( 0 \neq f \in PW_2^{2}(\mathbb{R})^{\alpha,\beta}; \) then \( F^{\alpha,\beta} f \in L_R(\mathbb{R})^{\alpha,\beta}. \) Rearranging (7) and an easy adaption of the proof of Lemma 5 shows that

\[
\lim_{n \to \infty} \left\{ \int_{\mathbb{R}} \left| (\Delta^{\alpha,\beta})^n f \right|^2 \left| \mathbb{F}^{\alpha,\beta} f(\lambda) \right|^2 \left| \mathbb{J}^{\alpha,\beta}(\lambda) \right|^{-2} \, d\lambda \right\}^{1/4n} = (R^2 + \rho^2)^{1/2}.
\]

Assume that \( \lim_{n \to \infty} \| (\Delta^{\alpha,\beta})^n f \|_2^{1/2n} < \infty. \) Then \( (-1)^n (\lambda^2 + \rho^2)^n \mathbb{F}^{\alpha,\beta} f(\lambda) = \mathbb{F}^{\alpha,\beta}((\Delta^{\alpha,\beta})^n f)(\lambda) \in L^2(\mathbb{R}^{\alpha,\beta}) \) for all \( n \in \mathbb{N}, \) and

\[
\lim_{n \to \infty} \left\{ \int_{\mathbb{R}} \left| \lambda|^n | \mathbb{F}^{\alpha,\beta} f(\lambda) |^2 \right| \left| \mathbb{J}^{\alpha,\beta}(\lambda) \right|^{-2} \, d\lambda \right\}^{1/4n} \leq \lim_{n \to \infty} \left\{ \int_{\mathbb{R}} (\lambda^2 + \rho^2)^n | \mathbb{F}^{\alpha,\beta} f(\lambda) |^2 \left| \mathbb{J}^{\alpha,\beta}(\lambda) \right|^{-2} \, d\lambda \right\}^{1/4n} = \lim_{n \to \infty} \| (\Delta^{\alpha,\beta})^n f \|_2^{1/2n} < \infty,
\]

that is, \( \mathbb{F}^{\alpha,\beta} f \) has compact support. \( \square \)

**Definition 10.** The Paley–Wiener space \( PW(\mathbb{R})^{\alpha,\beta} \) is defined as the space of all even functions \( f \in C^\infty(\mathbb{R})^{\alpha,\beta} \) such that

1. \( (1 + |t|^m) (\Delta^{\alpha,\beta})^n f \in L^2(J^{\alpha,\beta}) \) for all \( m, n \in \mathbb{N} \cup \{0\}, \)
2. \( R_f^2 = \lim_{n \to \infty} \| (\Delta^{\alpha,\beta} + \rho^2)^n f \|_2^{1/2n} < \infty, \)

and \( PW_R(\mathbb{R})^{\alpha,\beta} := \{ f \in PW(\mathbb{R})^{\alpha,\beta} \mid R_f^\Delta = R \} \) for \( R \geq 0. \)
We notice that the only difference between \( PW_2(\mathbb{R})^{\alpha,\beta} \) and \( PW(\mathbb{R})^{\alpha,\beta} \) is the extra requirement of polynomial decay, to help ensure that \( F^{\alpha,\beta} f \in C^\infty(\mathbb{R}) \).

Let also \( C^\infty_c(\mathbb{R})^{\text{even}} \) := \{ \varphi \in C^\infty_c(\mathbb{R}) | R \varphi = R \} \).

The real Paley–Wiener theorem for the inverse Jacobi transform is the following

**Theorem 11.** The inverse Jacobi transform \( T^\alpha,\beta \) is a bijection of \( C^\infty_c(\mathbb{R})^{\text{even}} \) onto \( PW(\mathbb{R})^{\alpha,\beta} \), mapping \( C^\infty_c(\mathbb{R})^{\text{even}} \) onto \( PW(\mathbb{R})^{\alpha,\beta} \).

**Proof.** Let \( g \in C^\infty_c(\mathbb{R})^{\text{even}} \). Then \( T^\alpha,\beta g \in PW_2(\mathbb{R})^{\alpha,\beta} \) by Theorem 7. We recall that \( (\Delta^{\alpha,\beta})^n(T^\alpha,\beta g) = T^\alpha,\beta((-1)^n(\lambda^2 + \rho^2)^n)g \) belongs to \( L^2(\mathbb{R})^{\alpha,\beta} \) for all \( n \in \mathbb{N} \cup \{ 0 \} \), so it only remains to show that \( T^\alpha,\beta g \) satisfies the polynomial decay condition for any \( g \in C^\infty_c(\mathbb{R})^{\text{even}} \).

Let \( t > 0 \). Using (5) we obtain

\[
T^\alpha,\beta g(t) = \frac{1}{4\pi} \int_\mathbb{R} \left\{ e^{\alpha,\beta}(u\lambda) \phi^\alpha,\beta_{\alpha,\beta}(t)g(\lambda)e^{\alpha,\beta}(-\lambda)^{-1} \right\} d\lambda.
\]

The interchange of integral and sum is allowed by the estimates of Lemma 2. Let \( G_n(\lambda) := \Gamma_n^{\alpha,\beta}(\lambda) e^{\alpha,\beta}(-\lambda)^{-1} g(\lambda) \). Then \( G_n \in C^\infty_c(\mathbb{R}) \) and the integral

\[
F^{-1} G_n(t) := \frac{1}{2\pi} \int_\mathbb{R} G_n(\lambda)e^{ij\lambda} d\lambda
\]

is the classical inverse Fourier transform of \( G_n \). We note that \( |(d^j/d\lambda^j) \Gamma_n^{\alpha,\beta}(\lambda)| \leq C(1 + n)^d \) for all \( \lambda \in \mathbb{R}, n \in \mathbb{N} \cup \{ 0 \} \) and \( j \in \mathbb{N} \cup \{ 0 \} \), by Lemma 2 and Cauchy’s theorem, yielding uniform estimates of the derivatives of \( G_n(\lambda) \). The equality

\[
t^j T^\alpha,\beta g(t) = \frac{1}{2\pi} e^{-\rho t} \sum_{n=0}^\infty e^{-nt} t^j F^{-1}\left\{ (-i)^j \frac{d^j}{d\lambda^j} G_n \right\}(t),
\]

for \( t > 0 \), then implies that \( T^\alpha,\beta g \) satisfies condition (1) in the definition of \( PW(\mathbb{R})^{\alpha,\beta} \).

Let \( f \in PW(\mathbb{R})^{\alpha,\beta} \subset PW_2(\mathbb{R})^{\alpha,\beta} \). Then \( F^{\alpha,\beta} f \in C^\infty(\mathbb{R}) \), since \( f \) has polynomial decay, and \( \phi^\alpha,\beta_{\alpha,\beta}(t) \) is bounded by \( (1 + t)e^{-nt} \). Theorem 7 finally implies that \( F^{\alpha,\beta} f \) has compact support with \( R_{F^{\alpha,\beta} f} = R \). \( \square \)
5. Other examples and further results

5.1. The classical Fourier transform

Let \( \mathcal{F} \) denote the Fourier transform on \( \mathbb{R}^k \),

\[
\mathcal{F} f(\lambda) := \int_{\mathbb{R}^k} f(x) e^{-i\lambda \cdot x} \, dx,
\]
defined for nice functions \( f \). Let \( \Delta := d^2/dx_1^2 + \cdots + d^2/dx_k^2 \) denote the Laplacian on \( \mathbb{R}^k \) and let \( \mathcal{S} (\mathbb{R}^k) \) denote the Schwartz space. Then \( \mathcal{F}(\Delta f)(\lambda) = -\|\lambda\|^2 \mathcal{F} f(\lambda) \) for all \( f \in \mathcal{S} (\mathbb{R}^k) \), and the Fourier transform is an isomorphism of \( \mathcal{S} (\mathbb{R}^k) \) onto itself, with inverse given by

\[
\mathcal{F}^{-1} g(x) = (2\pi)^{-k} \int_{\mathbb{R}^k} g(\lambda) e^{i\lambda \cdot x} \, d\lambda
\]
for \( g \in \mathcal{S} (\mathbb{R}^k) \). Parseval’s formula states that

\[
\int_{\mathbb{R}^k} f(x) \overline{f_1(x)} \, dx = (2\pi)^{-k} \int_{\mathbb{R}^k} \mathcal{F} f(\lambda) \overline{\mathcal{F} f_1(\lambda)} \, d\lambda
\]
for \( f, f_1 \in \mathcal{S} (\mathbb{R}^k) \), which for \( f = f_1 \) specialises to \( \|f\|_2 = \|\mathcal{F} f\|_2 \), and it is seen that the Fourier transform extends to an isometry from \( L^2 (\mathbb{R}^k) \) onto itself.

Parseval’s formula and a density argument again imply that \( \mathcal{F}(\Delta^n f)(\lambda) = (-1)^n \|\lambda\|^{2n} \mathcal{F} f(\lambda) \) a.e., for all \( f \in C^\infty (\mathbb{R}^k) \) such that \( \Delta^n f \in L^2 (\mathbb{R}^k) \) for all \( n \in \mathbb{N} \cup \{0\} \), and

\[
\int_{\mathbb{R}^k} |\Delta^n f(x)|^2 \, dx = \int_{\mathbb{R}^k} \|\lambda\|^{4n} |\mathcal{F} f(\lambda)|^2 \, d\lambda
\]
for all \( n \in \mathbb{N} \cup \{0\} \).

The \( L^2 \)-Paley–Wiener space \( PW^2 (\mathbb{R}^k) \), the image of the functions on \( \mathbb{R}^k \) with compact support under the (inverse) Fourier transform, can be characterised as

\[
\begin{align*}
(1) & \quad \Delta^n f \in L^2 (\mathbb{R}^k) \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}, \\
(2) & \quad \lim_{n \to \infty} \|\Delta^n f\|_2^{1/2n} < \infty,
\end{align*}
\]

and functions in the Paley–Wiener space \( PW (\mathbb{R}^k) \), the image of \( C^\infty (\mathbb{R}^k) \) under the (inverse) Fourier transform, furthermore satisfy \( (1 + |x|)^m f \in L^2 (\mathbb{R}^k) \) for all \( m \in \mathbb{N} \cup \{0\} \).

5.2. The Chébli–Trimèche transform

We refer to [2] for details and references. Consider the following more general second order differential operator on \( \mathbb{R}_+ := (0, \infty) \):

\[
\Delta^A := A(x)^{-1} \frac{d}{dx} \left( A(x) \frac{d}{dx} \right) = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx},
\]
where the (Chébli–Trimèche) function $A$ is continuous on $[0, \infty)$, twice continuously differentiable on $(0, \infty)$ and satisfies the following conditions:

(i) $A(0) = 0$ and $A(x) > 0$ for $x > 0$,
(ii) $A$ is increasing and unbounded,
(iii) $A'(x)/A(x) = (2\alpha + 1)/x + B(x)$ on a neighbourhood of 0, where $\alpha > -1/2$ and $B$ is an odd function on $\mathbb{R}$,
(iv) $A'(x)/A(x)$ is a decreasing smooth function on $(0, \infty)$, whence $\rho := (1/2) \times \lim_{x \to \infty} (A'(x)/A(x))$ exists.

Assume furthermore (in order to have a nice Plancherel formula) that there exists $\delta > 0$ such that, for all $x \in (x_0, \infty)$,

$$
\frac{A'(x)}{A(x)} = \begin{cases} 
2\rho + e^{-\delta x} D(x) & \text{if } \rho > 0, \\
\frac{2\alpha+1}{x} + e^{-\delta x} D(x) & \text{if } \rho = 0,
\end{cases}
$$

where $D$ is a smooth function whose derivatives of any order are bounded.

Let $\phi^A_\lambda$ denote the solutions of the differential equation

$$
\Delta^A \phi^A_\lambda = -(\lambda^2 + \rho^2) \phi^A_\lambda , \quad \phi^A_\lambda(0) = 1, \quad (\phi^A_\lambda)'(0) = 0,
$$

and define for $f \in C^\infty_c(\mathbb{R})$ even the Chébli–Trimèche transform $F^A$ as

$$
F^A f(\lambda) := \int_{\mathbb{R}_+} f(x) \phi^A_\lambda(x) A(x) dx.
$$

The inverse $I^A = (F^A)^{-1}$ is given by

$$
I^A g(x) = \int_{\mathbb{R}_+} g(\lambda) \phi^A_\lambda(x) |c^A(\lambda)|^{-2} d\lambda
$$

for nice even functions $g$, where $|c^A(\lambda)|^{-2}$ is continuous on $[0, \infty)$. Parseval’s formula for $F^A$ has the same form as for the Jacobi transform and $F^A$ again extends to an $L^2$ isometry. We also note that $\Delta^A$ is self-adjoint with respect to the measure $A(x) \, dx$.

The Chébli–Trimèche transform $F^A$ reduces to the Jacobi transform $F^{\alpha, \beta}$ when $A = J^{\alpha, \beta}$, $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$; and to the Hankel transform when $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$.

The real Paley–Wiener theorems for the (inverse) Chébli–Trimèche transform can be formulated as follows (with the obvious definitions).

**Theorem 12.** (1) The inverse Chébli–Trimèche transform $I^A$ is a bijection of $L^c_c(\mathbb{R})^{even}$ onto $PW^2(\mathbb{R})^A$, mapping $L^c_R(\mathbb{R})^{even}$ onto $PW^2_R(\mathbb{R})^A$.

(2) The inverse Chébli–Trimèche transform $I^A$ is a bijection of $C^\infty_c(\mathbb{R})^{even}$ onto $PW(\mathbb{R})^A$, mapping $C^\infty_R(\mathbb{R})^{even}$ onto $PW_R(\mathbb{R})^A$.

**Proof.** The proof of (1) is the same as for Theorem 7.
However, in order to prove (2), we need a little more work—and we need the classical Paley–Wiener theorem for $\mathcal{F}^A$. It is shown in [2] that $\mathcal{F}^A$ is an isomorphism between generalised $L^2$-Schwartz spaces, but the proof relies on estimates coming from the classical Paley–Wiener theorem.

So let now $g \in C_c^\infty(\mathbb{R})$ even. Then $(1 + |x|)^m (\Delta^A)^n \mathcal{F}^A g \in L^2(A(x) \, dx)$ even for all $m, n \in \mathbb{N} \cup \{0\}$ by [2, Theorem 4.27] and the definition [2, Definition 4.3] of the $L^2$-Schwartz space $\mathcal{S}_2(\mathbb{R})$. Conversely, using estimates from [2], we can show that $\mathcal{F}^A f$ is smooth if $f$ satisfies the polynomial $L^2$-growth estimate. ☐

We finally note that we have studied real Paley–Wiener theorems for the Fourier transform on Riemannian symmetric spaces $G/K$ of the noncompact type in [1].

References