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Rational Parametrization of Surfaces

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The parametrization problem asks for a parametrization of an implicitly given surface, in terms of rational functions in two variables. We give an algorithm that decides if such a parametric representation exists, based on Castelnuovo's rationality criterion. If the answer is yes, then we compute such a parametric representation, using the concept of adjoint functions.

1. Introduction

The parametrization problem is the following. Given is the implicit representation of a surface, in terms of an equation in three variables. Wanted are three rational functions in two variables, such that the image of the rational map defined by this triple is a two-dimensional subset of the surface.

For instance, consider the the unit sphere, with equation $x^2 + y^2 + z^2 - 1 = 0$. A parametrization is

$$(x, y, z) = \left(\frac{2s}{s^2 + t^2 + 1}, \frac{2t}{s^2 + t^2 + 1}, \frac{s^2 + t^2 - 1}{s^2 + t^2 + 1}\right).$$

The reverse problem (given the three rational functions, find the equation) is called the implicitization problem. It has been investigated in Canny and Manocha (1992), Gao and Chou (1992), Kalkbrener (1990) and Chionh and Goldman (1992). An algorithm can be found in the CASA package (Tran and Winkler, 1997).

There are tasks in computational geometry, for which the parametric representation is more convenient, and others for which the implicit representation is more convenient. A task in the second group is to decide whether a given point is lying on the surface. The first group contains tasks in which many points have to be produced fast. The parametric representation, especially parametrization by NURBSs (i.e. piecewise rational functions) is used for many applications in computer aided design and manufacture, such as reliable surface plotting and display, motion display (computing transformations), computing cutter offset surfaces, computing curvatures for shading and colouring, and many others (see also Böhm *et al.* (1984), Qiulin and Davies (1987) and Farin (1988)). Therefore, the parametrization problem is important.

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If the rational parametrization map is birational, then we call the parametric representation *proper*. The image of a proper parametrization is almost all of the surface, at most a one-dimensional subset of X is missing.

For instance, the above example is a proper parametrization, with the stereographic projection

$$(s,t) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

as inverse.

In the curve case, the situation is as follows. By a theorem of Riemann, a curve has a parametrization iff it has a proper parametrization iff its *genus* (see Shafarevich (1974) for a definition of this notion) vanishes. Algorithmically, the parametrization problem for curves has been solved in the following sense. First, the genus can be computed (Walker, 1978). If it is zero, then there are algorithms (Walker, 1978; Abhyankar and Bajaj, 1987; Sendra and Winkler, 1991; Schicho, 1992; van Hoeij, 1994; Mñuk *et al.*, 1996; Mñuk, 1996; Sendra and Winkler, 1997; van Hoeij, 1997) that compute a proper parametrization.

In the surface case, the theory is well understood, too. By a theorem of Castelnuovo (1939), a surface has a parametrization iff it has a proper parametrization iff the arithmetical genus p_a and the second plurigenus P_2 are both zero (see Shafarevich (1974) for a definition of these notions). Castelnuovo's theorem holds for algebraically closed ground fields of characteristic zero.

Algorithmically, the problem is much more difficult. Before Schicho (1995), there were no parametrization algorithms except for very limited classes of surfaces, like quadric and cubic surfaces (Abhyankar and Bajaj, 1987; Sederberg and Snively, 1987) or canal surfaces (Pottmann, 1996). Parametrization of rational surfaces was posed as an 'open problem in computational algebraic geometry' in Eisenbud (1993).

Our main tool for parametrization is adjoints. These were introduced in the last century by Clebsch (1868) and Nöether (1871). Adjoints have played a fundamental role in the surface theory of the Italian school (see Enriques (1949)), similar to the role of the canonical divisor in more modern treatments (Shafarevich, 1965; Kurke, 1982).

Adjoints computation is a difficult problem. The only algorithm is the one in Schicho (1995); it is based on an algorithm for the resolution of singularities by Hironaka (1984).

In this paper, we will not go into technical details for adjoint computation. We will just briefly sketch an algorithm and use adjoints as a black box for the rest of the paper. There is a two-fold reason for this. First, the techniques for adjoints computation look rather different from the techniques for parametrization using adjoints. Second, one can do other things than parametrization using adjoints. Computation of adjoints will, therefore, be the topic of another paper (we refer to Schicho (1995) in the meantime).

Once we have the adjoints, we compute p_a and P_2 to decide the existence of a parametrization (possibly over an algebraic extension of the ground field). The formula for P_2 was known to the classical Italians. The formula for p_a first appeared in Schicho (1995), it improves a result in Blass and Lipman (1979).

If both numbers vanish, then we compute a birational map from the given surface to another surface, which is easier to parameterize. This other surface is one of the following.

1. The projective plane.
2. A quadric surface in $\mathbf{P}^3$.
3. A rational scroll (a surface with a pencil of lines).

4. A surface with a pencil of conics.
5. A Del Pezzo surface.

Theoretically, the possibility for a reduction of the parametrization problem to the parametrization of one of the types above was shown in Enriques (1895). The main result of this paper is an algorithm for this reduction (algorithm *ParameterizeWithAdjoints*). The algorithm is also contained in Schicho (1995), but this paper is smaller and more easy to read.

In 1 above, the parametrization problem is already solved. We give algorithms to parameterize the surfaces 2 and 3 (these are quite easy). In 4, we give a new parametrization algorithm for algebraically closed ground fields of characteristic $\neq 2$. Also, 5 is solved for algebraically closed ground fields; here, we refer to the literature (Conforto, 1939; Manin, 1974; Schicho, 1995).

The final step is the inversion of a birational map to the plane. Here, we use the general method of Gröbner bases (Buchberger, 1965, 1985; Becker and Weispfenning, 1993). Alternatively, one also might apply resultants (Collins, 1967; Brown and Traub, 1971; Buchberger *et al.*, 1982; Mishra, 1993).

2. The Problem

We work in the projective setting, i.e. the surface equation is a homogeneous polynomial in four variables over some ground field k. Note that homogenization and dehomogenization are trivial computations.

We assume that the given equation is absolutely irreducible.

Here is the precise specification of the parametrization problem.

Input: A homogeneous polynomial $F \in k[x, y, z, w]$, representing a surface.
Output: A quadruple $(X : Y : Z : W) \in k[s, t]^4$ of bivariate polynomials,
representing a rational parametrization of the surface, if exists.
`NotExist` otherwise.

A quadruple $(X : Y : Z : W)$ is a rational parametrization of the surface F iff the following two conditions hold.

$$F(X : Y : Z : W) = 0 \qquad \text{in } k[s, t].$$

The rank of the matrix

$$\begin{pmatrix} X & Y & Z & W \\ \partial_s X & \partial_s Y & \partial_s Z & \partial_s W \\ \partial_t X & \partial_t Y & \partial_t Z & \partial_t W \end{pmatrix}$$

over the field $k(s, t)$ is 3.

The second condition ensures that the image of the $(X : Y : Z : W)$ in $\mathbf{P}^3$ is not just a point or a curve on the surface.

EXAMPLE 2.1. A rational parametrization of the unit sphere $x^2 + y^2 + z^2 - w^2 = 0$ is given by

$$(X : Y : Z : W) = (2s : 2t : s^2 + t^2 - 1 : s^2 + t^2 + 1).$$

In some of our subalgorithms, we have to assume that k is algebraically closed. I do not want to assume this from the beginning, because some subalgorithms work for a larger class of fields, and others can maybe be generalized to work for a larger class of fields.

Surfaces which have a parametrization are called unirational, and surfaces which have a proper parametrization are called rational (see Shafarevich (1974)). In the case where k has characteristic zero and is algebraically closed, the two properties coincide by Castelnuovo's theorem (Castelnuovo, 1939). There are irrational surfaces, for instance the K3-surface with equation $x^4 + y^4 + z^4 - w^4 = 0$ (see Shafarevich (1965)). However, the rational surfaces form an interesting and important class of surfaces (see Conforto (1939) for a decent theoretical introduction to this topic).

3. Preliminary Techniques

In this section, we introduce some useful techniques and subalgorithms for later perusal in the parametrization algorithm. Except for the algorithm *FindPoint*, the material is basically known. The section also contains references to other algorithms for the same problems, which are more general or more efficient (but more cumbersome to describe) than the ones given here.

3.1. quadric surfaces

As one of the base cases in the algorithm, we will have to deal with quadric surfaces in $\mathbf{P}^3$. A parametrization algorithm for this case can be found in Abhyankar and Bajaj (1987).

For our purpose, we need to compute a birational map from a given quadric surface $F \in \mathbf{P}^3$ to $\mathbf{P}^2$. This is easy: if p is a smooth point of F, then the projection from p is birational onto $\mathbf{P}^2$. Note that a quadric surface has at most one point which is not smooth, namely the vertex of a quadric cone.

Input: A homogeneous polynomial $F \in k[x, y, z, w]$ of degree 2, describing a quadric surface in $\mathbf{P}^3$.
Output: Three linear forms $(S : T : U)$ in x, y, z, w, representing a birational map from the surface to $\mathbf{P}^2$.

Algorithm *QuadricSurface*(F):
    $p :=$ a smooth point on F;
    $\{S, T, U\} :=$ three linearly independent linear forms vanishing at p;
    return $(S : T : U)$

Example 3.1. Let $F = x^2 + y^2 + z^2 - w^2$, the equation of the unit sphere. For p, we choose the north pole $(0 : 0 : 1 : 1)$. The resulting map is the stereographic projection $(x : y : w - z)$.

3.2. inversion of birational maps

In many cases, it is easier to compute the inverse of a parametrization. To compute a parametrization, we invert the inverse. Here is an algorithm for doing this.

Input: A homogeneous polynomial $F \in k[x,y,z,w]$, representing the surface.
A triple of homogeneous polynomials $(S:T:U) \in k[x,y,z,w]^3$ of the same degree, representing a birational map from the surface to $\mathbf{P}^2$.
Output: A quadruple $(X:Y:Z:W) \in k[s,t]^4$ of bivariate polynomials, representing the inverse of $(S:T:U)$.

Algorithm *InvertBirational*$(F,(S:T:U))$:
Cancel common factors in S, T, U;
$G := \{S-s, T-t, U-1, F\}$;
$G' :=$ a Gröbner basis of G over $k(s,t)[x,y,z,w]$ with total degree order;
$G'' :=$ the set of all elements on G' of degree 1;
$(X,Y,Z,W) :=$ a nontrivial solution of G'
(considered as a linear equation system in (x,y,z,w));
return $(X:Y:Z:W)$;

Example 3.2. Let $F = x^2+y^2+z^2-w^2$, the equation of the unit sphere. We will invert the stereographic projection $(x:y:w-z)$.

We compute a Gröbner base of

$$G = \{x-s, y-t, z-w-1, x^2+y^2+z^2-w^2\}$$

(total degree order, s and t are constants). It consists entirely of linear elements:

$$G' = G'' = \{x-s, y-t, 2z+1-s^2-t^2, 2w-1-s^2-w^2\}.$$

We solve for x,y,z,w and obtain

$$(X:Y:Z:W) = \left(s:t:\frac{s^2+t^2-1}{2}:\frac{s^2+t^2+1}{2}\right).$$

Theorem 3.1. *The algorithm InvertBirational is correct.*

Proof. Let $(X':Y':Z':W') \in k[s,t]^4$ be an inverse of $(S:T:U)$. Let $q=(s:t:1)$ be a generic point of $\mathbf{P}^2$, and let p be its unique preimage. Let k' be the algebraic closure of $k(s,t)$. The set G generates a radical ideal in $I < k'[x,y,z,w]$, whose zeros are on the line corresponding to the projective point p. Moreover, the set G has at least one zero, namely $(X'/D, Y'/D, Z'/D, W'/D)$, where D is an nth root of $U(X',Y',Z',W')$, where n is the common degree of S, T, U. Therefore, all solutions of the *linear* equations in I correspond to the projective point p, hence are inverse to $(S:T:U)$.

By a well-known property of Gröbner bases, the linear equations in I are generated by the linear equations in G'.

The first step (cancellation of common factors) is not really necessary, but it speeds up the rest of the computation.

3.3. Parametrization of a generic fibre

Sometimes, we find the necessary parameters sequentially. In an intermediate step, we will have a rational map

$$\pi : F \rightarrow \mathbf{P}^1, (x:y:z:w) \mapsto (T:U)$$

with two polynomials T, U of the same degree, such that the generic fibre $F_t := \pi^{-1}(t:1)$ is a rational curve. Since F_t is a projective space curve over the ground field $k(t)$, a parametrization of F_t would be of the form $(X : Y : Z : W)$, where X, Y, Z, W are polynomials in $k(t)[s]$. Since we are in the projective case, we can clear denominators. By changing the semantics of the t from a generic constant to a variable, we obtain a parametrization of the surface F.

In our situation, we will have a birational map of the generic fibre F_t to a projective curve $C_t \in \mathbf{P}^r$, coming from a birational map of F to a surface in $\mathbf{P}^r$. Moreover, C_t will be either a line or a conic.

It remains to find a parametrization of C_t.

3.3.1. LINES

If C_t is a line, then the problem is easy: the second parameter is simply a ratio of two projective coordinates. Here is an algorithm for this easy case.

Input: A homogeneous polynomial $F \in k[x, y, z, w]$, representing the surface.
A pair of homogeneous polynomials $(T : U) \in k[x, y, z, w]^2$ of the same degree, representing a rational map from the surface to $\mathbf{P}^1$.
An $(r+1)$-tuple of homogeneous polynomials $(P_0 : \ldots : P_r) \in k[x, y, z, w]^{r+1}$ of the same degree, representing a birational map which maps the generic fibre of $(T : U)$ to a line in $\mathbf{P}^r$.
Output: A quadruple $(X : Y : Z : W) \in k[s, t]^4$ of bivariate polynomials, representing a parametrization of F.

Algorithm *FiberIsLine*$(F, (T : U), (P_0, \ldots, P_r))$:
$i := 1$;
while $P_i \in k(\frac{T}{U})P_0$ do
$i := i + 1$;
$(S, T', U') := (P_i U, P_0 T, P_0 U)$;
return *InvertBirational*$(F, (S : T' : U'))$;

EXAMPLE 3.3. Let

$$F = (x^4 + z^4)w^2 + (xy + z^2)^3,$$

$$(T : U) = (x : z),$$

$$(P_0 : P_1 : P_2 : P_3 : P_4) = (z^2 w : xzw : x^2 w : xyz + z^3 : x^2 y + x^2 z).$$

Since P_1 and P_2 are in $k(\frac{x}{z})P_0$ and P_3 is not, we have $i = 3$ after the while loop. We find

$$(S : T' : U') = (xwz^2 : xyz^2 + z^4 : z^3 w),$$

and apply *InvertBirational*. The result is

$$(X : Y : Z : W) = (-s^3 t^2 : t^3 + s^4 t + t : -st^3 : s^5 + s).$$

The correctness of the algorithm *FiberIsLine* is obvious.

3.3.2. CONICS

If C_t is a conic, then the problem is more difficult. The idea is to project the conic isomorphically to the plane, then find a point on the conic, and then project from this point. This yields a birational map from the conic to the projective line. Here is an algorithm.

Input: A homogeneous polynomial $F \in k[x,y,z,w]$, representing the surface.
A pair of homogeneous polynomials $(T:U) \in k[x,y,z,w]^2$ of the same degree, representing a rational map from the surface to $\mathbf{P}^1$.
An $(r+1)$-tuple of homogeneous polynomials $(P_0 : \ldots : P_r) \in k[x,y,z,w]^{r+1}$ of the same degree, representing a birational map which maps the generic fibre of $(T:U)$ to a conic in $\mathbf{P}^r$.
Output: A quadruple $(X:Y:Z:W) \in k[s,t]^4$ of bivariate polynomials, representing a parametrization of F.

Algorithm *FiberIsConic*$(F,(T:U),(P_0,\ldots,P_r))$:
$i := 1$;
while $P_i \in k(\frac{T}{U})P_0$ do
$i := i+1$;
$j := i+1$;
while $P_j \in k(\frac{T}{U})P_0 + k(\frac{T}{U})P_i$ do
$j := j+1$;
$C(p_0,p_1,p_2) :=$ find the relation between P_0, P_i, P_j ;
(C is a homogeneous element of $k(t)[p_0,p_i,p_j]$ of degree 2.)
$(Q_0 : Q_i : Q_j) :=$ *FindPoint*(C);
(Note that $Q_0, Q_i, Q_j \in k(t)$.)
$S := \frac{Q_0(\frac{T}{U})P_i - Q_i(\frac{T}{U})P_0}{Q_0(\frac{T}{U})P_j - Q_j(\frac{T}{U})P_0}$
$(S',T',U') :=$ clear denominators in $(SU : T : U)$;
return *InvertBirational*$(F,(S':T':U'))$;

EXAMPLE 3.4. Let

$$F = x^2y^2 + 8x^3y + 4x^4 + xyz^2 - x^2z^2 - y^2w^2 - 7xyw^2 + 8x^2w^2,$$

$$(T:U) = (y:x),$$

$$(P_0 : P_1 : P_2 : P_3) = (x:y:z:w).$$

Since P_1 is in $k(\frac{y}{x})P_0$ and P_2 is not, we have $i=2$ after the first while loop. Since P_3 is not in $k(\frac{y}{x})P_0 + k(\frac{y}{x})P_2$, we have $j=3$. The quadratic relation between P_0, P_2, P_3 is

$$C = (t^2+8t+4)p_0^2 + (t-1)p_2^2 + (-t^2-7t+8)p_3^2 = 0.$$

We do not have the subalgorithm *Findpoint* yet, but it is easily checked that

$$(Q_0 : Q_2 : Q_3) = (-t+1 : 5t+34 : t+12)$$

is a point on C. Therefore, we find the second parameter

$$S = \frac{(-\frac{y}{x}+1)z - (5\frac{y}{x}+34)x}{(-\frac{y}{x}+1)w - (\frac{y}{x}+12)x}$$

$$= \frac{-yz + xz - 5xy - 34x^2}{-yw + xw - xy - 12x^2}.$$

By clearing denominators, we find

$$(S' : T' : U') = (-xyz + x^2z - 5x^2y - 34x^3 : -y^2w + xyw \\ - xy^2 - 12x^2y : -xyw + x^2w - x^2y - 12x^3).$$

We apply *InvertBirational* and find

$$(X : Y : Z : W) = (t^2 + 7t - s^2t + s^2 - 8 : t^3 + 7t^2 - s^2t^2 + s^2t - 8t : \\ -5t^2 + 2st^2 - 74t - 5s^2t + 40st + 192s - 34s^2 - 272 : t^2 \\ -10st + 20t + s^2t + 12s^2 - 68s + 96).$$

The above algorithm is obviously correct, but it leaves us with the problem of finding a point on the conic C' defined over $k(t)$ (subalgorithm *FindPoint*).

If k is algebraically closed, then every conic over $k(t)$ has a point (see Greenberg (1969)). For an efficient algorithm, we use an idea of Clebsch (1868): we find a solution modulo $t = t_i$ for sufficiently many numbers t_i, at which the equation of the conic splits into linear factors. We also assume *char* $k \neq 2$, to be able to eliminate the mixed terms by suitable linear transformations.

Input: A homogeneous polynomial $G \in k(t)[x, y, z]$ of degree 2.
Output: A nontrivial solution $(X : Y : Z) \in k[t]^3$.

Algorithm *FindPoint*:
  eliminate the mixed terms by suitable linear transformations;
  clear denominators;
  eliminate multiple factors of the coefficients by suitable linear transformations;
  eliminate common factors of the coefficients:
    if two coefficients have a common factor P then
      multiply the equation with P;
      eliminate the double factors;
  (Now, $G = Ax^2 + By^2 + Cz^2$ and ABC is squarefree.)
  if two polynomials of A, B, C have degree 0 then
    $(X : Y : Z) :=$ two suitable constants and 0;
  else
    Make an ansatz with undetermined coefficients:
      $b_x := [\frac{\deg B + \deg C - 1}{2}]$;
      $X := x_0 + \cdots + x_{b_x} t^{b_x}$;
      $b_y := [\frac{\deg A + \deg C - 1}{2}]$;
      $Y := y_0 + \cdots + y_{b_y} t^{b_y}$;
      $b_z := [\frac{\deg A + \deg B - 1}{2}]$;
      $Z := z_0 + \cdots + z_{b_z} t^{b_z}$;
    $n := b_x + b_y + b_z + 2$;
    $(t_1, \ldots, t_n) := n$ different zeros of ABC;
    for i from 1 to n do
      $L(x, y, z) :=$ a linear factor of $G(t_i, x, y, z)$;
      $E_i := L(X(t_i), Y(t_i), Z(t_i))$;

(E_i is linear in $x_0, \ldots, z_{b_z}$.)
$x_0, \ldots, z_{b_z}$:= a nontrivial solution of $(E_1, \ldots, E_n)$;
transform the solution back;
return $(X : Y : Z)$

Example 3.5. Let

$$G = (t^2 + 8t + 4)x^2 + (t-1)y^2 + (-t^2 - 7t + 8)z^2$$

as in the previous example. There are no mixed terms and no multiple zeros, but $t = 1$ is a common zero of two of the coefficients. We eliminate the common zero and obtain

$$G' = (t^2 + 8t + 4)(t-1)x^2 + y'^2 - (t+8)z'^2$$

with $y' = (t-1)y$, $z' = (t-1)z$. We find $b_x = 0$, $b_y = b_z = 1$ and make the ansatz

$$x = x_0, y' = y_0 + y_1 t, z' = z_0 + z_1 t.$$

We compute linear equations using the zeros

$$t_1 = 2\sqrt{3} - 4, \quad t_2 = -2\sqrt{3} - 4, \quad t_3 = 1, \; t_4 = -8$$

of the coefficients.

To find an equation using t_1, we substitute $t = t_1$ in G' and factor:

$$G'(t_1) = y'^2 - (2\sqrt{3} + 4)z'^2 = (y' - (1 - \sqrt{3})z')(y' + (1 - \sqrt{3})z').$$

We substitute the ansatz into the first factor and obtain the equation

$$y_0 + (2\sqrt{3} - 4)y_1 - (1 - \sqrt{3})(z_0 + (2 + \sqrt{3})z_1) = 0.$$

The other equations are

$$y_0 + (-2\sqrt{3} - 4)y_1 - (1 + \sqrt{3})(z_0 + (2 - \sqrt{3})z_1) = 0,$$

$$y_0 + y_1 - 3(z_0 + z_1) = 0,$$

$$6x_0 - y_0 + 8y_1 = 0.$$

We find the nontrivial solution

$$x_0 = -1, \quad y_0 = -34, \quad y_1 = 5, \quad z_0 = 12, \quad z_1 = 1,$$

which yields the point

$$(X : Y' : Z') = (-1 : 5t + 34 : t + 12).$$

We transform back and obtain

$$(X : Y : Z) = \left(-1 : \frac{5t + 34}{t - 1} : \frac{t + 12}{t - 1}\right) = (-t + 1 : 5t + 34 : t + 12).$$

Theorem 3.2. *The algorithm FindPoint is correct.*

Proof. The case that two of A, B, C have degree 0 is trivial.

Assume that at most one of A, B, C has degree 0. We have either $n = \deg A + \deg B + \deg C$ or $n = \deg A + \deg B + \deg C - 1$, depending on the parities of the degrees. Therefore, we can find n different zeros of ABC.

Because of the inequality

$$\max(\deg A + 2b_x, \deg B + 2b_y, \deg C + 2b_z) \leq n-1,$$

the polynomial $AX^2 + BY^2 + CZ^2$ has degree $\leq n-1$. Since it vanishes at $t_1, \ldots, t_n$, it vanishes identically.

Remark 3.1. The problem of finding a point on a conic over $k(t)$ was recently solved by Hillgarter (1996) for a large class of fields k. However, it may happen that our quadratic equation C has no zeros defined over $k(t)$, even in the case where the input surface has a parametrization.

3.4. rational normal curves

In order to find the first parameter of a parametrization, we will sometimes be concerned with particular rational curves, namely *rational normal curves*. These are curves of degree n in $\mathbf{P}^n$ (not in some linear subspace), for some n. We want to construct a birational map from the curve to the projective line.

If $C \in \mathbf{P}^n$ is a rational normal curve, and $p_1, \ldots, p_{n-1}$ are $n-1$ different points on C, then the projection from the linear $(n-2)$-plane spanned by $p_1, \ldots, p_{n-1}$ is birational onto $\mathbf{P}^1$ (see Walker (1978)).

Input: A set Γ of homogeneous polynomials in $k[x_0, \ldots, x_n]$, describing a rational normal curve in $\mathbf{P}^n$.
Output: Two linear forms $(T : U)$ in $x_0, \ldots, x_n$, representing a birational map from the curve to $\mathbf{P}^1$.

Algorithm *RationalNormalCurve*(Γ):
  PointSet := $n-1$ points in the zero set of Γ;
  $\{T, U\}$:= two linearly independent linear forms vanishing at *PointSet*;
  return $(T : U)$

It can be shown that a rational normal curve can be described by a set of polynomials of degree 2 (see Shafarevich (1974)).

Example 3.6. Let Γ be the set

$$\{x_0x_1 + x_2x_3, x_1^2 + x_2^2 - x_1x_3 - x_0x_2, x_0^2 - x_3^2 + x_1x_3 - x_0x_2\},$$

which represents a rational normal curve in $\mathbf{P}^3$. As *PointSet*, we take

$$\{(1:0:1:0), (0:1:0:1)\}.$$

We find the solution

$$(T : U) = (x_0 - x_2 : x_1 - x_3).$$

If k is algebraically closed, then it is obvious that a *PointSet* can be found.

Remark 3.2. If k is an arbitrary computable field, then we can solve the problem if we can find points on a conic (see Sendra and Winkler (1991) and Schicho (1992)).

3.5. Del Pezzo surfaces

As another base case, we will have to deal with *Del Pezzo surfaces*. These are surfaces of degree n in $\mathbf{P}^n$ (not in some linear subspace) for some n, which satisfy the following conditions.

The generic hyperplane section is an elliptic curve. $p_a = P_2 = 0$.

For instance, a smooth cubic surface in $\mathbf{P}^3$ is a Del Pezzo surface.

Theorem 3.3. *The largest possible degree of a Del Pezzo surface is 9.*

Proof. See Shafarevich (1965) and Manin (1974).

For $n = 1$ or 2, Del Pezzo surfaces of degree n are constructed in a nonstandard way, adapting the definition appropriately. We start with a generalization of the notion of projective space.

Definition 3.1. Let k be a field. Let $w_0, \ldots, w_n$ be positive integers. The *weighted projective space* over k with weights $w_0, \ldots, w_n$ is the set of all elements in $k^{n+1} - \{0\}$ modulo the following equivalence relation. Two elements $(p_0, \ldots, p_n)$ and $(q_0, \ldots, q_n)$ are equivalent iff there exists an element λ in the algebraic closure of k, such that

$$(p_0, \ldots, p_n) = (\lambda^{w_0} q_0, \ldots, \lambda^{w_n} q_n).$$

If all weights are equal, then the weighted projective space is just the old $\mathbf{P}^n$. In general, the weighted projective space is a projective variety of dimension n, which may have singularities.

Example 3.7. Let $(w_0, w_1, w_2) = (1, 1, 2)$. This weighted projective space is isomorphic to a quadratic cone in $\mathbf{P}^3$. An isomorphism to the cone with equation $xz - y^2 = 0$ is given by

$$(x_0 : x_1; x_2) \mapsto (x : y : z : w) = (x_0^2 : x_0x_1 : x_1^2 : x_2),$$

$$(x : y : z : w) \mapsto (x_0 : x_1; x_2) = (x : y; xw) = (y : z; zw).$$

The weighted projective space has a singular point $(0 : 0; 1)$, which is the vertex of the cone.

A hypersurface in weighted projective space can be described by a weighted homogeneous polynomial, with weights $w_1, \ldots, w_n$. The 'hyperplanes' are the zero sets of weighted homogeneous polynomials of weighted degree 1, i.e. the linear forms in the coordinates with weight 1.

3.5.1. Del Pezzo surfaces of degree 2

We consider the weighted projective space with weights $(w_0, w_1, w_2, w_3) = (1, 1, 1, 2)$. Let F be a hypersurface (i.e. a surface) given by a weighted homogeneous polynomial of weighted degree 4. For suitable choices of F, the following are true (see Manin (1974)).

There is a natural projection of F to $\mathbf{P}^2$, which is a map of degree 2. The generic 'hyperplane section' is an elliptic curve. $p_a = P_2 = 0$.

In this situation, we say that F is a Del Pezzo surface of degree 2.

Example 3.8. Let F be given by the equation

$$x_0^4 + x_1^4 - x_2^4 - x_3^2 = 0.$$

The surface F is a two-sheeted branched covering of $\mathbf{P}^2$, and the branch curve is a smooth quartic with equation $x_0^4 + x_1^4 - x_2^4 = 0$.

Let C be the intersection with a generic 'hyperplane' $x_0 + sx_1 + tx_2 = 0$. We have a degree 2 map from C to the projective line with four branch points (the intersections with the branch curves). By the Hurwitz genus formula (see Hartshorne (1977)), we see that C is elliptic.

3.5.2. Del Pezzo surfaces of degree 1

We consider the weighted projective space with weights $(w_0, w_1, w_2, w_3) = (1, 1, 2, 3)$. Let F be a surface given by a weighted homogeneous polynomial of weighted degree 6. For suitable choices of F, the following are true (see Manin (1974)).

There is a natural projection of F to $\mathbf{P}^1$.
The generic 'hyperplane section' is an elliptic curve.
The number of intersections of two generic 'hyperplane sections' is 1.
$p_a = P_2 = 0$.

In this situation, we say that F is a Del Pezzo surface of degree 1.

Example 3.9. Let F be given by the equation

$$x_0^5 x_1 + x_2^3 + x_3^2 = 0.$$

The surface F is a two-sheeted branched covering of the weighted homogeneous space with weights $(w_0, w_1, w_2) = (1, 1, 2)$. The branch locus consists of the the curve with equation $x_0^5 x_1 + x_2^3 = 0$ and an isolated branch point $(0 : 0; 1)$ (which is the singular point on the weighted projective space).

The generic 'hyperplane section' C has equations

$$x_0^5 x_1 + x_2^3 + x_3^2 = x_0 + tx_1 = 0.$$

We have a degree 2 map from C to the projective line with four branch points, namely the isolated branch point and three intersection points with the branch curve. By the Hurwitz genus formula, we see that C is elliptic.

3.5.3. parametrization of Del Pezzo surfaces

If k is algebraically closed, then the Del Pezzo surfaces allow a proper parametrization (see Conforto (1939) and Shafarevich (1965)).

For our purpose, we will need an algorithm that computes the inverse of a proper parametrization.

Input: A set Γ of homogeneous polynomials in $k[x_0, \ldots, x_n]$, $3 \le n \le 9$, describing a Del Pezzo surface of degree n;
Or a weighted homogeneous polynomial with weights $(1, 1, 1, 2)$ of weighted degree 4, describing a Del Pezzo surface of degree 2;
Or a weighted homogeneous polynomial with weights $(1, 1, 2, 3)$ of weighted degree 6, describing a Del Pezzo surface of degree 1.

Output: Three (weighted) homogeneous polynomials $(S : T : U)$ of the same degree, representing a birational map from the Del Pezzo surface to $\mathbf{P}^2$.

A Del Pezzo surface of degree ≥ 4 can be described by a set of polynomials of degree 2 (see Griffiths and Harris (1978) for the case $r = 4$; the general case can be reduced to this case).

Example 3.10. Let F be given by the equation

$$x_0^5 x_1 + x_2^3 + x_3^2 = 0.$$

A birational map from F to $\mathbf{P}^2$ is given by

$$(x_0 : x_1; x_2; x_3) \mapsto (s : t : u) = (x_0 x_2 : x_3 : x_0^3).$$

Its inverse is

$$(s : t : u) \mapsto (x_0 : x_1; x_2; x_3) = (u^3 : -s^3 - t^2 u : su^5 : tu^8).$$

Theorem 3.4. *There exists an algorithm DelPezzo for the specification above, which is of 'formula type': it requires only the computation of the roots of a polynomial of constant degree, a constant number of field operations.*

Proof. We have an upper bound for the degree of the Del Pezzo surface (Theorem 3.3). Moreover, there exists always a solution with polynomials of degree ≤ 6 (see Conforto (1939) and Schicho (1995)). The number of involved variables is also bounded. Therefore, we have to find the solution of a system of algebraic equations and inequalities with bounded degree in a constant number of unknowns (the coefficients of the solution). Obviously, such a problem has always a solution of 'formula type'.

For explicit algorithms, we refer to Conforto (1939), Manin (1974) and Schicho (1995).

Remark 3.3. If k is not algebraically closed, then the situation is as follows. If we can find a single point on a Del Pezzo surface of degree n, then we can compute a proper parametrization for $5 \leq n \leq 9$, and an improper parametrization for $2 \leq n \leq 4$. There are Del Pezzo surfaces of degrees 1, 2, 3 and 4 with enough points (i.e. the points do not all lie on a curve), which have no proper parametrization. It is not known whether there exists a Del Pezzo surface of degree 1 with enough points having no improper parametrization (see also Manin (1974) and Schicho (1995)).

4. Adjoints

In this section, we introduce the main tool for parametrization, namely adjoint functions. Using adjoints, one can reduce the parametrization problem to one of the problems in the previous section.

The theory of adjoints is classical (Enriques, 1949; Zariski, 1971). The first adjoints computation algorithm has appeared in Schicho (1995). Here, we explain the basic idea of this algorithm.

Finally, we show how to compute the arithmetical genus, the geometrical genus and the plurigeni, using adjoints.

4.1. what are adjoints

Let $F{\in}k[x,y,z,w]$ be a homogeneous polynomial of degree d, representing a surface. (As always, we assume that F is absolutely irreducible.) A *form* on the surface is an equivalence class of polynomials modulo multiples of F. We will represent forms by homogeneous polynomials which are reduced modulo F, with respect to the lexicographical term order.

For $m{\geq}0$, a form G is an m-adjoint iff it vanishes with order $\geq m(r-1)$ at each r-fold curve singularity of F and with order $\geq m(r-2)$ at each r-fold point singularity of F. Obviously, any form is a 0-adjoint. Singularities must be computed over the algebraic closure of k. 'Infinitely near singularities' (see Walker (1978)) must also be taken into account.

Example 4.1. Let $F := x^4 + y^4 - z^2w^2$, $m = 1$. The surface has two isolated double points $(0:0:0:1)$ and $(0:0:1:0)$, which impose no conditions for the adjoints. But each of the two double points has a double line in the infinitely near (this can be seen when the points are blown up). The 1-adjoints have to vanish at the these double lines, and therefore also on the isolated double points.

Using the algorithm in the next subsection, one can show vice versa that all forms vanishing at the two double points are 1-adjoints.

The m-adjoints of fixed degree form a finite-dimensional vectorspace over k. The vectorspace of all m-adjoints of dimension $n+m(d-4)$ is denoted by $V_{n,m}$, and its dimension is denoted by $v_{n,m}$.

Here are some basic properties of the adjoints.

Proposition 4.1. *The m-adjoints have the following properties.*

1. *If F is smooth, then every form is an m-adjoint.*
2. *The question if G is an m-adjoint can be decided locally.*
3. *We have the inclusion $V_{n,m}{\cdot}V_{n',m'}{\subset}V_{n+n',m+m'}$.*

Proof. Obvious.

4.2. adjoints computation

The algorithm *AdjointSpace* has the following specification.

Input: A homogeneous polynomial $F{\in}k[x,y,z,w]$ of degree d, representing a surface. Two integers n, m.
Output: A basis for the vectorspace $V_{n,m}$ of m-adjoints of degree $n + m(d-4)$.

To compute the 'infinitely near singularities' which occur in the definition of adjoints, we have to compute a resolution of the singularities by blowing ups (see Hartshorne (1977)). Here, we use the resolution method described in Hironaka (1984).

For convenience of the recursive description of the algorithm, we change the specification slightly. It is obvious how to reduce the original problem to the one below.

Input: A surface F in 3-space, given implicitly.
A vectorspace V of forms on X.
An integer $m > 0$.
Output: A basis for the subvectorspace of m-adjoints.

Algorithm *AdjointSubspace*(F, V, m):
  if F is smooth then
    return V;
  else
    choose a singularity S;
    $r :=$ the multiplicity of F at S;
    if S is a curve then
      $p := m(r-1)$;
    else (S is a point)
      $p := m(r-2)$;
    $V_1 :=$ the subvectorspace of the elements that vanish with order $\geq p$ at S;
    $(\tilde{F}, e) := (\mathit{BlowUp}(F, S)$, equation of the exceptional divisor);
    $f : V_1 \to \{$forms on $\tilde{F}\} :=$ division by $e^p \circ$ pullback;
    $V_2 := \mathit{AdjointSubspace}(\tilde{F}, f(V_1), m)$;
    return $f^{-1}(V_2)$;

The singularity S must be chosen according to the following rules.

1. S must be either a point or a smooth curve.
2. If S is a point, then it must not lie on a smooth curve with the same multiplicity.
3. If S is a curve, then it all its points must have the same multiplicity.

The rules ensure termination (Hironaka, 1984).

Example 4.2. Let $F := x^4 + y^4 - z^2w^2$, as in the previous example. We compute $V_{1,1}$, the vectorspace of 1-adjoints of degree 1.

In Schicho (1995), the blowing up is realized as a union of affine charts. The decomposition business starts with the input surface, so we dehomogenize with respect to w and restrict our attention to this patch for the moment. The input to *AdjointSubspace* is

$$x^4 + y^4 - z^2, \langle x, y, z, 1\rangle, 1.$$

The only singularity is the origin. It is a double point, so $p = 0$, and the subspace V_1 is equal to the input vectorspace.

In the blowing-up patch with affine coordinates $x, \tilde{y} = \frac{y}{x}, \tilde{z} = \frac{z}{x}$, the blown up surface has the equation $\tilde{F}_1 = x^2 + x^2\tilde{y}^4 - \tilde{z}^2$. The equation of the exceptional divisor is x. We find

$$f(V_1) = \langle x, x\tilde{y}, x\tilde{z}, 1\rangle.$$

There are also other patches of $\tilde{F}$, but we restrict our attention for the moment.

We call *AdjointSubspace* with input

$$x^2 + x^2\tilde{y}^4 + \tilde{z}^2, \langle x, x\tilde{y}, x\tilde{z}, 1\rangle, 1.$$

As the singularity, we choose the double line $x = \tilde{z} = 0$. Then $p = 1$, and $V_1 = \langle x, x\tilde{y}, x\tilde{z}\rangle$.

The next blowing up is smooth. Therefore, the next recursion does not make the vectorspace any smaller, and we obtain the result

$$\langle x, x\tilde{y}, x\tilde{z}\rangle.$$

Now, we have to compute the inverse image (in the outer recursion). The result is

$$\langle x, y, z\rangle.$$

Checking also the other patches, one finds that the whole vectorspace $\langle x, y, z\rangle$ consists of 1-adjoints of the affine surface $x^4 + y^4 - z^2 = 0$. A new restriction comes from the affine patch obtained by dehomogenization with respect to z $(x^4 + y^4 - w^2 = 0)$. The final result is

$$V_{1,1} = \langle x, y\rangle.$$

For technical details, especially how to perform the blowing ups of curves, consult (Schicho, 1995). Another blowing-up method can be found in Kurke (1982).

Obviously, one can always find a singularity S matching the rules, provided that F is not smooth. Computationally, this amounts to solving a system of algebraic equations over the algebraic closure of k. This can be done with Gröbner bases (Buchberger, 1965, 1985; Becker and Weispfenning, 1993) as suggested in Schicho (1995) or with resultants (Collins, 1967; Brown and Traub, 1971; Buchberger *et al.*, 1982; Mishra, 1993), or with characteristic sets (Wu, 1986; Pfalzgraf and Wang, 1995). An equation solver specialized to three unknowns can be found in Kalkbrener (1995).

If S is not defined over the coefficient field k (but in some algebraic extension), then the subvectorspace V_1 cannot be generated by elements with coefficient in k. However, singularities which are not defined over k come in conjugated sets. Therefore, it is always possible to eliminate the algebraic coeffients in basis elements at some later step.

The precise meaning of "infinitely near points" is implicitly contained in the algorithm: a singularity in the infinitely near to a singularity S is a singularity on the blowing up along S. Therefore, we see that the algorithm "matches the definition". For a correctness proof, we refer to Schicho (1995).

Remark 4.1. A weakness of the algorithm *AdjointSubspace* is that we cannot give a bound for the worst-case complexity. The reason is that nobody knows a bound for the number of blowing ups required by Hironaka's resolution algorithm.

A consolation is the observation that surfaces are typically far less singular than they could be. Generically, there will only be a few, rather simple, singularities (see also Teitelbaum (1990)).

Remark 4.2. It is worthwhile trying improve the algorithm by choosing a different resolution method, as in Jung (1908), Walker (1935), Zariski (1939), Abhyankar (1969), Hironaka (1984), Lipman (1978), Villamayor (1989) and Bierstone and Milman (1991). For quasi-ordinary singularities, one might think of computing adjoints using two-dimensional Puiseaux-expansions (Alonso *et al.*, 1988).

4.3. simple applications

Here are formulae for the arithmetical genus p_a and for the plurigeni P_m, including the geometrical genus $p_g = P_1$ (see Shafarevich (1974)) for a discussion of these numbers).

Theorem 4.1. *For any surface in $\mathbf{P}^3$ of degree d, the following hold.*

(a) $p_a = d + 2v_{1,1} - v_{2,1} - 1$.
(b) $P_m = v_{0,m}$.

Proof. For (b) see Enriques (1949). For (a) the function $n \mapsto v_{n,1}$ is a polynomial for sufficiently large n, whose constant term is $p_a + 1$ (see Blass and Lipman (1979)). In the last section, we will show that the function coincides with the polynomial for $n \geq 1$, that the degree of the polynomial is 2, and that the leading coefficient is $\frac{d}{2}$. Therefore, the constant term is $d + 2v_{1,1} - v_{2,1}$.

Using these formulae and Castelnuovo's rationality criterion, it is easy to decide the existence of a parametrization (possibly over an algebraic extension of k).

Input: A homogeneous polynomial $F \in k[x, y, z, w]$ of degree d, representing a surface.
Output: `Exist` if there is a parametrization of F over the algebraic closure of k, `NotExist` otherwise.

Algorithm *DecideRationality*(F):
  $v_{1,1} := \#$ *AdjointSpace*$(F, 1, 1)$;
  $v_{2,1} := \#$ *AdjointSpace*$(F, 2, 1)$;
  $v_{0,2} := \#$ *AdjointSpace*$(F, 0, 2)$;
  if $d + 2v_{1,1} - v_{2,1} - 1 = v_{0,2} = 0$ then
    return `Exist`;
  else
    return `NotExist`;

The correctness of the algorithm *DecideRationality* follows immediately by Castelnuovo's criterion $p_a = P_2 = 0$ for the existence of a parametrization, and from the correctness of the subalgorithm *AdjointSpace* (the dimension $v_{n,m}$ is the cardinality of the basis returned by *AdjointSpace*).

5. The Parametrization Algorithm

Using the subalgorithm *AdjointSpace*, we construct a birational map $\phi : F \to B$, where B is one of the following.

1. The projective plane.
2. A quadric surface in $\mathbf{P}^3$.
3. A rational scroll (a surface with a pencil of lines).
4. A surface with a pencil of conics.
5. A Del Pezzo surface.

The map ϕ is represented by a basis Λ of $V_{n,m}$ (i.e. by the Output of *AdjointSpace*), where n, m are suitable integers. The choice of the basis is not important: a different basis leads to a variety B' which is projectively isomorphic to B. We say that ϕ is defined by $V_{n,m}$.

In 1 above, all we have to do is to invert the birational map ϕ.

In order to compute the equations of B in 2 and 5, we have to compute the algebraic relations between the forms in Λ. In general, these relations can be found with methods from elimination theory, for instance with Gröbner bases. In our situation, we know the degree of the relations in advance. Therefore, it suffices to compute the linear relations in the set Λ^d, where d is the degree. Once we have the equations of B, we compute a birational map $\psi : B \rightarrow \mathbf{P}^2$ by *QuadricSurface* or *DelPezzo*. Then, we invert the composed map $F \rightarrow B \rightarrow \mathbf{P}^2$.

In 3 and 4, it is not necessary to compute the equations of B. Instead, we construct an auxiliary rational map $\phi' : F \rightarrow C$, where C is a rational normal curve, such that ϕ maps the generic fibre of ϕ' to a line or conic. The map ϕ' is also defined by $V_{n',m'}$, for suitable integers n', m'. Then, we construct a birational map $\tau : C \rightarrow \mathbf{P}^1$ using the algorithm *RationalNormalScroll*, and apply the algorithm *FiberIsLine* or *FiberIsConic*.

5.1. The weighted projective case

The case where weighted projective spaces occur (Del Pezzo surfaces of degree 2 and 1) is an exception. Here, more than one vectorspace $V_{n,m}$ is involved in the definition of the map $\phi : F \rightarrow B$.

Suppose that there are integers n, m, such that the following hold.

We have $v_{n,m} = 3$. If $\{X_0, X_1, X_2\}$ is a basis for $V_{n,m}$, then X_0, X_1, X_2 are algebraically independent. We have $v_{2n,2m} = 7$. There is an element $X_3 \in V_{2n,2m}$, such that

$$X_0^2, X_0X_1, X_0X_2, X_1^2, X_1X_2, X_2^2, X_3$$

is a basis. The forms X_0, X_1, X_2, X_3 fulfill an algebraic relation of weighted degree 4, where $(w_0, w_1, w_2, w_3) = (1, 1, 1, 2)$.

We say that the map $(X_0 : X_1 : X_2; X_3)$ is defined by the pair $(V_{n,m}; V_{2n,2m})$. The image is a surface in weighted projective space with an equation of weighted degree 4. This could be a Del Pezzo surface of degree 2.

Del Pezzo surfaces of degree 1 are obtained in a similar way. Here we need a triple of vectorspaces $(V_{n,m}; V_{2n,2m}; V_{3n,3m})$ to define the map.

5.2. How to find suitable integers

This question is answered by a group of lemmas. We globally assume that the surface has positively passed the test *DecideRationality*.

The proofs for the lemmas appear in the Appendix.

Lemma 5.1. *There is an integer m such that $v_{1,m} = 0$.*

Throughout, let μ be the smallest integer with $v_{1,\mu+1} = 0$. Note that μ cannot be -1, because $v_{1,0} = 4$.

Lemma 5.2. (Case 1) *Suppose that $v_{1,\mu} = 3$ and $v_{2,2\mu} = 6$. Then $V_{1,\mu}$ defines a birational map to $\mathbf{P}^2$.*

Lemma 5.3. (Case 2) *Suppose that $v_{2,2\mu+1} = 1$. Then $v_{1,\mu} = 4$, and $V_{1,\mu}$ defines a birational map to a quadric surface.*

Lemma 5.4. (Case 3) *Suppose that* $v_{2,2\mu+1}\geq 2$*. Let* ϕ' *be the map defined by* $V_{2,2\mu+1}$*. Then one of the following two cases holds.*

(a) The image of the map ϕ' *is a rational normal curve. The map defined by* $V_{1,\mu}$ *is birational and maps the generic fibre of* ϕ' *to a line.*
(b) We have $v_{2,2\mu+1} = 3$*, and* ϕ' *is birational onto* $\mathbf{P}^2$*.*

Lemma 5.5. (Case 4) *Suppose that* $v_{1,\mu}\geq 2$*, and* $v_{2,2\mu+1} = 0$*, and either* $v_{1,\mu}\neq 3$ *or* $v_{2,2\mu} \neq 6$*. Let* ϕ' *be the map defined by* $V_{1,\mu}$*. Then the map defined by* $V_{2,2\mu-1}$ *is birational and maps the generic fibre of* ϕ' *to a conic.*

Lemma 5.6. (Case 5a) *Suppose that* $\mu\geq 2$ *and* $v_{1,\mu} = 1$*. Then* $2\leq v_{1,\mu-1}\leq 10$*, and one of the following holds.*

(a) If $v_{1,\mu-1}\geq 4$*, then* $V_{1,\mu-1}$ *defines a birational map to a Del Pezzo surface of degree* $v_{1,\mu-1} - 1$*.*
(b) If $v_{1,\mu-1} = 3$*, then* $(V_{1,\mu-1}; V_{2,2\mu-2})$ *defines a birational map to a Del Pezzo surface of degree 2.*
(c) If $v_{1,\mu-1} = 2$*, then* $(V_{1,\mu-1}; V_{2,2\mu-2}; V_{3,3\mu-3})$ *defines a birational map to a Del Pezzo surface of degree 1.*

Lemma 5.7. Case 5b *Suppose that* $\mu\leq 1$ *and* $v_{1,\mu} = 1$*. Then* $\mu = 1$*, and* $4\geq v_{2,1}\geq 10$*, and* $V_{2,1}$ *defines a birational map to a Del Pezzo surface of degree* $v_{2,1} - 1$*.*

As one can easily check, the above lemmas cover all possibilities. Here is an algorithm that tries to avoid unnecessary calls of the subalgorithm *AdjointSpace*.

Input: A homogeneous polynomial $F\in k[x,y,z,w]$, representing a surface.
Output: A quadruple $(X:Y:Z:W)\in k[s,t]^4$ of bivariate polynomials, representing a rational parametrization of the surface.

Algorithm *ParameterizeWithAdjoints*(F):
  $m := 0$;
  while *AdjointSpace*$(F,1,m+1)$ is not zero do
    $m := m+1$;
  $V :=$ *AdjointSpace*$(F,1,m)$; $r := \#\ V\ -1$;
  if $r = 0$ (Case 5) then
    $V' :=$ *AdjointSpace*$(F,1,m-1)$; $r' := \#\ V'\ -1$;
    if $m = 1$ (Case 5b) then
      $V' :=$ *AdjointSpace*$(F,2,1)$; $r' := \#\ V\ -1$;
    $d := 3$ for $r = 3$, 2 otherwise;
    if $r' = 2$ then
      $X_3 :=$ an element in *AdjointSpace*$(F,2,2m-2)$, linearly independent of V^2;
      append X_3 to V'; $r' := 3$;
      $d := 4$;
    if $r' = 1$ then
      $X_2 :=$ an element in *AdjointSpace*$(F,2,2m-2)$, linearly independent of V^2;
      $X_3 :=$ an element in *AdjointSpace*$(F,3,3m-3)$, linearly independent of V^3,

$V{\cdot}X_2$;
append X_2, X_3 to V'; $r' := 3$;
$d := 6$;
$G(x_0, \ldots, x_r) :=$ find the relations in V' (degree d);
$(S : T : U) :=$ *DelPezzo*(G);
substitute $(x_0, \ldots, x_r) := V'$ in $(S : T : U)$;
$(X : Y : Z : W) :=$ *InvertBirational*$(F, (S : T : U))$;
else if $r = 2$ and $\#$ *AdjointSpace* $(F, 2, 2m) = 6$ (Case 1) then
$(X : Y : Z : W) :=$ *InvertBirational*(F, V);
else if *AdjointSpace*$(F, 2, 2m + 1)$ is not zero do
$V' :=$ *AdjointSpace*$(F, 2, 2m + 1)$; $r' := \# V' - 1$;
if $r' = 0$ then (Case 2)
$Q(x_0, \ldots, x_3) :=$ find the relation in V (degree 2);
$(S : T : U) :=$ *QuadricSurface*(Q);
substitute $(x_0, \ldots, x_3) := V$ in $(S : T : U)$;
$(X : Y : Z : W) :=$ *InvertBirational*$(F, (S : T : U))$;
else
$\Gamma(x_0, \ldots, x'_r) :=$ find the relations in V' (degree 2);
if $r' = 2$ and Γ is empty then (Case 1)
$(X : Y : Z : W) :=$ *InvertBirational*(F, V');
else (Case 3)
$(T : U) :=$ *RationalNormalCurve*(Γ);
substitute $(x_0, \ldots, x'_r) := V'$ in $(T : U)$;
$(X : Y : Z : W) :=$ *FiberIsLine*$(F, (T : U), V)$;
else (Case 4)
$\Gamma(x_0, \ldots, x_r) :=$ find the relations in V (degree 2);
$(T : U) :=$ *RationalNormalCurve*(Γ);
substitute $(x_0, \ldots, x_r) := V$ in $(T : U)$;
$V' :=$ *AdjointSpace*$(F, 2, 2m - 1)$;
$(X : Y : Z : W) :=$ *FiberIsConic*$(F, (T : U), V')$;
return $(X : Y : Z : W)$;

EXAMPLE 5.1. Let $F = x^2w^3 + y^3w^2 + z^5$.

After the while loop, we have $m = 3$, because $V_{1,1}$, $V_{1,2}$, $V_{1,3}$ are not zero and $V_{1,4}$ is zero. Since $v_{1,3} = 1$, we have $r = 0$, and we are in the case (5). We have

$$V' = V_{1,2} = \langle zw^2, w^3\rangle$$

and $r = 1$, so we have a Del Pezzo surface of degree 1.

An element in $V_{2,4}$ which is not in $V_{1,2}^2$ is $X_2 = yw^5$. An element in $V_{3,6}$ which is not in $V_{2,4}{\cdot}V_{1,3}$ is $X_3 = xw^8$. The algebraic relation between $X_0 = zw^2$,$X_1 = w^3$,X_2,X_3 is

$$G(x_0, x_1, x_2, x_3) = x_0^5x_1 + x_2^3 + x_3^2 = 0.$$

We apply the algorithm *DelPezzo* and obtain the birational map $(x_0x_2 : x_3 : x_0^3)$ (see Example 3.10). By substitution, we obtain the composed map

$$(S : T : U) = (yzw^7 : xw^8 : z^3w^6) = (yzw : xw^2 : z^3).$$

We apply *InvertBirational* and obtain

$$(X : Y : Z : W) = (t : -s^5 - s^2t^2 : 1 : -(s^3 + t^2)^3).$$

EXAMPLE 5.2. Let $F = (x^4 + z^4)w^2 + (xy + z^2)^3$.

After the while loop, we have $m = 1$, because $V_{1,1}$ is not zero and $V_{1,2}$ is zero. Then,

$$V = V_{1,1} = \langle z^2w, xzw, x^2w, xyz + z^3, x^2y + x^2z\rangle$$

and $r = 4$.

Next, we find that $V_{2,3}$ is not zero. We have

$$V' = V_{2,3} = \langle xz^5w^2, z^6w^2\rangle$$

and $r' = 1$. Therefore, we are in Case 3. The image of the map $\phi' = (xz^5w^2 : z^6w^2)$ is a rational normal curve C, and its generic fibre is mapped to a line by the birational map

$$\phi = (z^2w : xzw : x^2w : xyz + z^3 : x^2y + x^2z).$$

In this case, the rational normal curve is already $\mathbf{P}^1$, so the algorithm *RationalNormalCurve* just returns its input. We apply the algorithm *FiberIsLine* to the input

$$F, (xz^5w^2 : z^6w^2) = (x : z), \phi$$

and obtain

$$(X : Y : Z : W) = (-s^3t^2 : t^3 + s^4t + t : -st^3 : s^5 + s)$$

(see Example 3.3).

THEOREM 5.1. *The Algorithm ParameterizeWithAdjoints is correct.*

PROOF. Termination follows from Lemma 5.1. The correctness follows from Lemmas 5.2–5.7, and from the correctness of the subalgorithms. The proof of all these lemmas is the main content of the appendix.

REMARK 5.1. Sometimes, we can save calls of *AdjointSpace* by using the fact

$$V_{n,m} \cdot V_{n',m'} \subset V_{n+n',m+m'}$$

(see Section 4.1).

In Case 5, when the Del Pezzo surface has degree 2, we need an element in $V_{2,2\mu-2}$, which is not contained in $V_{1,\mu-1}^2$. Since $\mu \geq 2$ must hold, we have already computed $V_{1,\mu-2}$ and $V_{1,\mu}$ before. It might be the case that their product contains such an element. A similar thing is possible for in the case of Del Pezzo surfaces of degree 1.

In Case 4, we use the vectorspace $V_{2,2\mu-1}$ as an input for *FiberIsConic*. But the subalgorithm *FiberIsConic* does not use all basis elements in general: it just looks for three elements fulfilling a certain independence condition. Now, we have already computed $V_{1,\mu-1}$ and $V_{1,\mu}$ before. It might be the case that their product contains three such elements.

Another option is always to compute the image of the map defined by $V_{1,\mu}$, in order to tell the case in which we are in. This might save the computation of $V_{2,2\mu}$ or $V_{2,2\mu+1}$.

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References

Abhyankar, S. S. (1969). Resolution of singularities of algebraic surfaces. In *Algebraic Geometry*, pp. 1–11. Oxford University Press.

Abhyankar, S. S., Bajaj, B. (1987). Automatic parametrization of curves and surfaces III. *Comp. Aided Geom. Des.*, **5-4**, 309–323.

Alonso, M. E., Luengo, I., Raimondo, M. (1988). An algorithm on quasi-ordinary polynomials. *Appl. Algebra Eng. Commun. Comput.*, **6**, 59–73.

Becker, T., Weispfenning, V. (1993). *Gröbner Bases - A Computational Approach to Commutative Algebra*. Graduate Texts in Mathematics. Springer Verlag.

Bierstone, E., Milman, P. (1991). A simple constructive proof of canonical resolution of singularities. In Lopez, A. C., Macarro, L. N., eds, *Effective Methods in Algebraic Geometry*, pp. 11–30. Birkhauser.

Blass, P., Lipman, J. (1979). Remarks on adjoints and arithmetic genera of algebraic varieties. *Am. J. Math.*, **101**, 331–336.

Böhm, W., Farin, G., Kahmann, J. (1984). A survey of curve and surface methods in cagd. *Comp. Aided Geom. Design*, **1**, 1–60.

Brown, W. S., Traub, J. F. (1971). On Euclid's algorithm and the theory of subresultants. *Journal ACM*, **18**, 505–514.

Buchberger, B. (1965). An Algorithm for Finding a Basis for the Residue Class Ring of a Zero-Dimensional Polynomial Ideal. PhD thesis, Universitat Innsbruck, Institut fur Mathematik. German.

Buchberger, B. (1985). Groebner bases: an algorithmic method in polynomial ideal theory. In Bose, N. K., ed., *Recent Trends in Multidimensional Systems Theory*, chapter 6. D. Riedel Publ. Comp.

Buchberger, B., Collins, G. E., Loos, R., eds, (1982). *Computer Algebra: Symbolic and Algebraic Computation*. Springer, Vienna-New York, 2nd edn.

Canny, J., Manocha, D. (1992). Implicit representation of rational parametric surfaces. *J. Symb. Comput.*, **13**(5).

Castelnuovo, G. (1939). Sulle superficie di genere zero. In *Memoria Scelte*, pp. 137–187. Zanichelli.

Chionh, E.-W., Goldman, R. (1992). Degree, multiplicity and inversion formulae for rational surfaces using u-resultants. *Comp. Aided Geom. Des.*, **9**, 93–108.

Clebsch, A. (1868). Über die Abbildungen algebraischer Flächen, insbesondere der vierten und fünften Ordnung. *Math. Ann.*, **1**, 253–316.

Collins, G. E. (1967). Subresultants and reduced polynomial remainder sequences. *Journal ACM*, **14**, 128–142.

Conforto, F. (1939). *La Superfici Razionali*. Zanichelli.

Eisenbud, D. (1993). Open problems in computational algebraic geometry. In *Cortona 1991*, volume 34 of *Proc. Symp. Math.*, pp. 49–70.

Enriques, F. (1895). Sulle irrazionalita da cui puo farsi dipendere la risoluzione d'un equazione f(xyz)=0 con funzioni razionali di due parametri. *Math. Ann.*, 1–23.

Enriques, F. (1949). *La Superfici Algebriche*. Zanichelli.

Farin, G. F. (1988). *Curves and Surfaces for Computer Aided Geometric Design: A practical guide*. Academic Press.

Gao, X.-S., Chou, S.-C. (1992). Implicitization of rational parametric equations. *J. Symb. Comput.*, **14**, pp. 459–470.

Grauert, H., Riemenschneider, O. (1970). Verschwindungssätze für analytische Kohomologiegruppen. *Invent. Math.*, **11**, 263–292.

Greenberg, M. J. (1969). *Lectures on Forms in Many Variables*. Benjamnin.

Griffiths, P., Harris, J. (1978). *Principles of Algebraic Geometry*. John Wiley.

Hartshorne, R. (1977). *Algebraic Geometry*. Springer–Verlag.

Hillgarter, E. (1996). Rational points on conics. Master's thesis, RISC Linz.

Hironaka, H. (1984). Desingularization of excellent surfaces. In Gossart, V., Giraud, J., Orbanz, U., eds, *Resolution of Surface Singularities*, pp. 99–132. Springer.

Jung, H. (1908). Darstellung der Funktionen eines algebraischen Körpers zweier unabhängiger Veränderlicher. *J. Reine Angew. Math.*, **133**, 289–314.

Kalkbrener, M. (1990). Implicitization of Rational Curves and Surfaces. In Sakata, ed., *Lect. Notes in Comp. Sci. 508, AAECC-8*, Tokyo, Japan.

Kalkbrener, M. (1995). An algorithm for solving systems of algebraic equations in three variables. In Pfalzgraf, J., Wang, D., eds, *Automated Practical Reasoning*, pp. 81–90. Springer.

Kurke, H. (1982). *Vorlesungen über Algebraische Flächen*. Teubner.

Lipman, J. (1978). Desingularization of two-dimensional schemes. *Ann. Math.*, **107**, 151–207.

Mñuk, M. (1996). Algebraic parametrization of rational curves. PhD thesis, RISC Linz.

Mñuk, M., Sendra, J. R., Winkler, F. (1996). On the complexity of parametrizing curves. *Beitr. Algebra und Geometrie*, **37/2**.

Manin, Y. (1974). *Cubic Forms*. North-Holland.

Mishra, B. (1993). *Algorithmic Algebra*. Springer, New York.

Nöether, M. (1871). Zur Theorie eineindeutigen Entsprechens höherer Raumgebilde. *Math.Ann.*, **4**.
Pfalzgraf, J., Wang, D. (1995). *Automated Practical Reasoning*. Springer.
Pottmann, H. (1996). Applications of laguerre geometry in cagd. Technical Report 30,31, Techn. Univ. Wien.
Qiulin, D., Davies, B. J. (1987). *Surface Engineering Geometry for Computer-aided Design and Manufacture*. John Wiley.
Reider, I. (1988). Vector bundles of rank 2 and linear systems on algebraic surfaces. *Ann.Math.*, **127**, 309–316.
Schicho, J. (1992). On the choice of pencils in the parametrization of curves. *J. Symb. Comput.*, **14**, 557–576.
Schicho, J. (1995). Rational Parametrization of Surfaces. PhD thesis, RISC Linz.
Sederberg, T. W., Snively, J. P. (1987). Parametrization of cubic algebraic surfaces. In *The Mathematics of Surfaces*, pp. 299–341. Clarendon Press.
Sendra, J., Winkler, F. (1991). Symbolic parametrization of curves. *J. Symb. Comput.*, **12**, 607–632.
Sendra, J., Winkler, F. (1997). Parametrization of algebraic curves over optimal field extensions. *J. Symb. Comput.*, **23**, 191–208.
Shafarevich, I. R., ed. (1965). *Algebraic Surfaces*. Proc. Steklov Inst. Math. transl. by AMS 1967.
Shafarevich, I. R. (1974). *Algebraic Geometry*. Springer.
Teitelbaum, J. (1990). The computational complexity of the resolution of plane curve singularities. *Math. Comp.*, **54**.
Tran, Q.-N., Winkler, F. (1997). *CASA Reference Manual*. http://www.risc.uni-linz.ac.at/software/casa.
van Hoeij, M. (1994). Computing parametrizations of rational algebraic curves. In *ISSAC-94*, pages 187–190. ACM Press.
van Hoeij, M. (1997). Rational parametrizations of rational algebraic curves. *J. Symb. Comput.*, to appear.
Villamayor, O. (1989). Constructiveness of Hironaka's resolution. *Ann. Scient. Ecole Norm. Sup. 4*, **22**, 1–32.
Volcheck, E. (1995). How to compute the adjoint divisor of a curve over a finite field. preprint.
Walker, R. J. (1935). Reduction of the singularities of an algebraic surface. *Ann. Math.*, **36**, 336–365.
Walker, R. J. (1978). *Algebraic Curves*. Springer.
Wu, W. T. (1986). Basic principles of mechianical theorem proving in elementary geometries. *J. Autom. Reasoning*, **2**, 221–252.
Zariski, O. (1939). The reduction of singularities of an algebraic surface. *Ann. Math.*, **40**, 639–689.
Zariski, O. (1971). *Algebraic Surfaces*. Springer, 2nd edn.

Appendix: The Correctness Proof

In order to show the correctness of the algorithm *ParameterizeWithAdjoints*, we have to prove Lemmas 5.1–5.7. We also have to finish the proof of Lemma 4.1, which is needed for the correctness of *DecideRationality*.

For the proof of all lemmas, we may assume that k is algebraically closed, since all involved statements are stable under field extensions.

Most proofs are written in the language of divisors on smooth surfaces. In two exceptions (Lemma 4.1 and Theorem 5.2), it is necessary to use sheaf theory.

Theorem 5.2. *Let $F\in\mathbf{P}^3$ be a surface of degree d. Let $\pi : \tilde{F}\to F$ be a desingularization of F. Let $H\in\mathrm{Cl}\tilde{F}$ be the class of the pullback of a plane section. Let $K\in\mathrm{Cl}\tilde{F}$ be the canonical class. Let $m > 0$, $n\geq 0$ be integers.*

Then $\dim|nH + mK| = v_{n,m} - 1$. If this number is ≥ 0, and ϕ is the map defined by $V_{n,m}$, then $\phi\circ\pi$ is associated to the linear system $|nH + mK|$.

Proof. Let $\mathcal{A}_m$ be the ideal sheaf generated by the m-adjoints. The formula

$$\pi_*(\mathcal{O}_{\tilde{F}}(mK))\cong\mathcal{A}_m\otimes\mathcal{O}_F(m(d-4))$$

is well known (see Blass and Lipman (1979)). By the projection formula, we have

$$\pi_*(\mathcal{O}_{\tilde{F}}(nH + mK))\cong\mathcal{A}_m(n + m(d-4)).$$

The theorem is proved if we can show that all global sections of the right-hand side are forms (of degree $n+m(d-4)$).

Let $f:\mathcal{O}_{\mathbf{P}^3}\to\mathcal{O}_F$ be the natural map, and let $\mathcal{A}'_m := f^{-1}(\mathcal{A})$, so that we have an exact sequence

$$0\to\mathcal{O}_{\mathbf{P}^3}(n+m(d-4)-d)\to\mathcal{A}'_m(n+m(d-4))\to\mathcal{A}_m(n+m(d-4))\to 0$$

of sheafs on $\mathbf{P}^3$. Since the kernel sheaf has no cohomology in degree 1, every global section of the quotient sheaf comes from a global section of the middle sheaf.

Proof of Theorem 4.1. Let $h:\mathbf{N}\to\mathbf{N}$ be the function $n\mapsto v_{n,1}$. We have to show that h is a polynomial for $n\geq 1$, that the degree of the polynomial is 2, and that the leading coefficient is $\frac{d}{2}$.

Let $\tilde{F}$, H and K as in Theorem 5.2. Since H is semi-ample and $H^2=d>0$, higher cohomology of $\mathcal{L}(nH+K)$ vanishes for $n\geq 1$, by the Grauert–Riemenschneider vanishing theorem Grauert and Riemenschneider (1970). Hence

$$v_{n,1}=\dim|nH+K|-1=\frac{d}{2}n^2+\frac{H\cdot K}{2}n+1+p_a$$

by the Riemann–Roch theorem.□

Proof of Lemma 5.1. By Castelnuovo's criterion, there is a proper parametrization $\sigma:\mathbf{P}^2\to F$. Let n be the degree of the polynomials defining σ.

By resolving the base points of σ, we obtain a desingularization $\sigma':\mathbf{Y}\to F$ of F. Let L be the pullback of the class of lines in $\mathbf{P}^2$, let H be the class of a pullback of a plane section in F, and let K be the canonical class. Then, we have $L\cdot H=n$, $L\cdot K=-3$, and L is numerically eventually free (nef) (a divisor L is nef iff $L\cdot C\geq 0$ for every curve C). If m is such that $3m>n$, then

$$(H+mK)\cdot L=n-3m<0,$$

hence $|H+mK|$ is empty. Therefore, $v_{1,m}=0$ by Theorem 5.2.□

From now on, we assume the vanishing of p_a and P_2. Note that $P_2=0$ implies $p_g=0$, where p_g is the geometrical genus (see Shafarevich (1965)).

We study pairs (X,D), where X is a smooth projective surface over an algebraically closed field, and D is a divisor class of X.

A curve E is called a -1-*curve* iff $E^2=E\cdot K=-1$. A pair (X,D) is called *minimal* iff there exists no -1-curve E such that $D\cdot E=0$. If (X,D) is not minimal, then we construct a minimalization of (X,D) recursively by blowing down some -1-curve E with $D\cdot E=0$. Eventually, we arrive at a minimal pair (X_0,D_0), and a birational regular minimalization map $\pi:X\to X_0$, such that $\pi^*D_0=D$.

Let (X,D) be a minimal pair. Let K be the canonical class of X. A minimalization (X',D') of $(X,D+K)$ is called an *adjoint pair* of (X,D).

A curve C is called *rigid* iff $\dim|C|=0$ (the only divisor in $|C|$ is C itself).

Lemma 5.8. *Let C be a rigid curve. Then $C\cdot K\geq -1$, and equality holds iff C is a -1-curve.*

Proof. The genus formula gives $C\cdot K+C^2\geq -2$. The Riemann–Roch formula gives $C\cdot K-C^2\geq 0$. This shows the inequality. In case of equality, we have $C^2=C\cdot K=-1$, i.e. C is a -1-curve.

LEMMA 5.9. *Let (X, D) a minimal pair, and let (X', D') be an adjoint pair. Suppose that D is nef, and the linear systems $|D|$, $|D'|$ are not empty. Then D' is also nef.*

PROOF. Minimalization does not change the nef property and the dimension. Hence, we may equivalently show that $D + K$ is nef under the assumption that $|D + K|$ is not empty.

Assume, indirectly, that there is a curve C with $(D + K){\cdot}C < 0$. Then C must be a fixed component of the nonempty linear system $|D + K|$. By Lemma 5.8 and nefness of D, C is a -1-curve and $C{\cdot}D = 0$. But this contradicts minimality of (X, D).

LEMMA 5.10. *Let (X, D) a minimal pair such that $p_a = p_g = 0$. Let (X', D') be an adjoint pair. Suppose that D is nef, $D^2 > 0$, the linear systems $|D|$, $|D'|$, $|D' + K'|$ are not empty (where K' is the canonical class of X'). Then $D'^2 > 0$.*

PROOF. The direct image of D is $D' - K'$. By factoring the map $X{\to}X'$ into blowing downs, one shows easily that $D' - K'$ is nef and $(D' - K')^2{\geq}D^2 > 0$.

By Lemma 5.9, D' is nef. Hence,

$$0 \leq D'{\cdot}(D' + K') + D'{\cdot}(D' - K') = 2D'^2.$$

Assume, indirectly, that equality holds. Then $D'{\cdot}(D' - K') = 0$, hence D' is numerically zero by the Hodge index theorem. But $|D'|$ is not empty, hence $D' = 0$. But then, $|K'|$ is not empty, contradicting $p_g = 0$.

From now on, we fix the following notation. Starting from the input surface $F{\in}\mathbf{P}^3$, we construct a chain of minimal pairs in the following way.

Let $\pi : X^0{\to}F$ be the minimal desingularization of F. Let D^0 be the class of the pullback of a plane section. Obviously, (X^0, D^0) is minimal.

For $i{\geq}0$, define (X^{i+1}, D^{i+1}) as an adjoint pair of (X^i, D^i). Denote by $\tau_i : X^0{\to}X^i$ the composite of the minimalization maps $\sigma_i : X^{i-1}{\to}X^i$. Define $\pi_i : X^i{\to}F$ as $\pi{\circ}\tau_i^{-1}$. Finally, K^i is the canonical class of X^i.

The following lemma is a refinement of Theorem 5.2.

LEMMA 5.11. *Let $n{\geq}0$, $q{\geq}0$, $r{\geq}0$, such that $m := qn + r > 0$.*

(a) $\dim|nD^q + rK^q| = v_{n,m} - 1$.
(b) If the integer above is ≥ 0, and ϕ is the map defined by $V_{n,m}$, then the rational map $\phi{\circ}\pi_q$ is associated to the linear system $|nD^q + rK^q|$.

PROOF. For fixed n, m we proceed by induction on q. For $q = 0$, the statement follows from Theorem 5.2. For the induction step, we have to consider the linear system $|nD^{q+1} + (r - n)K^{q+1}|$, where $r - n{\geq}0$. Its pullback along σ_{q+1} is

$$|n(\sigma_{q+1})^*D^{q+1} + (r - n)(\sigma_{q+1})^*K^{q+1}| = |nD^q + nK^q + (r - n)(\sigma_{q+1})^*K^{q+1}|.$$

By induction hypotheses, it suffices to show that the pullback system and the linear system $|nD^q + rK^q|$ have the same dimension and, if the dimension is nonnegative, the same associated map. (Note that also (c) follows, because the composition of regular maps is regular.)

More general, let $f : X{\to}Y$ be a birational regular map. Let D be a class on Y

that lies in the image of f^*. Then we show the linear systems $\Gamma_1 := |D + rK(X)|$ and $\Gamma_2 := |D + rf^*K(Y)|$ have same dimension and, if the dimension is nonnegative, same associated map.

First, we assume that f is the blowing down of a -1-curve E. Then $\Gamma_1 = \Gamma_2 + rE$. But E is an r-fold fixed component of Γ_1, because $D{\cdot}E = 0$ and $E{\cdot}K(X) = E^2 = -1$.

In the general case, f is a composite of blowing downs (see Shafarevich (1974)). Since the statement holds for each component, it also holds for f.

Lemma 5.12. *For $i{\le}\mu$, D^i is nef. For $i < \mu$, $(D^i)^2 > 0$.*

Proof. This follows immediately by induction on i, using Lemmas 5.9–5.10.

Lemma 5.13. *Let (X, D) a minimal pair, such that $|D|$ is not empty, D is nef, $D^2 > 0$, and $\dim|D + K| = 0$. Then $D = -K$.*

Proof. By the Riemann–Roch formula, we have $D{\cdot}(D + K){\le}0$. Since D is nef and $|D + K|$ is not empty, we have $D{\cdot}(D + K) = 0$. By the Hodge index theorem, $D + K$ is numerically zero. But $|D + K|$ is not empty, hence $D + K = 0$.

Proof of Lemmas 5.6 and 5.7. Suppose that $v_{1,\mu} = 1$. Then $\mu{\ge}1$ because $v_{1,0} = 4$. We have $\dim|D^{\mu-1} + K^{\mu-1}| = 0$ by Lemma 5.11 ($n = 1$, $q = \mu - 1$, $r = 1$). By Lemma 5.13, $D^{\mu-1} = -K^{\mu-1}$. Moreover, $D^{\mu-1}$ is nef and $n := (D^{\mu-1})^2 > 0$ by Lemma 5.9 and Lemma 5.10.

If X is a surface with $p_a = P_2 = 0$ and K nef and $K^2 > 0$, then the map associated with $| - K|$ (in the case $K^2{\ge}3$), $| - 2K|$ (in the case $K^2 = 2$) or $| - 3K|$ (in the case $K^2 = 1$) is an embedding except for the blowing down of -2-curves, and the image is a Del Pezzo surface of degree K^2 (see Manin (1974) and Kurke (1982)).

If $\mu{\ge}2$, then we apply Lemma 5.11 ($n = 1$, $q = \mu - 1$, $r = 0$), and Lemma 5.6 follows.

If $\mu = 1$, then we apply Lemma 5.11 ($n = 2$, $q = 0$, $r = 1$). If $V_{1,1}$ is generated by G, then $V_{2,1}$ has at least four linearly independent elements xG, yG, zG, wG. Whence Lemma 5.7.□

Lemma 5.14. *Let (X, D) a minimal pair, and let (X', D') be an adjoint pair. Suppose that D is nef, $D^2 > 0$, $D'^2 = 0$, $D'{\ne}0$, and the linear systems $|D|$, $|D'|$ are not empty. Then $D' = nP$ for some $n > 0$ and P, $P^2 = 0$, $P{\cdot}K' = -2$. The map associated with $|D' = nP|$ is equal to the map associated with $|P|$ (with image $\mathbf{P}^1$) composed with the embedding of $\mathbf{P}^1$ as a rational normal curve in $\mathbf{P}^n$.*

Proof. By Lemma 5.9, D' is nef. By Lemma 5.10, $|D' + K'|$ is empty.

Let C be any prime component of a divisor in $|D'|$. Then $D'{\cdot}C = 0$ since D' is nef.

We show that C is not rigid. Assume the contrary. By Lemma 5.8 and minimality of (X', D'), we have $C{\cdot}K'{\ge}0$. The class $D' - K'$ is nef (since it is the direct image of D), hence $C{\cdot}K' = 0$. Moreover, $(D' - K')^2{\ge}D^2 > 0$. Hence, C is numerically zero by the Hodge index theorem. But this is absurd, because C is positive.

Because C is not rigid, we have $C^2{\ge}0$. This holds for any component of a divisor in D'. But $D'^2 = 0$, hence $C_1{\cdot}C_2 = 0$ for any two components of two divisors in $|D'|$. This implies that all possible components lie in one pencil $|P|$, every divisor in $|P|$ is irreducible and $P^2 = 0$. This implies the assertion on the associated map.

It remains to show $P{\cdot}K' = -2$. Suppose, indirectly, that this were not the case. Since the divisors in $|P|$ are irreducible, $P{\cdot}K'\geq 0$ by the genus formula. Then $D'{\cdot}K'\geq 0$ and $|D' + K'|$ is not empty by Riemann–Roch. But this contradicts Lemma 5.10.

Lemma 5.15. *Let (X, D) be a minimal pair such that X is a rational surface, D is nef, $D^2 > 0$, $|D|$ is not empty, and $|D + K|$ is empty. Then $|D|$ has no base points, and the associated map is birational. Moreover, one of the following is true.*

(a) $X = \mathrm{P}^2$, and $D = L$ (the class of lines).
(b) $X = \mathrm{P}^2$, and $D = 2L$.
(c) X is a rational ruled surface. There is a ruling $|P|$, such that $P^2 = 0$, $P{\cdot}K = -2$, and $2D + K = nP$ for some $n\geq 0$.

Proof. We use the following facts from the theory of rational surfaces (see Kurke (1982) and Schicho (1995)).

> Let X be a rational surface. Then $K(X)^2\leq 9$. Equality holds only if $X = \mathbf{P}^2$. If $K^2 = 8$, then X is a ruled.
>
> Let C be a class on a rational ruled surface. If C is nef, then $|C|$ has no base points. If C is nef and $C^2 > 0$, then the associated map is birational. If $C^2 = 0$ and $|C|$ is not empty, then $C = nP$ for a rational pencil P and $n\geq 0$.

By Riemann–Roch and the assumption that $|D+K|$ is empty, we have $D^2+D{\cdot}K\leq -2$, and also $D{\cdot}K\leq -3$.

Let $m > 1$ be the integer uniquely defined by the inequality

$$0\leq mD^2 + D{\cdot}K\leq D^2 - 1.$$

By Riemann–Roch, $|mD + K|$ is not empty. mD is nef, hence also $mD + K$ is nef by Lemma 5.9. Then

$$\begin{aligned}
0 &\leq (mD + K)^2 = m^2D^2 + mD{\cdot}K + mD{\cdot}K + K^2 \\
&\leq mD^2 - m + D{\cdot}K + (m-1)D{\cdot}K + K^2 \\
&\leq D^2 - 1 - m + D{\cdot}K + (m-2)D{\cdot}K + K^2 \\
&\leq -2 - 1 - m + (m-2)(-3) + K^2 = 3 - 4m + K^2 \leq 12 - 4m,
\end{aligned}$$

hence $m\leq 3$.

If $m = 3$, then we must have equality everywhere: $D^2 = 1$, $D{\cdot}K = -3$, $K^2 = 9$. Obviously, (a) holds, and the map associated to $|D|$ is the identity $X\to\mathbf{P}^2$.

If $m = 2$, then

$$0 \leq (2D + K)^2 = 4(D^2 + D{\cdot}K) + K^2 \leq -8 + K^2.$$

The right inequality must be an equation: $D^2 + D{\cdot}K = -2$ (because $K^2 < 12$).

If $K^2 = 9$, then $D^2 = 4$, $D{\cdot}K = -6$. Obviously, (b) holds, and the map associated to $|D|$ is the Veronese embedding $X = \mathbf{P}^2\hookrightarrow\mathbf{P}^5$.

If $K^2 = 8$, then $(2D + K)^2 = 0$. Hence X is ruled, $|D|$ has no base points, the map associated is birational, and (c) holds.

Proof of Lemma 5.2. Suppose that $v_{1,\mu} = 3$ and $v_{2,2\mu} = 6$. Then $\mu \geq 1$, since $v_{1,0} = 4$. By Lemma 5.11 ($n = 1$, $q = \mu$, $r = 0$), we have $\dim|D^\mu| = 2$. We distinguish two cases.

If $(D^\mu)^2 = 0$, then $D_\mu = 2P$, and $\dim|2D^\mu| = \dim|4P| = 4$. By Lemma 5.11 ($n = 2$, $q = \mu$, $r = 0$), we obtain $v_{2,2\mu} = 5$, a contradiction. This case is therefore impossible.

If $(D^\mu)^2 > 0$, then the map associated to $|D^\mu|$ is birational to $\mathbf{P}^2$ (by Lemma 5.15). Then the map defined by $V_{1,\mu}$ is also birational.□

Proof of Lemmas 5.3 and 5.4. Suppose that $v_{2,2\mu+1} \geq 1$. By Lemma 5.11 ($n = 2$, $q = \mu$, $r = 1$), we have $\dim|2D^\mu + K^\mu| \geq 0$. This implies $D^\mu \neq 0$. We distinguish two cases.

If $(D^\mu)^2 = 0$, then $D^\mu = nP$ with $P \cdot K = -2$ by Lemma 5.14. Then $P \cdot (2D^\mu + K^\mu) = -2$. This is a contradiction, because P is nef. This case is therefore impossible.

If $(D^\mu)^2 > 0$, then the map associated with $|D^\mu|$ is birational, and one of the subcases (a), (b) or (c) in Lemma 5.15 holds. We treat the subcases separately.

In Subcase (a), $2D^\mu + K^\mu = -L$, which contradicts $\dim|2D^\mu + K^\mu| \geq 0$.

In Subcase (b), $2D^\mu + K^\mu = L$. Therefore, the map associated with $|2D^\mu + K^\mu|$ is the identity. By Lemma 5.11 again, the map ϕ' defined by $V_{2,2\mu+1}$ is birational to $\mathbf{P}^2$.

In Subcase (d), we have $2D^\mu + K^\mu = nP$ and $n = \dim|2D^\mu + K^\mu| = v_{2,2\mu+1} - 1$. If $n = 0$, then then $(D^\mu)^2 = (K^\mu)^2/4 = 2$, and the map associated with $|D^\mu|$ maps X^μ birationally to a quadric surface in $\mathbf{P}^3$. If $\mu \geq 1$, then we apply Lemma 5.11 ($n = 1$, $q = \mu$, $r = 0$), and find that the map defined by $V_{1,\mu}$ maps birationally to a quadric surface to $\mathbf{P}^3$. If $\mu = 0$, then F must have been a quadric surface itself, because X^0 is a desingularization, D^0 is the class of the pullback of a plane section, and $(D^0)^2 = 2$. Thus, we have proved Lemma 5.3.

If $n \geq 1$, then the image map associated with $|2D^\mu + K^\mu| = |nP|$ is a rational normal curve. The fibres are the divisors in $|P|$. The map associated to $|D^\mu|$ maps these divisors to projective curves of degree

$$D^\mu \cdot P = \frac{(nP - K^\mu) \cdot P}{2} = 1,$$

i.e. to lines. In the case $\mu \geq 1$, we conclude Lemma 5.4 from Lemma 5.11 ($n = 1$, $q = \mu$, $r = 0$). In the case $\mu = 0$, we have $D^0 \cdot P = 1$, where D^0 is the class of the pullback of a hyperplane section. Therefore, the divisors in $|P|$ are mapped to lines on F. This finishes the proof of Lemma 5.4.□

Proof of Lemma 5.5. Suppose that $v_{1,\mu} \geq 2$, and $v_{2,2\mu+1} = 0$, and either $v_{1,\mu} \neq 3$ or $v_{2,2\mu} \neq 6$. By Lemma 5.11, we have that $|2D^\mu + K^\mu|$ is empty, $\dim|D^\mu| \geq 2$, and either $\dim|D^\mu| \neq 2$ or $\dim|2D\mu| \neq 5$. Thus, we can rule out the three cases with $(D^\mu)^2 > 0$ in Lemma 5.15; it remains only the case $D^\mu \neq 0$ and $(D^\mu)^2 = 0$.

By Lemma 5.14, $D^\mu = nP$ for some $n \geq 1$ and P with $P^2 = 0$, $P \cdot K^\mu = -2$. Let P' be the pullback of P on $X^{\mu-1}$. We distinguish two cases.□

Case 1: $\mu \geq 2$. Then $D^{\mu-1} - K^{\mu-1}$ is nef, and $(D^{\mu-1} - K^{\mu-1})^2 = 4(D^{\mu-1})^2 - 8n > 0$ by Lemma refseries. Hence $(D^{\mu-1})^2 > 2$ and $(2D^{\mu-1})^2 \geq 11$. By a theorem of Reider (1988), $|2D^{\mu-1} + K^{\mu-1}|$ has no base points and the associated map is birational (see also Schicho (1995)). Since $P' \cdot (2D^{\mu-1} + K^{\mu-1}) = 2$, the generic divisor of $|P'|$ is mapped to a conic. By Lemma 5.11 ($n = 2$, $q = \mu - 1$, $r = 1$), the map defined by $V_{2,2\mu-1}$ is birational and maps the generic fibre of ϕ' to a conic.

Case 2: $\mu = 1$. Since $2D^0 + K = D^0 + nP'$, the map associated to $|D^0|$ is birational, and both $|D^0|$ and $|nP'|$ do not have base points, we conclude that $|2D^0 + K|$ does not have base points and the associated map is birational. As in Case 1, the generic divisor in $|P'|$ is mapped to a conic.

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