Topological properties of attractors for dynamical systems

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Abstract

For a dynamical system \{S_t\} on a metric space \(X\), we examine the question whether the topological properties of \(X\) are inherited by the global attractor \(A\) (if it exists). When \{S_t\} is jointly continuous, we prove that the Čech–Alexander–Spanier cohomology groups of \(A\) are isomorphic to the corresponding cohomology groups of \(X\). The same conclusion is obtained in the case where \{S_t\} is a group and \(A\) has a bounded neighborhood which is a deformation retract of \(X\).

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1. Introduction

Dynamical systems (or semigroups) are a fundamental tool in the description and in the study of many important problems of natural sciences.

In Banach spaces, continuous semigroups arise in a natural way e.g. from the Cauchy problem for the autonomous differential equation \(u'(t) = f(u(t))\), provided that, for all initial data \(u(0) = u_0\), there exists a unique global solution for all positive times, which depends continuously on \(u_0\).

In metric spaces discrete semigroups arise simply by considering the successive powers of any continuous map \(f: X \to X\), i.e. by setting \(S^n(x) := f^n(x)\). In this case “\(f\) generates \(S^n\)”. In many applications, discrete semigroups are used as numerical approximations of continuous ones: note that digital computers always consider discrete dynamical systems, even when solving differential equations.

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One of the most fascinating problems of the theory of semigroups is the so-called asymptotical dynamic, i.e. the long-term behaviour of the system \( t \to + \infty \). Some aspects of the asymptotic flow can be explained by the existence of the global attractor, to which trajectories of bounded sets converge as \( t \to + \infty \).

Mathematical literature provides results of existence of the global attractor for large classes of dynamical systems [4,6,13], but it is in general very hard to describe the structure of this attractor, which may sometimes be a strange fractal. Great interest has been devoted to measure theoretic properties of the global attractor: for example in [2,6,13] estimates of its fractal and Hausdorff dimension are provided for dynamical systems arising from certain classes of partial differential equations.

In this paper, which is intended as a continuation of [3], we deal with topological properties of the global attractor. In particular, we examine the question whether the topological invariants of \( X \) are inherited by the global attractor or not.

Trivial examples show that in the general case the answer is negative (cf. Example 9.1). So we must restrict our attention to two particular classes of semigroups:

- jointly continuous semigroups;
- groups.

Čech–Alexander–Spanier cohomology theory, due to its tautness and continuity properties, is the fundamental tool of our analysis. From now on, we denote by \( \tilde{H}^q(X, G) \) the \( q \)-dimensional Čech–Alexander–Spanier cohomology group of \( X \) with coefficients in \( G \) (see Section 3 for more details).

For jointly continuous semigroups we establish that:

1. if there exists the global attractor \( A \), then the restriction homomorphism \( i_A^q : \tilde{H}^q(X, G) \to \tilde{H}^q(A, G) \) is an isomorphism for all \( q \geq 0 \) and all coefficient groups \( G \) (Theorem 6.3).

If the semigroup \( \{S_t\} \) is not jointly continuous, then \( i_A^q \) may fail to be injective and/or surjective, even if \( \{S_t\} \) is a group (cf. Examples 9.3 and 9.4).

However, if \( \{S_t\} \) is a group (and there exists the global attractor \( A \)), then we establish that:

2. if \( A \) has a bounded neighborhood which is a retract of \( X \), then the restriction homomorphism \( i_A^q : \tilde{H}^q(X, G) \to \tilde{H}^q(A, G) \) is surjective (Theorem 7.3), but not necessarily injective (Example 9.3);

3. if \( A \) has a bounded neighborhood which is a deformation retract of \( X \), then the restriction homomorphism \( i_A^q : \tilde{H}^q(X, G) \to \tilde{H}^q(A, G) \) is an isomorphism (Theorem 7.4).

Finally, we point out that our results about Čech–Alexander–Spanier cohomology groups of global attractors cannot be extended to singular cohomology groups (cf. Example 9.2).

This paper is organized as follows: in Section 2 we provide basic notations and definitions from the theory of semigroups; in Section 3 we recall the main properties of Čech–Alexander–Spanier cohomology groups; in Section 4 we prove a topological lemma; in Section 5 we establish some relations between the cohomology of a subset \( B \subseteq X \) and the cohomology of its \( \omega \)-limit \( \omega(B) \); in Section 6 we prove our assertion (1) for jointly continuous semigroups; in Section 7 we prove (2) and (3) for groups of continuous operators; in Section 8 we present some examples where the results of Sections 6 and 7 can be applied; in Section 9 we collect all the counterexamples quoted in this introduction and in the following sections.
2. Preliminaries

In this section we give notations and we recall basic definitions from the theory of semigroups of continuous operators. Throughout this paper, unless otherwise stated, $X$ will denote a generic (not necessarily complete) metric space, sometimes called phase space, with distance function $d$. For any $A \subseteq X$, we denote by $\overline{A}$ the closure of $A$ in $X$, and for any $\varepsilon > 0$ we denote by $A_\varepsilon$ the open $\varepsilon$-neighborhood of $A$ in $X$, i.e.

$$A_\varepsilon := \left\{ x \in X : \inf_{a \in A} d(x, a) < \varepsilon \right\}.$$ 

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{C}$, respectively, the set of nonnegative integers, integers, real numbers, nonnegative real numbers, complex numbers.

In order to give a unified treatment of continuous and discrete semigroups, we give here a rather general definition of semigroup, very similar to the definition given in [5,6] (cf. [3]).

**Definition 2.1.** A subset $P \subseteq \mathbb{R}$ is said to be a parameter space if:

- there exists $\varepsilon > 0$ such that $\{0, \varepsilon\} \subseteq P$;
- $P$ is additively closed, i.e. for all $t \in P$, $s \in P$ we have that $t + s \in P$.

**Remark 2.2.** We will hereafter assume, without loss of generality, that $\varepsilon = 1$, hence $\mathbb{N} \subseteq P$.

**Definition 2.3.** Let $X$ be a metric space and let $P \subseteq \mathbb{R}$ be a parameter space. A semigroup of continuous operators on $X$, parameterized by $P$, is a family of maps $\{S_t\}_{t \in P}$, satisfying:

- $S_t : X \to X$ is continuous, for every $t \in P$;
- $S_0$ is the Identity on $X$;
- $S_{t+s} = S_t \circ S_s$, for every $t \in P$, $s \in P$.

When in addition $P$ is an additive subgroup of $\mathbb{R}$, we call $\{S_t\}_{t \in P}$ a group of continuous operators.

**Definition 2.4.** Let $\{S_t\}_{t \in P}$ be a semigroup of continuous operators on a metric space $X$.

- We call $\{S_t\}_{t \in P}$ a discrete semigroup (resp. a discrete group) provided that $P \cap \mathbb{R}_{\geq 0} = \mathbb{N}$ (resp. $P = \mathbb{Z}$).
- We call $\{S_t\}_{t \in P}$ a time-continuous semigroup (resp. a time-continuous group) provided that $P \supseteq \mathbb{R}_{\geq 0}$ (resp. $P = \mathbb{R}$) and, for every $x \in X$, the function $t \to S_t(x)$ is continuous on $P$.
- We call $\{S_t\}_{t \in P}$ a jointly continuous semigroup (resp. a jointly continuous group) provided that $P \supseteq \mathbb{R}_{\geq 0}$ (resp. $P = \mathbb{R}$) and the function $(t,x) \to S_t(x)$ is continuous on $P \times X$.

Sometimes we will use the expression “arbitrary semigroup” to emphasize that we are dealing with a semigroup in the sense of Definition 2.3, i.e. without any further assumption on $P$. 

Definition 2.5. Let $\{S_t\}_{t \in P}$ be an arbitrary semigroup of continuous operators on a metric space $X$, and let $A \subseteq X$.

- The $\omega$-limit of $A$ is defined as
  \[ \omega(A) := \bigcap_{s \geq 0} \bigcup_{t \geq s} S_t(A); \]

- $A$ is positively invariant if and only if $S_t(A) \subseteq A$, for every $t \geq 0$, $t \in P$;
- $A$ is invariant if and only if $S_t(A) = A$, for all $t \in P$.

For a detailed discussion of the properties of the $\omega$-limit operator, the reader is referred to the wide literature on this subject [1,4,6,13].

Definition 2.6. Let $\{S_t\}_{t \in P}$ be an arbitrary semigroup of continuous operators on a metric space $X$, and let $A \subseteq X$, $B \subseteq X$. We say that $A$ attracts $B$ if and only if, for every $\varepsilon > 0$, there exists $t_* \geq 0$ such that:

\[ S_t(B) \subseteq A_\varepsilon, \quad \forall t \geq t_*, t \in P. \]

Definition 2.7. Let $\{S_t\}_{t \in P}$ be an arbitrary semigroup of continuous operators on a metric space $X$. A subset $A \subseteq X$ is a global attractor if and only if:

- $A$ is compact;
- $A$ is invariant;
- $A$ attracts any bounded subset of $X$.

The global attractor, when it exists, is necessarily unique: it turns out to be the maximal compact invariant set, and the minimal closed set which attracts any bounded subset of $X$. The reader interested in existence results for the global attractor under suitable assumptions on $X$ and $\{S_t\}$ is referred to [4,6,13].

In Section 5 we will often need the following lemma.

Lemma 2.8 ([4, Lemma 3.1.1]). Let $\{S_t\}_{t \in P}$ be an arbitrary semigroup of continuous operators on a metric space $X$, and let $B \subseteq X$. Let us assume that $\omega(B)$ is compact and attracts $B$.

Then $\omega(B)$ is invariant.

3. The Alexander cohomology theory

In this section we recall some basic properties of the Čech–Alexander–Spanier cohomology. There are at least two approaches to this theory: one is based on the Alexander construction [11, Chapter 6, Section 4], the other is based on the Čech construction [11, Chapter 6, Section 7]. In the case of paracompact Hausdorff spaces, e.g. metric spaces, these constructions give the same result...

For any topological space $X$, any integer $q \geq 0$, and any abelian group $G$, there is defined the $q$-dimensional Čech–Alexander–Spanier cohomology group with coefficients in $G$, which we denote by $\tilde{H}^q(X, G)$. We denote by $\tilde{H}^* (X, G)$ the graded group $\{\tilde{H}^q(X, G)\}_{q \in \mathbb{N}}$.

A continuous map $f: X \to Y$ induces a homomorphism $f^*: \tilde{H}^q(Y, G) \to \tilde{H}^q(X, G)$ for each $q$ and each $G$. This correspondence is functorial, i.e. if $g: Y \to Z$ is another continuous map, then $(g \circ f)^* = f^* \circ g^*$, and if $id: X \to X$ is the identity on $X$, then $id^*$ is the identity on $\tilde{H}^q(X, G)$. If moreover $G$ is a field, then $\tilde{H}^q(X, G)$ is a $G$-vector space and $f^*$ is a $G$-linear application.

In order to avoid exceptional cases in many statements, it is useful to set $\tilde{H}^q(X, G) = 0$ for $q < 0$, and define $f^*$ in the only way possible for $q < 0$.

Čech–Alexander–Spanier cohomology theory satisfies the four axioms in the definition of a cohomology theory with coefficients in $G$ ([11, p. 240]); in this paper we need only the following two axioms.

- **Dimension axiom.** If $P$ is a one-point space and $G$ is any abelian group, then:

  \[
  \tilde{H}^q(P, G) \cong \begin{cases} \mathbb{G} & \text{if } q = 0, \\ 0 & \text{if } q \geq 1. \end{cases}
  \]

- **Homotopy axiom.** Let $f_0, f_1: X \to Y$ be two homotopic maps, i.e. there exists a continuous map $\Phi: [0, 1] \times X \to Y$ such that $\Phi(0, x) = f_0(x)$ and $\Phi(1, x) = f_1(x)$, for each $x \in X$. Then

  \[ f_0^* = f_1^*: \tilde{H}^*(Y, G) \to \tilde{H}^*(X, G). \]

In order to describe more subtle properties of the Čech–Alexander–Spanier cohomology theory, we will make an extensive use of the notion of direct limit (see the appendix in [7]).

Let $A$ be a subset of a topological space $X$. The family of all neighborhoods of $A$ in $X$ is directed downward by inclusion. Hence $\{\tilde{H}^q(U, G)\}$, where $U$ ranges over all neighborhoods of $A$ in $X$, is a direct system of groups (the homomorphisms are those induced by inclusions, of course). The restriction maps $\tilde{H}^q(U, G) \to \tilde{H}^q(A, G)$ define a homomorphism

\[
\lim\text{dir } \tilde{H}^q(U, G) \to \tilde{H}^q(A, G).
\]

The subset $A$ is said to be **tautly imbedded in** $X$ (or simply **taut in** $X$), with respect to Čech–Alexander–Spanier cohomology, if this homomorphism is an isomorphism for all $q \geq 0$ and all coefficient groups $G$. We recall that a group homomorphism is called **monomorphism**, **epimorphism**, or **isomorphism**, respectively, if it is injective, surjective or bijective.

The definition of tautness can be formulated for any cohomology theory. One of the major differences between singular and Čech–Alexander–Spanier cohomology is this matter of tautness. In general, tautness is more likely to hold with respect to the Čech–Alexander–Spanier theory then with respect to the singular theory, and this is the reason why our results about Čech–Alexander–Spanier groups of attractors are in general not true for the corresponding singular cohomology groups.
In this paper we need the following result, which is a trivial consequence of [12, Theorem 1].

**Theorem 3.1.** Let $A$ be an arbitrary subset of a metric space $X$.
Then $A$ is taut in $X$ with respect to the Čech–Alexander–Spanier cohomology theory, i.e.

$$
\tilde{H}^q(A, G) \cong \text{dir lim} \tilde{H}^q(U, G),
$$

where $U$ ranges over all neighborhoods of $A$ in $X$.

There are examples of compact subsets of $\mathbb{R}^2$ that are not tautly imbedded with respect to the singular cohomology ([11, Examples 6.1.8 and 6.6.4] and our Example 9.2).

For a precise comparison of singular and Čech–Alexander–Spanier cohomology, the reader is referred to [11, chapter 4, Section 9]; however, if $X$ is any locally contractible space, in particular any open subset of a Banach space or any manifold, then its singular cohomology groups coincide with its Čech–Alexander–Spanier cohomology groups.

The following theorem characterizes connectedness and path connectedness by means of zero-dimensional cohomology groups.

**Theorem 3.2.** Let $X$ be a nonempty topological space. Then:

1. $\tilde{H}^0(X, G)$ is isomorphic to the group of locally constant functions from $X$ to $G$. In particular, $X$ is connected if and only if $\tilde{H}^0(X, G) \cong G$ for all abelian groups $G$.
2. The singular cohomology group $H^0(X, G)$ is isomorphic to $\bigoplus_{i \in I} G$, where $I$ is the set of path components of $X$. In particular, $X$ is path connected if and only if $H^0(X, G) \cong G$ for all abelian groups $G$.

In the case $X = \mathbb{R}^n$, we recall the following duality result.

**Theorem 3.3** (Alexander Duality). If $A$ is a compact subset of $\mathbb{R}^n$, then, for all $q$ and all coefficient groups $G$, we have that:

$$
\tilde{H}_q(\mathbb{R}^n \setminus A, G) \cong \tilde{H}^{n-q-1}(A, G),
$$

where $\tilde{H}_q$ denotes the $q$-dimensional reduced homology group.

For more informations about reduced homology, see [8,11]; we need Alexander duality only in Section 8.

**4. Inverse limits and cohomology**

The main result of this section is the topological Lemma 4.5. In the proof of this lemma we need the notion of inverse limit and of its first derived functor, denoted by “$\text{lim}^1$”. For the convenience of the reader, we recall here the basic properties of inverse limits and of $\text{lim}^1$ (for more details see the appendix in [7]).
For each inverse sequence of groups

\[ M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots \]

there is defined the group \( \lim^1 M_n \). This is a derived functor in the sense of homological algebra, i.e. the following result holds true.

**Theorem 4.1** ([7, Theorem A.14]). *The short exact sequence of inverse sequences of groups*

\[
0 \to \{M'_n\} \to \{M_n\} \to \{M''_n\} \to 0
\]

gives rise to the long exact sequence

\[
0 \to \text{(inv lim } M'_n) \to \text{(inv lim } M_n) \to \text{(inv lim } M''_n) \to \lim^1 M'_n \to \lim^1 M_n \to \lim^1 M''_n \to 0.
\]

In many applications it is useful to know that \( \lim^1 M_n = 0 \). The following lemma gives a simple sufficient condition.

**Lemma 4.2** ([7, Lemma A.15]). *Let \( \{M_n\} \) be an inverse sequence of groups. Let us assume that each homomorphism \( M_{n+1} \to M_n \) is surjective.*

*Then \( \lim^1 M_n = 0 \).*

The following theorem measures the extent to which the passage to the inverse limit fails to commute with the taking of cohomology. Since the theory of cochain complexes is isomorphic to that of chain complexes, this result is an immediate consequence of [7, Theorem A.19] for homology groups.

**Theorem 4.3.** *Let us consider the inverse sequence*

\[
K(1) \leftarrow K(2) \leftarrow K(3) \leftarrow \cdots ,
\]

*where each \( K(n) = \{K(n), \delta_n\} \) is a cochain complex.*

*Let \( K(\infty) = \text{inv lim } K(n) \). Let us assume that \( \lim^1 K_i(n) = 0 \) for each integer \( i \).*

*Then for each \( q \) there exists a short exact sequence of cohomology groups*

\[
0 \to \lim^1 H^{q-1}(K(n)) \to H^q(K(\infty)) \xrightarrow{z} \text{inv lim } H^q(K(n)) \to 0,
\]

*where \( z \) is the homomorphism induced by the compatible family of cochain morphisms \( \{K(\infty) \to K(n)\} \).*

In the proof of the following lemma, we use some notations of [7], which we recall for the convenience of the reader. For any topological space \( X \), any integer \( q \geq 0 \), and any abelian group \( G \), we consider the following abelian groups:

\[
\Phi^q(X, G) = \{ \varphi : X^{q+1} \to G \},
\]

\[
\Phi^q_0(X, G) = \{ \varphi \in \Phi^q(X, G) : |\varphi| = 0 \},
\]

\[
\tilde{C}^q(X, G) = \Phi^q(X, G)/\Phi^q_0(X, G),
\]
where $|\varphi| \subseteq X$ is defined by: $x \notin |\varphi|$ if and only if there exists a neighborhood $U$ of $x$ such that

$$(x_0, x_1, \ldots, x_q) \in U^{q+1} \Rightarrow \varphi(x_0, x_1, \ldots, x_q) = 0.$$ 

Hereafter we denote by $\check{C}^*(X, G)$ the cochain complex $\{\check{C}^q(X, G), \delta\}$, where $\delta: \check{C}^q(X, G) \to \check{C}^{q+1}(X, G)$ is the coboundary operator defined in [7,11]. With these notations $\check{H}^q(X, G) = H^q(\check{C}^*(X, G))$.

**Lemma 4.4.** Let $X$ be a topological space, and let $\{C_n\}_{n \in \mathbb{N}}$ be a family of closed subsets of $X$ such that:

(i) $C_n \subseteq C_{n+1}$ for each $n \in \mathbb{N}$;

(ii) $\bigcup_{n \in \mathbb{N}} C_n = X$, where $C_n$ denotes the interior part of $C_n$ in $X$.

Then, for every $q \geq 0$ and every coefficient group $G$, there is an exact sequence

$$0 \to \lim^1 \check{H}^{q-1}(C_n, G) \to \check{H}^q(X, G) \xrightarrow{\delta^*} \invlim \check{H}^q(C_n, G) \to 0,$$

where $\delta^*$ is the homomorphism induced by the compatible family of inclusions $C_n \subseteq X$.

**Proof.** By (ii) we have that

$$\Phi^q(X, G) = \invlim \Phi^q(C_n, G). \quad (4.1)$$

Furthermore, the restriction homomorphism $\Phi^q(C_{n+1}, G) \to \Phi^q(C_n, G)$ is surjective for each $n$, hence by Lemma 4.2

$$\lim^1 \Phi^q(C_n, G) = 0. \quad (4.2)$$

By (ii) we have that

$$\Phi^q_0(X, G) = \invlim \Phi^q_0(C_n, G). \quad (4.3)$$

Furthermore, since each $C_n$ is a closed set, the restriction homomorphism $\Phi^q_0(C_{n+1}, G) \to \Phi^q_0(C_n, G)$ is surjective for each $n$, hence by Lemma 4.2

$$\lim^1 \Phi^q_0(C_n, G) = 0. \quad (4.4)$$

Applying Theorem 4.1 to the short exact sequence of inverse sequences

$$0 \to \{\Phi^q_0(C_n, G)\} \to \{\Phi^q(C_n, G)\} \to \{\check{C}^q(C_n, G)\} \to 0,$$

and making use of (4.1)–(4.4), we have that

$$\check{C}^q(X, G) = \invlim \check{C}^q(C_n, G), \quad (4.5)$$

and

$$\lim^1 \check{C}^q(C_n, G) = 0. \quad (4.6)$$
Since (4.6) holds true for every integer $q$, we can apply Theorem 4.3 to the inverse sequence of cochain complexes $\{\tilde{C}^n_*(G, \mathbb{G})\}$. By (4.5) we obtain a short exact sequence

$$0 \to \lim^1 H^{q-1}(\tilde{C}^*(G, \mathbb{G})) \to H^q(\tilde{C}^*(X, G)) \overset{x^*}{\to} \text{inv lim } H^q(\tilde{C}^*(G, \mathbb{G})) \to 0,$$

which coincides with the required short exact sequence by definition of Čech–Alexander–Spanier cohomology groups. □

The following lemma will be crucial in the proof of Theorem 6.3.

**Lemma 4.5.** Let $X$ be a metric space, let $K \subseteq X$ be a compact set, and let $G$ be an abelian group. Let us assume that there exists a family $\{C_n\}_{n \in \mathbb{N}}$ of closed subsets of $X$ such that:

(i) $C_n \subseteq C_{n+1}$ for every $n \in \mathbb{N}$;
(ii) $\bigcup_{n \in \mathbb{N}} \tilde{C}_n = X$, where $\tilde{C}_n$ denotes the interior part of $C_n$ in $X$;
(iii) $K \subseteq C_n$ and the restriction homomorphism $i^n_! : \tilde{H}^*(G, \mathbb{G}) \to \tilde{H}^*(K, \mathbb{G})$ is an isomorphism for every $n \in \mathbb{N}$.

Then the restriction homomorphism $i^K_! : \tilde{H}^*(X, G) \to \tilde{H}^*(K, G)$ is an isomorphism.

**Proof.** Step 1: By (i) the family $\{\tilde{H}^*(C_n, \mathbb{G})\}_{n \in \mathbb{N}}$ with the restriction homomorphisms $j^n_! : \tilde{H}^*(C_{n+1}, \mathbb{G}) \to \tilde{H}^*(C_n, \mathbb{G})$ is an inverse sequence of groups. We claim that $j^n_!$ is an isomorphism for all $n$.

Indeed let us consider the following commutative diagram:

$$\begin{array}{ccc}
\tilde{H}^*(C_{n+1}, G) & \xrightarrow{j^n_!} & \tilde{H}^*(C_n, G) \\
\downarrow{i^n_!} & & \downarrow{i^n_!} \\
\tilde{H}^*(K, G) & & \\
\end{array}$$

Since by (iii) the maps $i^n_{n+1}$ and $i^n_!$ are isomorphisms, it follows that $j^n_!$ is an isomorphism. Therefore the homomorphisms

$$\chi^n_! : \text{inv lim } \tilde{H}^*(C_n, \mathbb{G}) \to \tilde{H}^*(C_n, \mathbb{G}),$$

given by the definition of inverse limit, are isomorphisms.

Step 2: We show that the homomorphism

$$\chi^* : \tilde{H}^*(X, G) \to \text{inv lim } \tilde{H}^*(C_n, \mathbb{G}),$$

induced by the compatible family of inclusions $C_n \subseteq X$, is an isomorphism.

Indeed, by Lemma 4.4 we have, for any $q \geq 0$, an exact sequence

$$0 \to \lim^1 \tilde{H}^{q-1}(C_n, \mathbb{G}) \to \tilde{H}^q(X, G) \overset{x^*}{\to} \text{inv lim } \tilde{H}^q(C_n, \mathbb{G}) \to 0.$$

By Step 1, $j^n_! : \tilde{H}^{q-1}(C_{n+1}, G) \to \tilde{H}^{q-1}(C_n, G)$ is an isomorphisms for all $n$. By Lemma 4.2 it follows that $\lim^1 \tilde{H}^{q-1}(C_n, G) = 0$, hence $\chi^*$ is an isomorphism.
Step 3: It is easy to verify that, for every $n \in \mathbb{N}$, the restriction homomorphism $i^n_K$ may be factorized as follows:

$$\tilde{H}^*(X, G) \overset{z_n^*}{\rightarrow} \text{inv lim} \tilde{H}^*(C_n, G) \overset{z_n^*}{\rightarrow} \tilde{H}^*(C_n, G) \overset{i_n^*}{\rightarrow} \tilde{H}^*(K, G).$$

Since $z_n^*$, $z_n^*$, and $i_n^*$ are isomorphisms, it follows that $i_K^*$ is an isomorphism. \(\square\)

5. Technical results

In this section we establish some relations between the cohomology of a set $B \subseteq X$ and the cohomology of the $\omega$-limit set $\omega(B)$.

**Proposition 5.1.** Let $\{S_t\}_{t \in P}$ be an arbitrary semigroup of continuous operators on a metric space $X$ and let $B \subseteq X$. Let $G$ be any coefficient group and $q \geq 0$ any integer.

Let us assume that:

(i) $\omega(B)$ is compact and attracts $B$;
(ii) $\omega(B) \subseteq B$;
(iii) $(S_t|_{\omega(B)})^*$ is the identity for any $t \in P$, where $(S_t|_{\omega(B)})^*$ is the homomorphisms induced on $\tilde{H}^q(\omega(B), G)$ by the restriction of $S_t$ to the invariant set $\omega(B)$.

Let $i_B^*: \tilde{H}^q(B, G) \rightarrow \tilde{H}^q(\omega(B), G)$ be the homomorphism induced by the inclusion $i_B: \omega(B) \rightarrow B$.

Then $i_B^*$ is surjective.

**Proof.** Let us fix $G$ and $q$, and let us set $M := \omega(B)$.

**Step 1:** By Theorem 3.1, $M$ is taut in $X$. Furthermore, since $M$ is compact, its $\varepsilon$-neighborhoods $M_\varepsilon$ are cofinal in the class of all neighborhoods of $M$; hence, by well known properties of direct limits, we have that:

$$\tilde{H}^q(M, G) = \text{dir lim} \tilde{H}^q(M_\varepsilon, G) = \bigcup_{\varepsilon > 0} \text{Imm}(i^*_M),$$

where $i^*_M: \tilde{H}^q(M_\varepsilon, G) \rightarrow \tilde{H}^q(M, G)$ is induced by the inclusion $M \subseteq M_\varepsilon$, and “Imm” denotes the image (of a homomorphism).

**Step 2:** Let us fix $\varepsilon > 0$. By (i) and Lemma 2.8, $M$ is an invariant set which attracts $B$, so by (ii) there exists $t \in P$ such that $M \subseteq S_t(B) \subseteq M_\varepsilon$. Let us consider the following commutative diagram:
where $i_B, i_M, i_{S(B)}, j$ are inclusions, and $S|M, S|B$ are the restrictions of $S_t$ to $M$ and $B$, respectively. Considering cohomology groups, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\bar{H}^q(M, G) & \xleftarrow{i_M^*} & \bar{H}^q(M_e, G) \\
S_t^* & \downarrow{i_M^*} & \bar{H}^q(S_t(B), G) \\
\bar{H}^q(M, G) & \xleftarrow{i_B^*} & \bar{H}^q(B, G)
\end{array}
\]

From this diagram it follows that
\[S|M^* \circ i_M^* = i_B^* \circ S|B^* \circ j^*, \quad (5.2)\]
for all $\varepsilon > 0$ and all $t \in P$ such that $M \subseteq S_t(B) \subseteq M_\varepsilon$.

**Step 3:** Since $S|M^*$ is the identity, from (5.2) it follows that, for all $\varepsilon > 0$:
\[\text{Imm}(i_M^*) \subseteq \text{Imm}(i_B^*). \quad (5.3)\]

Owing (5.1) and (5.3) we have that
\[\bar{H}^q(M, G) = \bigcup_{\varepsilon > 0} \text{Imm}(i_M^*) \subseteq \text{Imm}(i_B^*) \subseteq \bar{H}^q(M, G),\]
and therefore $\text{Imm}(i_B^*) = \bar{H}^q(M, G)$, i.e. $i_B^*$ is surjective. \(\square\)

We will prove in Section 6 that hypothesis (iii) of Proposition 5.1 is automatically satisfied if the semigroup is jointly continuous.

**Remark 5.2.** If instead of assumption (iii) we only assume that $(S|_{\omega(B)})^*$ is an isomorphism, then $i_B^*$ may fail to be surjective, even if $\bar{H}^q(B, G)$ is finitely generated (cf. Example 9.5).

However, using (5.1), (5.2), and the properties of direct limits, it is possible to prove the surjectivity of $i_B^*$ when (iii) is replaced by any one of the following assumptions:

(iii-1) $(S|_{\omega(B)})^*$ is surjective for any $t \in P$, and $\bar{H}^q(\omega(B), G)$ is finitely generated;
(iii-2) $(S|_{\omega(B)})^*$ is injective for any $t \in P$, and $\bar{H}^q(B, G) = 0$;
(iii-3) $(S|_{\omega(B)})^*$ is an isomorphism for any $t \in P$, $G$ is a field, and $\bar{H}^q(B, G)$ is a finite dimensional $G$-vector space.

**Proposition 5.3.** Let $\{S_t\}_{t \in P}$ be an arbitrary semigroup of continuous operators on a metric space $X$, and let $B \subseteq X$. Let $G$ be any coefficient group and $q \geq 0$ any integer.
Let us assume that:

(i) \( \omega(B) \) is compact and attracts \( B \);

(ii) \( \omega(B) \subseteq \overline{B} \), where \( \overline{B} \) denotes the interior part of \( B \) in \( X \);

(iii) \( B \) is positively invariant;

(iv) \( S_t|_B^* \) is injective for any \( t \in P \), where \( S_t|_B^* \) denotes the homomorphism induced on \( \tilde{H}^q(B, G) \) by the restriction of \( S_t \) to the positively invariant set \( B \).

Let \( i_B^* : \tilde{H}^q(B, G) \rightarrow \tilde{H}^q(\omega(B), G) \) be the homomorphism induced by inclusion \( i_B : \omega(B) \rightarrow B \).

Then \( i_B^* \) is injective.

**Proof.** Let us fix \( G \) and \( q \), and let us set \( M := \omega(B) \).

**Step 1:** By (iii) we have that \( S_t(B) \subseteq B \), for each \( t \in P \cap \mathbb{R}_{\geq 0} \). Let us denote by \( i \) the inclusion map.

We show that \( i_B^* : \tilde{H}^q(B, G) \rightarrow \tilde{H}^q(S_t(B), G) \) is injective. Indeed let us consider the following commutative diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{S_t|_B} & B \\
\downarrow{\overline{S}_t|_B} & & \downarrow{i} \\
S_t(B) & \end{array}
\]

where \( S_t|_B \) and \( \overline{S}_t|_B \) denote the restrictions of \( S_t \) to \( B \) with target spaces \( B \) and \( S_t(B) \), respectively.

For cohomology groups we have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{H}^q(B, G) & \xrightarrow{S_t|_B^*} & \tilde{H}^q(S_t(B), G) \\
\downarrow{\overline{S}_t|_B^*} & & \downarrow{i_t^*} \\
\tilde{H}^q(S_t(B), G) & \end{array}
\]

By (iv) the map \( S_t|_B^* \) is injective, hence \( i_t^* \) is injective.

**Step 2:** Let us assume that \( M_{\epsilon} \subseteq B \) for some \( \epsilon > 0 \), and let us denote by \( j_\epsilon \) the inclusion map. We show that \( j_\epsilon^* : \tilde{H}^q(B, G) \rightarrow \tilde{H}^q(M_{\epsilon}, G) \) is injective.

By (i) there exists \( t \in P \cap \mathbb{R}_{\geq 0} \) such that \( S_t(B) \subseteq M_{\epsilon} \). Let us consider the following commutative diagram of inclusions:

\[
\begin{array}{ccc}
S_t(B) & \xrightarrow{i_t} & B \\
\downarrow{j_\epsilon} & & \downarrow{} \\
M_{\epsilon} & \end{array}
\]
Considering cohomology groups the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{H}^q(S_1(B), G) & \xrightarrow{i^*_1} & \tilde{H}^q(B, G) \\
\downarrow{i^*_1} & & \downarrow{j^*_1} \\
\tilde{H}^q(M, G) & & \\
\end{array}
\]

Since in Step 1 we have proved that \(i^*_t\) is injective, it follows immediately that \(j^*_t\) is injective.

**Step 3:** By (ii), \(B\) is a neighborhood of \(M\). Let us assume by contradiction that \(i^*_t\) is not injective. Since \(M\) is taut in \(X\), and the \(M_i\)'s are cofinal among the neighborhoods of \(M\), by well known properties of direct limits there exists \(\varepsilon_0 > 0\) such that \(M_{\varepsilon_0} \subseteq B\) and \(j_{\varepsilon_0}\) is not a monomorphism.

This is inconsistent with what proved in Step 2. \(\square\)

### 6. Jointly continuous semigroups

In this section we show that for jointly continuous semigroups the Čech–Alexander–Spanier cohomology groups of the global attractor \(A\) are isomorphic to the corresponding groups of the phase space \(X\).

In order to prove this result, we first apply the results of Section 5 to show that the cohomology groups of \(A\) are isomorphic to the corresponding cohomology groups of any bounded positively invariant neighborhood of \(A\) (Proposition 6.2). This allows to construct a sequence of closed subsets of \(X\) which satisfy the assumptions of Lemma 4.5 with \(K := A\).

The following lemma shows that hypothesis (iii) of Proposition 5.1, and hypothesis (iv) of Proposition 5.3 are verified whenever the semigroup is jointly continuous.

**Lemma 6.1.** Let \(\{S_t\}_{t \in \mathbb{R}}\) be a jointly continuous semigroup on a metric space \(X\), and let \(M \subseteq X\) be a positively invariant set. For any \(t \geq 0\) and any coefficient group \(G\), let us denote by \(S_t|_M^*\) the homomorphism induced on \(\tilde{H}^q(M, G)\) by the restriction of \(S_t\) to \(M\).

Then \(S_t|_M^*\) is the identity for all \(t \geq 0\) and all \(G\).

**Proof.** Let us fix \(t \geq 0\) and \(G\). Since the semigroup is jointly continuous, the map \(\Phi : [0, 1] \times M \to M\) defined by \(\Phi(\tau, x) = S_{\tau \cdot t}(x)\) is a homotopy between \(S_0|_M\) and \(S_t|_M\). Since \(S_0|_M\) is the identity on \(M\), by the homotopy axiom it follows that \(S_t|_M^* = S_0|_M^* = \text{identity.} \quad \square\)

The above Lemma allows us to apply the results of Section 5 to study cohomology groups of attractors for jointly continuous semigroups.
Proposition 6.2. Let \( \{S_i\}_{i \in \mathbb{R}} \) be a jointly continuous semigroup on a metric space \( X \), and let \( B \subseteq X \).

Let us assume that:

(i) there exists the global attractor \( A \) for \( \{S_i\} \);
(ii) \( B \) is positively invariant;
(iii) \( A \subseteq B \), where \( B \) denotes the interior part of \( B \) in \( X \);
(iv) \( A \) attracts \( B \).

For any coefficient group \( G \), let us denote by \( i_B^*: \tilde{H}^*(B, G) \rightarrow \tilde{H}^*(A, G) \) the homomorphism induced by inclusion (iii).

Then \( i_B^* \) is an isomorphism.

Proof. By (iii) and (iv) we have that \( A = \omega(B) \).

From Lemma 6.1 with \( M := \omega(B) \) and Proposition 5.1 it follows that \( i_B^* \) is surjective. From Lemma 6.1 with \( M := B \) and Proposition 5.3 it follows that \( i_B^* \) is injective. \( \square \)

We can now prove the main result of this section.

Theorem 6.3. Let \( \{S_i\}_{i \in \mathbb{R}} \) be a jointly continuous semigroup on a metric space \( X \). Let us assume that there exists the global attractor \( A \) for \( \{S_i\} \).

Then the restriction homomorphism \( i_A^*: \tilde{H}^*(X, G) \rightarrow \tilde{H}^*(A, G) \) is a homomorphism for all abelian groups \( G \).

Proof. For every \( n \in \mathbb{N} \), let us denote by \( A_n \) the open neighborhood of \( A \) with radius \( n + 1 \). Let \( C_n \) denote the topological closure of the positive orbit of \( A_n \), i.e.:

\[
C_n := \bigcup_{t \geq 0} S_t(A_n).
\]

Thus \( C_n \) is a closed positively invariant set, which contains \( A \) in its interior and is attracted by \( A \). By Proposition 6.2 the restriction homomorphisms \( \tilde{H}^*(C_n, G) \rightarrow \tilde{H}^*(A, G) \) are isomorphisms for all coefficient groups \( G \).

Therefore the family \( \{C_n\}_{n \in \mathbb{N}} \) satisfies all the hypotheses of Lemma 4.5 with \( K := A \). From this lemma it follows that \( i_A^* \) is an isomorphism. \( \square \)

7. Semigroups without time continuity assumptions

This section is devoted to cohomology groups of global attractors for semigroups without any time-continuity assumption. As usual, we denote by \( X \) the phase space, by \( A \) the global attractor, and by \( i_A^*: \tilde{H}^*(X, G) \rightarrow \tilde{H}^*(A, G) \) the restriction homomorphism.

In Example 9.3 (resp. Example 9.4) we exhibit groups \( \{S_i\} \) such that \( i_A^* \) fails to be injective (resp. surjective).

However, \( i_A^* \) turns out to be an isomorphism for groups defined on a large class of phase spaces. In order to state the precise results, we first need a definition.
Definition 7.1. A subset $Y$ of a topological space $X$ is a retract of $X$ if there exists a continuous map $r : X \to Y$ such that $r(y) = y$ for every $y \in Y$.

A subset $Y$ of a topological space $X$ is a deformation retract of $X$ if there exists a continuous map $\Phi : [0,1] \times X \to X$ such that
\[
\begin{align*}
\Phi(0,x) &= x, \quad \forall x \in X, \\
\Phi(1,x) &= y, \quad \forall x \in X, \\
\Phi(t,y) &= y, \quad \forall (t,y) \in [0,1] \times Y.
\end{align*}
\]

Remark 7.2. From the functorial properties of Čech–Alexander–Spanier cohomology groups, it easily follows that:

- if $Y$ is a retract of $X$, then the restriction homomorphism $\tilde{H}^*(X, G) \to \tilde{H}^*(Y, G)$ is surjective for all coefficient groups $G$;
- if $Y$ is a deformation retract of $X$, then the restriction homomorphism $\tilde{H}^*(X, G) \to \tilde{H}^*(Y, G)$ is an isomorphism for all coefficient groups $G$.

We are now ready to state and prove the two main results of this section.

Theorem 7.3. Let $\{S_t\}_{t \in P}$ be a group of continuous operators on a metric space $X$. Let us assume that:

(i) there exists the global attractor $A$ for $\{S_t\}$;
(ii) there exists a bounded neighborhood $B$ of $A$ which is a retract of $X$.

Then the restriction homomorphism $i^*_A : \tilde{H}^*(X, G) \to \tilde{H}^*(A, G)$ is surjective for all coefficient groups $G$.

Proof. Let us fix the coefficient group $G$.

Step I: For each $t \in P$, the restriction homomorphism
\[
i^*_t : \tilde{H}^*(X, G) \to \tilde{H}^*(S_t(B), G)
\]
is an epimorphism.

Indeed, let us consider the following commutative diagram of cohomology groups:
\[
\begin{array}{ccc}
\tilde{H}^*(X, G) & \xrightarrow{S^*_t} & \tilde{H}^*(X, G) \\
\downarrow{i^*_0} & & \downarrow{i^*_t} \\
\tilde{H}^*(B, G) & \xleftarrow{S^*_t} & \tilde{H}^*(S_t(B), G)
\end{array}
\]  
(7.1)

Since $B$ is a retract of $X$, the map $i^*_0$ is surjective; furthermore the horizontal arrows are isomorphisms since $S_t$ is a homeomorphism. It follows that $i^*_t$ is necessarily surjective.
Step 2: Since $A$ is the global attractor and $B$ is bounded, it follows that $A$ attracts $B$. Therefore, since $B$ is a neighborhood of $A$, there exists $T \in P$, $T > 0$ such that $S_t(B) \subseteq B$. Let us set, for each $n \in \mathbb{N}$:

$$B_n := S_{nT}(B).$$

Thus $\{B_n\}_{n \in \mathbb{N}}$ turns out to be a nested sequence of neighborhoods of $A$, which is cofinal in the class of all neighborhoods of $A$. Since $A$ is taut in $X$, the inclusions $A \subseteq B_n$ induce an isomorphism

$$\lambda : \text{dir lim} \tilde{H}^*(B_n, G) \to \tilde{H}^*(A, G).$$

Step 3: The restriction maps $i^{\#}_{n,T} : \tilde{H}^*(X, G) \to \tilde{H}^*(B_n, G)$ define a homomorphism

$$\mu : \tilde{H}^*(X, G) \to \text{dir lim} \tilde{H}^*(B_n, G).$$

Since each $i^{\#}_{n,T}$ is surjective (Step 1) and direct limits preserve epimorphisms, $\mu$ is an epimorphism. Since $i^{\#}_{A} = \lambda \circ \mu$, we conclude that $i^{\#}_{A}$ is surjective. □

**Theorem 7.4.** Let $\{S_t\}_{t \in P}$ be a group of continuous operators on a metric space $X$. Let us assume that:

(i) there exists the global attractor $A$ for $\{S_t\}$;
(ii) there exists a bounded neighborhood $B$ of $A$ which is a deformation retract of $X$.
Then the restriction homomorphism $i^{\#}_{A} : \tilde{H}^*(X, G) \to \tilde{H}^*(A, G)$ is an isomorphism for all coefficient groups $G$.

**Proof.** The argument is very similar to the proof of Theorem 7.3. In this case in diagram (7.1) we have that $i^{\#}_{n,T}$ is an isomorphism, hence $i^{\#}_{T}$ is an isomorphism. In Step 3 we have that $\mu$ is an isomorphism, since direct limits preserve isomorphisms. It follows that $i^{\#}_{A} = \lambda \circ \mu$ is an isomorphism. □

Assumption (ii) in Theorem 7.3 (resp. Theorem 7.4) is automatically satisfied if every compact set $K \subseteq X$ has a bounded neighborhood which is a retract (resp. deformation retract) of $X$.

We note also that the assumption “$S_t$ is a group” cannot be weakened to “$S_t$ is injective for all $t \in P$” (sometimes called in the literature “backward uniqueness”), as Example 9.5 shows.

8. Examples

The following two results provide a simple example of the application of the theory developed in Sections 6 and 7. First of all, we examine a contractible space.

**Theorem 8.1.** Let $X$ be a star-like subset of a Banach space, and let $\{S_t\}_{t \in P}$ be a semigroup of continuous operators on $X$. Let us assume that the semigroup satisfies at least one of the following assumptions:

- $\{S_t\}$ is jointly continuous;
- $\{S_t\}$ is a group.
Moreover, let us assume that there exists the global attractor $A$ for $\{S_i\}$.

Then, for any group $G$, we have that:

$$\tilde{H}^q (A, G) \cong \begin{cases} G & \text{if } q = 0, \\ 0 & \text{if } q \geq 1. \end{cases}$$

**Proof.** Since a star-like subset of a Banach space is contractible, its cohomology groups coincide with the cohomology groups of a one-point space.

Therefore, if $\{S_i\}$ is jointly continuous, then the result follows from Theorem 6.3.

Moreover every compact set $K \subseteq X$ is contained in a bounded deformation retract of $X$ (e.g. the intersection of $X$ with a large enough ball in the Banach space). Therefore, if $\{S_i\}$ is a group, the result follows from Theorem 7.4.

**Remark 8.2.** For jointly continuous semigroups (or groups) defined in a Banach space, the above theorem provides a great limitation to the topology of the global attractor. For example, this attractor cannot be homeomorphic to a solid torus, or to a spherical surface, or more in general to any manifold of dim $\geq 1$.

Moreover, in the case $X = \mathbb{R}^n$, we can obtain informations on the topology of $\mathbb{R}^n \setminus A$ combining Theorem 8.1 and Alexander duality (Theorem 3.3). For example, if $n \geq 2$ it turns out that $\tilde{H}_0 (\mathbb{R}^n \setminus A, G) = 0$ for every $G$. By well known properties of restricted homology, this implies that $\mathbb{R}^n \setminus A$ is path connected.

In order to give another example, we now consider as phase space the complementary set of an open ball $U$ in $\mathbb{R}^n$. Roughly speaking, in this case we show that the global attractor for a jointly continuous semigroup (or any group) is a set $A$ which “surrounds” $U$.

**Theorem 8.3.** Let $X$ be the complementary set of an open ball $U$ in $\mathbb{R}^n$, and let $\{S_i\}_{i \in \mathbb{P}}$ be a semigroup of continuous operators on $X$. Let us assume that the semigroup satisfies at least one of the following assumptions:

- $\{S_i\}$ is jointly continuous;
- $\{S_i\}$ is a group.

Moreover, let us assume that there exists the global attractor $A$ for $\{S_i\}$.

Then $\mathbb{R}^n \setminus A$ has exactly two connected components, one bounded and one unbounded. Moreover, $U$ is contained in the bounded connected component of $\mathbb{R}^n \setminus A$.

**Proof.** It is easy to verify that $\tilde{H}^{n-1} (X, \mathbb{Z}) \cong \mathbb{Z}$. We also claim that $\tilde{H}^{n-1} (A, \mathbb{Z}) \cong \mathbb{Z}$.

Indeed if $\{S_i\}$ is jointly continuous this follows from Theorem 6.3, while if $\{S_i\}$ is a group this follows from Theorem 7.4, since each compact subset of $X$ has a bounded neighborhood which is a deformation retract of $X$.

By Alexander duality (Theorem 3.3), we have that $\tilde{H}_0 (\mathbb{R}^n \setminus A) \cong \mathbb{Z}$, where $\tilde{H}_0$ denotes the zero-dimensional reduced homology. By well known properties of reduced homology, this implies that $\mathbb{R}^n \setminus A$ consists of exactly two path components. Since $\mathbb{R}^n \setminus A$ is an open subset of $\mathbb{R}^n$, it is locally path connected, and therefore its path components coincide with its connected components.
Since \( A \) is bounded, one of these two connected components is necessarily bounded. However, since the restriction homomorphism \( i_*^A : \tilde{H}^*(X, G) \to \tilde{H}^*(A, G) \) is an isomorphism, \( U \) must be contained in the unbounded component of \( \mathbb{R}^n \setminus A \). \( \square \)

9. Counter examples

In this section we collect all the counterexamples quoted in this paper.

**Example 9.1.** Let us consider the discrete semigroup \( \{S_n\} \) on \( \mathbb{R}^2 \) generated by the function
\[
f(x, y) = (\cos(10x), \sin(10x)), \quad \forall (x, y) \in \mathbb{R}^2.
\]
Let us set \( A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \). Since \( f(\mathbb{R}^2) = A \) and \( f(A) = A \), it is clear that \( A \) is the global attractor for \( \{S_n\} \).

In this example the phase space is contractible, but the global attractor is not simply connected.

**Example 9.2.** We construct a jointly continuous semigroup on a contractible space \( X \), which admits a global attractor \( A \) with two path components.

Let us set
\[
Y := \bigcup_{n \in \mathbb{N}} \left[ \frac{1}{2^n + 1}, \frac{1}{2^n} \right] \subseteq [0,1];
\]
\[
X := [0, 1] \times [0, 1];
\]
\[
A := (\partial Y \times [0,1]) \cup ([0] \times Y) \cup ([1] \times ([0,1] \setminus Y)) \subseteq X,
\]
where \( \partial Y \) denotes the boundary of \( Y \) in \([0,1]\).

Let us consider the function \( f: [0,1] \to \mathbb{R}_{\geq 0} \) defined by
\[
f(x) := \text{dist}(x, \partial Y), \quad \forall x \in [0,1],
\]
where \( \text{dist} \) denotes the usual distance on the real line. Let us define a semigroup \( \{S_t\}_{t \geq 0} \) on \( X \) by
\[
S_t(x, y) = \begin{cases} (x, e^{-f(x)y}) & \text{if } (x, y) \in Y \times [0,1], \\ (x, 1 - e^{-f(x)(1 - y)}) & \text{otherwise}. \end{cases}
\]

It turns out that \( \{S_t\} \) is a jointly continuous semigroup on \( X \), and \( A \) is the global attractor for \( \{S_t\} \). Since \( A \) is a connected space with two path components, namely \( A_1 = \{0\} \times [0,1] \) and \( A_2 = A \setminus A_1 \), then, by Theorem 3.2, for any coefficient group \( G \) we have that:
\[
\tilde{H}^0(A, G) \cong G, \quad H^0(A, G) \cong G \oplus G.
\]

In an analogous way, it is not difficult, but rather cumbersome, to construct a jointly continuous group on \( \mathbb{R}^2 \) for which \( A \) is the global attractor. This example shows that singular and Čech–Alexander–Spanier cohomology groups of attractors may not coincide, even for jointly continuous (semi)groups in Banach spaces.
Example 9.3. Let $X := \{0\} \cup \{2^z : z \in \mathbb{Z}\}$, and let us consider the discrete group $\{S_t\}_{t \in \mathbb{Z}}$ on $X$ generated by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2^{-1} & \text{if } x = 2^z. \end{cases}$$

It turns out that $A = \{0\}$ is the global attractor for $\{S_t\}$. Furthermore $\hat{H}^0(A, \mathbb{Z}) \approx \mathbb{Z}$, while $\hat{H}^0(X, \mathbb{Z})$ is not finitely generated.

In this case the restriction homomorphism $i_A^* : \hat{H}^0(X, \mathbb{Z}) \to \hat{H}^0(A, \mathbb{Z})$ is not injective.

Example 9.4. Let $X$ and $\{S_t\}_{t \in \mathbb{Z}}$ be the space and the discrete group defined in Section 4 of [3]. Since $X$ is connected and $A$ has an infinite number of connected components, we have that $\hat{H}^0(X, \mathbb{Z}) \approx \mathbb{Z}$, while $\hat{H}^0(A, \mathbb{Z})$ is not finitely generated.

Therefore, the restriction homomorphism $i_A^* : \hat{H}^0(X, \mathbb{Z}) \to \hat{H}^0(A, \mathbb{Z})$ is not surjective.

Example 9.5. (2-adic solenoid). Let us set

$$D := \{z \in \mathbb{C} : ||z|| \leq 1\},$$

$$S^1 := \{z \in \mathbb{C} : ||z|| = 1\}.$$  

and let us consider $X := D \times S^1$. The space $X$ is homeomorphic to a solid torus.

Let us consider the function $f : X \to X$ defined by

$$f(z, w) := (z/4 + w/2, w^2),$$

and let $\{S_n\}$ be the discrete semigroup on $X$ generated by $f$. It is easy to check that $\{S_n\}$ is a semigroup of injective operators.

Intuitively, $f$ takes the solid torus $X$, stretches it, makes it thinner, and folds it in such a way that its image $f(X)$ winds twice around the central hole of $X$ (see [10] for a pictorial description of this phenomenon). More rigorously: each $f^n(X)$ is homeomorphic to $X$, hence $\hat{H}^1(f^n(X), \mathbb{Z}) \approx \mathbb{Z}$, and the inclusion map $i_n^* : f^{n+1}(X) \to f^n(X)$ induces a homomorphism

$$i_n^* : \hat{H}^1(f^{n}(X), \mathbb{Z}) \to \hat{H}^1(f^{n+1}(X), \mathbb{Z})$$

of degree 2.

The global attractor $A = \omega(X)$ is equal to the intersection of the forward images $f^n(X)$. Since the family $\{f^n(X)\}$ is cofinal among all neighborhoods of $A$, it turns out that $\hat{H}^1(A, \mathbb{Z})$ is the direct limit of $\{\hat{H}^1(f^n(X), \mathbb{Z})\}$ with homomorphisms $\{i_n^*\}$, which is isomorphic to the additive group of all rational fractions whose denominator is a power of 2.

Therefore $\hat{H}^1(A, \mathbb{Z})$ is not finitely generated, hence the restriction homomorphism $i_A^* : \hat{H}^1(X, \mathbb{Z}) \to \hat{H}^1(A, \mathbb{Z})$ is not surjective.

The set $A$ is called in the literature the “2-adic solenoid”: it is of great historical importance in algebraic topology (for further informations see [7, p. 113–114] and the references quoted therein), and in the theory of “strange attractors” (see the discussion in Appendix 3 of [9]).
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References