Note

Partitions of Reals: Measurable Approach

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We prove that if $X$ has positive Lebesgue measure in each interval $(0, a)$, then there exists a sequence $r_0, r_1, \ldots$ of positive real numbers such that each sum of these numbers belongs to $X$. This yields a measurable version of the theorem of Prömel and Voigt [3] on partitions of $[0, 1]$. © 1991 Academic Press, Inc.

1. INTRODUCTION

In [3] Prömel and Voigt considered partitions of the unit interval $[0, 1]$ and asked whether there exist infinitely many real numbers with all their sums, finite and infinite but without repetitions, that belong to the same part. They proved that for each finite partition consisting of sets with the Baire property the answer is yes. We prove a measurable version of their theorem.

2. THE MAIN THEOREM

Let $\lambda(S)$ denote the Lebesgue measure of $S$. A point $x \in S$ is said a density point of $S$ if

$$\lim_{h \to 0} \frac{\lambda((x-h, x+h) \cap S)}{2h} = 1,$$

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The Lebesgue density theorem says that if $S$ is a measurable set, then its measure, measure means Lebesgue measure, equals the measure of the set of its density points. Let us observe that

\begin{itemize}
\item[(\ast)] If $x \neq 0$ and $y$ are real numbers, then $z$ is a density point of the set $(1/x)(X - y)$ iff $z \cdot x + y$ is a density point of $X$.
\item[(\ast\ast)] A real number $z$ is a density point of sets $A_0, \ldots, A_n$ iff $z$ is a density point of the intersection of those sets.
\end{itemize}

We shall use the following theorem of Ruziewicz [4] in our proof.

**Theorem.** If a subset $P$ of reals has positive measure, then for each sequence $t_0, \ldots, t_n$ of non-negative real numbers there exists $\varepsilon > 0$ such that if $0 < \delta < \varepsilon$, then all points $x, x + \delta t_0, \ldots, x + \delta t_n$ belong to $P$ for some $x \in P$.

In order to make the paper self-contained, we insert the proof of this theorem. Namely, let $X \subset P$ be a compact subset with positive measure and let $U \supset X$ be an open set such that $(n + 1) \lambda(U) < (n + 2) \lambda(X)$. Take $\varepsilon > 0$ such that if $0 < \delta < \varepsilon$, then the sets $X - \delta t_0, \ldots, X - \delta t_n$ are contained in $U$. We have

$$\lambda(X \cap X - \delta t_0 \cap \cdots \cap X - \delta t_n) > \lambda(X) - (n + 1)(\lambda(U) - \lambda(X)) > 0.$$ 

Thus if $x \in X \cap X - \delta t_0 \cap \cdots \cap X - \delta t_n$, then all points $x, x + \delta t_0, \ldots, x + \delta t_n$ belong to $X \subset P$.

**Lemma.** Let $X$ be a compact subset of $[0, \infty)$ having positive measure in each interval $(0, \varepsilon)$. There exists a sequence of finite sequences

\begin{enumerate}
\item[(1)] $r_0^{(m)}, \ldots, r_m^{(m)}, m = 0, 1, \ldots$ of elements of $X$ such that
\item[(2)] for every $m$ all sums $\sum_{n \in \mathcal{U}} r_n^{(m)}, \emptyset \neq \mathcal{U} \subset \{0, \ldots, m\}$, belong to $X$,
\item[(3)] $r_k^{(k)} < r_{k+1}^{(k+1)} < \cdots$, for every $k$.
\end{enumerate}

**Proof.** We shall require more for sequences (1), namely, that their elements and sums of their elements are density points of $X$. We begin with letting $r_0^{(0)} \in X$ be a density point of $X$. Assume that sequences (1) satisfying (2) are defined for $m \leq n$ and that sums from (2) are density points of $X$. To define a sequence (1) for $m = n + 1$ consider sets

$$X(V) = \frac{1}{|V|} \left\{ x - \sum_{i \in V} r_i^{(m)} : x \in X \right\},$$

where $V$ is a non-empty subset of $\{0, \ldots, n\}$ and $|V|$ stands for the cardinality of $V$. By (\ast) the point 0 is a density point of every set $X(V)$ and by (\ast\ast) it is a density point of the intersection $S$ of sets $X(V)$.

Let $P$ be the set of density points of $S \cap [0, \infty)$. By the Lebesgue density
theorem the set \( P \) has positive measure. We apply the theorem of Ruziewicz for \( P \) and for the sequence \( 1, 1/2, \ldots, 1/n \). So there exists \( \varepsilon > 0 \) such that if \( 0 < \delta < \varepsilon \), then points \( x + \delta, x + \delta/2, \ldots, x + \delta/n \) belong to \( P \) for some \( x \in P \). Let \( t, 0 < t < \varepsilon \), be a density point of the set \( X \). Such a point \( t \) exists because it was assumed that \( X \cap (0, \varepsilon) \) has positive measure. We take \( r_{n+1}^{(n+1)} = t \) and \( r_{i}^{(n+1)} = r_{i}^{(n)} + x \), where \( x \) is such that points \( x, x + t, x + t/2, \ldots, x + t/n \) belong to \( P \).

We show that points \( \sum_{i \in V} r_{i}^{(n+1)} \) are density points of \( X \) and therefore belong to \( X \). Indeed, if \( \emptyset = V \subset \{0, \ldots, n\} \), then

\[
\sum_{i \in V} r_{i}^{(n+1)} = \sum_{i \in V} r_{i}^{(n)} + |V| \cdot x
\]

and

\[
\sum_{i \in V} r_{i}^{(n+1)} + r_{n+1}^{(n+1)} = \sum_{i \in V} r_{i}^{(n)} + |V| \cdot x + t.
\]

From (4) and (5) we see that

\[
x = \frac{1}{|V|} \left( \sum_{i \in V} r_{i}^{(n+1)} - \sum_{i \in V} r_{i}^{(n)} \right)
\]

and

\[
x + \frac{t}{|V|} = \frac{1}{|V|} \left( \sum_{i \in V} r_{i}^{(n+1)} + r_{n+1}^{(n+1)} - \sum_{i \in V} r_{i}^{(n)} \right),
\]

but \( x \) and \( x + t/|V| \) belong to \( P \) as \( 0 < t < \varepsilon \). By (*) sums \( \sum_{i \in V} r_{i}^{(n+1)} \) and \( \sum_{i \in V} r_{i}^{(n+1)} + r_{n+1}^{(n+1)} \) are density points of \( X \) and, whence, belong to \( X \), i.e., (2) is fulfilled for \( m = n + 1 \).

The inductive construction of a sequence of sequences (1) is performed satisfying (2). Because \( t \) and \( x \) were positive real numbers we infer that (3) is also fulfilled.

**Theorem 1.** If \( Y \) is a measurable subset of reals having positive measure in each interval \((0, \varepsilon)\), then there exists a sequence \( r_0, r_1, \ldots \) of positive real numbers such that each sum of these numbers belong to \( Y \).

**Proof.** Let \( X_n \subset Y \cap (0, 1/n) \) be a compact subset with positive measure. Take \( X = \{0\} \cup X_1 \cup X_2 \cup \ldots \). The set \( X \) satisfies hypotheses of the lemma. Thus we obtain sequences \( r_0^{(m)}, \ldots, r_m^{(m)} \) satisfying (2) and (3). Let \( r_n = \lim_{m \to \infty} r_n^{(m)} \); this limit exists by (3) and is finite as \( X \) is compact and numbers \( r_n^{(m)} \) belong to \( X \). Since the set \( X \) is compact and all sums \( \sum_{i \in V} r_{i}^{(n)} \) belong to \( X \) we infer that numbers \( r_0, r_1, \ldots \) and all their sums belong to \( X \), but \( X \subset Y \cup \{0\} \) what finishes the proof.
The hypothesis that \( Y \) has positive measure in each interval \((0, \varepsilon)\) is important. If \( Y \) is a perfect set which is independent over rationals, clearly the measure of \( Y \) is zero, then no sequence \( r_0, r_1, \ldots \) of positive real numbers has the property that all sums of these numbers belong to \( Y \); between these sums are numbers dependent over rationals, for instance, \( r_0, r_1, \) and \( r_0 + r_1 \).

Let us notice that if \( Y \) is a measurable subset of \( n \)-dimensional Euclidean space \( R^n \) having positive \( n \)-dimensional Lebesgue measure in each ball with the center 0 and the radius \( \varepsilon \), then there exists a sequence \( r_0, r_1, \ldots \) of vectors from \( R^n \) such that each sum of these vectors belongs to \( Y \). The proof of this is the same as the proof of Theorem 1 because all results used there are true in \( n \)-dimensional versions.

### 3. Partitions of Reals

We will prove the measurable version of the theorem of Prömel and Voigt [3].

**Theorem 2.** If \( U \) is a finite partition of reals consisting of measurable sets, then there exists infinitely many real numbers with all their sums belonging to the same part of \( U \).

**Proof.** There exists \( Y \in U \) which satisfies hypotheses of Theorem 1. Thus by this theorem we obtain the proof.

Let us observe that if in Theorems 1 or 2 we consider sums with repetitions, then we obtain some falsehood. For sums with repetitions the set \( Y \) has to be dense and it is clear that there exist partitions of reals consisting of finitely many measurable sets which are not dense.

For any \( x \in (0, 1] \) let \( D_x \) be the infinite subset of natural numbers such that \( x = \sum_{n \in D_x} 2^{-n-1} \). If \( U \) is a partition of \((0, 1]\), then let \( DU = \{DV = \{Dx: x \in V\}: V \in U\} \). We can ask whether there exists a sequence \( r_0, r_1, \ldots \) of real numbers such that all unions of \( Dr_0, Dr_1, \ldots \) belong to the same part of \( DU \). If \( U \) is a finite partition consisting of sets with the Baire property, then one can obtain a sequence \( r_0, r_1, \ldots \) such that sets \( Dr_0, Dr_1, \ldots \) are pairwise disjoint and all unions of these sets belong to some \( DV \in DU \), as was observed in Plewik [2].

If \( U \) consists of measurable sets, then one cannot obtain sets \( Dr_0, Dr_1, \ldots \) that are pairwise disjoint, because the set \( \{x \in (0, 1]: D_x \} \) has the asymptotic density \( 2^{-1} \) has full measure and there are no three pairwise disjoint subsets in it. Thus via the standard diagonalization argument one can construct an appropriate partition.

However, Moran and Strauss [1] proved that if a set \( C \subset (0, 1] \) has
positive measure, then there exists \( x \in C \) and an infinite subset \( A \) of natural numbers such that \( A \cap D_x = \emptyset \) and, for any \( B \subseteq A \), the union \( D_x \cup B \) belongs to \( DC \). Thus if \( A_0, A_1, \ldots \) are subsets of \( A \), then the sequence \( D_x \cup A_0, D_x \cup A_1, \ldots \) is such that any union of its elements belongs to \( DC \). If \( U \) is a finite partition of \( (0, 1] \) consisting of measurable sets, then some element of \( U \) has positive measure and we have the positive answer to our question for unions.

REFERENCES