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# The Oberwolfach Problem and Factors of Uniform Odd Length Cycles

BRIAN ALSPACH*

*Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, Canada*

P. J. SCHELLENBERG*

*Department of Combinatorics and Optimization University of Waterloo, Waterloo, Ontario, Canada*

D. R. STINSON*

*Department of Computer Science University of Manitoba, Winnipeg, Manitoba, Canada*

AND

DAVID WAGNER†

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

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Let $m \geqslant 3$ be an odd integer. In this paper it is shown that if $n \geqslant m$ is odd and $m$ divides $n$, then the edge-set of the complete graph $K_n$ can be partitioned into 2-factors each of which is comprised of $m$-cycles only. Similarly, if $n$ is an even multiple of $m$, $n \neq 4m$ and $n > 6$, then the edge-set of the complete graph on $n$ vertices with a 1-factor removed can also be partitioned into 2-factors each of which is comprised of $m$-cycles. © 1989 Academic Press, Inc.

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## 1. Introduction and Terminology

The complete graph with $n$ vertices will be denoted $K_n$ and the complete digraph with $n$ vertices will be denoted $DK_n$. If $G$ is a graph, then $|G|$ will denote the number of vertices in $G$. The notation $V(G)$ will be used for the vertex-set of $G$.

If $H$ is a subgraph of $G$, then $G \backslash H$ denotes the subgraph of $G$ that is obtained by deleting $V(H)$ from $V(G)$ and all edges incident with any vertex of $H$.

An *m-cycle* in a graph $G$ is a sequence of $m$ distinct vertices $u_1, u_2, ..., u_m$ such that $u_i$ is adjacent to $u_{i+1}$ and $u_m$ is adjacent to $u_1$. Edges are denoted by juxtaposition so that an $m$-cycle is denoted by $u_1 u_2 \cdots u_m u_1$. An *m-dicycle* in a digraph $G$ is a sequence of $m$ distinct vertices $u_1, u_2, ..., u_m$ so that there is an arc from $u_i$ to $u_{i+1}$ and from $u_m$ to $u_1$. An $m$-dicycle will be denoted $u_1 u_2 \cdots u_m u_1$ as well. The notation $C_m$ will be used for the cycle of length $m$, that is, with $m$ edges and $m$ vertices. Often $H_i$ will be used as a notation for a Hamilton cycle when it is desirable to index the Hamilton cycles for listing purposes.

A *spanning* subgraph $H$ of $G$ is one for which $V(H) = V(G)$. A *2-factor* of $G$ is a spanning subgraph that is regular of degree 2. Consequently, every component of a 2-factor is a cycle. A *2-factorization* of a graph $G$ is a partition of the edge-set $E(G)$ into 2-factors. Thus, $G$ must be regular of even degree. An $\{r_1, r_2, ..., r_t\}$ *resolvable cycle decomposition*, denoted $\{r_1, r_2, ..., r_t\}$-RCD, is a 2-factorization of $G$ so that every cycle that occurs in any of the 2-factors has length in $\{r_1, r_2, ..., r_t\}$.

The Oberwolfach problem was first formulated by Ringel and first mentioned in [3]. It asks: Given integers $r_1, r_2, ..., r_t$ all at least 3 and $\sum_{i=1}^{t} r_i = n$ odd, is it possible to 2-factorize $K_n$ so that each 2-factor consists of cycles of lengths $r_1, r_2, ..., r_t$? When it comes to cycle decomposition problems, the complete graph on an even number $n$ of vertices with a 1-factor removed, denoted $K_n - I$, plays the same role as $K_{2m+1}$. Consequently, the Oberwolfach problem now usually includes the obvious analogous question for even $n$. The notation $\mathrm{OP}(r_1^{a_1} r_2^{a_2} \cdots r_t^{a_t})$ will be used for the Oberwolfach problem when there are required to be $a_i$ cycles of length $r_i$ for $i = 1, 2, ..., t$. Of course, $n = \sum_{i=1}^{t} a_i r_i$ and the parity of $n$ determines whether $K_n$ or $K_n - I$ is under discussion.

This paper concentrates on the case when all cycles have the same length. The notation of [4] will be employed wherein $D(m) = \{mk \in \mathbb{N} : \mathrm{OP}(m^k) \text{ has a solution}\}$ was used and $\mathbb{N}$ denotes the set of positive integers.

There are some graph notations that must be presented. If $G$ is a graph, then $dG$ will denote the graph with $d$ components each of which is isomorphic to $G$. If $G$ and $H$ are two graphs so that $V(G) = V(H)$ but they have no edges in common, then $G \oplus H$ denotes the graph with the same vertex-set and $E(G \oplus H) = E(G) \cup E(H)$. Finally, $\bar{G}$ denotes the complement of $G$ and $K_n = G \oplus \bar{G}$.

The *wreath product* $G \wr H$ is obtained by replacing each vertex of $G$ with a copy of $H$ and joining two vertices in different copies of $H$ with an edge if and only if the corresponding vertices of $G$ are adjacent. That is, for $V(G) = \{w_i : i = 1, 2, ..., |G|\}$ and $V(H) = \{v_i : i = 1, 2, ..., |H|\}$, $V(G \wr H) = \{u_{ij} : i = 1, 2, ..., |G|$ and $j = 1, 2, ..., |H|\}$ and $u_{ij}u_{rs} \in E(G \wr H)$ if and only if either $i = r$ and $v_j v_s \in E(H)$ or $i \neq r$ and $w_i w_r \in E(G)$.

If $G$ and $H$ are two graphs, then $G \cup H$ is the graph satisfying $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. In almost all occurrences of $G \cup H$ throughout this paper, $V(G)$ and $V(H)$ are disjoint.

If $G$ and $H$ are two graphs, then the *direct product* $G \times H$ satisfies $V(G \times H) = V(G) \times V(H)$ and $u_{ij}u_{rs} \in E(G \times H)$ if and only if both $w_i w_r \in E(G)$ and $v_j v_s \in E(H)$.

Let $S \subseteq \{1, 2, ..., n\}$ satisfy the property that $i \in S$ if and only if $n - i \in S$. The *circulant graph* Circ$(n; S)$ has vertex-set $\{u_0, u_1, ..., u_{n-1}\}$ and $u_i$ adjacent to $u_j$ if and only if $j - i$ is in $S$ modulo $n$. The *length* of an edge $u_i u_j$ is the minimum of the two elements of $S$ congruent to $j - i$ and $i - j$ modulo $n$.

Let $V(K_n) = \{u_1, u_2, ..., u_n\}$ and $H$ be a subgraph of $K_n$. If $\sigma$ is a permutation of $\{1, 2, ..., n\}$, then $\sigma(H)$ denotes the subgraph of $K_n$ with vertex-set $\sigma(V(H))$ and edge-set $\{\sigma(u_i)\,\sigma(u_j) : u_i u_j \in E(H)\}$. In this paper, $H$ is always a spanning subgraph so that $H$ and $\sigma(H)$ have the same vertex-set.

Early papers on the Oberwolfach problem are [4, 6, 7]. A more recent paper [5] contains a good history of the earlier work together with improvements. However, most of the results have now been superseded by the present paper and [1]. In the latter paper, it is shown that $D(m) = m\mathbb{N}$ for all even $m \geqslant 4$. In the present paper, it is shown that $D(m) \supseteq m\mathbb{N} \setminus \{4m\}$ for all $m \geqslant 5$. Ray-Chaudhuri and Wilson have shown [9] that there exists a Kirkman triple system of order $3m$ for every odd integer $m$. It is also known [10] that there is a nearly Kirkman triple system of order $3m$ for every even integer $m > 4$. Nearly Kirkman triple systems of orders 6 and 12 do not exist. Thus, $D(3) = 3\mathbb{N} \setminus \{6, 12\}$.

## 2. Main Result and Outline of Its Proof

The main result of this paper, Theorem 1 below, essentially settles the Oberwolfach problem for 2-factors all of whose cycles have the same length in that only one small case is left unsettled.

THEOREM 1. *For $m$ odd and $m \geqslant 5$, $D(m) \supseteq m\mathbb{N} \setminus \{4m\}$. In addition, $D(3) = 3\mathbb{N} \setminus \{6, 12\}$.*

The proof of Theorem 1 is long. An outline of the proof is now presented in order to consolidate the details of the remaining sections.

The first essential step is to break $K_{dm}$ or $K_{dm}-I$ into $d$ blocks of cardinality $m$ and to work with the resulting subgraphs. That is, write

$$K_{dm}=dK_m \oplus (K_d \wr \bar{K}_m)$$

when $d$ is odd and when $d$ is even, write

$$K_{dm}-I=\frac{d}{2}(K_{2m}-I) \oplus ((K_d-I) \wr \bar{K}_m).$$

Since $dK_m$ and $(d/2)(K_{2m}-I)$ can both easily be decomposed into the desired 2-factors, the proof concentrates on $K_d \wr \bar{K}_m$ and $(K_d-I) \wr \bar{K}_m$.

If $G$ has a 2-factorization $F_1 \oplus F_2 \oplus \cdots \oplus F_r$, then

$$G \wr \bar{K}_m=(F_1 \wr \bar{K}_m) \oplus (F_2 \wr \bar{K}_m) \oplus \cdots \oplus (F_r \wr \bar{K}_m).$$

Furthermore, if $F_i$ is a union of cycles $C_{i_1}, C_{i_2}, ..., C_{i_s}$, then

$$F_i \wr \bar{K}_m=(C_{i_1} \wr \bar{K}_m) \cup (C_{i_2} \wr \bar{K}_m) \cup \cdots \cup (C_{i_s} \wr \bar{K}_m).$$

In Sections 4 and 5 it is shown that $K_d$ has a $\{3, 5\}$-RCD when $d$ is odd and $d \notin \{1, 7, 11\}$, and $K_d-I$ has a $\{3, 5\}$-RCD when $d$ is even and $d \notin \{2, 4, 6, 12\}$. In Section 3, it is shown that for $k$ an odd integer, $C_k \wr \bar{K}_p$ can be decomposed into 2-factors made up entirely of $p$-cycles whenever $p \geqslant k$ is a prime. In Section 6, these results are used to show that $K_d \wr \bar{K}_p$ or $(K_d-I) \wr \bar{K}_p$ can be decomposed into 2-factors each of which is composed of $p$-cycles. The proof of Theorem 1 is completed by directly verifying it for the few small values of $dm$ not covered by the previous arguments.

## 3. A Direct Construction

Definition. Consider the graph $C_s \wr \bar{K}_t$ where $t \geqslant s \geqslant 3$ and both $s$ and $t$ are odd. Let $H=u_{j_1}u_{j_2}\cdots u_{j_t}u_{j_1}$ be a Hamilton dicycle of $DK_t$. The *i-projection* of $H$ onto $C_s \wr \bar{K}_t$ is the $t$-cycle $u_{i,j_1}u_{i+1,j_2}u_{i+2,j_3}\cdots u_{i+s-1,j_s}u_{i,j_{s+1}}u_{i+1,j_{s+2}}u_{i,j_{s+3}}\cdots u_{i+1,j_t}u_{i,j_1}$, where the first subscript is reduced modulo $s$.

Lemma 2. *If $H=u_{j_1}u_{j_2}\cdots u_{j_t}u_{j_1}$ is a Hamilton dicycle of $DK_t$, then the i-projections of $H$ onto $C_s \wr \bar{K}_t$, $i=1, 2, ..., s$, yield a 2-factor composed of $s$ cycles all of length $t$. Furthermore, if the edge $u_{i,j}u_{i+1,l}$ appears in one of the* *t-cycles of the 2-factor, then the edge* $u_{k,j}u_{k+1,l}$ *for* $k = 1, 2, ..., s$ *appears in the 2-factor.*

*Proof.* The lemma follows immediately from the definition of $i$-projection of $H$ onto $C_s \wr \bar{K}_t$. ▮

The following lemma is an immediate consequence of the definitions of direct product, wreath product, and circulant graph but it is useful to explicitly state it.

LEMMA 3. *Let* $s$ *and* $t$ *be odd with* $t \geqslant 5$ *and* $s \geqslant 3$. *Then* $C_s \wr \bar{K}_t = (C_s \times \mathrm{Circ}(t; \{3, 4, ..., t-3\})) \oplus (C_s \times \mathrm{Circ}(t; \{1, 2, t-2, t-1\})) \oplus \{u_{ij}u_{i+1,j}: i = 1, 2, ..., s$ *and* $1 \leqslant j \leqslant t\}$.

We now wish to show that $C_s \wr \bar{K}_t$ can be decomposed into 2-factors all of whose cycles have length $t$ when $s \leqslant t$ and both are odd. The idea is to prove it separately for the two subgraphs arising from Lemma 3. The following lemma provides a tool for handling one of the subgraphs.

It is worth pointing out that the following two results, Lemma 4 and Theorem 5, are true for all odd $t \geqslant 7$ and odd $t \geqslant s$, respectively. However, all that is needed later in this paper for the main result is for $t$ to be an odd prime. Since the proof of Lemma 4 without the restriction that $t$ is a prime is technically much more involved, it is omitted here. Similarly, Theorem 6 is true if $t$ is not restricted to being a prime.

LEMMA 4. *The circulant graph* $\mathrm{Circ}(t; \{3, 4, ..., t-3\})$ *has a Hamilton decomposition when* $t \geqslant 7$ *and* $t$ *is a prime.*

*Proof.* The proof is immediate because all the edges of the same length form a Hamilton cycle. ▮

THEOREM 5. *Let* $s$ *be an odd integer and* $t$ *be a prime so that* $3 \leqslant s \leqslant t$. *Then* $C_s \wr \bar{K}_t$ *has a 2-factorization so that each 2-factor is composed of* $s$ *cycles of length* $t$. *Moreover, the number of 2-factors is* $t$ *which is independent of* $s$.

*Proof.* By Lemma 3, $C_s \wr \bar{K}_t$ is the edge-disjoint union of two graphs one of which is $C_s \times \mathrm{Circ}(t; \{3, 4, ..., t-3\})$. The latter graph has a Hamilton decomposition for $t \geqslant 7$. For $t = 5$, there is only one graph in the edge-disjoint union of Lemma 3, namely, $(C_s \times \mathrm{Circ}(t; \{1, 2, 3, 4\})) \oplus \{u_{ij}u_{i+1,j}: 1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant 5\}$ which will be treated shortly. When $t = 3$, $s = 3$ must hold and the result is true in this case by simply considering a Kirkman triple system on nine elements.

For $t \geqslant 7$ consider a Hamilton decomposition of $\mathrm{Circ}(t; \{3, 4, ..., t-3\})$. For a given Hamilton cycle $H$, give it an arbitrary orientation producing a Hamilton dicycle $H_1$. The $i$-projections of $H_1$ onto $C_s \wr \bar{K}_t$ for $i = 1, 2, ..., s$ yield a 2-factor of $C_s \wr \bar{K}_t$ of the desired kind. Then give $H_1$ the opposite orientation producing another Hamilton dicycle $H_2$. Again use Lemma 2 to show that the $i$-projections of $H_2$ onto $C_s \wr \bar{K}_t$ produce an appropriate 2-factor. Doing this for each Hamilton cycle in the decomposition of $\mathrm{Circ}(t; \{3, 4, ..., t-3\})$ gives a 2-factorization of $C_s \times \mathrm{Circ}(t; \{3, 4, ..., t-3\})$ into the appropriate 2-factors.

Let $G$ denote the graph $(C_s \times \mathrm{Circ}(t; \{1, 2, t-2, t-1\})) \oplus \{u_{ij}u_{i+1,j}: i = 1, 2, ..., s$ and $1 \leqslant j \leqslant t\}$. It remains to find a decomposition of $G$ into 2-factors each of which is composed of $s$ cycles of length $t$. Let the vertices have the coordinates $(i, j)$, $0 \leqslant i \leqslant s-1$ and $0 \leqslant j \leqslant t-1$. Define the following sets of edges for each $i$, $0 \leqslant i < s$:

$$A_i = \begin{cases} (i, j)(i+1, j-1) & \text{if } 0 \leqslant j < s \text{ or } s \leqslant j \leqslant t-1 \text{ and } j \text{ is odd} \\ (i, j)(i+1, j+1) & \text{if } s \leqslant j \leqslant t-1 \text{ and } j \text{ is odd} \end{cases}$$

$$B_i = \begin{cases} (i, j-1)(i+1, j) & \text{if } 0 \leqslant j < s \text{ or } s \leqslant j \leqslant t-1 \text{ and } j \text{ is odd} \\ (i, j+1)(i+1, j) & \text{if } s \leqslant j \leqslant t-1 \text{ and } j \text{ is odd} \end{cases}$$

$$D_i = \begin{cases} (i, j)(i+1, j) & \text{if } 0 \leqslant j < s \text{ or } s \leqslant j \leqslant t-1 \text{ and } j \text{ is odd} \\ (i, j)(i+1, j+2) & \text{if } s \leqslant j \leqslant t-1 \text{ and } j \text{ is odd} \end{cases}$$

$$E_i = \begin{cases} (i, j+1)(i+1, j-1) & \text{if } 0 \leqslant j < s \text{ or } s \leqslant j \leqslant t-1 \text{ and } j \text{ is odd} \\ (i, j+1)(i+1, j+1) & \text{if } s \leqslant j \leqslant t-1 \text{ and } j \text{ is odd} \end{cases}$$

$$F_i = \begin{cases} (i, j-1)(i+1, j+1) & \text{if } 0 \leqslant j < s \text{ or } s \leqslant j \leqslant t-1 \text{ and } j \text{ is odd} \\ (i, j+2)(i+1, j) & \text{if } s \leqslant j \leqslant t-1 \text{ and } j \text{ is odd.} \end{cases}$$

The sets of edges $A_i$, $B_i$, $D_i$, $E_i$, $F_i$ for $i = 0, 1, ..., s-1$ are pairwise disjoint and partition the edges of $G$. Now let

$$R_1 = A_0 \cup D_1 \cup E_2 \cup D_3 \cup E_4 \cup \cdots \cup D_{s-2} \cup E_{s-1},$$

$$R_2 = D_0 \cup E_1 \cup D_2 \cup E_3 \cup D_4 \cup \cdots \cup D_{s-3} \cup E_{s-2} \cup A_{s-1},$$

$$R_3 = E_0 \cup A_1 \cup A_2 \cup \cdots \cup A_{s-3} \cup A_{s-2} \cup D_{s-1},$$

$$R_4 = B_0 \cup B_1 \cup \cdots \cup B_{s-1}, \text{ and}$$

$$R_5 = F_0 \cup F_1 \cup \cdots \cup F_{s-1}.$$

Each of the subgraphs $R_1$, $R_2$, $R_3$, $R_4$, and $R_5$ is a 2-factor made up of $s$ cycles of length $t$ (an example is shown in Fig. 1 with $s = 5$ and $t = 9$). This completes the proof of Theorem 5. ∎

THEOREM 6. *If $s$ is an odd integer and $t$ is a prime so that $t \geqslant s$, then $st \in D(t)$.*

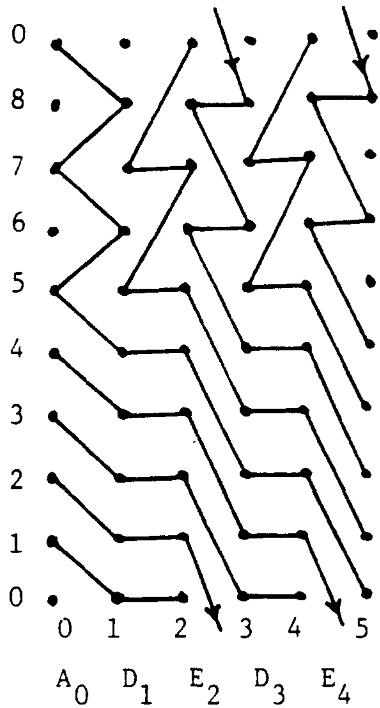


FIGURE 1

*Proof.* Write $K_{st} = K_s \wr \bar{K}_t \oplus sK_t$. Each $K_t$ has a Hamilton decomposition so that $sK_t$ can be easily decomposed into $(t-1)/2$ 2-factors of the appropriate type. Next, $K_s$ has a Hamilton decomposition $H_1, H_2, ..., H_{(s-1)/2}$ and $K_s \wr \bar{K}_t = H_1 \wr \bar{K}_t \oplus H_2 \wr \bar{K}_t \oplus \cdots \oplus H_{(s-1)/2} \wr \bar{K}$. Each $H_i \wr \bar{K}_t$ can be decomposed into the appropriate 2-factors by Theorem 5 and the result follows. ∎

THEOREM 7. *If st is odd and $st \in D(t)$, then $stl \in D(tl)$ for every positive integer l.*

*Proof.* Since $st \in D(t)$, there is a 2-factorization of $K_{ts}$, say $F_1, F_2, ..., F_r$, in which every cycle has length $t$.

If $l$ is odd, then

$$K_{stl} = K_{st} \wr K_l = F_1 \wr K_l \oplus F_2 \wr \bar{K}_l \oplus \cdots \oplus F_r \wr \bar{K}_l$$

and, if $l$ is even, then

$$K_{stl} - I = K_{st} \wr (K_l - I) = F_1 \wr (K_l - I) \oplus F_2 \wr \bar{K}_l \oplus \cdots \oplus F_r \wr \bar{K}_l.$$

Each component of $F_1 \wr K_l$, $F_1 \wr (K_l - I)$, $F_i \wr \bar{K}_l$, $i = 2, 3, ..., r$ is isomorphic to $C_t \wr K_l$, $C_t \wr (K_l - I)$, $C_t \wr \bar{K}_l$, respectively.

Now Baranyai and Szász have shown [2] that if $G$ and $H$ can be decomposed into Hamilton cycles, then so can $G \wr H$. Hence, each of $C_t \wr K_l$, $C_t \wr (K_l - I)$, and $C_t \wr \bar{K}_l$ can be decomposed into Hamilton cycles, which implies $stl \in D(tl)$. ▮

## 4. A Recursive Construction for Odd $n$

The major result of this section is the following and its proof is developed in the section. A definition is needed for the statement.

Let $J$ be a set of positive integers. An *$(n, J)$-resolvable cycle design* (denoted $(n, J)$-RCD) is a 2-factorization of $K_n$, $n$ odd, or $K_n - I$, $n$ even, such that the length of any cycle in the 2-factorization is from $J$.

Theorem 8. *For $n$ odd there exists an $(n, \{3, 5\})$-RCD if and only if $n \neq 7$ or 11.*

Let $G$ be a complete multipartite graph and let $J$ be a set of positive integers. *A $(G, J)$-cycle frame* is an edge decomposition of $G$, say $\mathscr{F} = \{F_1, F_2, ..., F_r\}$, such that

(1) every $F_i$ is a 2-factor of $G \backslash P$ for some part $P$ of the multipartition of $G$, and

(2) for every cycle $C \in F_i$, $i = 1, 2, ..., r$, $|C| \in J$.

From this definition, it follows that $r = |G|/2$ and that for each part $P$ of $G$, $|P|/2$ of the $F_i$'s are 2-factors of $G \backslash P$.

Suppose $\mathscr{F}$ is an $(n, J)$-RCD and let $H$ be a subgraph $K_t$ or $K_t - I$ (depending on the parity of $t$) of $K_n$ or $K_n - I$. If $\mathscr{F}$ restricted to $H$ induces a $(t, J)$-RCD on $H$, then this $(t, J)$-RCD is called a *subdesign* of $\mathscr{F}$ and is designated as a sub-$(t, J)$-RCD.

Clearly there is no $(7, \{3, 5\})$-RCD. In [8] it is shown that the Oberwolfach problem $\mathrm{OP}(3^2 5)$ has no solution. This implies there is no $(11, \{3, 5\})$-RCD. In the remainder of this section it will be shown that $(n, \{3, 5\})$-RCDs exist for all other odd $n \geqslant 3$.

Note that an $(n, \{3, 5\})$-RCD $\mathscr{F}$ has a sub-$(3, \{3, 5\})$-RCD if and only if there is a 3-cycle in some 2-factor of $\mathscr{F}$. It has a sub-$(5, \{3, 5\})$-RCD if and only if there are two 5-cycles of the form $u_1 u_2 u_3 u_4 u_5 u_1$ and $u_1 u_3 u_5 u_2 u_4 u_1$ in the 2-factors of $\mathscr{F}$. Also, any vertex gives rise to a sub-$(1, \{3, 5\})$-RCD.

Resolvable cycle designs are constructed by first constructing cycle frames and then "filling in holes." The main recursive construction for cycle frames follows the next collection of definitions.

A *group divisible design* is a triple $(X, \mathcal{G}, \mathcal{A})$, where

(1) $X$ is a set of points,

(2) $\mathcal{G}$ is a class of nonempty subsets of $X$ (called groups) which partition $X$,

(3) $\mathcal{A}$ is a class of subsets of $X$ (called blocks), each containing two or more points,

(4) no block meets a group in more than one point, and

(5) each pair of points not contained in a group is contained in precisely one block.

A *transversal design*, or $\mathrm{TD}(t, n)$, is a group divisible design with $tn$ points, $t$ groups of $n$ points each and such that every block has cardinality $t$. A *resolvable* transversal design, or $\mathrm{RTD}(t, n)$, is a $\mathrm{TD}(t, n)$ where the blocks can be partitioned into parallel classes.

A group divisible design can be thought of as an edge partition of a complete multipartite graph into complete subgraphs. A $\mathrm{TD}(t, n)$ can be thought of as an edge partition of $K_t \wr \bar{K}_n$ into $K_t$'s and an $\mathrm{RTD}(t, n)$ as an edge partition of $K_t \wr \bar{K}_n$ into subgraphs isomorphic to $nK_t$.

*Construction* 1 (fundamental cycle frame construction). Let $(X, \mathcal{G}, \mathcal{A})$ be a group divisible design, let $w: X \to \mathbb{N}$ ($w$ is called a weighting), and let $J$ be a set of positive integers. For every block $A \in \mathcal{A}$, let $G(A)$ be the complete multipartite graph with parts $\{x\} \times \{1, 2, ..., w(x)\}$, $x \in A$. Suppose there is a $(G(A), J)$-cycle frame for every $A \in \mathcal{A}$. Then there is a $(G(x), J)$-cycle frame, where $G(x)$ is a complete multipartite graph having parts $\{(x, i): 1 \leqslant i \leqslant w(x), x \in G\}$, $G \in \mathcal{G}$.

A multipartite graph (a cycle frame) has type $t_1^{a_1} t_2^{a_2} ...$ if there are $a_i$ parts of cardinality $t_i$, $i = 1, 2, .....$ The next construction provides a means of producing RCDs given certain cycle frames.

*Construction* 2 (filling in holes). Suppose there exists a $(G, J)$-cycle frame of type $t_1^{a_1} \cdots t_l^{a_l}$ and let $w \geqslant 1$ be odd.

(1) Suppose that there exists a $(t_i + w, J)$-RCD which contains a sub-$(w, J)$-RCD for $i = 1, 2, ..., l$. Then there is a $(\sum_{i=1}^{l} t_i a_i + w, J)$-RCD which contains a sub-$(w, J)$-RCD.

(2) Suppose that $a_l = 1$, there exists a $(t_l + w, J)$-RCD, and for $1 \leqslant i \leqslant l-1$, there is a $(t_i + w, J)$-RCD which contains a sub-$(w, J)$-RCD. Then there is a $(\sum_{i=1}^{l} t_i a_i + w, J)$-RCD.

There are two useful cycle frames for the following recursive constructions.

LEMMA 9. *The cycle frames* $(K_{2,2,2,2}, \{3\})$ *and* $(K_{2,2,2,2,4}, \{3, 5\})$ *both exist.*

*Proof.* A $(K_{2,2,2,2}, \{3\})$-cycle frame is obtained by deleting a point from a Kirkman triple system on nine points. A $(K_{2,2,2,2,4}, \{3, 5\})$-cycle frame is presented in Fig. 2. ▮

COROLLARY 10. *Suppose there exists a* TD(5, $m$), *a* $(2m+1, \{3, 5\})$-RCD *and a* $(4t+1, \{3, 5\})$-RCD *for some* $0 \leqslant t \leqslant m$. *Then there exists an* $(8m+4t+1, \{3, 5\})$-RCD *which contains a* sub-$(3, \{3, 5\})$-RCD.

*Proof.* Delete $m-t$ points from one group $G$ of a TD(5, $m$). Give every point remaining in $G$ weight 4 and all other points weight 2. Now apply Construction 1 using the frames of types $2^4$ and $2^4 4^1$ in Lemma 9. An $(8m+4t, \{3, 5\})$-cycle frame of type $2m^4 4t^1$ is obtained.

Now apply part (1) of Construction 2, with $w=1$, to produce an $(8m+4t+1, \{3, 5\})$-RCD. It contains a sub-$(3, \{3, 5\})$-RCD because the input frames for the construction each contained a 3-cycle. ▮

COROLLARY 11. *If, for some* $0 \leqslant t \leqslant m$, *there exists a* TD(5, $m$), *a* $(2m+3, \{3, 5\})$-RCD *with a* sub-$(3, \{3, 5\})$-RCD, *and a* $(4t+3, \{3, 5\})$-RCD, *then there exists an* $(8m+4t+3, \{3, 5\})$-RCD *which contains a* sub-$(3, \{3, 5\})$-RCD.

*Proof.* As in the proof of Corollary 10, construct an $(8m+4t, \{3, 5\})$-cycle frame of type $2m^4 4t^1$. Then apply part 2 of Construction 2 with $w=3$. ▮

A variety of useful small RCDs are now constructed by direct methods.

LEMMA 12. *There is an* $(n, \{3\})$-RCD *for all* $n \equiv 3 \pmod 6$.

*Proof.* These are just Kirkman triple systems and were constructed by Ray-Chaudhuri and Wilson [9]. ▮

Lemma 12 means that $n \in D(3)$ for all $n \equiv 3 \pmod 6$.

| Part | Factors |
|---|---|
| $u_1, u_8$ | $u_3u_5u_9u_6u_{11}u_3$, $u_2u_{12}u_4u_7u_{10}u_2$ |
| $u_2, u_4$ | $u_1u_7u_9u_3u_{10}u_1$, $u_5u_{12}u_6u_8u_{11}u_5$ |
| $u_3, u_6$ | $u_1u_5u_{10}u_4u_9u_1$, $u_2u_8u_{12}u_7u_{11}u_2$ |
| $u_5, u_7$ | $u_2u_6u_{10}u_8u_9u_2$, $u_1u_{12}u_3u_4u_{11}u_1$ |
| $u_9, u_{10}, u_{11}, u_{12}$ | $u_1u_4u_6u_1$, $u_2u_7u_3u_8u_5u_2$, |
| | $u_1u_2u_3u_1$, $u_4u_5u_6u_7u_8u_4$ |

FIGURE 2

LEMMA 13. *There is a* (5, {5})-RCD.

*Proof.* The cycles $u_1u_2u_3u_4u_5u_1$ and $u_1u_3u_5u_2u_4u_1$ provide the desired decomposition. ▌

LEMMA 14. *There is a* (13, {3, 5})-RCD.

*Proof.* Apply the permutation $\sigma$ and $\sigma^2$ to the two 2-factors $xu_2w_2x$, $u_0u_1v_2z_1z_0u_0$, $v_0v_1w_0z_2w_1v_0$ and $xz_0v_2x$, $u_0v_0u_1z_2w_2u_0$, $w_0w_1v_1z_1u_2w_0$, where

$$\sigma = (u_0u_1u_2)(v_0v_1v_2)(w_0w_1w_2)(z_0z_1z_2),$$

to obtain six 2-factors that partition $K_{13}$. ▌

Notice that the above factorization does more than provide a (13, {3, 5})-RCD. In fact, it solves OP($3^15^2$). Also, Lemma 14 can be proved from Corollary 10 by choosing $m = 1$ and $t = 1$. The above explicit construction is needed later.

LEMMA 15. *There is a* ($K_{4,4,4,4}$, {3})*-cycle frame and hence there exists a* (17, {3, 5})-RCD.

*Proof.* The frame is easily described in Fig. 3. In each of the four diagrams, let the four vertices correspond to the partition sets of $K_{4,4,4,4}$. Suppose that the vertices of each partition set are labelled 0, 1, 2, and 3. Then each of the 3-dicycles in the diagrams of Fig. 3 give rise to four 3-cycles in $K_{4,4,4,4}$ in two different ways. If $x$, $y$, $z$ are the labels on the arcs of the 3-dicycle made up of the first (second) elements of each pair, then let vertex $i$ be adjacent to vertex $i + x$ in the next partition set which in turn is adjacent to $i + x + y$ in the next partition set which is then adjacent to $i + x + y + z = i$ in the original parition set since $x + y + z \equiv 0 \pmod 4$ and all arithmetic is done modulo 4.

Construction 2 is used to deduce that there is a (17, {3, 5})-RCD. ▌

A ($K_{4,4,4,4}$, {3})-cycle frame is presented in [12] and can be gleaned from Piotrowski's work [8]. The above factorization also solves OP($3^45^1$).

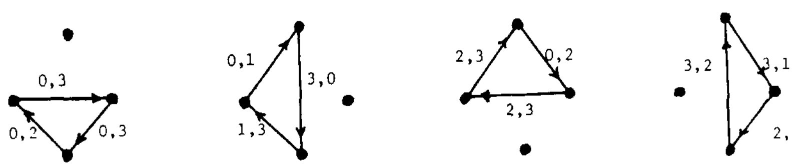


FIGURE 3

LEMMA 16. *There is a* $(19, \{3, 5\})$-RCD *that is also a solution to* $OP(3^3 5^2)$.

*Proof.* The 2-factor

$$x u_0 v_0 x, \quad u_2 u_7 v_8 u_2, \quad v_2 v_7 u_8 v_2, \quad v_1 v_3 v_6 u_4 u_5 v_1, \quad u_1 u_3 u_6 v_4 v_5 u_1$$

under the permutation $(u_0 u_1 \cdots u_8)(v_0 v_1 \cdots v_8)$ and its powers produce the desired factorization. ∎

LEMMA 17 (E. Seah [11]). *There is a* $(23, \{3, 5\})$-RCD *that is also a solution to* $OP(3^6 5^1)$.

*Proof.* The 2-factor

$$u_0 u_1 v_0 v_1 v_3 u_0, \quad u_2 u_4 u_{10} u_2, \quad v_5 u_8 v_{10} v_5, \quad v_2 u_6 v_6 v_2,$$
$$u_3 u_7 v_8 u_3, \quad v_4 v_7 u_9 v_4, \quad x u_5 v_9 x$$

under the permutation $(u_0 u_1 \cdots u_{10})(v_0 v_1 \cdots v_{10})$ and its powers produce the desired factorization. ∎

LEMMA 18 (E. Seah [11]). *There is a* $(K_{4,4,4,4,4,4}, \{3, 5\})$*-cycle frame and hence a* $(25, \{3, 5\})$-RCD.

*Proof.* Let the parts be $\{u_i, u_{6+i}, v_i, v_{i+6} : 0 \leqslant i \leqslant 5\}$. The 2-regular subgraph $u_5 v_8 u_9 v_7 v_9 u_5$, $u_1 u_2 u_4 u_1$, $v_1 v_2 v_5 v_1$, $u_3 u_{10} v_{11} u_3$, $v_4 u_7 u_{11} v_4$, $v_3 u_8 v_{10} v_3$ under the permutation $(u_0 u_1 \cdots u_{11})(v_0 v_1 \cdots v_{11})$ and its powers produces the cycle frame. The RCD comes from Construction 2. ∎

LEMMA 19. *There is a* $(K_{4,4,4,4,4,4,4}, \{3\})$*-cycle frame and hence there is a* $(29, \{3, 5\})$-RCD *that is also a solution to* $OP(3^8 5^1)$.

*Proof.* The cycle frame is presented in Fig. 4 and by Construction 2, the remaining conclusions follows. To obtain the cycle frame from Fig. 4, obtain seven digraphs from $A$ by cyclically rotating it through each of its seven positions. For each such digraph, place $B$ around the three parts corresponding to the vertices of the 3-dicycle $a$, placing the column $x$ in place of vertex $x$ and following $B$ in the direction of the arrow above it. This replaces $a$ with four edge-disjoint 3-cycles. Do the same with $C$ around 3-dicycle $b$. Then repeat with $D$ and $E$. ∎

LEMMA 20 (E. Seah [11]). *There is a* $(31, \{3, 5\})$-RCD *that is also a solution to* $OP(3^2 5^5)$.

*Proof.* The 2-factor $u_0 u_1 v_0 v_1 v_3 u_0$, $u_2 u_4 v_2 v_5 v_9 u_2$, $u_5 v_7 u_{12} u_7 u_{13} u_5$, $u_8 v_{13} u_9 u_6 v_{14} u_8$, $u_3 v_4 v_{11} v_6 v_{12} u_3$, $u_{10} v_{10} u_{14} u_{10}$, $x v_8 u_{11} x$ under the permuta-

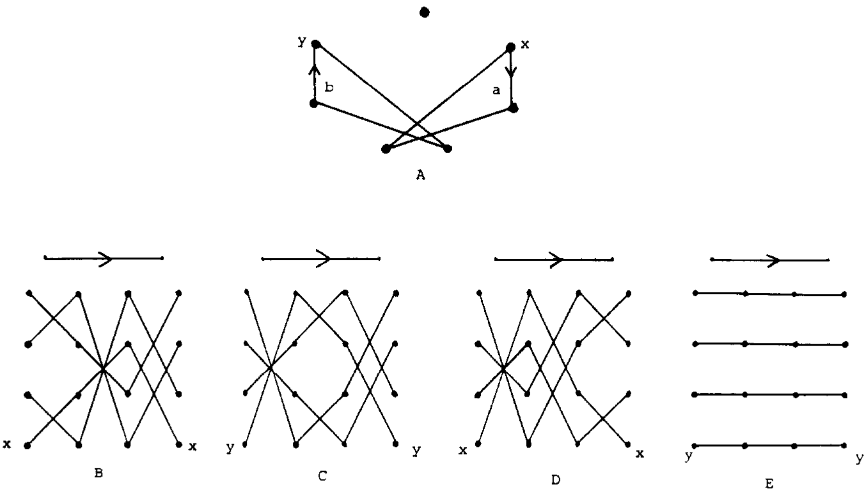


FIGURE 4

tion $(u_0 u_1 \cdots u_{14})(v_0 v_1 \cdots v_{14})$ and its powers gives the appropriate 2-factorization. ▮

LEMMA 21. *There is a* $(35, \{3, 5\})$-RCD *that is also a solution to* $\mathrm{OP}(3^{10}5^1)$.

*Proof.* The 2-factor $xu_0v_0v_9v_5x$, $u_1u_2v_3u_1$, $u_5u_7v_{12}u_5$, $u_8u_{11}v_2u_8$, $u_{12}u_{16}v_{11}u_{12}$, $u_9u_{14}v_6u_9$, $u_4u_{10}v_{14}u_4$, $u_3u_{13}v_{16}u_3$, $u_6u_{15}v_4u_6$, $v_{10}v_{13}v_{15}v_{10}$, $v_1v_7v_8v_1$ under the permutation $(u_0u_1 \cdots u_{16})(v_0v_1 \cdots v_{16})$ and its powers gives a desired 2-factorization. ▮

LEMMA 22. *There exists a* $(43, \{3, 5\})$-RCD.

*Proof.* This is an application of Construction 1. Add a point to the groups of a TD(4, 5), producing a pairwise balanced design on 21 points with blocks of sizes 4 and 6. Consider this pairwise balanced design to be a group divisible design with groups of size 1 and give every point weight 2. As input frames, the $(K_{2,2,2,2}, \{3\})$-cycle frame constructed earlier and a $(K_{2,2,2,2,2,2}, \{5\})$-cycle frame obtained by deleting the vertex $x$ from the RCD of Lemma 14 are needed. A frame of type $2^{21}$ results. The desired RCD is obtained by applying Conctruction 2. ▮

LEMMA 23. *There is a* $(47, \{3, 5\})$-RCD *that is also a solution to* $\mathrm{OP}(3^{14}5^1)$.

*Proof.* The 2-factor $xu_0v_0v_{10}v_8x$, $u_1u_2v_{13}u_1$, $u_3u_5v_4u_3$, $u_4u_7v_{17}u_4$, $u_{13}u_{17}v_{15}u_{13}$, $u_{11}u_{16}v_2u_{11}$, $u_8u_{14}v_{11}u_8$, $u_{15}u_{22}v_7u_{15}$, $u_{10}u_{18}v_{14}u_{10}$,

$u_{12}u_{21}v_5u_{12}$, $u_6u_{19}v_1u_6$, $u_9u_{20}v_3u_9$, $v_{18}v_{19}v_{22}v_{18}$, $v_9v_{16}v_{21}v_9$, $v_6v_{12}v_{20}v_6$ under the permutation $(u_0u_1 \cdots u_{22})(v_0v_1 \cdots v_{22})$ and its powers yields an appropriate 2-factorization. ▮

LEMMA 24. *There exists a* (53, {3, 5})-RCD.

*Proof.* Obtain a cycle frame of type $10^4 8^1$ as in the beginning of the proof of Corollary 10 with $m = 5$ and $t = 2$. Now apply the first part of Construction 2 with $w = 5$. In order to do this, a (13, {3, 5})-RCD is needed and it is provided by Lemma 14. Additionally, a (15, {3, 5})-RCD containing a sub-(5, {3, 5})-RCD is needed. Such a RCD is easy to find by using Theorem 5 with $s = 3$ and $t = 5$. ▮

The groundwork has now been prepared for the proof of Theorem 8. In preparation for a recursive attack, the necessary small resolvable cycle designs have been constructed above. The results are summarized in the following result.

LEMMA 25. *For $n$ odd,* $3 \leqslant n \leqslant 53$ *and* $n \neq 5, 7, 11, 37, 41$, *or* 49, *there exists an* ($n$, {3, 5})-RCD *which contains a* sub-(3, {3, 5})-RCD. *Also, there exists a* (5, {5})-RCD *and there exists neither a* (7, {3, 5})-RCD *nor an* (11, {3, 5})-RCD.

*Proof.* It suffices to observe that all the RCDs of Lemmas 12 through 24 contain at least one $C_3$ except for the (5, {5})-RCD. It is clear for those constructed directly. For those constructed recursively, observe that there is an input cycle frame containing a $C_3$. ▮

The proof of Theorem 8 is by induction on $n$ with two cases distinguished: $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$. It should be pointed out that a TD(5, $m$) exists whenever there are three mutually orthogonal Latin squares of order $m$. Their existence for the values of $m$ required in this proof are given in [13, 14]. First, let $n \equiv 1 \pmod 4$. It may be assumed that $n \geqslant 37$, $n \neq 53$. Corollary 10 may be applied as long as $m \neq 2, 3, 5, 6, 10$ and $0 \leqslant t \leqslant m$. Any $n \geqslant 37$, $n \neq 53$, can be written as $n = 8m + 4t + 1$ where $m$ and $t$ satisfy the above conditions.

Now let $n \equiv 3 \pmod 4$. It may be assumed that $n \geqslant 55$. Corollary 11 can be applied for $m \neq 2, 3, 4, 6, 10$ and one of $t = 0$ or $3 \leqslant t \leqslant m$. The stated $n$ can be written as $n = 8m + 4t + 3$ with $m$ and $t$ as above.

The result now follows by induction.

## 5. A RECURSIVE CONSTRUCTION FOR EVEN $n$

The major result of this section is the analogue of Theorem 8 for $n$ even. It is now stated.

THEOREM 26. *If $n$ is even, then there exists an* $(n, \{3, 5\})$-RCD *if and only if* $n \neq 4$, 6, *or* 12.

It is easy to see that there can be no $(n, \{3, 5\})$-RCD when $n=4$ or $n=6$. A $(12, \{3, 5\})$-RCD would have to be a solution to $OP(3^4)$ which is known to not exist [6]. An $(n, \{3, 5\})$-RCD will be constructed for all other even $n$.

A construction analogous to Construction 2 for filling in the holes of cycle frames for even $n$ is necessary. Any $(n, J)$-RCD has a trivial sub-$(2, J)$-RCD when $n$ is even. No $(n, \{3, 5\})$-RCD has a sub-$(4, \{3, 5\})$-RCD since the subdesign does not exist. As a consequence, it is a useful concept to allow RCDs to be missing subdesigns, since it does not matter if a missing subdesign exists or not. The final definition is somewhat messy, but is more easily understood if one keeps in mind that what one has is what remains when a subdesign is removed.

Let $G$ be the graph $K_n - (K_m \cup I)$ where $n$ and $m$ are both even, and $I$ is a 1-factor of $K_n$ which contains a 1-factor of the $K_m$. An $(n, J)$-RCD *missing* a sub-$(m, J)$-RCD is a set $\mathscr{F} = \{F_1, \ldots, F_{n/2-1}\}$ which satisfies:

(1) $F_1, F_2, \ldots, F_{m/2-1}$ are 2-factors of $G \setminus V(K_m)$,

(2) $F_{m/2}, \ldots, F_{n/2-1}$ are 2-factors of $G$,

(3) the $F_i$'s are mutually edge-disjoint and $\bigcup_{i=1}^{n/2-1} E(F_i) = E(G)$, and

(4) for every cycle $C \in F_i$, $i = 1, 2, \ldots, n/2-1$, $|C| \in J$.

*Construction* 3 (filling in holes). Suppose there exists a $(G, J)$-cycle frame of type $t_1^{a_1} \cdots t_l^{a_l}$ (where the $t_i$'s are all even) and let $w \geqslant 2$ be even. For $1 \leqslant i \leqslant l$, suppose that there exists a $(t_i + w, J)$-RCD missing a sub-$(w, J)$-RCD. Also, suppose that for some $j$, $1 \leqslant j \leqslant l$, there exists a $(t_j + w, J)$-RCD. Then there exists a $(\sum_{i=1}^{l} t_i a_i + w, J)$-RCD.

There are again some special cases that must be done directly as they do not follow from general results.

LEMMA 27 (Huang, Kotzig, and Rosa [6]). *There exist* $(n, \{3, 5\})$-RCDs *for* $n = 8$, 10, 14 *and* 16 *that are also solutions for* $OP(3^1 5^1)$, $OP(5^2)$, $OP(3^3 5^1)$, *and* $OP(3^2 5^2)$, *respectively.*

LEMMA 28. *There is a* $(20, \{3, 5\})$-RCD *that is also a solution to* $OP(3^5 5^1)$.

*Proof.* The 2-factor $x u_0 y v_0 v_7 x$, $v_2 v_3 v_6 v_2$, $u_1 u_3 v_8 u_1$, $u_2 u_5 v_4 u_2$, $u_4 u_8 v_5 u_4$, $u_6 u_7 v_1 u_6$ under the permutation $(u_0 u_1 \cdots u_8)(v_0 v_1 \cdots v_8)$ and its powers gives the appropriate 2-factorization of $K_{20} - I$. The missing 1-factor is $xy$ together with $u_i v_i$, $i = 0, 1, \ldots, 8$. ∎

LEMMA 29. *There is a* (22, {3, 5})-RCD *that is also a solution to* OP($3^4 5^2$).

*Proof.* The two 2-factors $u_3w_4w_2v_4z_4u_3$, $u_4v_1w_3z_2z_3u_4$, $xu_0v_0x$, $yw_0z_0y$, $u_1u_2w_1u_1$, $z_1v_2v_3z_1$ and $u_2u_4w_1z_3v_1u_2$, $v_0v_3z_4u_1w_4v_0$, $xw_0z_1x$, $yu_3v_4y$, $v_2w_2w_3v_2$, $u_0z_0z_2u_0$ under the permutation $(u_0u_1 \cdots u_4)(v_0v_1 \cdots v_4)$ $(w_0w_1 \cdots w_4)(z_0z_1 \cdots z_4)$ and its powers provide the appropriate factorization. The missing 1-factor is $xy$, $u_iv_{i+3}$, $w_iz_{i+3}$, $i=0, 1, 2, 3, 4$, where the subscripts are reduced modulo 5. ∎

LEMMA 30. *There exists a* (28, {3, 5})-RCD *that is also a solution to* OP($3^6 5^2$).

*Proof.* The 2-factor $u_0xv_0v_3yu_0$, $v_1v_8v_7v_{11}v_9v_1$, $u_1u_{12}v_6u_1$, $u_2u_{11}v_{12}u_2$, $u_3u_{10}v_5u_3$, $u_4u_9v_2u_4$, $u_5u_8v_4u_5$, $u_6u_7v_{10}u_6$ under the permutation $(u_0u_1 \cdots u_{12})(v_0v_1 \cdots v_{12})$ and its powers yields the appropriate 2-factorization of $K_{28}-I$. The missing 1-factor is $xy$, $u_iv_i$, where $i=0, 1, ..., 12$. ∎

LEMMA 31. *There exists a* (40, {3, 5})-RCD *that is also a solution to* OP($3^{10} 5^2$).

*Proof.* The 2-factor $u_7xv_0yu_9u_7$, $u_5u_{17}u_6u_{16}u_{11}u_5$, $u_0u_1u_4u_0$, $u_2v_3v_4u_2$, $u_8v_{12}v_{16}u_8$, $u_{13}v_7v_{10}u_{13}$, $u_{14}v_2v_9u_{14}$, $u_{18}v_8v_{17}u_{18}$, $u_{15}v_{11}v_{13}u_{15}$, $u_3v_6v_{14}u_3$, $u_{12}v_5v_{18}u_{12}$, $u_{10}v_1v_{15}u_{10}$ under the permutation $(u_0u_1 \cdots u_{18})$ $(v_0v_1 \cdots v_{18})$ and its powers yields the appropriate 2-factorization. ∎

Various congruences classes of $n$ in Theorem 26 can now be eliminated.

LEMMA 32. *If* $n \equiv 0$ (mod 6), *there exists an* ($n$, {3, 5})-RCD *if and only if* $n \geqslant 18$.

*Proof.* The non-existence of a (6, {3, 5})-RCD and (12, {3, 5})-RCD was noted earlier. A solution to OP($3^{n/3}$) for $n \geqslant 18$ is given in [10]. ∎

LEMMA 33. *If* $n \equiv 2$ (mod 6), *there is an* ($n$, {3, 5})-RCD *that is also a solution to* OP($3^{(n-5)/3} 5^1$).

*Proof.* The values $n=8$, 14, and 20 were done above so that $n \geqslant 26$ may be assumed. By the results in [12], there exists a cycle frame of type $6^{(n-2)/6}$ with cycles of length 3. Apply Construction 3 with $w=2$ using solutions to OP($3^1 5^1$). ∎

LEMMA 34. *For* $n \equiv 10$ (mod 24), *there exists an* ($n$, {3, 5})-RCD *that is also a solution to* OP($3^{(n-10)/3} 5^2$).

*Proof.* The case $n=10$ was done earlier. For $n \geqslant 34$, start with a cycle-frame of type $8^{(n-2)/8}$ and cycles of length 3 which exists by [12]. Apply Construction 3 with $w=2$ using solutions to OP($5^2$). ∎

A particular RCD missing a sub-(4, $\{3, 5\}$)-RCD is necessary for the next class of RCDs. It is constructed in the next lemma and used in the lemma immediately following.

LEMMA 35. *There exists a* (16, $\{3, 5\}$)-RCD *missing a* sub-(4, $\{3, 5\}$)-RCD.

*Proof.* Let the vertex-set $V=\{x_1, x_2, x_3, x_4\} \cup \{u_{ij}: 1 \leqslant i \leqslant 4$ and $0 \leqslant j \leqslant 2\}$ with $W=\{x_1, x_2, x_3, x_4\}$. The rest of the missing 1-factor outside of $W$ is $\{u_{1j}u_{4j}$ and $u_{2j}u_{3,j+1}: j=0, 1, 2\}$. The 2-factor of $V \backslash W$ is $u_{10}u_{11}u_{12}u_{10}$, $u_{20}u_{21}u_{22}u_{20}$, $u_{30}u_{31}u_{32}u_{30}$, $u_{40}u_{41}u_{42}u_{40}$. The remaining six 2-factors of $V$ are obtained by taking the two 2-factors $x_1u_{10}u_{20}u_{30}u_{40}x_1$, $x_2u_{21}u_{42}u_{11}u_{32}x_2$, $x_3u_{31}u_{12}x_3$, $x_4u_{41}u_{22}x_4$ and $x_1u_{20}u_{11}u_{40}u_{31}x_1$, $x_2u_{10}u_{21}u_{30}u_{41}x_2$, $x_3u_{22}u_{42}x_3$, $x_4u_{12}u_{32}x_4$ under the powers of the permutation $(u_{10}u_{11}u_{12})(u_{20}u_{21}u_{22})(u_{30}u_{31}u_{32})(u_{40}u_{41}u_{42})$. ∎

LEMMA 36. *For* $n \equiv 4$ (mod 12), *there is an* ($n$, $\{3, 5\}$)-RCD *if and only if* $n \geqslant 16$.

*Proof.* It was observed earlier that no such RCD exists when $n=4$ and showed that they do exist when $n=16$, 28, and 40. Assume $n \geqslant 52$. The results of [12] yield a cycle frame of type $12^{(n-4)/12}$ with cycles of length 3. Apply Construction 3 with $w=4$, filling in (16, $\{3, 5\}$)-RCDs missing a sub-(4, $\{3, 5\}$)-RCD and one (16, $\{3, 5\}$)-RCD. ∎

LEMMA 37. *For* $n \equiv 22$ (mod 24), *there exists an* ($n$, $\{3, 5\}$)-RCD.

*Proof.* Proceed by induction on $n$. The value $n=22$ was done in Lemma 29. Do the following when $n \neq 94$. By the methods of the proof of Corollary 10, construct a cycle frame of type $2m^4 12^1$. Since $t=3$, $m=(n-14)/8 \equiv 1$ (mod 3), there is a TD(5, $m$). Now apply Construction 3 with $w=2$; filling in ($2m+2$, $\{3, 5\}$)-RCDs and (14, $\{3, 5\}$)-RCDs. (Note that $2m+2$ may be congruent to 22 modulo 24 so that the induction hypothesis is required.)

The exceptional value $n=94$ is handled as above but with $m=9$ and $t=5$. The RCDs filled in are (20, $\{3, 5\}$)-RCDs and (22, $\{3, 5\}$)-RCDs. ∎

All the possibilities have been covered and the proof of Theorem 26 is complete.

## 6. Proof of the Main Result

We now combine the results of Sections 3 and 4 and exhibit three particular 2-factorizations to establish the following result.

LEMMA 38. *If $d$ is an odd integer and $p$ is an odd prime, there is a 2-factorization of $K_d \wr \bar{K}_p$ which consists entirely of $p$-cycles.*

*Proof.* For $p=3$, removing any single resolution class of a Kirkman triple system of order $3d$ yields a $\{3\}$-RCD of $K_d \wr \bar{K}_3$. Since it is well known [9] that such triple systems exist for every odd integer $d$, the result holds for $p=3$.

Now consider prime $p$ such that $p \geqslant 5$. If $d \notin \{7, 11\}$, there exists a $(d, \{3, 5\})$-RCD by Theorem 8. Let $F_1, F_2, ..., F_r$, where $r=(d-1)/2$, be the 2-factors of this design. Hence,

$$K_d \wr \bar{K}_p = F_1 \wr \bar{K}_p \oplus F_2 \wr \bar{K}_p \oplus \cdots \oplus F_r \wr \bar{K}_p;$$

furthermore, each component of $F_i \wr \bar{K}_p$, $i=1, 2, ..., r$, is isomorphic to $C_3 \wr \bar{K}_p$ or $C_5 \wr \bar{K}_p$. Since $p \geqslant 5$, Theorem 5 implies that each of these graphs can be decomposed into $p$ 2-factors consisting entirely of $p$-cycles. Hence, there is a $\{p\}$-RCD of $K_d \wr \bar{K}_p$.

If $d \in \{7, 11\}$ and $p$ is a prime satisfying $p \geqslant d$, then Theorem 6 implies $dp \in D(p)$. Observe that in the proof of Theorem 6 it is shown that $K_d \wr \bar{K}_p$ has a $\{p\}$-RCD.

To complete the proof, we exhibit a 2-factorization of $K_d \wr \bar{K}_p$ consisting entirely of $p$-cycles for $(d, p)=(7, 5)$, $(11, 5)$, and $(11, 7)$.

Designate the vertices of $K_{35}$ by $x_1 x_2, ..., x_5, u_0, u_1, ..., u_{14}, v_0, v_1, ..., v_{14}$. Obtain a graph isomorphic to $K_7 \wr \bar{K}_5$ by removing the edges of the seven vertex-disjoint $K_5$'s induced by the vertex sets

$$\{x_1, x_2, x_3, x_4, x_5\},$$

$$\{u_i : i \equiv k \pmod 3\} \quad \text{for} \quad k=0, 1, 2$$

and

$$\{v_i : i \equiv k \pmod 3\} \quad \text{for} \quad k=0, 1, 2.$$

The required fifteen 2-factors are obtained by taking the 2-factor

$$u_0 u_1 u_8 u_6 u_{10} u_0, \qquad v_0 v_1 v_8 v_6 v_{10} v_0,$$

$$x_1 u_2 v_3 u_{14} v_4 x_1, \qquad x_2 v_2 u_3 v_{14} u_4 x_2,$$

$$x_3 u_5 v_{12} u_9 v_{11} x_3, \qquad x_4 v_5 u_{12} v_9 u_{11} x_4,$$

$$x_5 u_7 v_{13} u_{13} v_7 x_5$$

under the powers of the permutation

$$(u_0 u_1 \cdots u_{14})(v_0 v_0 \cdots v_{14}).$$

Designate the vertices of $K_{55}$ by $x_1, x_2, ..., x_5, u_0, u_1, ..., u_{24}, v_0, v_1, ..., v_{24}$. As in the above case, remove the edges of the eleven vertex-disjoint $K_5$'s induced by the vertex sets,

$$\{x_1, x_2, x_3, x_4, x_5\},$$

$$\{u_i : i \equiv k \pmod 5\} \qquad \text{for} \quad k = 0, 1, 2, 3, 4,$$

and

$$\{v_i : i \equiv k \pmod 5\} \qquad \text{for} \quad k = 0, 1, 2, 3, 4,$$

to obtain a graph isomorphic to $K_{11} \wr \bar{K}_5$. The required twenty-five 2-factors are obtained by taking the 2-factors

$$u_0 u_3 u_{12} u_{14} u_1 u_0, \qquad v_0 v_3 v_{12} v_{14} v_1 v_0,$$

$$u_2 u_9 u_5 u_{24} v_{10} u_2, \qquad v_2 v_9 v_5 v_{24} u_{10} v_2,$$

$$u_{15} u_{23} v_{22} u_{20} v_{11} u_{15}, \qquad v_{15} v_{23} u_{22} v_{20} u_{11} v_{15},$$

$$u_6 v_{13} u_7 v_{17} x_1 u_6, \qquad v_6 u_{13} v_7 u_{17} x_2 v_6,$$

$$u_8 v_{21} u_{16} v_{19} x_3 u_8, \qquad v_8 u_{21} v_{16} u_{19} x_4 v_8,$$

$$u_4 u_{18} v_{18} v_4 x_5 u_4$$

under the permutation

$$(u_0 u_1 \cdots u_{24})(v_0 v_1 \cdots v_{24})$$

and its powers.

Designate the vertices of $K_{77}$ by $x_1, x_2, ..., x_7; u_0, u_1, ..., u_{34}; v_0, v_1, ..., v_{34}$. Remove the edges of the eleven vertex-disjoint $K_7$'s induced by the vertex sets

$$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\},$$

$$\{u_i : i \equiv k \pmod 5\} \qquad \text{for} \quad k = 0, 1, 2, 3, 4,$$

and

$$\{v_i : i \equiv k \pmod 5\} \qquad \text{for} \quad k = 0, 1, 2, 3, 4,$$

to obtain a graph isomorphic to $K_{11} \wr \bar{K}_7$. The required 2-factorization is obtained by letting the permutation

$$(u_0 u_1 \cdots u_{34})(v_0 v_1 \cdots v_{34})$$

and its powers act on the 2-factors

$$u_0u_1u_{34}u_2u_{33}u_4u_{21}u_0, \qquad v_0v_1v_{34}v_2v_{33}v_4v_{21}v_0,$$

$$u_3u_{22}u_9u_{20}u_{32}u_{23}u_{30}u_3, \qquad v_3v_{22}v_9v_{20}v_{32}v_{23}v_{30}v_3,$$

$$u_{17}v_{13}u_{18}v_{12}u_{19}v_{11}x_1u_{17}, \qquad v_{17}u_{13}v_{18}u_{12}v_{19}u_{11}x_2v_{17},$$

$$u_{26}v_{28}u_{15}v_{27}u_{16}v_6x_3u_{26}, \qquad v_{26}u_{28}v_{15}u_{27}v_{16}u_6x_4v_{26},$$

$$u_{14}v_{29}u_{10}v_{24}u_{25}v_7x_5u_{14}, \qquad v_{14}u_{29}v_{10}u_{24}v_{25}u_7x_6v_{14},$$

$$u_8v_5u_{31}v_{31}u_5v_8x_7u_8.$$

This completes the proof. ▮

In a similar vein, we establish the following result for even integers $d$.

LEMMA 39. *If $d \in 2\mathbb{N} \setminus \{4, 6\}$ and $p$ is a prime, $p \geqslant 5$, then $(K_d - I) \wr \bar{K}_p$ has a $\{p\}$-RCD.*

*Proof.* By Theorem 26, there exists a $(d, \{3, 5\})$-RCD for $d \in 2\mathbb{N} \setminus \{4, 6, 12\}$. Let $F_1, F_2, ..., F_r$, for $r = (d-2)/2$, be the 2-factors of such a decomposition. Then

$$(K_d - I) \wr \bar{K}_p = F_1 \wr \bar{K}_p \oplus F_2 \wr \bar{K}_p \oplus \cdots \oplus F_r \wr \bar{K}_p,$$

where each component of $F_i \wr \bar{K}_p$, $i = 1, 2, ..., r$, is isomorphic to $C_3 \wr \bar{K}_p$ or $C_5 \wr \bar{K}_p$. As in Lemma 38, Theorem 5 implies $(K_d - I) \wr \bar{K}_p$ has a $\{p\}$-RCD.

Next we consider $(K_{12} - I) \wr \bar{K}_p$ for $p$ a prime, $p \geqslant 7$. Huang, Kotzig, and Rosa have shown [6] that there is a $(12, \{5, 7\})$-RCD. Let its 2-factors be $F_1, F_2, ..., F_5$. Then

$$(K_{12} - I) \wr \bar{K}_p = F_1 \wr \bar{K}_p \oplus F_2 \wr \bar{K}_p \oplus \cdots \oplus F_5 \wr \bar{K}_p,$$

where each component of $F_i \wr \bar{K}_p$, $i = 1, 2, ..., 5$, is isomorphic to $C_5 \wr \bar{K}_p$ or $C_7 \wr \bar{K}_p$. Since $p \geqslant 7$ then, as above, Theorem 5 implies that $(K_{12} - I) \wr \bar{K}_p$ has a $\{p\}$-RCD.

Finally, we exhibit a $\{5\}$-RCD of $(K_{12} - I) \wr \bar{K}_5$. Designate the vertices of $K_{60}$ by $x_1, x_2, ..., x_{10}, u_0, u_1, ..., u_{24}, v_0, v_1, ..., v_{24}$. To obtain a graph isomorphic to $(K_{12} - I) \wr \bar{K}_5$, remove the edges of the six vertex-disjoint $K_{10}$'s induced by the vertex sets

$$\{x_i : i = 1, 2, ..., 10\}$$

and

$$\{u_i, v_i : i \equiv k \pmod 5\} \qquad \text{for} \quad k = 0, 1, 2, 3, 4.$$

The required 2-factorization is obtained by letting the permutation

$$(u_0 u_1 \cdots u_{24})(v_0 v_1 \cdots v_{24})$$

and its powers act on

$$\begin{array}{ll} u_0 u_1 u_3 u_6 u_{14} u_0, & v_0 v_1 v_3 v_6 v_{14} v_0, \\ x_1 u_{13} u_7 v_{15} v_{22} x_1, & x_2 v_{13} v_7 u_{15} u_{22} x_2, \\ x_3 u_8 u_{17} v_{16} v_{12} x_3, & x_4 v_8 v_{17} u_{16} u_{12} x_4, \\ x_5 u_{20} v_{23} u_2 v_4 x_5, & x_6 v_{20} u_{23} v_2 u_4 x_6, \\ x_7 u_9 v_{21} u_{10} v_{19} x_7, & x_8 v_9 u_{21} v_{10} u_{19} x_8, \\ x_9 u_5 u_{18} v_{11} v_{24} x_9, & x_{10} u_{24} v_5 u_{11} v_{18} x_{10}. \end{array}$$

This completes the proof. ∎

Next, the result of Lemma 39 is extended to include $d = 6$.

LEMMA 40. *If $m$ is an odd integer, $m > 5$, then $(K_6 - I) \wr \bar{K}_m$ has an $\{m\}$-RCD.*

*Proof.* Observe that $(K_6 - I) \wr \bar{K}_m$ is isomorphic to $K_3 \wr \bar{K}_{2m}$. Designate the vertices of $K_{6m}$ by $x_0, x_1, ..., x_{2m-1}, u_0, u_1, ..., u_{2m-1}, v_0, v_1, ..., v_{2m-1}$. To obtain a graph isomorphic to $K_3 \wr \bar{K}_{2m}$, remove the edges of the vertex disjoint $K_{2m}$'s induced by the three vertex sets

$$X = \{x_i : i = 0, 1, ..., 2m-1\},$$
$$U = \{u_i : i = 0, 1, ..., 2m-1\},$$
$$V = \{v_i : i = 0, 1, ..., 2m-1\}.$$

The required 2-factorization is obtained by letting the permutation

$$(u_0 u_1 \cdots u_{2m-1})(v_0 v_1 \cdots v_{2m-1})$$

and its powers act on an initial 2-factor. All that remains is to describe an appropriate initial 2-factor.

*Case* 1. $m = 2r + 1 \equiv 1 \pmod 4$, $m > 5$. Let two $m$-cycles of the desired initial 2-factor be

$$u_3 v_{2m-3} u_4 v_{2m-4} \cdots u_r v_{2m-r} x_0 u_m x_1 v_{r+1} x_2 u_3,$$
$$v_3 u_{2m-3} v_4 u_{2m-4} \cdots v_r u_{2m-r} x_3 v_0 x_4 u_{r+1} x_5 v_3.$$

The remaining four $m$-cycles include, as four consecutive vertices,

$$u_1 v_{2m-1} u_2 v_{2m-2},$$

$$v_1 u_{2m-1} v_2 u_{2m-2},$$

$$u_0 v_m u_{m-1} v_{m+4},$$

$$u_{m+3} v_{m+3} u_{m+4} v_{m-1},$$

respectively. These four $m$-cycles are then completed by alternating vertices of $X$ with vertices of $U \cup V$, with the requirement that each $x_i \in X$ must be adjacent to one vertex of $U$ and one of $V$.

For example, for $m = 13$, an appropriate initial 2-factor is

$$u_3 v_{23} u_4 v_{22} u_5 v_{21} u_6 v_{20} x_0 u_{13} x_1 v_7 x_2 u_3,$$

$$v_3 u_{23} v_4 u_{22} v_5 u_{21} v_6 u_{20} x_3 v_0 x_4 u_7 x_5 v_3,$$

$$u_1 v_{25} u_2 v_{24} x_6 u_8 x_7 v_9 x_8 u_{10} x_9 v_{11} x_{10} u_1,$$

$$v_1 u_{25} v_2 u_{24} x_{11} v_8 x_{12} u_9 x_{13} v_{10} x_{14} u_{11} x_{15} v_1,$$

$$u_0 v_{13} u_{12} v_{17} x_{16} u_{14} x_{17} v_{14} x_{18} u_{18} x_{19} v_{18} x_{20} u_0,$$

$$u_{16} v_{16} u_{17} v_{12} x_{21} u_{15} x_{22} v_{15} x_{23} u_{19} x_{24} v_{19} x_{25} u_{16}.$$

*Case* 2. $m = 2r + 1 \equiv 3 \pmod 4$, $m > 3$. Let two $m$-cycles of the desired initial 2-factor be

$$u_1 v_{2m-1} u_2 v_{2m-2} \cdots u_r v_{2m-r} x_0 u_1,$$

$$v_1 u_{2m-1} v_2 u_{2m-2} \cdots v_r u_{2m-r} x_1 v_1.$$

The remaining four $m$-cycles include, as adjacent vertices,

$$u_0 v_m, \qquad v_{r+1} u_{r+1}, \qquad u_{r+2} v_{r+3}, \qquad \text{and} \qquad v_{r+2} u_{r+3},$$

respectively. These four $m$-cycles are completed just as in Case 1.

As an example, an appropriate initial 2-factor for $m = 7$ is

$$u_1 v_{13} u_2 v_{12} u_3 v_{11} x_0 u_1,$$

$$v_1 u_{13} v_2 u_{12} v_3 u_{11} x_1 v_1,$$

$$u_0 v_7 x_2 u_7 x_3 v_8 x_4 u_0,$$

$$v_4 u_4 x_5 v_0 x_6 u_8 x_7 v_4,$$

$$u_5 v_6 x_8 u_9 x_9 v_{10} x_{10} u_5,$$

$$v_5 u_6 x_{11} v_9 x_{12} u_{10} x_{13} v_5.$$

This completes the proof of Lemma 40. ∎

Before proceeding with the proof of Theorem 1, we recall the following result of Huang, Kotzig, and Rosa.

THEOREM 41 [6]. *For any integer* $m > 3$, $2m \in D(m)$.

Now let us proceed with the proof of Theorem 1. For any odd integer $d$ and any odd prime $p$,

$$K_{dp} = dK_p \oplus K_d \wr \bar{K}_p.$$

Since $K_p$ has a decomposition into Hamilton cycles, there is a $\{p\}$-RCD of $dK_p$. By Lemma 38, there is a $\{p\}$-RCD of $K_d \wr \bar{K}_p$. Hence, $dp \in D(p)$. By Theorem 7, it follows that for any odd integers $d$ and $m$, $dm \in D(m)$.

Now consider any even integer $d$ and any odd prime $p$. Let $r = d/2$ and consider the decomposition

$$K_{dp} - I = r(K_{2p} - I) \oplus (K_d - I) \wr \bar{K}_p.$$

Observe that for $p = 3$, this decomposition is not effective for decomposing $K_{3d} - I$ into 2-factors of 3-cycles since it is known that $K_6 - I$ cannot be decomposed into 2-factors of 3-cycles. However, as mentioned in the Introduction, it is known that $D(3) = 3\mathbb{N} \setminus \{6, 12\}$. Hence, we consider primes $p \geqslant 5$.

By Theorem 41, there is a $\{p\}$-RCD of $K_{2p} - I$ and, hence, a $\{p\}$-RCD of $r(K_{2p} - I)$. Lemmas 39 and 40 imply there is a $\{p\}$-RCD of $(K_d - I) \wr \bar{K}_p$ for all $d \in 2\mathbb{N} \setminus \{4\}$ and all primes $p \geqslant 5$ except possibly for $(d, p) = (6, 5)$. Hence, $dp \in D(p)$ for all such $d$ and $p$. A direct construction establishes that $30 \in D(5)$. Let the permutation

$$(u_0 u_1 \cdots u_{13})(v_0 v_1 \cdots v_{13})$$

and its powers act on the initial 2-factor,

$$\begin{array}{ll} u_0 v_2 v_{13} v_3 v_5 u_0, & v_0 u_2 u_{12} u_3 u_4 v_0, \\ u_1 v_7 x_1 u_8 v_8 u_1, & v_1 u_6 x_2 v_4 u_7 v_1, \\ u_5 v_9 u_{10} u_{13} u_{11} u_5, & v_{10} u_9 v_{12} v_6 v_{11} v_{10}, \end{array}$$

to give the necessary 2-factorization.

Since $2m \in D(m)$ for all $m > 3$, $D(3) = 3\mathbb{N} \setminus \{6, 12\}$ and $D(p) \supseteq p\mathbb{N} \setminus \{4p\}$ for all primes $p \geqslant 5$, Theorem 7 implies that, for any odd integer $m > 3$,

$$D(m) \supseteq m\mathbb{N} \setminus \{4m\};$$

furthermore, $D(3) = 3\mathbb{N} \setminus \{6, 12\}$. This establishes Theorem 1.

We close this paper by observing that the result of Baranyai and Szász referred to in Theorem 7 permits us to strengthen the results of Lemmas 38, 39, and 40 as follows.

THEOREM 42. (a) *If d and m are both odd integers, there is a 2-factorization of* $K_d \wr \bar{K}_m$ *which consists entirely of m-cycles.*

(b) *If* $d \in 2\mathbb{N} \setminus \{4\}$ *and m is any odd number which is not a power of* 3 *and* $(d, m) \neq (6, 5)$, *then there is a 2-factorization of* $(K_d - I) \wr \bar{K}_m$ *which consists entirely of m-cycles.*

## REFERENCES

1. B. ALSPACH AND R. HÄGGKVIST, Some observations on the Oberwolfach problem, *J. Graph Theory* **9** (1985), 177–187.
2. Z. BARANYAI AND GY. R. SZÁSZ, Hamiltonian decomposition of lexicographic product, *J. Combin. Theory Ser. B* **31** (1981), 253–261.
3. R. K. GUY, Unsolved combinatorial problems, *in* "Combinatorial Mathematics and Its Applications, Proceedings Conf. Oxford 1967" (D. J. A. Welsh, Ed.), p. 121, Academic Press, New York, 1971.
4. P. HELL, A. KOTZIG, AND A. ROSA, Some results on the Oberwolfach problem, *Aequationes Math.* **12** (1975), 1–5.
5. J. D. HORTON, B. K. ROY, P. J. SCHELLENBERG, AND D. R. STINSON, On decomposing graphs into isomorphic uniform 2-factors, *Ann. Discrete Math.* **27** (1985), 297–319.
6. C. HUANG, A. KOTZIG, AND A. ROSA, On a variation of the Oberwolfach problem, *Discrete Math.* **27** (1979), 261–277.
7. E. KÖHLER, Über der Oberwolfacher Problem, *Beiträge Geom. Algebra Basel* (1977), 189–201.
8. W. PIOTROWSKI, Untersuchungen über das Oberwolfacher Problem, preprint.
9. D. K. RAY-CHAUDHURI AND R. M. WILSON, Solution of Kirkman's schoolgirl problem, *in* Proc. Symp. Pure Math. Vol. 19, pp. 187–204, Amer. Math. Soc., Providence, RI, 1971.
10. R. REES AND D. R. STINSON, On resolvable group-divisible designs with block size 3, *Ars Combin.* **23** (1987), 107–120.
11. E. SEAH, personal communication.
12. D. R. STINSON, Frames for Kirkman triple system, *Discrete Math.* **65** (1987), 289–300.
13. D. TODOROV, Three mutually orthogonal Latin squares of order 14, *Ars Combin.* **20** (1985), 45–47.
14. W. D. WALLIS, Three orthogonal Latin squares, *Congr. Numer.* **42** (1984), 69–86.