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Duality Theory and Slackness Conditions in Multiobjective Linear Programming

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Abstract—The aim of this paper is to develop a duality theory for linear multiobjective programming verifying similar properties as in the scalar case. We use the so-called "strongly proper optima" and we characterize such optima and its associated dual solutions by means of some complementary slackness conditions. Moreover, the dual solutions can measure the sensitivity of the primal optima.

Keywords—Multiobjective linear programming, Duality, Strongly proper optimum, Slackness conditions.

1. INTRODUCTION

The aim of the present paper is to develop a duality theory for Linear Multiobjective Programming that generalizes the usual properties of the scalar case.

The importance of duality techniques in scalar Linear Programming is well known because of the many relationships existing between primal and dual solutions, the most important of which is, no doubt, the interpretation of the dual solutions in terms of the sensitivity of the primal program. Many authors have tried to extend such properties to the multiobjective case, starting from different definitions of dual programs. For example, in [1,2] Zowe has defined a dual program whose solutions measure the primal sensitivity in a similar way as in the scalar case. However, Zowe defines the optima of the multiobjective program as strong optima instead of Pareto optima, as most authors do. To be precise, if the problem is a minimization program, f is the objective function and x_0 is a feasible solution, then Zowe defines x_0 as an optimal solution if it verifies that $f(x_0) \leq f(x)$ for every x feasible, while it is usual to define it as an optimal solution if there is no x feasible verifying that $f(x) \leq f(x_0)$ and $f(x) \neq f(x_0)$. From Zowe's work, we can conclude that it is possible to measure primal sensitivity from the corresponding dual solution when the multiobjective optima are strong optima instead of Pareto optima. This fact has motivated the introduction of a new kind of Pareto optima, the so-called *T-optima*, defined as those Pareto optima that can be transformed into strong optima by the composition of the objective function with a certain linear transformation T. As we will see in the following

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pages, it is possible to define a dual program whose optimal solutions are related to the primal T-optimal solutions by means of some conditions. In many practical examples such primal-dual relations allow the calculation of the primal T-optima knowing their associated dual solutions (see, the example below). Under certain conditions it is also possible to calculate the whole efficient set of the primal program. Moreover, such dual solutions can be used to measure the primal sensitivity with respect to changes in the independent term of the restrictions. Finally, our dual program reduces to the usual one in the case of scalar programs.

Perhaps the most famous duality theory for multiobjective linear programs is due to Isermann (see [3,4]). Isermann's dual solutions also verify some primal-dual relations, and measure the primal sensitivity if the Pareto solutions of the program are restricted to be basic feasible solutions of the primal feasible set. It is not difficult to prove that our dual program reduces to Isermann's dual program if the linear transformation T is real valued: our duality theory generalizes Isermann's duality. Moreover, there is a strong relationship between the basic feasible solutions and our T-optimal solutions.

Finally, some authors have extended the duality techniques to arbitrary dimensional programs, so that they can also be applied to dynamic problems. The classical reference is [5]. The scalar linear case is specifically treated by Anderson and Nash in [6]. And the multiobjective case is treated by the authors in [7] (where Isermann's duality theory is generalized to arbitrary dimensions) and [8] (devoted to convex programming). Our present paper also studies arbitrary dimensional programs, although the corresponding finite dimensional results are also discussed.

The paper is divided into four sections. In Section 2, we define the concept of T-optimal solution and its relationship with the usual Pareto solution. In Section 3, we define a dual program for linear multiobjective primal programs, and we prove the existence of some primal-dual relations between both programs, allowing in some cases the calculation of the efficient set of the primal program. In Section 4, we show the strong relationship existing between the obtained dual solutions and the primal sensitivity in relation to changes in the independent term of the constraints. Finally, the last section is devoted to the complete resolution of an example showing how our duality theory works out in practice.

2. STRONGLY PROPER OPTIMA IN LINEAR MULTIOBJECTIVE PROGRAMMING

Let us suppose that X and Z are Banach spaces ordered by the pointed, closed, and convex cones X^+ and Z^+, respectively (as particular cases we can consider $\mathbb{R}^n$ and $\mathbb{R}^m$). Let us consider two linear and continuous functions $f : X \to \mathbb{R}^p$, $g : X \to Z$. Finally, let $z_0 \in Z$ be a constant. We consider the linear multiobjective primal program (P) "$\min f(x)$, s.t. $x \in F$", where $F = \{x \in X^+ : g(x) \leq z_0\}$ is the primal feasible set (for simplicity, in this section we shall suppose that $f(F)$ is bounded). We define the optimal solutions of program (P) as its usual Pareto minima.

Let $T : \mathbb{R}^p \to \mathbb{R}^q$ be a linear surjective mapping such that $q \leq p$ and T is positive, that is, $T(u) > 0$ if $u > 0$ (the notation $x > y$ means $x \geq y$ and $x \neq y$). We say that a point $x_0 \in F$ is a *proper T-optimum* of program (P) if $Tf(x) \geq Tf(x_0)$ holds for every $x \in F$. If, in addition, the transformation T is bijective ($q = p$), we say that x_0 is a *strongly proper T-optimum*. Finally, a point is just a (*strongly*) *proper optimum* if it is a (strongly) proper T-optimum for some T.

It is almost evident that proper optima must also be Pareto optima of program (P), but the converse is not necessarily true. Therefore, the following question is raised. When is a Pareto optimum also a proper optimum? For the most important case of the strongly proper optima, a sufficient condition is given in the following theorem, whose demonstration can be found in [9].

THEOREM 1. *A point $x_0 \in F$ is a strongly proper optimum of program (P) if the set of nonnegative linear mappings $\alpha : \mathbb{R}^p \to \mathbb{R}$ such that $\alpha f(x_0) \leq \alpha f(x)$, $\forall x \in F$, distinguishes points of $\mathbb{R}^p$*

(that is, if $u \neq 0$ in $\mathbb{R}^p$, then there exists a nonnegative linear mapping α verifying the previous condition and such that $\alpha(u) \neq 0$ in $\mathbb{R}$).

In practice, it is often easier to study another equivalent sufficient condition: $x_0 \in F$ is a strongly proper optimum if there exist p linearly independent and nonnegative real valued linear mappings $\alpha_1, \dots, \alpha_p$ such that $\alpha_i(f(x_0)) \leq \alpha_i(f(x))$, $\forall\, x \in F$, $\forall\, i = 1, \dots, p$.

As we show in the following two theorems, the strongly proper optima are also related to the basic feasible solutions (extreme points) of the set $f(F)$.

THEOREM 2. *If x_0 is a strongly proper optimum for program (P), then $f(x_0)$ is an extreme point of $f(F)$.*

PROOF. Let x_0 be a strongly proper T-optimum for program (P). If $f(x_0)$ is not an extreme point of $f(F)$, then there exist $\lambda \in (0,1)$ and $f(x_1), f(x_2) \in f(F)$ $(f(x_1) \neq f(x_2))$ such that $f(x_0) = \lambda f(x_1) + (1-\lambda) f(x_2)$ and $Tf(x_0) = \lambda Tf(x_1) + (1-\lambda) Tf(x_2)$.

From $Tf(x_1), Tf(x_2) \geq Tf(x_0)$, we deduce that $Tf(x_1) = Tf(x_2) = Tf(x_0)$. Finally, since T is bijective we obtain the contradiction $f(x_1) = f(x_2) = f(x_0)$. ∎

THEOREM 3. *Let us suppose that $f(F)$ is a compact set, and let x_0 be a Pareto optimum of program (P) obtained as a solution of a 'scalarized program' (that is, there exists a positive, real valued linear mapping α such that $\alpha(f(x_0)) \leq \alpha(f(x))$ $\forall\, x \in F$). Let us suppose, in addition, that $f(x_0)$ is an extreme point of $f(F)$, and that the convex hull of the set of those extreme points which are distinct from $f(x_0)$ is closed. Then, x_0 is a strongly proper optimum of program (P).*

PROOF. We begin with some notations: let $\mathbb{R}^p_+$ be the set of vectors $\alpha \in \mathbb{R}^p$ such that $\alpha_i \geq 0$, $\forall\, i = 1, \dots, p$; let $\overset{0}{\mathbb{R}^p_+}$ be the set of vectors $\alpha \in \mathbb{R}^p$ such that $\alpha_i > 0$, $\forall\, i = 1, \dots, p$; let A be the set the vectors $\alpha \in \mathbb{R}^p_+$ such that $\alpha f(x_0) \leq \alpha f(x)$, $\forall\, x \in F$; and let C be the convex hull of the set of those extreme points of $f(F)$ which are distinct from $f(x_0)$.

Since x_0 is an optimal solution of a 'scalarized program', it is clear that there exists some $\alpha_1 \in \overset{0}{\mathbb{R}^p_+} \cap A$, and therefore, the set A is nonvoid. In order, to get Theorem 3, we have to prove (see Theorem 1) that the set A distinguishes points.

Let $\alpha \in \mathbb{R}^p$, $\alpha \neq 0$, and let us suppose that $\alpha_1 \alpha = 0$.

Since C is closed and $f(x_0) \notin C$, from a separation theorem (see [5]) we deduce that there exists $\alpha_2 \in \mathbb{R}^p$, $\alpha_2 \neq 0$, such that $\alpha_2 f(x_0) < \alpha_2 f(x)$, $\forall\, f(x) \in C$; moreover, we have that

$$\inf\{\alpha_2 f(x) - \alpha_2 f(x_0) \mid f(x) \in C\} = d > 0.$$

If we multiply α_2 by a sufficiently small number, it is possible to obtain that $\alpha_1 + \alpha_2 \in \overset{0}{\mathbb{R}^p_+}$; moreover, it is easy to prove that $(\alpha_1 + \alpha_2) f(x_0) \leq (\alpha_1 + \alpha_2) f(x)$, $\forall\, x \in F$; therefore, we have that $(\alpha_1 + \alpha_2) \in A$ and, if $(\alpha_1 + \alpha_2)\alpha \neq 0$, then we have finished our proof. But let us suppose that $(\alpha_1 + \alpha_2)\alpha = 0$, and let us consider $\alpha_3 \in \mathbb{R}^p$ such that $\alpha_3 \alpha \neq 0$.

If we multiply α_3 by a sufficiently small number, it is possible to obtain that $\alpha_1 + \alpha_2 + \alpha_3 \in \overset{0}{\mathbb{R}^p_+}$. Moreover, since $f(F)$ is a compact set we know that $|\alpha_3(f(x) - f(x_0))| \leq M$, $\forall\, x \in F$, and again it is possible to obtain that $M \leq d$.

Let us take $x \in F$. Under all these conditions, we have

$$(\alpha_1 + \alpha_2 + \alpha_3)(f(x) - f(x_0)) = \alpha_1(f(x) - f(x_0)) + \alpha_2(f(x) - f(x_0)) + \alpha_3(f(x) - f(x_0)), \quad (1)$$

$$\alpha_1(f(x) - f(x_0)) \geq 0. \quad (2)$$

Since $f(F)$ is a compact and convex set, it coincides with the convex hull of its extreme points. Therefore, $f(x) = \lambda_0 f(x_0) + \sum_{i=1}^n \lambda_i f(x_i)$, where $f(x_1), \dots, f(x_n)$ are extreme points of $f(F)$ and distinct from $f(x_0)$. Thus, we have that

$$\begin{aligned} \alpha_2(f(x) - f(x_0)) &= \alpha_2 \left(\lambda_0 f(x_0) + \sum \lambda_i f(x_i) - \lambda_0 f(x_0) - \sum \lambda_i f(x_0) \right) \\ &= \alpha_2 \left(\sum \lambda_i (f(x_i) - f(x_0)) \right) = \sum \lambda_i \alpha_2 (f(x_i) - f(x_0)) \geq \left(\sum \lambda_i \right) d. \end{aligned} \quad (3)$$

Finally,

$$\begin{aligned}|\alpha_3(f(x)-f(x_0))| &= \left|\alpha_3\left(\lambda_0 f(x_0)+\sum\lambda_i f(x_i)-\lambda_0 f(x_0)-\sum\lambda_i f(x_0)\right)\right| \\ &= \left|\alpha_3\left(\sum\lambda_i(f(x_i)-f(x_0))\right)\right| = \left|\sum\lambda_i(\alpha_3(f(x_i)-f(x_0)))\right| \\ &\le \sum\lambda_i|\alpha_3(f(x_i)-f(x_0))| \le \left(\sum\lambda_i\right)d.\end{aligned} \tag{4}$$

As a consequence of (1)–(4), we have that $\alpha_1+\alpha_2+\alpha_3 \in A$. Moreover, $(\alpha_1+\alpha_2+\alpha_3)\alpha = \alpha_3\alpha \neq 0$. Therefore, the set A distinguishes points. ∎

Intuitively, the hypothesis relative to the convex hull is verified if the extreme points are 'isolated', as occurs in finite dimensions. Moreover, in this case the whole set of Pareto optima can be obtained as solutions of 'scalarized' programs (see [10]). Therefore, in finite dimensions the Pareto optima whose images by the objective function are extreme points of $f(F)$ are necessarily strongly proper solutions. In this case, Theorems 2 and 3 characterize the strongly proper optima as those Pareto optima whose images by f are extreme points of $f(F)$. This property also holds in many infinite dimensional programs, like for example discrete-time optimal control problems (see the example below). As an easy consequence, the next theorem shows that, under similar conditions, the image of every Pareto optimum must be a convex combination of the images of strongly proper optima, and this fact allows us to calculate the whole set of Pareto optima if we know the strongly proper optima of program (P).

THEOREM 4. *Let us suppose that $f(F)$ is a compact set, and that every Pareto optimum of program (P) whose image by f is an extreme point of $f(F)$ must be 'isolated' and can be obtained as a solution of a 'scalarized' program (remember that these two last conditions hold in finite dimension). Then, the image of every Pareto optimum must be a convex combination of the images of strongly proper optima of program (P).*

PROOF. Let x_0 be a Pareto optimum of program (P). Since $f(F)$ is a compact and convex set, $f(F)$ must be the convex hull of its extreme points, and therefore, $f(x_0) = \sum_{i=1}^{n} t_i f(x_i)$, where $f(x_1), \dots, f(x_n)$ are extreme points of $f(F)$ and $t_i > 0$ for all i.

If for some j, the point x_j is not a Pareto optimum, then there exists $y \in F$ such that $f(x_j) > f(y)$, and

$$f(x_0) = \sum_{i=1}^{n} t_i f(x_i) > \sum_{i\neq j} t_i f(x_i) + f(y) = f\left(\sum_{i\neq j} t_i x_i + y\right) \in f(F),$$

which is a contradiction because x_0 is a Pareto optimum.

Therefore, $x_1, \dots, x_n$ are Pareto optima and we are under the conditions of Theorem 3, so that we can conclude that $x_1, \dots, x_n$ are strongly proper optima and the theorem holds. ∎

3. THE DUALITY THEORY

In this section, we define the *dual program* of program (P) and study the relationship between both programs.

Let q $(\le p)$ be a fixed natural number, and let Π be the set of the linear, surjective, and positive mappings from $\mathbb{R}^p$ onto $\mathbb{R}^q$. Let $\mathbb{L}(Z, \mathbb{R}^q)$ be the set of the linear and continuous mappings from Z to $\mathbb{R}^q$ ordered by the cone $\mathbb{L}^+(Z, \mathbb{R}^q)$. Such a cone is formed by all the nonnegative mappings of $\mathbb{L}(Z, \mathbb{R}^q)$, and therefore, if $z_1^*, z_2^* \in \mathbb{L}(Z, \mathbb{R}^q)$, then $z_1^* \ge z_2^*$ if $z_1^*(z) \ge z_2^*(z)$ holds for every $z \in Z$, $z \ge 0$. In a similar way we define $\mathbb{L}(Z, \mathbb{R}^p)$, $\mathbb{L}(X, \mathbb{R}^p)$, and $\mathbb{L}(X, \mathbb{R}^q)$.

We define the *dual program* (D) of program (P) in the following way: "$\max -G(z_0)$, s.t. $TGg \ge -Tf$, $TG \ge 0$, $T \in \Pi$, $G \in \mathbb{L}(Z, \mathbb{R}^p)$".

If the spaces X and Z are finite dimensional then, the dual variables (T, G) can be represented as matrices.

It is not difficult to prove that if $q = 1$, then program (D) reduces to the infinite dimensional Isermann's dual program presented in [7]. On the other hand, if $p = 1$, that is, if we are in the scalar case, then program (D) reduces to the infinite dimensional scalar dual program introduced in [6].

It is not difficult to prove that programs (P) and (D) verify certain *primal-dual relations*. According to the first property, the dual objective can never be larger than the primal.

THEOREM 5. *If x is primal feasible and (T, G) is dual feasible, we never have $-G(z_0) > f(x)$.*

PROOF. $-G(z_0) > f(x)$ implies $-TG(z_0) > Tf(x)$, and therefore, we have the contradiction $TG(z_0) < -Tf(x) \leq TGg(x) \leq TG(z_0)$. ∎

As a consequence of the previous result, if x is primal feasible, (T, G) is dual feasible and $-G(z_0) = f(x)$, then x and (T, G) must be primal and dual optimums, respectively (x being a proper T-optimum). In this case, we say that they are *associated solutions*. The existence of associated solutions can be proved under very general conditions, like the Slater condition (see [8]) or others similar to the obtained in [6]. Moreover, it is possible to characterize such associated solutions by means of some *complementary slackness conditions*.

THEOREM 6. *Let x and (T, G) be primal and dual feasible points, respectively, and let us take $q = p$. Then, x and (T, G) are associated solutions if and only if*

(i) $TG(g(x) - z_0) = 0$,
(ii) $(TGg + Tf)(x) = 0$.

PROOF. If (i) and (ii) hold, then $Tf(x) = -TGg(x) = -TG(z_0)$; therefore, we have that $f(x) = -G(z_0)$, since T is bijective. On the other hand, if x and (T, G) are associated solutions, they must verify $g(x) \leq z_0$, $(TGg + Tf) \geq 0$, $TG \geq 0$, and $f(x) = -G(z_0)$. Consequently, we have $(TG)(g(x)) \leq (TG)(z_0) = -Tf(x) \leq (TG)(g(x))$, and both (i) and (ii) hold. ∎

Notice, that matrix T can disappear of the expressions (i) and (ii), since $p = q$ and T is bijective.

Let us consider a measure μ defined over the Borel σ-field of the natural numbers and such that $\mu(n) > 0$ for each $n \in \mathbb{N}$. Let $l_r(\mu)$ be the space of sequences (u_n) such that $\sum_{n=1}^{\infty} u_n^r \mu(n)$ converges, is ordered by the usual cone. In the rest of this section we study, as an application of the previous theory, the particular form of the complementary slackness conditions when the spaces X or Z are equal to $l_r(\mu)$ for some $r \in \mathbb{N}$.

If $Z = l_r(\mu)$, then $G \in \mathbb{L}(l_r(\mu), \mathbb{R}^p)$ and hence,

$$G = \begin{pmatrix} G_{11} & G_{12} & \dots \\ \dots & \dots & \dots \\ G_{p1} & G_{p2} & \dots \end{pmatrix},$$

where the rows are vectors of $l_s(\mu)$, with $1/r + 1/s = 1$. The condition $TG(g(u_n) - z_0) = 0$ can be written as

$$\begin{pmatrix} t_{11} & \dots & t_{1p} \\ \dots & \dots & \dots \\ t_{p1} & \dots & t_{pp} \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} & \dots \\ \dots & \dots & \dots \\ G_{p1} & G_{p2} & \dots \end{pmatrix} \begin{pmatrix} (g(u_n) - z_0)_1 \mu(1) \\ (g(u_n) - z_0)_2 \mu(2) \\ \dots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The first equation we obtain is

$$(t_{11}G_{11} + \dots + t_{1p}G_{p1})(g(u_n) - z_0)_1 \mu(1) + (t_{11}G_{12} + \dots + t_{1p}G_{p2})(g(u_n) - z_0)_2 \mu(2) + \dots = 0.$$

Since we know that $TG \geq 0$ and $g(u_n) \leq z_0$, we conclude that

$$(t_{11}G_{11} + \cdots + t_{1p}G_{p1}(g(u_n) - z_0)_1 = 0,$$
$$(t_{11}G_{12} + \cdots + t_{1p}G_{p2}(g(u_n) - z_0)_2 = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

The last equation would be

$$(t_{p1}G_{11} + \cdots + t_{pp}G_{p1})(g(u_n) - z_0)_1\mu(1) + (t_{p1}G_{12} + \cdots + t_{pp}G_{p2})(g(u_n) - z_0)_2\mu(2) + \cdots = 0,$$

and again we conclude that

$$(t_{p1}G_{11} + \cdots + t_{pp}G_{p1}(g(u_n) - z_0)_1 = 0,$$
$$(t_{p1}G_{12} + \cdots + t_{pp}G_{p2}(g(u_n) - z_0)_2 = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Consequently, if $(g(u_n) - z_0)_1 \neq 0$, then

$$t_{11}G_{11} + \cdots + t_{1p}G_{p1} = 0, \ldots, t_{p1}G_{11} + \cdots + t_{pp}G_{p1} = 0,$$

that is,

$$\begin{pmatrix} t_{11} & \ldots & t_{1p} \\ \ldots & \ldots & \ldots \\ t_{p1} & \ldots & t_{pp} \end{pmatrix} \begin{pmatrix} G_{11} \\ \ldots \\ G_{p1} \end{pmatrix} = \begin{pmatrix} 0 \\ \ldots \\ 0 \end{pmatrix}.$$

Since T is a regular matrix, this implies that $G_{11} = \cdots = G_{p1} = 0$. The same reasoning shows that $(g(u_n) - z_0)_2 \neq 0$ implies $G_{12} = \cdots = G_{p2} = 0$ and, in general, $(g(u_n) - z_0)_k \neq 0$ implies $G_{1k} = \cdots = G_{pk} = 0$ for all $k \in \mathbb{N}$, that is, the product of every component of the vector $(g(u_n) - z_0)$ and the elements of the corresponding column of the matrix G must be zero.

We conclude that, if $Z = l_r(\mu)$, the condition $TG(g(u_n) - z_0) = 0$ is equivalent to the following equations:

$$(g(u_n) - z_0)_k G_{1k} = \cdots = (g(u_n) - z_0)_k G_{pk} = 0, \qquad \text{for all } k = 1, 2, \ldots.$$

On the other hand, if $X = l_r(\mu)$, then $H = Gg + f \in \mathbb{L}(l_r(\mu), \mathbb{R}^p)$ and therefore, H can be represented as a matrix with infinite columns

$$H = \begin{pmatrix} h_{11} & h_{12} & \ldots \\ \ldots & \ldots & \ldots \\ h_{p1} & h_{p2} & \ldots \end{pmatrix},$$

where the rows are vectors of $l_s(\mu)$. By means of similar arguments it is easy to demonstrate that the condition $(TGg + Tf)(u_n) = 0$ is equivalent to the equations

$$(u_n)_k h_{1k} = \cdots = (u_n)_k h_{pk} = 0, \qquad \text{for all } k = 1, 2, \ldots,$$

that is, the product of every component of the vector (u_n) and the elements of the corresponding column of the matrix H must be null.

It is also easy to demonstrate that, if X or Z are finite dimensional, the complementary slackness conditions have the same form but the number of equations is finite.

If $p = 1$, the complementary slackness conditions that we have obtained in Theorem 6 coincide with the well-known scalar conditions (see [6]). Moreover, the previous example shows another important relation between multiobjective and scalar slackness conditions: when we work in finite or infinite but countable dimensions, the multiobjective equations can be obtained by aggregating the scalar complementary slackness conditions of the p scalar programs associated with our multiobjective program by selecting only one component of the objective function. This fact makes easier the resolution of the equations in these two important particular cases. From all these properties, we conclude that our duality theory and our complementary slackness conditions become a useful generalization to the multiobjective case of the well-known theory in the scalar case.

4. SENSITIVITY ANALYSIS

The dual program defined in the previous section is related to the primal sensitivity. In this section, we show that such primal sensitivity is measured by the sum of a dual solution plus a new term that can be interpreted as the sensitivity of such a dual solution. This last term vanishes in the scalar case, and this is the reason why dual solutions measure exactly the primal sensitivity in scalar linear programs.

In this section we consider the 'perturbed' programs $(P)_z$ defined as "$\min f(x)$, s.t. $g(x) - z \leq z_0$", where z belongs to a certain neighbourhood V of $0 \in Z$ (notice that program (P) coincides with $(P)_0$). Since $(P)_z$ is a linear program, we consider its dual program for a certain q ($1 \leq q \leq p$). Let us suppose that for each z we find the associated solutions x_z and (T, G_z) (where T does not depend on z). Then, we can define two new functions

$$F : V \to \mathbb{R}^p, \quad F(z) = f(x_z); \qquad G : V \to \mathbb{L}(Z, \mathbb{R}^p), \quad G(z) = G_z.$$

If we assume that F and G are Frechet differentiable, then $\partial F(0, z)$ denotes the Frechet differential of F at 0 (which is a linear and continuous mapping from V to $\mathbb{R}^p$) particularized on $z \in V$, and therefore, $\partial F(0, z) \in \mathbb{R}^p$. In a similar way, $\partial G(0, z) \in \mathbb{L}(Z, \mathbb{R}^p)$, and therefore, $\partial G(0, z)(z_0) \in \mathbb{R}^p$. The following important result relates primal sensitivity and dual solutions.

THEOREM 7. *Under the previous hypothesis and notations, we have*

$$\partial F(0, z) = -T^* T G_0(z) - \pi \left(\partial G(0, z)(z_0) \right),$$

where T^* is the generalized inverse of T (that is, the inverse of the restriction of T to the orthogonal complement of $\ker(T)$*), and π is the projection from $\mathbb{R}^p$ onto* $\ker(T)$.

PROOF. See [8]. ∎

Notice, that if we work in the scalar case, or in the multiobjective case with strongly proper optima ($q = p$), then $T^* T$ is the identity and π vanishes, and therefore, the dual solution measures exactly the primal sensitivity when the independent term of the restrictions changes: $\partial F(0, z) = -G_0(z)$. Theorems 6 and 7 give us a special kind of Pareto optima, the strongly proper optima, for which duality properties similar to those obtained in the scalar case hold.

5. AN EXAMPLE

The purpose of this last section is to present a simple example showing how the developed theory works in practice. The example studies a linear program in infinite dimensional spaces, with two objective functions. We define its dual program and solve both, primal and dual programs, by means of the conditions stated in Theorem 6. To be more precise, we find the strongly proper optima and all their associated dual solutions, which are presented as primal sensitivity measures.

Let us consider a measure μ defined over the Borel σ-field of the natural numbers and such that $\mu(n) = e^{-n}$ for each $n \in \mathbb{N}$. Let $l_2(\mu)$ be the space of sequences (u_n) such that $\sum_{n=1}^{\infty} u_n^z e^{-n}$ is convergent, and let us consider the following primal program:

$$\min \left\{ -\left(\sum_{n=1}^{\infty} u_n e^{-n}, \sum_{n=1}^{\infty} \left(\frac{v_n}{n} \right) e^{-n} \right), \right.$$
$$\left. \text{s.t. } (u_n), (v_n) \in l_2(\mu), \; u_n + v_n \leq n, \; u_n, v_n \geq 0, \; n = 1, 2, \dots \right\},$$

where it is clear that the objective functions are given by convergent series, because they are the scalar product in $l_2(\mu)$ of the sequences $(u_n), (1)$ and $(v_n), (1/n)$, respectively. We also consider that the constraint is valued in $l_2(\mu)$.

From the results of the Section 2, we know that the dual variable is a pair (T, G), where T is a regular matrix with nonnegative terms and G is a linear function from $l_2(\mu)$ to $\mathbb{R}^2$; therefore, G may be represented as a matrix with an infinite number of columns and such that the rows are in $l_2(\mu)$

$$T = \begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix}, \qquad G = \begin{pmatrix} G_{11}, G_{12}, G_{13}, \dots \\ G_{21}, G_{22}, G_{23}, \dots \end{pmatrix}.$$

The dual program becomes

$$\max - \left(\sum_{n=1}^{\infty} n G_{1n} e^{-n}, \sum_{n=1}^{\infty} n G_{2n} e^{-n} \right),$$
$$\alpha, \beta, \chi, \delta \geq 0, \qquad \alpha\delta - \beta\chi \neq 0, \qquad \alpha G_{1n} + \beta G_{2n} \geq 0,$$
$$\chi G_{1n} + \delta G_{2n} \geq 0, \qquad \alpha G_{1n} + \beta G_{2n} \geq \alpha, \qquad \alpha G_{1n} + \beta G_{2n} \geq \frac{\beta}{n},$$
$$\chi G_{1n} + \delta G_{2n} \geq \chi, \qquad \chi G_{1n} + \delta G_{2n} \geq \frac{\delta}{n},$$

where the constraints must be verified for any $n = 1, 2, 3, \dots$. The first and second constraints come from the conditions on matrix T, the third and fourth come from the inequality $TG \geq 0$, and the rest of them may be easily obtained from $TGg \geq -Tf$.

From Condition (i) of Theorem 6, it may be proved that

$$u_n + v_n = n, \qquad n = 1, 2, \dots. \tag{5}$$

Condition (ii) leads in this case to

$$\begin{aligned} (\alpha G_{1n} + \beta G_{2n} - \alpha) u_n = 0, &\qquad n = 1, 2, \dots, \\ \left(\alpha G_{1n} + \beta G_{2n} - \frac{\beta}{n} \right) v_n = 0, &\qquad n = 1, 2, \dots, \end{aligned} \tag{6}$$

and the last equalities must hold as well if we change α and β by χ and δ, respectively. If for some n we had that $u_n \neq 0$ and $v_n \neq 0$, then from (6) we would have

$$\alpha = \frac{\beta}{n} \quad \text{and} \quad \chi = \frac{\delta}{n}, \tag{7}$$

and this is not possible because the determinant of T cannot be zero. Now, (5) gives us the following condition which must hold for any $n = 1, 2, \dots$, and any strongly proper optimum

$$(u_n = 0 \text{ and } v_n = n) \quad \text{or} \quad (u_n = n \text{ and } v_n = 0). \tag{8}$$

If $u_n \neq 0$, we obtain from the first equality in (6) that

$$\alpha G_{1n} + \beta G_{2n} = \alpha, \qquad \chi G_{1n} + \delta G_{2n} = \chi,$$

and therefore,

$$G_{1n} = 1 \quad \text{and} \quad G_{2n} = 0. \tag{9}$$

Similarly, if $v_n \neq 0$, then

$$G_{1n} = 0 \quad \text{and} \quad G_{2n} = \frac{1}{n}. \tag{10}$$

Finally, if we introduce (9) and (10) into the dual constraints, we obtain that the primal strongly proper optima and their associated dual solutions are, for every $m \in \mathbb{N}$,

$$\begin{aligned} &\text{if } n > m, \qquad \text{then } u_n = n, \quad v_n = 0, \quad G_{1n} = 1, \quad G_{2n} = 0, \\ &\text{if } n \leq m, \qquad \text{then } u_n = 0, \quad v_n = n, \quad G_{1n} = 0, \quad G_{2n} = \frac{1}{n}. \end{aligned}$$

Moreover, for any strongly proper optimum the associate dual solution gives the primal sensitivity in the sense that if we change the sequence (n) by $(z_n) \in l_2(\mu)$, then the change in the two objective functions is given by $\sum_{n=1}^{\infty} G_{in}(z_n - n)e^{-n}$, $i = 1, 2$.

REFERENCES

1. S. Zowe, A duality theorem for a convex programming problem in order complete vector lattices, *Journal of Mathematical Analysis and Applications* **50**, 273–287 (1975).
2. S. Zowe, The saddle point theorem of Kuhn and Tucker in ordered vector spaces, *Journal of Mathematical Analysis and Applications* **57**, 41–55 (1977).
3. H. Isermann, The relevance of duality in multiple objective linear programming, *TIMS Studies in the Management Sciences* **6**, 241–262 (1977).
4. H. Isermann, On some relations between a dual pair of multiple objective linear programs, *Zeitschrift für Operations Reserach* **22**, 33–41 (1978).
5. D.G. Luenberger, *Optimization by Vector Space Methods*, Wiley, (1969).
6. E.J. Anderson and P. Nash, *Linear Programming in Infinite Dimensional Spaces*, Wiley, (1987).
7. A. Balbas and A. Heras, Duality theory for infinite dimensional multiobjective linear programming, *European Journal of Operational Research* **68**, 379–388 (1993).
8. A. Balbas and P. Jimenez, Sensitivity for convex programming in abstract spaces, *Journal of Mathematical Analysis and Applications* **202**, 645–658 (1996).
9. A. Balbas, P. Jimenez and C. Nunez, Strongly proper optimums and maximal optimization in multiobjective programming, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales de Madrid* **86**, 289–295 (1992).
10. Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization*, Academic Press, (1985).