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Lower bounds for constant degree independent sets

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Abstract

Let α^* denote the maximum number of independent vertices all of which have the same degree. We provide lower bounds for α^* for graphs that are planar, maximal planar, of bounded degree, or trees.

1. Introduction

Given a graph G and an integer k, deciding whether the independence number of G is at least k [5, 8] is one of the classic NP-complete problems. The mathematical responses to this complexity result include fast heuristics to give (hopefully) large independent sets as well as bounds on the independence number for particular classes of graphs [6, 8]. Here we consider a restriction: specifically we seek an independent set in which every vertex has the same degree. Clearly, although this will not result in any computational simplification, we obtain bounds on the size of these sets. We were inspired by the following pseudo-Ramsey result.

Theorem 1.1. (Albertson [1]). *If G is any graph with $v \geqslant 6$, then either G or G^c contains a K_3 in which two vertices have the same degree.*

We extend the notation of [4]. Specifically, $v = v(G)$ will denote the number of vertices in a connected graph G; $\varepsilon = \varepsilon(G)$ the number of edges in G; $\Delta = \Delta(G)$ the maximum degree in G; $\alpha = \alpha(G)$ the independence number of G; G_j the subgraph of G induced by the vertices of degree j; and $N_j = v(G_j)$. The cardinality of the largest independent set of vertices in which all have degree j will be denoted by $\alpha_j = \alpha_j(G)$. The constant degree independence number, denoted by $\alpha^* = \alpha^*(G)$, will be the maximum value of α_j.

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We show that if G is planar, $\alpha^* \geqslant 2v/65$; while if G is maximal planar, $\alpha^* \geqslant 3v/61$. We exhibit graphs to show that these bounds are not as bad as they seem. Furthermore, we show that if G has bounded degree,

$$\alpha^* \geqslant \frac{v}{\binom{\Delta+1}{2}};$$

while if G is a tree, $\alpha^* \geqslant (v+2)/4$.

We exhibit infinite families of graphs to show that these bounds are sharp.

2. Trees

Theorem 2.1. *For any tree, $\alpha^* \geqslant (v+2)/4$.*

Proof. Excluding the case where $T = K_2$ (the theorem is trivially true here), all vertices of degree 1 in a tree must be independent. Therefore $\alpha_1 = N_1$. Since trees are bipartite, $\alpha_2 \geqslant N_2/2$. Note that $N_1 > N_j$ for all $j \geqslant 3$. This implies that $\alpha_1 > \alpha_j$ for all $j \geqslant 3$. Therefore, α^* must be either α_1 or α_2. Thus, it must be at least their average, i.e.

$$\alpha^* \geqslant \frac{\alpha_1 + \alpha_2}{2} \geqslant \frac{2N_1 + N_2}{4}.$$

For a tree, $2\varepsilon = 2v - 2 = \sum jN_j$. Combining this with $\sum N_j = v$ yields

$$\begin{aligned} N_1 &= N_3 + 2N_4 + \cdots + (\Delta G - 2)N_{\Delta G} + 2 \\ &\geqslant N_3 + N_4 + \cdots + N_{\Delta G} + 2. \end{aligned}$$

Substitution gives

$$\alpha^* \geqslant \frac{N_1 + N_2 + (N_3 + \cdots + N_{\Delta G} + 2)}{4} = \frac{v+2}{4}.$$

We construct an infinite family of trees to show that this bound is sharp. Let T' be any tree ($v(T') > 2$) whose vertices have degree 3 or 1. Make every edge of T' that is incident with a leaf into a path of length three. Call the resulting tree T. A sample is shown in Fig. 1.

By construction, T has

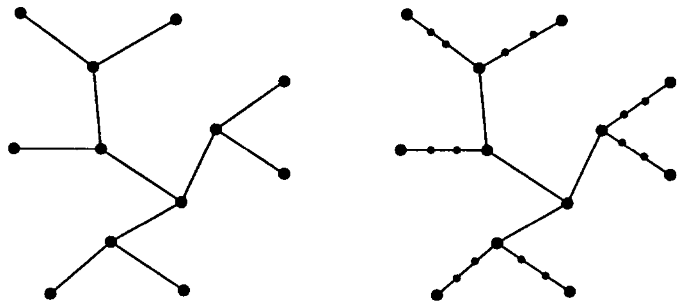
$$\alpha^* = N_1 = \alpha_1 = \alpha_2 = N_2/2$$

and

$$v = N_1 + N_2 + N_3 = 4N_1 - 2 = 4\alpha^* - 2.$$

Thus,

$$\alpha^* = \frac{v+2}{4}.$$



T′ T

Fig. 1.

This sort of bound cannot be extended to the set of bipartite graphs. It is straightforward to construct a bipartite graph with $v=2k$ such that $\alpha^*=\alpha_1=\alpha_2=\cdots=\alpha_k=2$.

3. Brooks' bounds

It is an immediate consequence of Brooks' theorem that if G is connected graph that is neither C_{2r+1} nor $K_{\Delta+1}$, then $\alpha \geqslant v/\Delta$. Happily, if G is a connected graph such that $G \neq K_{\Delta+1}$, C_{2r+1}, then each $G_j \neq K_{j+1}$, (or C_{2r+1} if $j=2$). Hence, we can apply the above inequality to each of the G_j's separately. Consequently, $\alpha_j \geqslant N_j/j$. Thus

$$\begin{aligned} v &= \sum N_j \leqslant \sum j\alpha_j \\ &\leqslant \sum j\alpha^* = \alpha^* \sum j = \alpha^* \binom{\Delta+1}{2}. \end{aligned}$$

Thus, we have arrived at the following theorem.

Theorem 3.1. $\alpha^* \geqslant v/\binom{\Delta+1}{2}$.

As in the case of trees, the lower bound is sharp. The construction will produce a graph in which

$$\alpha^*=2 \quad \text{and} \quad v=2\binom{\Delta+1}{2}.$$

We begin by constructing a path of cliques. Take a single vertex and join it to one vertex of a K_2. Join the other vertex of that K_2 to one vertex of a K_3. Join a different vertex of that K_3 to one vertex of a K_4. Continue. At the ith stage one vertex of K_i will be joined to one vertex of K_{i+1}. Finally one vertex of $K_{\Delta-1}$ will be joined to one vertex

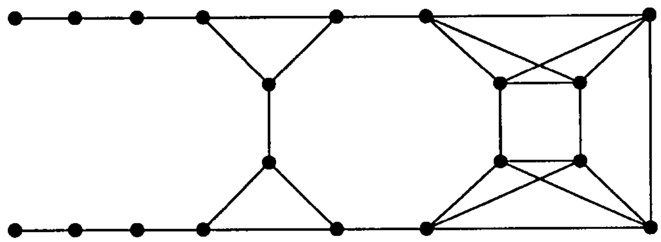


Fig. 2.

of K_Δ. At this stage we have $\binom{\Delta+1}{2}$ vertices. Take a duplicate of the above construction. Each such path of cliques has two vertices of its K_j having degree j (for $2 \leqslant j \leqslant \Delta - 1$) while the remaining are *deficient*, i.e. have degree $j-1$. Each copy of K_Δ has one vertex of degree Δ and $\Delta - 1$ vertices that are deficient. For each deficient vertex in one path of cliques find the corresponding deficient vertex in the other path of cliques and join these two vertices. In the resulting graph G each G_j consists of two copies of K_j together with some edges of a partial matching. An example with $\Delta = 4$ is shown in Fig. 2. For this graph $\alpha_1 = \alpha_2 = \cdots = \alpha_\Delta = \alpha^* = 2$ and $v = 2\binom{\Delta+1}{2}$.

4. Planar graphs

The situation for planar graphs is more unsettled. If G is a planar graph, then the Four Color Theorem implies that $\alpha \geqslant v/4$, and this bound is sharp even for graphs without K_4's. Using $2\varepsilon = \sum jN_j$, $v = \sum N_j$, and $2\varepsilon \geqslant 6v - 12$ (which follows immediately from Euler's formula) a standard argument reveals that

$$5N_1 + 4N_2 + 3N_3 + 2N_4 + N_5 \geqslant 12 + N_7 + 2N_8 + 3N_9 + \cdots + (\Delta G - 6)N_{\Delta G}.$$

For a maximal planar graph (i.e. a triangulation of the plane), since Euler's formula is an equality and since there are no vertices of degree 1 or 2, we get

$$3N_3 + 2N_4 + N_5 = 12 + N_7 + 2N_8 + 3N_9 + \cdots + (\Delta G - 6)N_{\Delta G}.$$

It is straightforward to verify that in a maximal planar graph (excluding the case where $G = K_4$ which easily satisfies our theorem), all vertices of degree 3 are independent. Therefore $\alpha_3 = N_3$. It can also be verified that each component of G_4 either contains no K_3's or is a K_3. Since Grötzsch's Theorem implies that G_4 is 3-colorable [7], we have

$$\alpha^* \geqslant \alpha_4 \geqslant N_4/3 \quad \text{or} \quad 3\alpha^* \geqslant N_4.$$

If $j \geqslant 5$, the Four Color Theorem [2, 3] implies that

$$\alpha^* \geqslant \alpha_j \geqslant N_j/4 \quad \text{or} \quad 4\alpha^* \geqslant N_j.$$

Combining these inequalities, we obtain,

$$13\alpha^* - 12 \geqslant N_7 + 2N_8 + 3N_9 + \cdots + (\Delta G - 6)N_{\Delta G}.$$

Let d_7 and d_8 be defined so that $N_7 = 4\alpha^* - d_7$ and $N_8 = 4\alpha^* - d_8$. These can be substituted into the preceding inequality to yield,

$$\frac{\alpha^* + d_7 + 2d_8 - 12}{3} \geqslant N_9 + \cdots + \frac{\Delta G - 6}{3} N_{\Delta G}.$$

We now compute an upper bound for v in terms of α^*.

$$\begin{aligned} v &= \sum_{j=3}^{\infty} N_j \\ &= \sum_{j=3}^{8} N_j + \sum_{j=9}^{\infty} N_j \\ &\leqslant (20\alpha^* - d_7 - d_8) + \frac{\alpha^* + d_7 + 2d_8 - 12}{3}. \end{aligned}$$

Therefore,

$$\frac{\alpha^*}{v} \geqslant \frac{3\alpha^*}{61\alpha^* - 2d_7 - d_8 - 12} \geqslant \frac{3}{61}.$$

We have proved the following theorem.

Theorem 4.1. *For any maximal planar graph,* $\alpha^* \geqslant (3/61)v$.

The preceding theorem implies that any maximal planar graph with more than 21 vertices must contain two independent vertices of the same degree. One can show by tedious case analysis that in such a graph, G_5 cannot contain a K_4, and G_6 cannot contain a K_4 unless $\alpha_3 \geqslant 4$. We then obtain the following corollary.

Corollary 4.2. *If G is a maximal planar graph and $v > 10$, then $\alpha^* > 1$.*

The graph in Fig. 3 shows that this result is sharp: specifically $v = 10$ and $\alpha^* = 1$. The proof of Theorem 4.1 can be readily modified to produce the following theorem.

Theorem 4.3. *For any planar graph,* $\alpha^* \geqslant (2/65)v$.

The analogue to Corollary 4.2 is the following.

Corollary 4.4. *If G is a planar graph and $v > 18$, then $\alpha^* > 1$.*

The graph in Fig. 4 shows that this result is sharp. This graph has $v = 18$ and $\alpha^* = 1$.

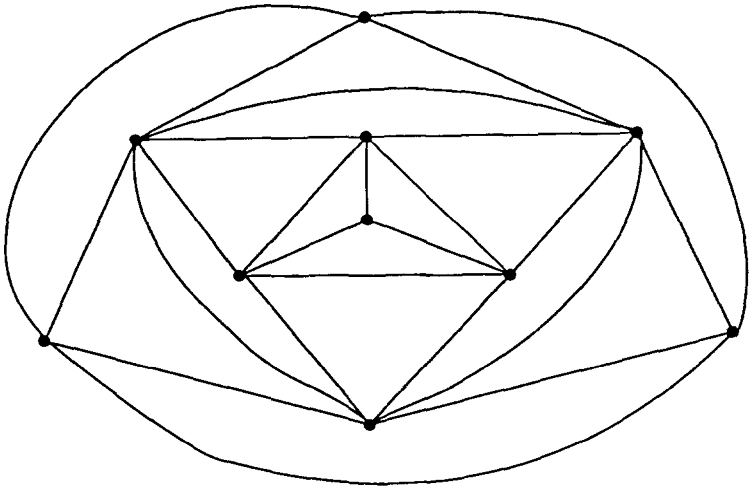


Fig. 3.

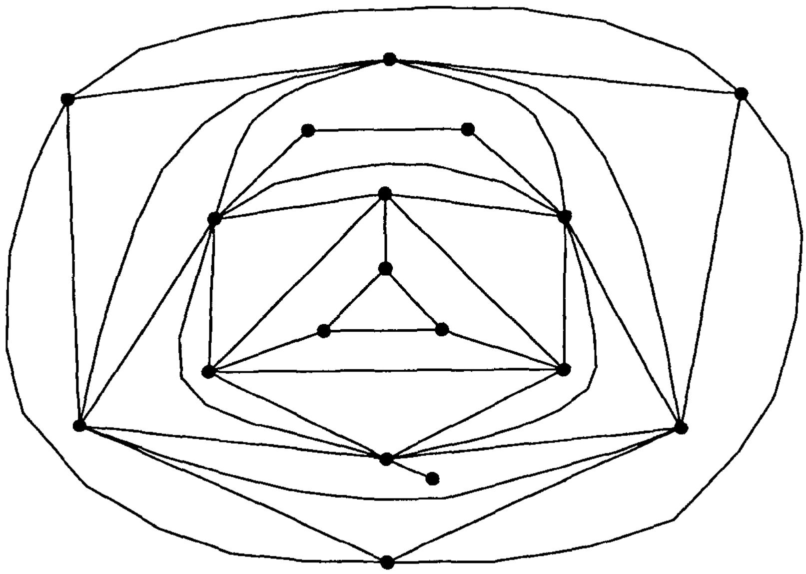


Fig. 4.

The principal open question arising from this work is what are the *right* numbers for Theorems 4.1 and 4.3. Perhaps if G is a maximal planar graph,

$$\alpha^* \geqslant (1/16)v.$$

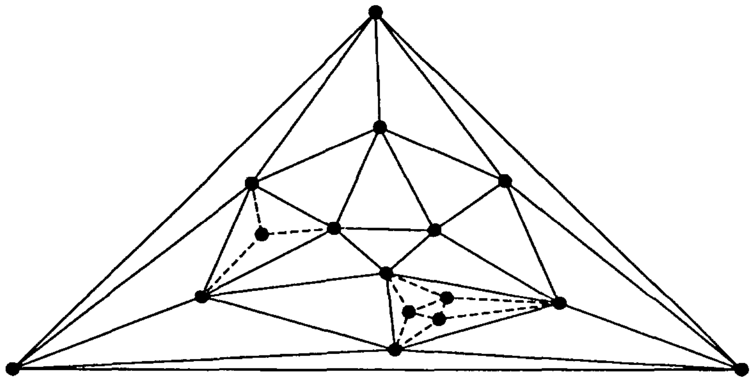


Fig. 5.

We conclude with a construction of an infinite family of maximal planar graphs in which α^* achieves the above bound. We begin with the icosahedron whose vertices can be partitioned into four triangles, each of which is a face boundary. We fix one of these as the exterior. Of the remaining three triangles one is unchanged, one has its interior triangulated with a vertex of degree 3, and one has its interior triangulated with three vertices of degree 4. Call the resulting graph H_1 (illustrated in Fig. 5).

Create M_1 by pasting a copy of H_1 into four independent triangles of the icosahedron. To obtain H_2 paste a copy of H_1 onto three independent triangles of the icosahedron (excluding the exterior triangle). Create M_2 by pasting a copy of H_2 onto four independent triangles of the icosahedron. In general H_j is obtained by pasting a copy of H_{j-1} onto three independent triangles of the icosahedron, while M_j is obtained by pasting a copy of H_j into four independent triangles of the icosahedron. In each case

$$\alpha^*(M_j) = \frac{1}{16} v(M_j).$$

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