ELSEVIER

Journal of Pure and Applied Algebra 141 (1999) 201–209

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

The rationality of certain moduli spaces associated to half-canonical extremal curves

G. Casnati [a,*], A. Del Centina [b,1]

[a] *Dipartimento di Matematica Pura ed Applicata, Università Degli Studi di Padova, via Belzoni 7, 35131 Padova, Italy*

[b] *Dipartimento di Matematica, Università Degli Studi di Ferrara, via Machiavelli 35, 44100 Ferrara, Italy*

Communicated by A.V. Geramita; received 14 July 1997; received in revised form 28 December 1997

Abstract

We prove the existence of a coarse, irreducible moduli space $\mathfrak{C}_g^{m-1}$ (resp. $\mathfrak{C}_{g,1}^{m-1}$) for (resp. pointed) subcanonical extremal curves of level $m-1$ and genus g in $\mathbb{P}^r_{\mathbb{C}}$. When $g=3r$, $r\geq 4$, $r\neq 5$, odd (resp. even) we show that $\mathfrak{C}_{3r}^2$ (resp. $\mathfrak{C}_{3r,1}^2$) is rational. © 1999 Elsevier Science B.V. All rights reserved.

MSC: 14H10; 14H45

0. Introduction

A classical result states that the canonical model of a non-hyperelliptic trigonal curve C of genus $g\geq 4$ lies on a unique rational normal scroll $X\subseteq\mathbb{P}^r_{\mathbb{C}}$ and $C\cap \mathrm{Sing}(X)=\emptyset$ (see [1, Ch. III]).

If C is general and its genus $g\geq 6$ is even then $X\cong\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, and $C\in|3f_1+((g+2)/2)f_2|$, where f_1, f_2 are two non-skew lines on X. Thus, with the above hypotheses, the trigonal locus $\mathfrak{T}_g\subseteq\mathfrak{M}_g$ inside the moduli space of smooth curves of genus g, is birationally equivalent to the quotient

$$\mathbb{P}(H^0(X,\mathcal{O}_X(3f_1+((g+2)/2)f_2)))/(SL_2\times SL_2).$$

* Corresponding author. Supported by the framework of the SCIENCE contract no. SCI-0398-C(A).
E-mail address: casnati@galileo.math.unipd.it (G. Casnati)

[1] Partially supported by MURST 40% funds.

PII: S0022-4049(98)00037-1

N.I. Shepherd-Barron in his paper [10] shows that if $g \equiv 2 \pmod 4$ and $g \geq 6$ the above quotient is rational.

The aim of this paper is to generalize the methods used in [10] to a class of curves whose geometry is quite similar to that of trigonal curves, namely the class of *subcanonical extremal curves*.

Let $C \subseteq \mathbb{P}^r_{\mathbb{C}}$ be a smooth, irreducible, connected curve of genus g and degree d. Then the following inequality, called *Castelnuovo's bound*, holds for C:

$$g \leq \pi(d,r) := \binom{m}{2}(r-1) + m\varepsilon \tag{0.1}$$

where $d = m(r-1) + \varepsilon + 1$ and $0 \leq \varepsilon \leq r-2$ (see [1, p. 116]). If equality holds in formula (0.1) then C is called *extremal curve*.

Notice that if C is extremal and $d = 2r$ then formula (0.1) gives $r = g-1$ ($m = 2$, $\varepsilon = 1$) and therefore C is a canonical curve since extremal curves are projectively normal (see [1, p. 117]).

Assume from now on that $r \geq 3$ and $d \geq 2r+1$. Then extremal curves $C \subseteq \mathbb{P}^r_{\mathbb{C}}$ of degree d do exist and, if $r \neq 5$ and $\varepsilon > 0$, each such curve is a smooth element of the linear system

$$|(m+1)H - (r - \varepsilon - 2)f| \tag{0.2}$$

on a rational normal scroll $X \subseteq \mathbb{P}^r_{\mathbb{C}}$ (H and f denote the hyperplane section and the general fibre of X: see [1, Ch. III, Theorem 2.5(iii)]. Moreover, we remark that the fibres of X cut out on C a g^1_{m+1}.

With the above conditions on r and ε it follows from Theorem 1.4(i) of [2] that the Zariski open subset $\mathscr{H}^0_{d,g,r}$ of the Hilbert scheme of curves of degree d and genus $g = \pi(d,r)$ in $\mathbb{P}^r_{\mathbb{C}}$, and whose points correspond to externmal curves of type 0.2, is irreducible and one has

$$\dim \mathscr{H}^0_{d,g,r} = (r-1)\left(\binom{m+1}{2} + r + 2\right) + (\varepsilon+2)(m+2) - 4. \tag{0.3}$$

If $\varepsilon = 1$ we also have that $|K_C| = |\mathscr{O}_C(m-1)|$ (see [1, Ch. III, Corollary 2.6(iii)]). According to Section 2 of [3] such curves will be called *subcanonical extremal curves of level* $m-1$.

The group PGL_{r+1} acts on $\mathscr{H}^0_{d,g,r}$, thus we get a PGL_{r+1}-equivariant rational map $h : \mathscr{H}^0_{d,g,r} \to \mathfrak{M}_g$ into the moduli space of smooth curves of genus $g = \pi(r,d)$. Since it follows from Proposition 2.1 of [3] that the g^r_d inducing the embedding $C \subseteq \mathbb{P}^r_{\mathbb{C}}$ is unique and complete, then the fibre over the point $[C] \in \mathfrak{M}_g$ representing C is exactly the PGL_{r+1}-orbit of $[C] \in \mathscr{H}^0_{d,g,r}$.

In particular, we can define in a natural way a coarse moduli space for subcanonical extremal curves of level $m-1$ and genus g, namely $\mathfrak{C}^{m-1}_g := \mathrm{im}(h) \subseteq \mathfrak{M}_g$.

The above remarks show that $\mathscr{H}^0_{d,g,r}/PGL_{r+1} \approx \mathfrak{C}^{m-1}_g$ (we will denote by $\cong$ isomorphisms and by $\approx$ birational equivalences). Since the action of PGL_{r+1} on $\mathscr{H}^0_{d,g,r}$ has finite stabilizer, formula (0.3) and the above discussion yields the following.

Proposition 0.4. *Let $r \geq 3$, $r \neq 5$. There exists a coarse, irreducible moduli space $\mathfrak{C}_g^{m-1}$ of dimension*

$$\dim \mathfrak{C}_g^{m-1} = (r-1)\left(\binom{m+1}{2} + r + 2\right) + 3m + 3 - (r+1)^2$$

for subcanonical extremal curves of level $m-1$ and genus g in $\mathbb{P}_{\mathbb{C}}^r$.

Let $\mathfrak{M}_{g,1}$ be the moduli space of smooth pointed curves of genus g, let $\mathfrak{m} : \mathfrak{M}_{g,1} \to \mathfrak{M}_g$ be the natural projection and set $\mathfrak{C}_{g,1}^{m-1} := \mathfrak{m}^{-1}(\mathfrak{C}_g^{m-1})$. Since the fibres of m are irreducible the same is true for $\mathfrak{C}_{g,1}^{m-1}$.

Corollary 0.5. *Let $r \geq 3$, $r \neq 5$. $\mathfrak{C}_{g,1}^{m-1}$ is a coarse irreducible moduli space for smooth pointed subcanonical extremal curves of level $m-1$ and genus g in $\mathbb{P}_{\mathbb{C}}^r$. Moreover* $\dim \mathfrak{C}_{g,1}^{m-1} = \dim \mathfrak{C}_g^{m-1} + 1$.

The aim of this paper is to prove the following theorem.

Theorem 0.6. *If $r \geq 4$, $r \neq 5$ is odd (resp. even) then $\mathfrak{C}_{3r}^2$ (resp. $\mathfrak{C}_{3r,1}^2$) is rational of dimension $5r+3$ (resp. $5r+4$).*

We remark that for $r=2$ half-canonical extremal curves are smooth plane quintic. The rationality of their coarse moduli space has been proved by N.I. Shepherd-Barron in [11].

1. A birational model of $\mathfrak{C}_{3r}^2$

In this section we describe a birational model of $\mathfrak{C}_{3r}^2$. Let C be as in the introduction. We already noticed that the curve C carries a g_{3r-1}^r which is unique and complete. Moreover, C lies on a unique rational normal scroll X (of minimal degree $r-1$ in $\mathbb{P}_{\mathbb{C}}^r$) and C does not intersect its singular locus Sing(X).

Blowing up X along Sing(X) we obtain a desingularization map $\Sigma_h := \mathbb{P}(\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1} \oplus \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(-h)) \xrightarrow{\varphi} X$ for some $h \geq 0$. Let $\pi_h : \Sigma_h \to \mathbb{P}_{\mathbb{C}}^1$ be the canonical projection and denote by f anyone of its fibres (notice that $f \cong \mathbb{P}_{\mathbb{C}}^1$ and $f^2 = 0$). If $h > 0$ (resp. $h = 0$) there exists a unique irreducible, smooth, rational curve (resp. pencil of curves) $D_0 \subseteq \Sigma_h$ (resp. $|D_0| \subseteq \mathrm{Div}(\Sigma_h)$) such that $D_0^2 = -h$ and $D_0 f = 1$. Moreover, the map $\Sigma_h \xrightarrow{\varphi} X$ is induced by the linear system $|D_0 + \ell f|$ for some $\ell \geq h$ and X is singular if and only if $\ell = h$. Since $r - 1 = \deg(X) = (D_0 + \ell f)^2$ then we also have

$$2\ell + 1 = h + r, \quad h \equiv r + 1 \pmod 2. \tag{1.1}$$

The curve C is embedded in Σ_h as a quadrisecant curve, more precisely $C \in |4D_0 + (r+2h+1)f|$ (see [4, Theorem 5.4 and its proof]). Conversely each smooth, connected curve $C\in|4D_0+(r+2h+1)f|$ is embedded in $\mathbb{P}^r_{\mathbb{C}}$ via $\varphi_{|C}$ as an extremal curve of degree $3r-1$. In particular, we can define rational maps

$$\varphi_h:|4D_0+(r+2h+1)f|\to\mathfrak{C}^2_{3r}.$$

Proposition 1.2. *Let $h=0,1$ and $r\geq 3$, $r\neq 5$. Then* $\mathrm{im}(\varphi_h)$ *is dense in $\mathfrak{C}^2_{3r}$ for $r\equiv h+1$* (mod 2).

Proof. It suffices to check that $\dim(\mathrm{im}(\varphi_h))=5r+3$, since $\mathfrak{C}^2_{3r}$ is irreducible of dimension $5r+3$. To this purpose let us consider the group $Aut(\Sigma_h)$. In any case $\dim(Aut(\Sigma_h))=6$ (see [12]).

As in the introduction notice that the embedding $C\subseteq\Sigma_h$ is induced by the unique g^r_{3r-1}. In particular, each automorphism of Σ_h leaving C fixed, carries such a g^r_{3r-1} into itself. It follows that the fibre of φ_h over a point $[C]\in\mathrm{im}(\varphi_h)$ are the G-orbits of the divisor C with respect to $G:=Aut(\Sigma_1)$ if $h=1$, $G:=Aut(\Sigma_0)\cong O_4$ if $h=0$ and $r=3$ or $G:=SO_4\subseteq Aut(\Sigma_0)\cong O_4$ if $h=0$ and $r\geq 7$. In any case $\dim(G)=6$ (see again [12]), hence

$$\dim(\mathrm{im}(\varphi_h))\geq\dim|4D_0+(r+2h+1)f|-6=5r+3$$

since $h^0(\Sigma_h,\mathcal{O}_{\Sigma_h}(C))=h^0(\mathbb{P}^1_{\mathbb{C}},\pi_{h*}\mathcal{O}_{\Sigma_h}(C))$ (see [7, Lemma V 2.4]). □

Now assume that $r\geq 3$ is odd. Then it follows from formulas (1.1) that h is even and $\ell\geq 1$ thus $X\cong\mathbb{P}^1_{\mathbb{C}}\times_{\mathbb{C}}\mathbb{P}^1_{\mathbb{C}}$ is smooth. Moreover Proposition 1.1 and its proof yield the birational equivalence

$$\mathfrak{C}^2_{3r}\approx|4D_0+(r+1)f|/G,$$

where $G\cong O_4$ if $r=3$ and $G=SO_4$ if $r\geq 7$. Notice that $|4D_0+(r+1)f|\cong\mathbb{P}(V_4\otimes V'_{r+1})$ where V_d (resp. V'_d) is the set of forms of degree d in the variables x_1 and x_2 (resp. y_1 and y_2). On the other hand, the representation of SO_4 on $V_4\otimes V'_{r+1}$ is equivalent to the natural representation of $SL_2\times SL_2$ on $V_4\otimes V'_{r+1}$. We summarize results in the following:

Theorem 1.3. *If $r=2k-1\geq 7$ then there exists a birational equivalence*

$$\mathfrak{C}^2_{3r}\approx\mathbb{P}(V_4\otimes V'_{2k})/(PSL_2\times PSL_2).$$

2. The rationality of $\mathbb{P}(V_4\otimes V'_{2k})/(PSL_2\times PSL_2)$

In this section, following the proof of Proposition 6 of [10], we will prove

Theorem 2.1. $\mathbb{P}(V_4\otimes V'_{2k})/(PSL_2\times PSL_2)$ *is rational if $k\geq 3$.*

A immediate consequence of Theorem 2.1 above is the rationality of $\mathfrak{C}^2_{3r}$ for odd $r\geq 7$, stated in Theorem 0.4 of the introduction.

Let $f \in V_4 \otimes V'_{2k}$. Using the symbolical notation (see [6]) we can write $f = a_x^4 \otimes A_y^{2k} = b_x^4 \otimes B_y^{2k}$ where $a_x := a_1x_1 + a_2x_2$, $b_x := b_1x_1 + b_2x_2$, $A_y := A_1y_1 + A_2y_2$, $B_y := B_1y_1 + B_2y_2$. We define a covariant

$$\varphi : V_4 \otimes V'_{2k} \to V_4 \otimes V'_0,$$

$$f \to (ab)^2(AB)^{2k}a_x^2b_x^2,$$

where, as usual, $(ab) := a_1b_2 - a_2b_1$, $(AB) := A_1B_2 - A_2B_1$. φ induces a rational $(PSL_2 \times PSL_2)$-equivariant map

$$\psi : \mathbb{P}(V_4 \otimes V'_{2k}) \to \mathbb{P}(V_4 \otimes V'_0).$$

Let $\gamma : \widetilde{\mathbb{P}} \to \mathbb{P}(V_4 \otimes V'_{2k})$ be the blow up of $\mathbb{P}(V_4 \otimes V'_{2k})$ along the base locus of ψ and let

$$\widetilde{\psi} : \widetilde{\mathbb{P}} \to \mathbb{P}(V_4 \otimes V'_0)$$

be the induced morphism. Notice that $\widetilde{\psi}$ is again a $(PSL_2 \times PSL_2)$-equivariant map.

The following proposition is an easy consequence of the definition of ψ and its proof is a trivial and tedious computation.

Proposition 2.2. *If* $f = \sum_{i=0}^{4}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\alpha_{i,j}x_1^{4-i}x_2^1 \otimes y_1^{2k-j}y_2^j$ *then*

$$\psi(f) = \sum_{h=0}^{4}\beta_h x_1^{4-h}x_2^h,$$

where

$$\beta_0 = 2\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}(\alpha_{0,j}\alpha_{2,2k-j} - \alpha_{1,j}\alpha_{1,2k-j}),$$

$$\beta_1 = 4\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}(\alpha_{0,j}\alpha_{3,2k-j} - \alpha_{1,j}\alpha_{2,2k-j}),$$

$$\beta_2 = 2\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}(\alpha_{0,j}\alpha_{4,2k-j} + 2\alpha_{1,j}\alpha_{3,2k-j} - 3\alpha_{2,j}\alpha_{2,2k-j}),$$

$$\beta_3 = 2\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}(\alpha_{4,j}\alpha_{1,2k-j} - \alpha_{2,j}\alpha_{3,2k-j}),$$

$$\beta_4 = 2\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}(\alpha_{4,j}\alpha_{2,2k-j} - \alpha_{3,j}\alpha_{3,2k-j}).$$

Corollary 2.3. *Both ψ and $\widetilde{\psi}$ are dominant.*

Proof. It suffices to prove that ψ is dominant. Let

$$F := \{f \in \mathbb{P}(V_4 \otimes V'_{2k}) \mid \alpha_{4,k} = \alpha_{0,k} = \lambda,\ \alpha_{2,k} = \mu,\ \alpha_{i,j} = 0 \text{ for } (i,j) \neq (4,k),(0,k),(2,k)\}.$$

It follows from Proposition 2.2 that $\psi(f) = \sum_{h=0}^{4} \beta_h x_1^{4-h} x_2^h$ where $\beta_1 = \beta_3 = 0$ and

$$\beta_0 = \beta_4 = 2(-1)^k \binom{2k}{k} \lambda\mu, \qquad \beta_2 = 2(-1)^k \binom{2k}{k} (\lambda^2 - 3\mu^2).$$

In particular, $\psi(F)$ is $(PSL_2 \times PSL_2)$-dense in $\mathbb{P}(V_4 \otimes V'_0)$ (see [9, Section 1]), whence for each general $\overline{f} \in \mathbb{P}(V_4 \otimes V'_0)$, there exists $g \in PSL_2 \times PSL_2$ such that $g(\overline{f}) \in \psi(F)$, thus there exists $f \in F$ such that $\psi(f) = g(\overline{f})$ hence $\overline{f} = \psi(g^{-1}(f))$. $\square$

Let

$$U := \mathbb{P}(\langle x_1^4 - x_2^4, x_1^3 x_2, x_1 x_2^3 \rangle \otimes V'_0),$$
$$V := \mathbb{P}(\langle x_1^4 + x_2^4, x_1^2 x_2^2 \rangle \otimes V'_0),$$

define

$$e := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad b := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and let $H \subseteq PSL_2$ be the subgroup generated by the classes of e, a, b. The normalizer N of H inside PSL_2 is the stabilizer of V (see [9, Section 1]), thus V is a $(PSL_2 \times PSL_2, N \times PSL_2)$-section of $\mathbb{P}(V_4 \otimes V'_0)$ in the sense of [8].

Now let $Y := \gamma(\widetilde{\psi}^{-1}(V))$.

Proposition 2.4. *If $k \geq 2$ then Y is irreducible.*

Proof. Let $L := \mathbb{P}(V_4 \otimes V'_{2k})$ the subspace defined by the equations

$$\alpha_{1,i} = \alpha_{3,j} = \alpha_{0,l} - \alpha_{4,l} = 0, \quad i,j,l = 0,\dots,2k.$$

The equations of $Y \subseteq \mathbb{P}(V_4 \otimes V'_{2k})$ are

$$\vartheta_1 := \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha_{4,j}\alpha_{1,2k-j} - \alpha_{2,j}\alpha_{3,2k-j}),$$

$$\vartheta_2 := \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha_{0,j}\alpha_{3,2k-j} - \alpha_{1,j}\alpha_{2,2k-j}),$$

$$\vartheta_3 = \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} ((\alpha_{0,j} - \alpha_{4,j})\alpha_{2,2k-j} - \alpha_{1,j}\alpha_{1,2k-j} + \alpha_{3,j}\alpha_{3,2k-j})$$

(see Proposition 2.2). In particular $L \subseteq Y$.

Let $Y_0 \subseteq Y$ be an irreducible component containing L and let Y_1 be another one not contained in Y_0. Obviously $Y_0 \cap Y_1 \subseteq \mathrm{Sing}(Y)$, hence $L \cap Y_1 \subseteq \mathrm{Sing}(Y)$.

Since Y is the intersection of three hypersurfaces one also has $\mathrm{codim}(Y_1) \leq 3$, thus

$$\mathrm{codim}(L \cap Y_1) \leq 3 + \mathrm{codim}(L) \leq 6k + 6. \tag{2.1}$$

On the other hand, the Jacobian matrix of the ϑ_i's is

$$J := \begin{pmatrix} 0 & \alpha_{3,i} & \alpha_{2,i} \\ \alpha_{4,j} & -\alpha_{2,j} & -2\alpha_{1,j} \\ -\alpha_{3,h} & -\alpha_{1,h} & \alpha_{0,h} - \alpha_{4,h} \\ -\alpha_{2,l} & \alpha_{0,l} & 2\alpha_{3,l} \\ \alpha_{1,m} & 0 & -\alpha_{2,m} \end{pmatrix}_{i,j,h,l,m=0,\dots,2k}.$$

In particular, $\mathrm{rk}(J) \leq 2$ at each point of $L \cap Y_1$, thus $L \cap Y_1 \subseteq X$ where $X \subseteq \mathbb{P}(V_4 \otimes V'_{2k})$ is defined by the equations

$$\begin{cases} \alpha_{1,i} = 0, \\ \alpha_{3,j} = 0, \\ \alpha_{0,l} - \alpha_{4,l} = 0, \\ (\alpha_{0,h}\alpha_{4,h} - \alpha_{2,h}^2)\alpha_{2,h} = 0, \end{cases}$$

where $i, j, l, h = 0, \dots, 2k$. With some easy computations one can check that the above system is equivalent to the system

$$\begin{cases} \alpha_{1,i} = 0, \\ \alpha_{3,j} = 0, \\ \alpha_{0,l} - \alpha_{4,l} = 0, \\ ((\alpha_{0,h} + \alpha_{4,h})^2 - 4\alpha_{2,h}^2)\alpha_{2,h} = 0. \end{cases} \tag{2.2}$$

Each polynomial in (2.2) involves a different set of variables hence they form a regular sequence. It follows that $\mathrm{codim}(X) = 4(2k+1)$ so that

$$\mathrm{codim}(L \cap Y_1) \geq 8k + 4. \tag{2.3}$$

Confronting (2.1) and (2.3) one should have $6k + 6 \geq 8k + 4$ which is possible only if $k = 0, 1$.

We conclude that $Y = Y_0$. □

Since Y is irreducible then it is a $(PSL_2 \times PSL_2, N \times PSL_2)$-section of $\mathbb{P}(V_4 \otimes V'_{2k})$ (see [8]). Thus we get the following result.

Corollary 2.5. $\mathbb{P}(V_4 \otimes V'_{2k})/(PSL_2 \times PSL_2) \approx Y/(N \times PSL_2)$.

Let $M \subseteq \mathbb{P}(V_4 \otimes V'_{2k})$ be the subspace defined by the equations

$$\alpha_{0,i} + \alpha_{4,i} = \alpha_{2,j} = 0,$$

where $i, j = 0, \ldots, 2k$. We denote by $\pi_L : Y \to M$ the projection from L. If we choose $\alpha_{1,j}$, $\alpha_{3,h}$, $\alpha_{0,l} - \alpha_{4,l}$ (i.e. if we fix $m \in M$) the equations ϑ_i specialized at these values becomes linear in the remaining variables.

Such equations are generically of maximal rank thus Y is generically a vector-bundle over M. Moreover π_L is $(N \times PSL_2)$-equivariant.

Lemma 2.6. *$M \cong \mathbb{P}(U)$ and the action of $N \times PSL_2$ on M is almost free for $k \geq 3$.*

Proof. Let $p \in \mathbb{P}(V'_{2k})$ such that the condition $g(p) = p$ implies $g = id \in PSL_2$ (such a p exists since $2k \geq 6$).

Let $f \in \mathbb{P}(\langle x_1^4 - x_2^4, x_1^3 x_2, x_1 x_2^3 \rangle)$ such that the condition $g(f) = f$ implies $g = id \in N$ (such an f exists since $N \cong \mathfrak{S}_4$ and the three-dimensional irreducible representations of $\mathfrak{S}_4$ are exact: see Section 1 of [9]).

It follows that the stabilizer of $f \otimes p$ is trivial. $\square$

Let $\overline{\pi}_L : Y/(N \times PSL_2) \to M/(N \times PSL_2)$ be induced by π_L. The method of irreducible representation (see [5]) implies that $\overline{\pi}_L$ is again a vector bundle.

Let

$$W := (\langle x_1^4 - x_2^4, x_1^3 x_2, x_1 x_2^3 \rangle \oplus \mathbb{C}) \otimes V'_0 \oplus V_0 \otimes V'_6.$$

W is an almost free linear representation of $N \times PSL_2$. Moreover

$$\mathbb{P}(W)/(N \times PSL_2) \approx \mathbb{P}(\langle x_1^4 - x_2^4, x_1^3 x_2, x_1 x_2^3 \rangle \oplus \mathbb{C})/N \times \mathbb{P}(V'_6)/PSL_2 \times \mathbb{P}^1_{\mathbb{C}}$$

which is rational (see [9]: for the first factor see the corollary in section 1, for the second one see Theorem 0.1 or 0.2).

Since $M \times \mathbb{P}(W)/(N \times PSL_2)$ is a vector bundle over $\mathbb{P}(W)/(N \times PSL_2)$ it is also rational. On the other hand, $M \times \mathbb{P}(W)/(N \times PSL_2)$ is also a vector bundle over $M/(N \times PSL_2)$ with typical fibre $\mathbb{P}(W) \approx \mathbb{C}^{10}$ hence $M/(N \times PSL_2) \times \mathbb{C}^{10}$ is rational too.

Since $\overline{\pi}_L^{-1}(m) \cong \mathbb{C}^{4k-1}$ it follows that $Y/(N \times PSL_2)$ is rational as soon as $4k - 1 \geq 10$ i.e. if $k \geq 3$.

3. Pointed curves

In this section we complete the proof of Theorem 0.6.

Proposition 3.1. *If $r \geq 4$ is even then $\mathfrak{C}^2_{3r,1}$ is rational.*

Proof. Let r be even and let C be a subcanonical extremal curve of level 2 and genus $3r$ contained in the ruled surface Σ_1 as an element of the linear system $|D_0 + (r+3)f|$ (for the notations see Section 1). Contracting D_0 to a point, we map C to a plane curve $C' \subseteq \mathbb{P}^2_{\mathbb{C}}$ of degree $r + 3$ with a point of multiplicity $r - 1$.

Fix $P, Q \in \mathbb{P}^2_{\mathbb{C}}$ and let W be the linear system of curves in $\mathbb{P}^2_{\mathbb{C}}$ through Q and having P as a $(r-1)$-fold point. We have a rational map

$$\varphi : W \to \mathfrak{C}^2_{3r,1}$$

sending each curve to its isomorphism class.

Let $SL_{3,P}$ and $SL_{3,Q}$ be the stabilizers inside SL_3 of P and Q respectively. The group $G := SL_{3,P} \cap SL_{3,Q} \subseteq SL_3$ acts on W (with finite stabilizer). Since φ is by construction G-equivariant then

$$\dim \varphi(W) = \dim W - \dim G = 5r + 4,$$

thus φ is dominant (see Corollary 0.5).

On the other hand, the orbits of G coincide with the fibres of φ thus $\mathfrak{C}^2_{3r,1} \approx W/G$ which is rational since G is triangular (see [13]). □

References

[1] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, Geometry of Algebraic Curves, vol. I, Springer, Berlin, 1985.

[2] C. Ciliberto, On the Hilbert scheme of curves of maximal genus in a projective space, Math. Z. 194 (1987) 351–363.

[3] C. Ciliberto, R. Lazarsfeld, On the uniqueness of certain linear series on some classes of curves, in: S. Greco, R. Strano (Eds.), Complete Intersections, Acireale, Lecture Notes in Math., vol. 1092, Springer, Berlin, 1983.

[4] A. Del Centina, Remarks on curves admitting an involution of genus ≥ 1 and some applications, Boll. Un. Mat Ital. 4—B (1985) 671–683.

[5] I.V. Dolgachev, Rationality of fields of invariants, in: S.J. Bloch (Ed.), Algebraic Geometry, Bowdoin, 1985, Proc. Symp. Pure Math., vol. 46, 1987.

[6] J.H. Grace, W.H. Young, The Algebra of Invariants, Chelsea, London, 1903.

[7] R. Hartshorne, Algebraic Geometry, Springer, Berlin, 1977.

[8] P.I. Katsylo, The rationality of the orbit spaces of irreducible representations of the group SL_2, Math. USSR Izvestiya 22 (1984) 23–32.

[9] P.I. Katsylo, The rationality of the moduli space of hyperelliptic curves, Math. USSR Izvestiya 25 (1985) 45–50.

[10] N.I. Shepherd-Barron, The rationality of certain spaces associated to trigonal curves, in: S.J. Bloch (Ed.), Algebraic Geometry, Bowdoin, 1985, Proc. Symp. Pure Math., vol. 46, 1987.

[11] N.I. Shepherd-Barron, The rationality of some moduli spaces of plane curves, Compositio Math. 67 (1988) 51–88.

[12] C. Turrini, Gli automorfismi delle rigate geometriche razionali, Ist. Lombardo (Rend. Sc.) 113 (1979) 210–223.

[13] E.B. Vinberg, Rationality of the field of invariants of a triangular group, Mosc. Univ. Math. Bul. 37 (1992) 27–29.