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5D SYM on 3D deformed spheres

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Abstract

We reconsider the relation of superconformal indices of superconformal field theories of class $\mathcal{S}$ with five-dimensional $\mathcal{N}=2$ supersymmetric Yang–Mills theory compactified on the product space of a round three-sphere and a Riemann surface. We formulate the five-dimensional theory in supersymmetric backgrounds preserving $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetries and discuss a subtle point in the previous paper concerned with the partial twisting on the Riemann surface. We further compute the partition function by localization of the five-dimensional theory on a squashed three-sphere in $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetric backgrounds and on an ellipsoid three-sphere in an $\mathcal{N}=1$ supersymmetric background.

1. Introduction

In the previous papers [1,2], we have attempted to give a physical proof for the conjecture of [3]. The conjecture states that the Schur limit of the superconformal index [4] of a four-dimensional $\mathcal{N}=2$ superconformal theory of class $\mathcal{S}$ [5,6] can be computed by two-dimensional q-deformed Yang–Mills theory [7]. The $\mathcal{N}=2$ superconformal theory of class $\mathcal{S}$ is defined in [6,5] as the infrared limit of M5-branes wrapped on a Riemann surface[1] Σ, and according to the conjecture, one may compute the Schur index of the theory by making use of the q-deformed Yang–Mill theory on Σ in the zero area limit.

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[1] The Riemann surface is commonly denoted by $\mathcal{C}$ in the recent literature and is often referred to as a UV curve.

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The superconformal index may be captured by the partition function of the four-dimensional theory compactified on $S^1 \times S^3$. (See [8] for the extension to Σ with nonzero area.) Since the infrared limit of M5-branes gives rise to the putative six-dimensional $\mathcal{N} = (2,0)$ superconformal theory, it is conceivable to obtain the index by computing the partition function of the $\mathcal{N} = (2,0)$ theory compactified on $S^1 \times S^3 \times \Sigma$ with a partial twisting on Σ. The idea[2] has been argued in [9] as a "top-down" approach to uncover the relation of a generic superconformal index of the theories of class $\mathcal{S}$ with a topological field theory. For a review of the superconformal indices of theories of class $\mathcal{S}$, see [15].

In the previous papers [1,2], we put the idea into practice by exchanging the order of the compactifications; regarding M5-branes wrapped on a circle as D4-branes, we compactified five-dimensional $\mathcal{N} = 2$ supersymmetric Yang–Mills theory on Σ with a partial twisting and further on the round S^3, in a somewhat ad hoc way. Computing the partition function of the compactified theory by localization, we have found that the fixed points give the fields of the q-deformed Yang–Mills theory, and that the one-loop contributions and the classical action at the fixed points yield the measure of the partition integral of it; namely, the partition function of the five-dimensional compactified theory is reduced to that of the two-dimensional q-deformed Yang Mill theory.

However, there was a confusion about the partial twisting in [1]. The supersymmetric background used in [1] – especially, the partial twisting – preserved only $\mathcal{N} = 1$ supersymmetry in four dimensions. Since the conjecture in [3] is concerned with the four-dimensional $\mathcal{N} = 2$ superconformal theories, the results in [1] seem to have nothing to do with the conjecture.

The construction of the $\mathcal{N} = 2$ superconformal theories by the twisted compactification of the $\mathcal{N} = (2,0)$ theory on Σ has been generalized for $\mathcal{N} = 1$ supersymmetric theories in four dimensions [16–19], which we will refer to as $\mathcal{N} = 1$ theories of class $\mathcal{S}$. We can see that the twisting used in [1] is identical to what is called[3] the $\mathcal{N} = 1$ twist in [16].

Superconformal indices of the $\mathcal{N} = 1$ superconformal theories of class $\mathcal{S}$ have been calculated in four dimensions in [20]. A simple comparison shows that the result in [1] is in good agreement with the Schur limit of the mixed Schur index in [20], as we will see later.

The questions we raise are two-fold; first, when the $\mathcal{N} = (2,0)$ theory compactified on Σ with the $\mathcal{N} = 2$ twisting so that $\mathcal{N} = 2$ supersymmetry remains unbroken in four dimensions, whether will we obtain the q-deformed Yang–Mills theory on Σ via localization? This was the original motivation in the previous paper [1,2].

Second, when replacing the round S^3 by a deformation of the S^3, such as a squashed S^3 and an ellipsoid S^3, as discussed in [21,22], whether will we obtain a deformation of the Schur index for the round S^3, like the mixed Schur index in [20]?

We will make an attempt to answer both of the questions in this paper, which is organized as follows: in Sections 2 and 3, we will begin with the construction of the five-dimensional supersymmetric Yang–Mills theory on a curved space, based on the idea of [23] that the fields of an off-shell supergravity multiplet are utilized as background fields to preserve supersymmetries of the field theory on a curved space. In fact, through the dimensional reduction of the six-dimensional $\mathcal{N} = (2,0)$ conformal supergravity in [25], on-shell supersymmetry transforma-

[2] Similar constructions have been used in [6,10] to explore four-dimensional $\mathcal{N} = 2$ superconformal field theories of class $\mathcal{S}$. See also [11] for $\mathcal{N} = 1$ supersymmetric theories. See [12,13] (also [14]) for related works on three-dimensional Chern–Simons theory from M5-branes.

[3] See Appendix D of [16] about the embedding of the spin connection on Σ to the R-symmetry group. The $\mathcal{N} = 1$ twist corresponds to the case of $l_1 = l_2$ in their Calabi–Yau construction of [19].

tions and an on-shell action of the five-dimensional theory compactified on a curved space have been derived in [24], following the idea [23].

Therefore, Sections 2, 3, and 4 are essentially devoted to a review of [24], up to a few points that we perform the dimensional reduction in the time direction of the six-dimensional theory, instead of the spatial direction as in [24]. And we obtain off-shell supersymmetry transformations and an off-shell action of the five-dimensional theory on a curved space in Section 6, which are necessary to carry out localization.

In Section 5, we will discuss the partial twistings mentioned above – the $\mathcal{N}=1$ twisting and the $\mathcal{N}=2$ twisting – in more details, in the language of the background gauge field of the R-symmetry group, and we will describe the supersymmetric background on a round S^3 in [1,2] in terms of supergravity background fields for the $\mathcal{N}=1$ twisting in Subsection 5.1, and give a supersymmetric background on the round S^3 for the $\mathcal{N}=2$ twisting in Subsection 5.2.

We will proceed to consider two supersymmetric backgrounds on a squashed S^3 – the analog of the background in [21] and of the one in [22] – in Subsections 5.3 and 5.4, respectively. Especially, for the former, we will give supersymmetry backgrounds for both of the twistings.

In Subsection 5.5, we will discuss a supersymmetric background for the $\mathcal{N}=1$ twisting on an ellipsoid S^3, in an analogous way to [21].

After the discussions about the off-shell formulation of the five-dimensional theory in Section 6, as mentioned above, we will explain our localization method in depth in Section 7. We will compute the partition functions by localization on the round and squashed S^3's in Section 8 for the background in Section 5.3 and that on the ellipsoid S^3 in Section 9 for the background in Section 5.5.

However, the computation of the partition function on the squashed S^3 for the background in Section 5.4 somewhat doesn't seem straightforward to be done by localization, and we will leave it as an open question. Finally, Section 10 is devoted to the summary and discussions of this paper.

Appendix A is a simple collection of our conventions about the (anti-)symmetrization of various indices and about differential forms, used in this paper, and the gamma matrices of the Lorentz groups in five and six dimensions are shown in our representation in Appendix B. The R-symmetry group of the six- and five-dimensional theories are commonly $\mathrm{Spin}(5)_R \simeq Sp(2)_R$ and the associating gamma matrices in our representation are given in Appendix C. The spinors in the theories are symplectic Majorana–Weyl spinors and in Appendix D, our convections about those spinors are explained.

After the dimensional reduction of the conformal supergravity, supersymmetry transforms of the fermionic fields in the supergravity multiplet (the Weyl multiplet) yield supersymmetry conditions on the background fields to preserve supersymmetries on the curved background. Besides the supersymmetry condition derived from the gravitino field, there is another supersymmetry condition from the fermionic auxiliary field in the Weyl multiplet and it is too long to write down explicitly in the text. Therefore, the explicit form of the supersymmetry condition is written in Appendix E.

In Appendix F, a few formulas which we think are useful to verify the invariance of the actions in Sections 3 and 4 under the supersymmetry transformations are given.

In Appendix G, Killing spinors and metrics are discussed on the round, squashed, and ellipsoid S^3, following [21,22].

Appendix H explains the difference among the notations used in [25], in [24], and in this paper, and further the difference between the notations used here and in the previous paper [1].

2. Euclidean 5D $\mathcal{N}=2$ SYM in SUGRA backgrounds

In this section, the dimensional reduction along the time direction will be performed for the six-dimensional $\mathcal{N}=(2,0)$ conformal supergravity derived in [25]. This section, Sections 3 and 4 are essentially a review of [24], but the spatial dimensional reduction was carried out there.

In Subsection 2.1, we will recapitulate the main results of [25], which we will need in this paper about the supergravity multiplet called the Weyl multiplet in the conformal tensor calculus.

In Subsection 2.2, we will discuss the dimensional reduction of the Weyl multiplet, which play roles of supersymmetric background fields to retain supersymmetries of the five-dimensional Yang–Mills theory on a curved space. Subsection 2.3 is just a small digression about the relation of Killing spinors with Killing vectors.

2.1. *Weyl multiplet in 6D $\mathcal{N}=(2,0)$ conformal supergravity*

In this paper, following [24], we will carry out dimensional reduction of the six-dimensional $\mathcal{N}=(2,0)$ supergravity in [25] to obtain a five-dimensional Euclidean maximally supersymmetric Yang–Mills theory in supergravity backgrounds. It has been discussed in [23] that the supergravity backgrounds provide a systematic method for supersymmetric compactifications of supersymmetric field theories. The construction of the supergravity in [25] is based on the conformal tensor calculus. (See the textbook [26] for the conformal tensor calculus and references therein.)

In this approach, one starts with a gauge field theory by gauging the six-dimensional $\mathcal{N}=(2,0)$ superconformal symmetry group $OSp(2,6|4)$, whose bosonic part consists of the conformal group $SO(2,6)$ and the R-symmetry group Spin(5). The symmetry group $OSp(2,6|4)$ is generated by

$$P_{\underline{a}}:\text{translation},\quad D:\text{translation},\quad M_{\underline{ab}}:\text{Lorentz},\quad K_{\underline{a}}:\text{special conformal},$$
$$R_{IJ}:R\text{-symmetry},\quad Q^{\alpha}:\text{supersymmetry},\quad S^{\alpha}:\text{conformal supersymmetry},$$

whose corresponding gauge fields are shown in Table 1.

Let us list the notations of the various indices on the generators and the gauge fields:

- $\underline{a},\underline{b}=0,1,\cdots,5$; the Lorentz indices,
- $\underline{\mu},\underline{\nu}=0,1,\cdots,5$; the coordinate frame indices,
- $I,J=1,\cdots,5$; the vector indices of the Spin$(5)_R$ symmetry,
- $\alpha,\beta=1,\cdots,4$; the spinor indices of the Spin$(5)_R$ symmetry.

The fermionic fields $\underline{\psi_{\mu}}^{\alpha}$ and $\underline{\phi_{\mu}}^{\alpha}$ are the gauge fields of the supersymmetry and the conformal supersymmetry, respectively. They are symplectic Majorana–Weyl spinors of positive and negative chirality, respectively. See Appendix D for our conventions about symplectic Majorana–Weyl spinors.

A straightforward manner of gauging translations doesn't lead to general coordinate transformations which is indispensable to a theory of gravity. To gain general coordinate transformations from translations in the conformal tensor calculus approach, auxiliary fields[4] in Table 2 are intro-

[4] They are referred to as 'matter fields' in [25,24]. It is not always necessary to introduce auxiliary fields for the deformation, and it depends on the numbers of supersymmetries and the dimensions of spacetime, *i.e.*, superconformal algebras. See [26] for more details.

Table 1
The gauge fields of the 6D $\mathcal{N} = (2,0)$ superconformal symmetry.

Gauge fields	Transformations	Restrictions	$\mathrm{Spin}(5)_R$	Weight
Boson				
$\underline{E}^{\underline{a}}{}_{\underline{\mu}}$	$P_{\underline{a}}$: translations	sechsbein	**1**	-1
$\underline{b}_{\underline{\mu}}$	D: dilatation		**1**	0
$\underline{V}_{\underline{\mu}}{}^{IJ}$	R_{IJ}: R-symmetry	$\underline{V}_{\underline{\mu}}{}^{IJ} = -\underline{V}_{\underline{\mu}}{}^{JI}$	**10**	0
Fermion				
$\underline{\psi}^{\alpha}{}_{\underline{\mu}}$	Q^{α}: supersymmetry	gravitini $\underline{\Gamma}^7\underline{\psi}^{\alpha}{}_{\underline{\mu}} = \underline{\psi}^{\alpha}{}_{\underline{\mu}}$	**4**	$-1/2$
Dependent gauge fields				
Boson				
$\underline{\Omega}_{\underline{\mu}}{}^{\underline{ab}}$	$M_{\underline{ab}}$: local Lorentz	spin connection	**1**	0
$\underline{f}^{\underline{a}}{}_{\underline{\mu}}$	$K_{\underline{a}}$: special conformal		**1**	$+1$
Fermion				
$\underline{\phi}^{\alpha}{}_{\underline{\mu}}$	S^{α}: conformal supersymmetry	$\underline{\Gamma}^7\underline{\phi}^{\alpha}{}_{\underline{\mu}} = -\underline{\phi}^{\alpha}{}_{\underline{\mu}}$	**4**	$+1/2$

Table 2
The auxiliary fields for the deformation of the superconformal symmetry.

Auxiliary fields	Symmetries	$\mathrm{Spin}(5)_R$	Weight
Bosonic fields			
$\underline{T}^{\alpha\beta}{}_{\underline{abc}}$	$\underline{T}^{\alpha\beta}{}_{\underline{abc}} = -\frac{1}{3!}\varepsilon_{\underline{abc}}{}^{\underline{def}}\underline{T}^{\alpha\beta}{}_{\underline{def}}$, $\underline{T}^{\alpha\beta}{}_{\underline{abc}} = -\underline{T}^{\beta\alpha}{}_{\underline{abc}}$, $\Omega_{\alpha\beta}\underline{T}^{\alpha\beta}{}_{\underline{abc}} = 0$.	**5**	1
$\underline{M}^{\alpha\beta}{}_{\gamma\delta}$	$\underline{M}^{\alpha\beta,\gamma\delta} = \underline{M}^{\gamma\delta,\alpha\beta} = -\underline{M}^{\beta\alpha,\gamma\delta} = -\underline{M}^{\alpha\beta,\delta\gamma}$, $\Omega_{\alpha\beta}\underline{M}^{\alpha\beta,\gamma\delta} = \Omega_{\gamma\delta}\underline{M}^{\alpha\beta,\gamma\delta} = \Omega_{\alpha\gamma}\Omega_{\beta\delta}\underline{M}^{\alpha\beta,\gamma\delta} = 0$.	**14**	2
Fermionic field			
$\underline{\chi}^{\alpha\beta}{}_{\gamma}$	$\underline{\Gamma}^7\underline{\chi}^{\alpha\beta}{}_{\gamma} = \underline{\chi}^{\alpha\beta}{}_{\gamma}$, $\underline{\chi}^{\alpha\beta}{}_{\gamma} = -\underline{\chi}^{\beta\alpha}{}_{\gamma}$, $\Omega_{\alpha\beta}\underline{\chi}^{\alpha\beta}{}_{\gamma} = \underline{\chi}^{\gamma\beta}{}_{\gamma} = 0$, $(\underline{\chi}^{\alpha\beta}{}_{\gamma})^{\dagger}\underline{\Gamma}^0 = (\underline{\chi}^{\alpha'\beta'}{}_{\gamma'})^T\underline{C}\,\Omega_{\alpha'\alpha}\Omega_{\beta'\beta}(\Omega^{-1})^{\gamma'\gamma}$.	**16**	3/2

duced and the transformation laws of the gauge fields are deformed by imposing some constraints on the gauge field strengths and the auxiliary fields such that the resulting transformation laws give a closed algebra, as explained in [26].

Furthermore, one requires the invertibility of the gauge field $\underline{E}^{\underline{a}}{}_{\underline{\mu}}$ of translations to solve the constraints, which allows us to regard it as the sechsbein. Solving the constraints makes the gauge fields $\underline{\Omega}_{\underline{\mu}}{}^{\underline{ab}}$, $\underline{f}^{\underline{a}}{}_{\underline{\mu}}$, and $\underline{\phi}^{\alpha}{}_{\underline{\mu}}$ dependent fields given in terms of the other gauge fields and the auxiliary fields. In fact, they are given by

$$\underline{\Omega}_{\underline{\mu}}{}^{\underline{ab}} = \underline{\omega}_{\underline{\mu}}{}^{\underline{ab}} + \underline{E}_{\underline{\mu}}{}^{\underline{a}}\underline{b}^{\underline{b}} - \underline{E}_{\underline{\mu}}{}^{\underline{b}}\underline{b}^{\underline{a}} + \cdots, \qquad \underline{\phi}^{\alpha}{}_{\underline{\mu}} = \cdots, \tag{1}$$

$$\underline{f}^{\underline{a}}{}_{\underline{\mu}} = \frac{1}{8}\underline{R}_{\underline{\mu}}{}^{\underline{a}}(\underline{\Omega}) - \frac{1}{80}\underline{E}_{\underline{\mu}}{}^{\underline{a}}\underline{R}_{\underline{c}}{}^{\underline{c}}(\underline{\Omega}) + \frac{1}{32}\underline{T}^{\alpha\beta}{}_{\underline{\mu cd}}\,\underline{T}_{\alpha\beta}{}^{\underline{acd}} + \cdots, \tag{2}$$

where the ellipses $\cdots$ denote the contributions from the fermionic fields. One can see that the spin connection $\underline{\Omega}_{\underline{\mu}}{}^{\underline{ab}}$ is a generalization of the Levi-Civita spin connection $\underline{\omega}_{\underline{\mu}}{}^{\underline{ab}}$ satisfying

$$d\underline{E}^{\underline{a}} + \underline{\omega}^{\underline{a}}{}_{\underline{b}} \wedge \underline{E}^{\underline{b}} = 0, \qquad \underline{E}^{\underline{a}} = \underline{E}_{\underline{\mu}}{}^{\underline{a}}\,d\underline{X}^{\underline{\mu}},$$

and $R_{\underline{\mu}}{}^{\underline{a}}(\underline{\Omega})$ is the Ricci tensor

$$\underline{R}_{\underline{\mu}}{}^{\underline{a}}(\underline{\Omega}) = \underline{\Theta}^{\underline{\nu}}{}_{\underline{b}}\, \underline{R}_{\underline{\nu\mu}}{}^{\underline{ba}}(\underline{\Omega}),$$

of the curvature tensor of the spin connection $\underline{\Omega}^{\underline{a}}{}_{\underline{b}}$,

$$\underline{R}^{\underline{a}}{}_{\underline{b}}(\underline{\Omega}) = \frac{1}{2}\underline{R}_{\underline{cd}}{}^{\underline{a}}{}_{\underline{b}}(\underline{\Omega})\, \underline{E}^{\underline{c}} \wedge \underline{E}^{\underline{d}} = d\underline{\Omega}^{\underline{a}}{}_{\underline{b}} + \underline{\Omega}^{\underline{a}}{}_{\underline{c}} \wedge \underline{\Omega}^{\underline{c}}{}_{\underline{b}},$$

where $\underline{\Theta}^{\underline{\nu}}{}_{\underline{b}}$ denotes the inverse of the sechsbein $\underline{E}^{\underline{a}}{}_{\underline{\mu}}$, *i.e.*, coframe.

After the deformation, one finds a closed algebra with the (covariant) general coordinate transformations. The remaining independent gauge fields and auxiliary fields form a multiplet called the Weyl multiplet including the graviton, the gravitini and the others. We show the resulting bosonic transformations of the independent gauge fields, except for the (covariant) general coordinate transformations,

$$\begin{aligned}
&\delta \underline{E}_{\underline{\mu}}{}^{\underline{a}} = -\Lambda_D\, \underline{E}_{\underline{\mu}}{}^{\underline{a}} + \Lambda^{\underline{a}}{}_{\underline{b}}\, \underline{E}_{\underline{\mu}}{}^{\underline{b}},\\
&\delta \underline{\psi}^{\alpha}{}_{\underline{\mu}} = -\frac{1}{2}\Lambda_D\, \underline{\psi}^{\alpha}{}_{\underline{\mu}} + \frac{1}{4}\Lambda_R{}^{IJ}\,(\rho_{IJ})^{\alpha}{}_{\beta}\, \underline{\psi}^{\beta}{}_{\underline{\mu}} + \frac{1}{4}\Lambda_{\underline{ab}}\, \underline{\Gamma}^{\underline{ab}}\, \underline{\psi}^{\alpha}{}_{\underline{\mu}},\\
&\delta \underline{V}_{\underline{\mu}}{}^{IJ} = \partial\Lambda_R{}^{IJ} + \Lambda_R{}^{I}{}_{K}\, \underline{V}_{\underline{\mu}}{}^{KJ} + \Lambda_R{}^{J}{}_{L}\, \underline{V}_{\underline{\mu}}{}^{IL}, \qquad \delta \underline{b}_{\underline{\mu}} = \partial_{\underline{\mu}}\Lambda_D - 2\Lambda_{K\underline{a}}\, \underline{E}_{\underline{\mu}}{}^{\underline{a}},
\end{aligned}$$

where Λ_D, Λ_{ab}, Λ_{Ka}, and $\Lambda_R{}^{IJ}$, are the parameters of dilatation, the Lorentz, special conformal, and R-symmetry transformations, respectively, and under the first four transformations, the auxiliary fields transform as

$$\begin{aligned}
&\delta \underline{T}^{\alpha\beta}{}_{\underline{\mu\nu\rho}} = \Lambda_D\, \underline{T}^{\alpha\beta}{}_{\underline{\mu\nu\rho}}, \qquad \delta \underline{M}^{\alpha\beta}{}_{\gamma\delta} = 2\Lambda_D\, \underline{M}^{\alpha\beta}{}_{\gamma\delta},\\
&\delta \underline{\chi}^{\alpha\beta}{}_{\gamma} = \frac{3}{2}\Lambda_D\, \underline{\chi}^{\alpha\beta}{}_{\gamma} + \frac{1}{4}\Lambda_{\underline{ab}}\, \underline{\Gamma}^{\underline{ab}}\, \underline{\chi}^{\alpha\beta}{}_{\gamma}.
\end{aligned}$$

Under the R-symmetry transformations, they transform in the representations shown in Table 2, respectively.

The resulting supersymmetry (Q-) transformations and superconformal (S-) transformations on the gauge fields and the auxiliary fields are given by

$$\begin{aligned}
&\delta \underline{E}_{\underline{\mu}}{}^{\underline{a}} = \frac{i}{2}\Omega_{\alpha\beta}\left(\underline{\epsilon}^{\alpha}\right)^{T} \underline{C\Gamma^{a}}\, \underline{\psi}^{\beta}{}_{\underline{\mu}},\\
&\delta \underline{\psi}^{\alpha}{}_{\underline{\mu}} = \underline{\mathcal{D}}_{\underline{\mu}}\underline{\epsilon}^{\alpha} + \frac{1}{4!}\, T^{\alpha\beta}{}_{\underline{abc}}\, \underline{\Gamma}^{\underline{abc}}\underline{\Gamma}_{\underline{\mu}}\, \underline{\epsilon}_{\beta} + \underline{\Gamma}_{\underline{\mu}}\, \underline{\eta}^{\alpha},\\
&\delta \underline{b}_{\underline{\mu}} = \frac{1}{2}\Omega_{\alpha\beta}\left(\underline{\epsilon}^{\alpha}\right)^{T} \underline{C\phi}^{\beta}{}_{\underline{\mu}} - \frac{1}{2}\Omega_{\alpha\beta}\left(\underline{\eta}^{\alpha}\right)^{T} \underline{C\psi}^{\beta}{}_{\underline{\mu}},\\
&\delta \underline{V}_{\underline{\mu}}{}^{IJ} = \left(\Omega\rho^{IJ}\right)_{\alpha\beta}\left[\left(\underline{\epsilon}^{\alpha}\right)^{T} \underline{C}\, \underline{\phi}^{\beta}{}_{\underline{\mu}} + \left(\underline{\eta}^{\alpha}\right)^{T} \underline{C}\, \underline{\psi}^{\beta}{}_{\underline{\mu}}\right] - \frac{1}{15}\left(\rho^{IJ}\right)^{\alpha}{}_{\beta}\Omega_{\gamma\delta}\left(\underline{\epsilon}^{\gamma}\right)^{T} \underline{\Gamma}_{\underline{\mu}}\, \underline{\chi}^{\delta\beta}{}_{\alpha},\\
&\delta \underline{T}^{\alpha\beta}{}_{\underline{abc}} = \frac{1}{16}\left(\underline{\epsilon}^{[\alpha}\right)^{T} \underline{C\Gamma^{de}\Gamma_{abc}}\, \underline{R}^{\beta]}{}_{\underline{de}}(Q) - \frac{1}{15}\left(\underline{\epsilon}^{\gamma}\right)^{T} \underline{C\Gamma_{abc}}\chi^{\alpha\beta}{}_{\gamma} - (\text{trace}),\\
&\delta \underline{M}^{\alpha\beta}{}_{\gamma\delta} = -\left(\underline{\epsilon}^{[\alpha}\right)^{T} \underline{C\Gamma^{\mu}\mathcal{D}_{\mu}\chi}_{\gamma\delta}{}^{\beta]} + 2\left(\underline{\eta}^{[\alpha}\right)^{T} \underline{C\chi}_{\gamma\delta}{}^{\beta]} - (\text{trace}),\\
&\delta \underline{\chi}^{\alpha\beta}{}_{\gamma} = \frac{5}{32}\, \underline{\Gamma}^{\underline{abc}}\underline{\Gamma}^{\underline{\mu}}\underline{\epsilon}_{\gamma}\, \underline{\mathcal{D}}_{\underline{\mu}}\underline{T}^{\alpha\beta}{}_{\underline{abc}} + \frac{15}{32}\, \underline{\Gamma}^{\underline{\mu\nu}}\underline{\epsilon}^{[\alpha}\, \underline{R}_{\underline{\mu\nu}\gamma}{}^{\beta]} - \frac{1}{4}D^{\alpha\beta}{}_{\gamma\delta}\, \underline{\epsilon}^{\delta}\\
&\qquad + \frac{5}{8}\, \underline{\Gamma}^{\underline{abc}}\underline{\eta}_{\gamma}\, \underline{T}^{\alpha\beta}{}_{\underline{abc}} - (\text{trace}),
\end{aligned} \tag{3}$$

with (trace) denoting necessary terms to give the same irreducible representations of the R-symmetry group as the fields on the left hand sides. The parameter $\underline{\epsilon}^{\alpha}$ of a supersymmetry transformation and $\underline{\eta}^{\alpha}$ of a superconformal transformation are symplectic Majorana–Weyl spinors of positive and negative chirality, respectively;

$$\underline{\Gamma}^7 \underline{\epsilon}^{\alpha} = \underline{\epsilon}^{\alpha}, \qquad \underline{\Gamma}^7 \underline{\eta}^{\alpha} = -\underline{\eta}^{\alpha}.$$

The operation T denotes transpose, and so $\left(\underline{\epsilon}^{\alpha}\right)^T$ and $\left(\underline{\eta}^{\alpha}\right)^T$ are the transposes of $\underline{\epsilon}^{\alpha}$ and $\underline{\eta}^{\alpha}$, respectively. The curvature $\underline{R}^{\alpha}{}_{\underline{ab}}(Q)$ is the field strength of the supersymmetry gauge field (gravitini) $\underline{\psi}^{\alpha}{}_{\underline{\mu}}$, whose exact form can be seen in [25], but it will not be necessary in this paper.

Here the covariant derivatives of $\underline{\epsilon}^{\alpha}$ and $\underline{T}^{\alpha\beta}{}_{\underline{abc}}$ are given by

$$\underline{\mathcal{D}}_{\underline{\mu}} \underline{\epsilon}^{\alpha} = \partial_{\underline{\mu}} \underline{\epsilon}^{\alpha} + \frac{1}{2} \underline{b}_{\underline{\mu}} \underline{\epsilon}^{\alpha} + \frac{1}{4} \left(\underline{\Omega}_{\underline{\mu}}\right)^{\underline{ab}} \underline{\Gamma}_{\underline{ab}} \underline{\epsilon}^{\alpha} - \frac{1}{4} \left(\underline{V}_{\underline{\mu}}\right)^{IJ} \left(\underline{\rho}_{IJ}\right)^{\alpha}{}_{\beta} \underline{\epsilon}^{\beta},$$

$$\underline{\mathcal{D}}_{\underline{\mu}} \underline{T}^{\alpha\beta}{}_{\underline{abc}} = \partial_{\underline{\mu}} \underline{T}^{\alpha\beta}{}_{\underline{abc}} + \left(\underline{\Omega}_{\underline{\mu}}\right)_{[\underline{a}}{}^{\underline{d}} \underline{T}^{\alpha\beta}{}_{\underline{bc}]\underline{d}} - \underline{b}_{\underline{\mu}} \underline{T}^{\alpha\beta}{}_{\underline{abc}} + \frac{1}{4} \underline{V}_{\underline{\mu}}{}^{IJ} \left(\rho_{IJ}\right)^{[\alpha}{}_{\gamma} \underline{T}^{\beta]\gamma}{}_{\underline{abc}}.$$

Here, the field strength of the R-symmetry gauge field $\underline{V}_{\underline{\mu}}{}^{IJ}$ is given by

$$\begin{aligned} \underline{R}_{\underline{\mu\nu}}{}^{\alpha}{}_{\beta} &= \frac{1}{2} \underline{R}_{\underline{\mu\nu}}{}^{IJ} \left(\rho_{IJ}\right)^{\alpha}{}_{\beta} \\ &= \frac{1}{2} \left[\partial_{\underline{\mu}} \underline{V}_{\underline{\nu}}{}^{IJ} - \partial_{\underline{\nu}} \underline{V}_{\underline{\mu}}{}^{IJ} - \underline{V}_{\underline{\mu}}{}^{I}{}_{K} \underline{V}_{\underline{\nu}}{}^{KJ} + \underline{V}_{\underline{\nu}}{}^{I}{}_{K} \underline{V}_{\underline{\mu}}{}^{KJ} \right] \left(\rho_{IJ}\right)^{\alpha}{}_{\beta}. \end{aligned}$$

2.2. Temporally dimensional reduction of the Weyl multiplet

In this subsection, the dimensional reduction of the Weyl multiplet along the time direction will be considered in the same way as the dimensional reduction along one spatial direction was performed in [24], where the strategy in [27] was followed.

For the usual ansatz for the metric

$$ds_6^2 = -\frac{1}{\alpha^2} (dt + C)^2 + ds_5^2 = -\underline{E}^0 \underline{E}^0 + ds_5^2 = \sum_{\underline{a}=0}^{5} \underline{E}^{\underline{a}} \underline{E}^{\underline{a}},$$

where

$$\underline{E}^0 = \frac{1}{\alpha} (dt + C), \qquad ds_5^2 = \sum_{a=1}^{5} \underline{E}^a \underline{E}^a,$$

as a gauge-fixing condition, the six-dimensional coframe $\underline{E}^{\underline{a}}{}_{\underline{\mu}}$ ($\underline{\mu} = t, 1, \cdots, 5$; $\underline{a} = 0, 1, 2, \cdots, 5$) can be taken by a local Lorentz transformation to be

$$\left(\underline{E}^{\underline{a}}{}_{\underline{\mu}}\right) = \begin{pmatrix} e^0{}_t & e^0{}_\mu \\ e^a{}_t & e^a{}_\mu \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & \alpha^{-1} C_\mu \\ 0 & e^a{}_\mu \end{pmatrix},$$

where $\mu = 1, 2, \cdots, 5$; $a = 1, 2, \cdots, 5$. Here, α is a scalar field (a.k.a. dilaton), which is sometimes denoted by $\sim \exp(-\varphi)$, but we will follow [27,24] to denote it by α. Therefore, one can see that

$$\underline{E}^0 = \frac{1}{\alpha} (dt + C), \qquad \underline{E}^a = e^a = e^a{}_\mu dx^\mu.$$

For the gauge field $C = C_\mu dx^\mu$, we define the field strength

$$G = dC = \frac{1}{2} G_{\mu\nu} dx^\mu \wedge dx^\nu .$$

Since the six-dimensional coframe $\underline{\Theta}^{\underline{\mu}}{}_{\underline{a}}$ is inverse to the sechsbein $\underline{E}^{\underline{a}}{}_{\underline{\mu}}$, it takes the form

$$\left(\underline{\Theta}^{\underline{\mu}}{}_{\underline{a}}\right) = \begin{pmatrix} \theta^t{}_0 & \theta^t{}_a \\ \theta^\mu{}_0 & \theta^\mu{}_a \end{pmatrix} = \begin{pmatrix} \alpha & -C_a \\ 0 & \theta^\mu{}_a \end{pmatrix},$$

under the gauge-fixing condition, with $C_a = \theta^\nu{}_a C_\nu$, where the funfbein $e^a{}_\mu$ and the five-dimensional coframe $\theta^\mu{}_b$ satisfy

$$e^a{}_\mu \theta^\mu{}_b = \delta^a{}_b, \qquad \theta^\mu{}_a e^a{}_\nu = \delta^\mu{}_\nu .$$

One then finds the Levi-Civita spin connection $\underline{\omega}^{\underline{ab}} = \underline{\omega}_{\underline{c}}{}^{\underline{ab}} \underline{e}^{\underline{c}}$ satisfying $d\underline{e}^{\underline{a}} + \underline{\omega}^{\underline{a}}{}_{\underline{b}} \wedge \underline{e}^{\underline{b}} = 0$,

$$\left(\underline{\omega}_0\right)^0{}_a = -\frac{1}{\alpha}\theta^\mu{}_a\, \partial_\mu \alpha, \qquad \left(\underline{\omega}_0\right)_{ab} = \left(\underline{\omega}_a\right)_{0b} = \frac{1}{2\alpha} G_{ab}, \qquad \left(\underline{\omega}_a\right)^{bc} = (\omega_a)^{bc},$$

with the five-dimensional Levi-Civita spin connection $\omega^{ab} = (\omega_c)^{ab}\, e^c$ satisfying $de^a + \omega^a{}_b \wedge e^b = 0$, and $G_{ab} = \theta^\mu{}_a \theta^\nu{}_b G_{\mu\nu}$.

As in [27,24], we will continue the partial gauge-fixing by using the conformal supersymmetry transformation S^α to set $\underline{\psi}^\alpha{}_{\underline{0}} = 0$; the special conformal transformations K_0 to set $\underline{b}_{\underline{0}} = 0$, and K_a to

$$\underline{b}_\mu = \frac{1}{\alpha}\partial_\mu \alpha, \qquad (\mu = 1, 2, \cdots, 5).$$

The latter condition makes the dilaton field α covariant constant [27];

$$\mathcal{D}_\mu \alpha = \partial_\mu \alpha - b_\mu \alpha = 0,$$

which will be convenient for the calculations below.

The partial gauge fixing conditions are summarized as

$$\boxed{\underline{E}^{\underline{a}}{}_t = 0, \qquad \underline{\psi}^\alpha{}_0 = 0, \qquad \underline{b}_0 = 0, \qquad \underline{b}_\mu = \alpha^{-1}\partial_\mu \alpha \quad (\mu = 1, 2, \cdots, 5).} \tag{4}$$

We will use b_μ as shorthand for $\alpha^{-1}\partial_\mu \alpha$ and $b_a = \theta_a{}^\mu b_\mu$.

Therefore, under the gauge fixing condition, one has the dependent gauge field $\underline{\Omega}_{\underline{\mu}}{}^{\underline{ab}}$ in (1)

$$\left(\underline{\Omega}_t\right)^0{}_a = 0, \qquad \left(\underline{\Omega}_t\right)_{ab} = \frac{1}{\alpha^2} G_{ab}, \qquad \left(\underline{\Omega}_\mu\right)^0{}_a = -\frac{1}{2\alpha} G_{\mu a},$$
$$\left(\underline{\Omega}_\mu\right)_{ab} = \left(\omega_\mu\right)_{ab} + \frac{1}{2\alpha^2} C_\mu G_{ab} + \left(e_{a\mu}\theta^\nu{}_b - e_{b\mu}\theta^\nu{}_a\right)\frac{1}{\alpha}\partial_\nu \alpha .$$

Among them, after the dimensional reduction, the component

$$\left(\underline{\Omega}_c\right)_{ab} = \underline{\Theta}^{\underline{\mu}}{}_c \left(\underline{\Omega}_{\underline{\mu}}\right)_{ab} = (\omega_c)_{ab} + \delta_{ac}\, b_b - \delta_{bc}\, b_a \equiv \theta^\mu{}_c \left(\Omega_\mu\right)_{ab}$$

often appears in the covariant derivatives, and we refer to it as $(\Omega_c)_{ab}$.

The auxiliary fields $\underline{V}_{\underline{a}}{}^{IJ}$, $\underline{T}^{\alpha\beta}{}_{\underline{abc}}$ are decomposed into five-dimensional fields S^{IJ}, $V_a{}^{IJ}$, $t^I{}_{ab}$ by

$$\underline{V}_{\underline{a}}{}^{IJ} = \begin{cases} \underline{V}_0{}^{IJ} \equiv S^{IJ}, \\ \underline{V}_a{}^{IJ} \equiv A_a{}^{IJ}, \end{cases} \qquad \underline{T}^{\alpha\beta}{}_{\underline{abc}} = \begin{cases} \underline{T}^{\alpha\beta}{}_{0ab} \equiv t^I{}_{ab}\left(\rho_I\Omega^{-1}\right)^{\alpha\beta}, \\ \underline{T}^{\alpha\beta}{}_{abc} = -(1/2)\varepsilon_{abc}{}^{de}\underline{T}^{\alpha\beta}{}_{0de}, \end{cases}$$

with $\varepsilon_{12345} = \varepsilon^{12345} = 1$. Note that the gauge field $A_\mu{}^{IJ}$ is given by

$$A_\mu{}^{IJ} = e^a{}_\mu A_a{}^{IJ} = e^a{}_\mu \underline{\Theta}^{\underline{\nu}}{}_a \underline{V}_{\underline{\nu}}{}^{IJ} = \underline{V}_\mu{}^{IJ} - C_\mu \underline{V}_t{}^{IJ} = \underline{V}_\mu{}^{IJ} - \frac{1}{\alpha} C_\mu S^{IJ}.$$

Let us remove the underline __ from $\underline{M}^{\alpha\beta}{}_{\gamma\delta}$ to denote its reduced one as $M^{\alpha\beta}{}_{\gamma\delta}$. It is sometimes convenient to replace the spinor indices α, β of $M^{\alpha\beta}{}_{\gamma\delta}$ by the vector indices I, J as

$$M^{\alpha\beta}{}_{\gamma\delta} = -M^{IJ}\left(\rho_I\Omega^{-1}\right)^{\alpha\beta}(\Omega\rho_J)_{\gamma\delta}.$$

The field M^{IJ} is in the representation **14** of the Spin$(5)_R$ group and enjoys the symmetry properties

$$M^{IJ} = M^{JI}, \qquad \delta_{IJ} M^{IJ} = 0.$$

The time component of the gravitini is set to zero by the gauge fixing condition (4); $\underline{\psi}^\alpha{}_t = 0$, and we will denote the remaining components $\underline{\psi}^\alpha{}_\mu$ $(\mu = 1, \cdots, 5)$ simply as

$$\underline{\psi}^\alpha{}_\mu = \begin{pmatrix} \psi^\alpha{}_\mu \\ 0 \end{pmatrix},$$

since it is of positive chirality, and our convention of the chirality is found in Appendix D.

Since the auxiliary spinor $\underline{\chi}^{\alpha\beta}{}_\gamma$ is also of positive chirality, we will take

$$\underline{\chi}^{\alpha\beta}{}_\gamma = \frac{15}{16}\begin{pmatrix} \chi^{\alpha\beta}{}_\gamma \\ 0 \end{pmatrix},$$

with the convenient coefficient 15/16 in [24].

The parameters $\underline{\epsilon}^\alpha$ and $\underline{\eta}^\alpha$ of supersymmetry and conformal supersymmetry transformations are of positive and negative chirality, respectively, and we will take

$$\underline{\epsilon}^\alpha = \begin{pmatrix} \epsilon^\alpha \\ 0 \end{pmatrix}, \qquad \underline{\eta}^\alpha = \begin{pmatrix} 0 \\ \eta^\alpha \end{pmatrix}.$$

The gauge fixing condition (4) is changed under the supersymmetry (Q-) transformation (3). In particular, the zeroth component of the gravitino transforms under the supersymmetry (Q) and the conformal supersymmetry (S) as

$$\delta\underline{\psi}^\alpha{}_0 = \frac{1}{8\alpha} G_{ab}\underline{\Gamma}^{ab}\underline{\epsilon}^\alpha - \frac{1}{4} S_{IJ}\left(\rho^{IJ}\right)^\alpha{}_\beta\underline{\epsilon}^\beta + \frac{1}{4} t^{\alpha\beta}{}_{ab}\underline{\Gamma}^{ab}\underline{\epsilon}_\beta + \underline{\Gamma}_0\underline{\eta}^\alpha, \tag{5}$$

under the gauge fixing condition (4). However, combining the supersymmetry (Q-) and the conformal supersymmetry (S-) transformations, one can find that one linear combination of them leaves the condition $\underline{\psi}^\alpha{}_0 = 0$ unchanged. For any $\underline{\epsilon}^\alpha$, one can see that the conformal supersymmetry transformation with the parameter

$$\eta^\alpha = \frac{1}{8\alpha} G_{ab}\gamma^{ab}\epsilon^\alpha - \frac{1}{4} S_{IJ}\left(\rho^{IJ}\right)^\alpha{}_\beta\epsilon^\beta + \frac{1}{4} t^{\alpha\beta}{}_{ab}\gamma^{ab}\epsilon_\beta, \tag{6}$$

compensates for the deviation (5) from the gauge fixing condition on the gravitini.

Among the other gauge fixing conditions in (4), the condition $\underline{E}^{\underline{a}}{}_t = 0$ remains unchanged under the supersymmetry (Q-) and the conformal supersymmetry (S-) transformations. But, the remaining gauge fixing conditions $\underline{b}_0 = 0$ and $\underline{b}_\mu = \alpha^{-1}\partial_\mu\alpha$ are changed under those transformations. However, the deviations can be canceled by the special conformal (K-) transformations with appropriate parameters $\Lambda_{K\underline{a}}$. Note here that $\underline{E}^{\underline{a}}{}_t$ and $\underline{\psi}^\alpha{}_0$ are left invariant under the special conformal (K-) transformations. Thus, one may define a supersymmetry transformation in the reduced five-dimensional theory as the linear combination of supersymmetry (Q-), conformal supersymmetry (S-), and special conformal (K-) transformations.

Following the ideas in [23], we are seeking for supersymmetric backgrounds of the reduced theory to obtain supersymmetric compactifications of the $\mathcal{N} = 2$ supersymmetric Yang–Mills theory in five dimensions. Since we would like to consider bosonic backgrounds, we will turn off background spinor fields, and we will find the supersymmetric bosonic backgrounds leaving the spinor fields $\psi^\alpha{}_\mu$, $\chi^{\alpha\beta}{}_\gamma$ unchanged under some of supersymmetry transformations in the reduced theory.

From the supersymmetry transformation of the gravitini

$$\begin{aligned}\delta_\epsilon\psi^\alpha{}_a = \theta^\mu{}_a\,\delta\psi^\alpha{}_\mu = \mathcal{D}_a\epsilon^\alpha - \frac{1}{4}S^{IJ}(\rho_{IJ})^\alpha{}_\beta\gamma_a\epsilon^\beta + \frac{1}{2\alpha}G_{ab}\gamma^b\epsilon^\alpha + \frac{1}{8\alpha}G_{bc}\gamma_a{}^{bc}\epsilon^\alpha \\ - \frac{1}{2}t^I{}_{bc}(\rho_I)^\alpha{}_\beta\gamma_a{}^{bc}\epsilon^\beta,\end{aligned}$$

with the covariant derivative of the supersymmetry parameter

$$\mathcal{D}_a\epsilon^\alpha = \theta^\mu{}_a\mathcal{D}_\mu\epsilon^\alpha = \theta^\mu{}_a\partial_\mu\epsilon^\alpha + \frac{1}{2}b_a\epsilon^\alpha + \frac{1}{4}(\Omega_a)^{bc}\gamma_{bc}\epsilon^\alpha - \frac{1}{4}A_a{}^{IJ}(\rho_{IJ})^\alpha{}_\beta\epsilon^\beta,$$

one can see that the supersymmetric bosonic backgrounds should obey

$$\boxed{\mathcal{D}_a\epsilon^\alpha = \frac{1}{4}S^{IJ}(\rho_{IJ})^\alpha{}_\beta\gamma_a\epsilon^\beta - \frac{1}{2\alpha}G_{ab}\gamma^b\epsilon^\alpha - \frac{1}{8\alpha}G_{bc}\gamma_a{}^{bc}\epsilon^\alpha + \frac{1}{2}t^I{}_{bc}(\rho_I)^\alpha{}_\beta\gamma_a{}^{bc}\epsilon^\beta.} \tag{7}$$

Under a supersymmetry transformation, the auxiliary spinor $\chi^{\alpha\beta}{}_\gamma$ transforms as

$$\begin{aligned}&\delta_\epsilon\chi^{\alpha\beta}{}_\gamma \\ &= -2\cdot\frac{1}{4!}\varepsilon^{abcde}\mathcal{D}^f t^{\alpha\beta}{}_{fa}\gamma_{bcde}\epsilon_\gamma + \frac{1}{4!}\varepsilon^{abcde}\mathcal{D}_a S_\gamma{}^{[\alpha}\gamma_{bcde}\epsilon^{\beta]} - t^{\alpha\beta}{}_{ab}t_{\gamma\delta cd}\gamma^{abcd}\epsilon^\delta \\ &\quad - \frac{1}{2}\varepsilon^{abcde}\mathcal{D}_a t^{\alpha\beta}{}_{bc}\gamma_{de}\epsilon_\gamma + 2\frac{1}{\alpha}G_a{}^c t^{\alpha\beta}{}_{bc}\gamma^{ab}\epsilon_\gamma - \frac{1}{2\alpha}G^{ab}S_\gamma{}^{[\alpha}\gamma_{ab}\epsilon^{\beta]} \\ &\quad - \frac{1}{2}S^{[\alpha}{}_\delta t^{\beta]\delta}{}_{ab}\gamma^{ab}\epsilon_\gamma - 2\,t^{\alpha\beta}{}_{ab}S_{\gamma\delta}\gamma^{ab}\epsilon^\delta + 4t^{\alpha\beta}{}_{ac}t_{\gamma\delta b}{}^c\gamma^{ab}\epsilon^\delta - \frac{1}{2}F^{ab}{}_\gamma{}^{[\alpha}\gamma_{ab}\epsilon^{\beta]} \\ &\quad - \frac{4}{15}M^{\alpha\beta}{}_{\gamma\delta}\epsilon^\delta - \frac{1}{\alpha}G^{ab}t^{\alpha\beta}{}_{ab}\epsilon_\gamma + 2t^{\alpha\beta}{}_{ab}t_{\gamma\delta}{}^{ab}\epsilon^\delta + \cdots,\end{aligned} \tag{8}$$

with $t^{\alpha\beta}{}_{ab} = t^I{}_{ab}(\rho_I\Omega^{-1})^{\alpha\beta}$, and $S^\alpha{}_\beta = (1/2)S^{IJ}(\rho_{IJ})^\alpha{}_\beta$, where the ellipse $\cdots$ denotes the necessary terms[5] to leave the right hand side in the representation **16** of the $\mathrm{Spin}(5)_R$ symmetry, since $\chi^{\alpha\beta}{}_\gamma$ is in the representation **16**. Here, the two covariant derivatives are given by

[5] In [24,25], they are denoted as (trace).

$$\mathcal{D}_\mu t^{\alpha\beta}{}_{ab} = \partial_\mu t^{\alpha\beta}{}_{ab} + \left(\Omega_\mu\right)_a{}^c t^{\alpha\beta}{}_{cb} + \left(\Omega_\mu\right)_b{}^c t^{\alpha\beta}{}_{ac} - b_\mu t^{\alpha\beta}{}_{ab} - \frac{1}{2} A_\mu{}^\alpha{}_\gamma t^{\gamma\beta}{}_{ab}$$
$$- \frac{1}{2} A_\mu{}^\beta{}_\gamma t^{\alpha\gamma}{}_{ab},$$
$$\mathcal{D}_\mu S^{\alpha\beta} = \partial_\mu S^{\alpha\beta} - b_\mu S^{\alpha\beta} - \frac{1}{2} A_\mu{}^\alpha{}_\gamma S^{\gamma\beta} - \frac{1}{2} A_\mu{}^\beta{}_\gamma S^{\alpha\gamma},$$

with $A_\mu{}^\alpha{}_\beta = (1/2) A_\mu{}^{IJ} (\rho_{IJ})^\alpha{}_\beta$, whose curvature tensor $F_{\mu\nu}{}^\alpha{}_\beta = (1/2) F_{\mu\nu}{}^{IJ} (\rho_{IJ})^\alpha{}_\beta$ is defined by

$$F_{\mu\nu}{}^I{}_J = \partial_\mu A_\nu{}^I{}_J - \partial_\nu A_\mu{}^I{}_J - A_\mu{}^I{}_K A_\nu{}^K{}_J + A_\nu{}^I{}_K A_\mu{}^K{}_J.$$

Therefore, the other condition for the supersymmetric backgrounds is that the right hand side of (8) should vanish. The explicit form (92) of the supersymmetry condition is given in Appendix E, because the equation is very lengthy to write it here.

Thus, (7) gives the Killing spinor equation, and supersymmetric backgrounds have to allow the existence of the solutions (the Killing spinors) to the equation. One may interpret that (92) determines the background field $M^{\alpha\beta}{}_{\gamma\delta}$, which will appear in the mass term of the scalar fields in the five-dimensional $\mathcal{N} = 2$ supersymmetric theory, as will be seen below.

2.3. *The Killing vectors and the Killing spinors*

The Killing spinors ϵ^α, η^α obeying the equation (7) form the bilinear

$$\xi^a = \left(\eta^\alpha\right)^T C\gamma^a \epsilon^\beta \Omega_{\alpha\beta} \equiv \bar{\eta} \cdot \gamma^a \epsilon,$$

and its covariant derivative

$$\mathcal{D}_\mu \xi^a \equiv \partial_\mu \xi^a + b_\mu \xi^a + \Omega_\mu{}^a{}_b \xi^b = \left[\left(\mathcal{D}_\mu \eta^\alpha\right)^T \cdot C\gamma^a \epsilon^\beta + \left(\eta^\alpha\right)^T C\gamma^a \mathcal{D}_\mu \epsilon^\beta \right] \Omega_{\alpha\beta}$$
$$= -\frac{1}{2} S_{IJ} \left(\bar{\eta} \cdot \rho^{IJ} \gamma_\mu{}^a \epsilon\right) - \frac{1}{\alpha} G_\mu{}^a \left(\bar{\eta} \cdot \epsilon\right)$$
$$+ \varepsilon_\mu{}^{abcd} \left[\frac{1}{4\alpha} G_{bc} \left(\bar{\eta} \cdot \gamma_d \epsilon\right) - t^I{}_{bc} \left(\bar{\eta} \cdot \rho_I \gamma_d \epsilon\right) \right],$$

satisfies $\mathcal{D}_a \xi_b + \mathcal{D}_b \xi_a = 0$. See Appendix D for the notations for the bilinears $\left(\bar{\eta} \cdot \rho_{I_1 \cdots I_n} \gamma^{a_1 \cdots a_m} \epsilon\right)$.

The vector field ξ^a obeys the conformal Killing vector equation

$$\nabla_a \xi_b + \nabla_b \xi_a = \frac{2}{5} \eta_{ab} \left(\nabla_c \xi^c\right), \tag{9}$$

with the covariant derivative $\nabla_\mu \xi^a \equiv \partial_\mu \xi^a + \omega_\mu{}^a{}_b \xi^b$, which is related to the previous covariant derivative as

$$\mathcal{D}_a \xi_b = \nabla_a \xi_b + \eta_{ab} \left(b_c \xi^c\right) + b_{[a} \xi_{b]}.$$

In fact, the equation $\mathcal{D}_a \xi_b + \mathcal{D}_b \xi_a = 0$ leads to

$$\nabla_a \xi_b + \nabla_b \xi_a = -2\eta_{ab} \left(b_c \xi^c\right),$$

which gives the conformal Killing vector equation (9).

Table 3
The tensor multiplet in the six-dimensional supergravity.

Tensor multiplet	Symmetries	$\mathrm{Spin}(5)_R$	Weight
Bosonic fields			
$\underline{B_{\mu\nu}}$	$\underline{B_{\mu\nu}} = -\underline{B_{\nu\mu}}$,	**1**	0
$\underline{\phi}^{\alpha\beta}$	$\underline{\phi}^{\alpha\beta} = -\underline{\phi}^{\beta\alpha}$, $\Omega_{\alpha\beta}\underline{\phi}^{\alpha\beta} = 0$,	**5**	2
Fermionic field			
$\underline{\chi}^{\alpha}$	$\underline{\Gamma}^7\underline{\chi}^{\alpha} = -\underline{\chi}^{\alpha}$,	**4**	5/2

3. Tensor multiplet in the supergravity theory

To the conformal supergravity, tensor multiplets can be added as matters, and after the dimensional reduction, they give rise to $\mathcal{N} = 2$ gauge multiplets in five dimensions. It therefore yields a five-dimensional $\mathcal{N} = 2$ supersymmetric Abelian theory in the supergravity background. It is the topic of this section.

A tensor multiplet $(\underline{B_{\mu\nu}}, \underline{\phi}^{\alpha\beta}, \underline{\chi}^{\alpha})$ of the $\mathcal{N} = (2,0)$ supergravity is listed in Table 3, and the field strength of the two-form $\underline{B}$ is given by

$$\underline{H} = \frac{1}{3!}\underline{H_{abc}}\,\underline{E}^a \wedge \underline{E}^b \wedge \underline{E}^c = d\underline{B}.$$

The transformation rules and the equations of motion of the tensor multiplet were derived in [25].

Under a fermionic transformation (supersymmetry+ conformal supersymmetry), the tensor multiplet transforms as

$$\begin{aligned}
\delta\underline{B_{\mu\nu}} &= i\left(\underline{\epsilon}^{\alpha}\right)^{\dagger}\underline{\Gamma}^0\underline{\Gamma_{\mu\nu}}\,\underline{\chi}^{\alpha}, \\
\delta\underline{\phi}^{\alpha\beta} &= -2i\left(\underline{\epsilon}_{\alpha}\right)^{\dagger}\underline{\Gamma}^0\underline{\chi}^{\beta} + 2i\left(\underline{\epsilon}_{\beta}\right)^{\dagger}\underline{\Gamma}^0\underline{\chi}^{\alpha} - i\Omega^{\alpha\beta}\left(\underline{\epsilon}^{\gamma}\right)^{\dagger}\underline{\Gamma}^0\underline{\chi}^{\gamma}, \\
\delta\underline{\chi}^{\alpha} &= \frac{1}{8}\cdot\frac{1}{3!}\underline{H}^{+}{}_{\underline{\mu\nu\rho}}\underline{\Gamma^{\mu\nu\rho}}\underline{\epsilon}^{\alpha} + \frac{1}{4}\underline{\mathcal{D}_{\mu}}\underline{\phi}^{\alpha\beta}\,\underline{\Gamma^{\mu}}\underline{\epsilon}_{\beta} - \underline{\phi}^{\alpha\beta}\underline{\eta}_{\beta},
\end{aligned} \tag{10}$$

where $\underline{H}^{\pm} = (1/2)\left(\underline{H} \pm \underline{*}\underline{H}\right)$. (See the definition of the Hodge dual $\underline{*}$ in Appendix A.) The covariant derivative of the scalar field $\underline{\phi}^{\alpha\beta}$ is

$$\underline{\mathcal{D}_{\mu}}\underline{\phi}^{\alpha\beta} = \partial_{\underline{\mu}}\underline{\phi}^{\alpha\beta} - 2\underline{b_{\mu}}\underline{\phi}^{\alpha\beta} - \frac{1}{4}\underline{V_{\mu}}^{IJ}\left(\rho_{IJ}\right)^{\alpha}{}_{\gamma}\underline{\phi}^{\gamma\beta} - \frac{1}{4}\underline{V_{\mu}}^{IJ}\left(\rho_{IJ}\right)^{\beta}{}_{\gamma}\underline{\phi}^{\alpha\gamma}.$$

The equations of motion of the tensor multiplet are given by

$$\underline{H}^{-} - \frac{1}{2}\underline{\phi}_{\alpha\beta}\underline{T}^{\alpha\beta} = 0, \tag{11}$$

$$\underline{\mathcal{D}^a}\underline{\mathcal{D}_a}\underline{\phi}_{\alpha\beta} - \frac{1}{15}\underline{M}_{\alpha\beta}{}^{\gamma\delta}\underline{\phi}_{\gamma\delta} + \frac{1}{3}\underline{H}^{+}{}_{\underline{abc}}\underline{T}_{\alpha\beta}{}^{\underline{abc}} = 0, \tag{12}$$

$$\underline{\Gamma^a}\underline{\mathcal{D}_a}\underline{\chi}^{\alpha} - \frac{1}{12}\underline{T}^{\alpha\beta}{}_{\underline{abc}}\underline{\Gamma^{abc}}\underline{\chi}_{\beta} = 0, \tag{13}$$

with the covariant derivatives

$$\underline{\mathcal{D}}^{\underline{a}}\underline{\mathcal{D}}_{\underline{a}}\underline{\phi}^{\alpha\beta} = \underline{\Theta}^{\underline{\mu}}{}_{\underline{a}}\left(\partial_{\underline{\mu}} - 3\underline{b}_{\underline{\mu}}\right)\underline{\mathcal{D}}^{\underline{a}}\underline{\phi}^{\alpha\beta} + \left(\underline{\Omega}_{\underline{a}}\right)^{\underline{ab}}\underline{\mathcal{D}}_{\underline{b}}\underline{\phi}^{\alpha\beta}$$
$$-\frac{1}{4}\underline{V}_{\underline{a}}{}^{IJ}\left(\rho_{IJ}\right)^{\alpha}{}_{\gamma}\underline{\mathcal{D}}^{\underline{a}}\underline{\phi}^{\gamma\beta} - \frac{1}{4}\underline{V}_{\underline{a}}{}^{IJ}\left(\rho_{IJ}\right)^{\beta}{}_{\gamma}\underline{\mathcal{D}}^{\underline{a}}\underline{\phi}^{\alpha\gamma} - \frac{1}{5}\underline{R}\left(\underline{\Omega}\right)\underline{\phi}^{\alpha\beta},$$
$$\underline{\mathcal{D}}_{\underline{\mu}}\underline{\chi}^{\alpha} = \left(\partial_{\underline{\mu}} - \frac{5}{2}\underline{b}_{\underline{\mu}} + \frac{1}{4}\left(\underline{\Omega}_{\underline{\mu}}\right)^{\underline{ab}}\underline{\Gamma}_{\underline{ab}}\right)\underline{\chi}^{\alpha} - \frac{1}{4}\underline{V}_{\underline{\mu}}{}^{IJ}\left(\rho_{IJ}\right)^{\alpha}{}_{\beta}\underline{\chi}^{\beta}.$$

3.1. Dimensional reduction of the tensor multiplet

From the six-dimensional Minkowski space to the five-dimensional Euclidean space, the dimensional reduction of the tensor multiplet gives rise to the five-dimensional abelian gauge multiplet $(A_\mu, \phi^I, \chi^\alpha)$,

$$\begin{aligned}
\underline{B}_{\underline{ab}} &\longrightarrow \underline{B}_{a0} \equiv \alpha\, A_a = \alpha\,\theta^{\mu}{}_{a}A_\mu \qquad (a = 1, 2, \cdots, 5)\\
\underline{\phi}^{\alpha\beta} &\longrightarrow \alpha\,\phi^{\alpha\beta} = \alpha\,\phi^I\left(\rho_I\Omega^{-1}\right)^{\alpha\beta},\\
\underline{\chi}^{\alpha} &\longrightarrow \frac{\alpha}{4}\begin{pmatrix}0\\ \chi^\alpha\end{pmatrix}.
\end{aligned}$$

The remaining components $\underline{B}_{ab}$ are described by A_μ and ϕ^I through the equation (11) of motion of $\underline{H}^-$, which is reduced to

$$\underline{H}^{-}{}_{ab0} = \frac{1}{2}\underline{\phi}_{\alpha\beta}\,\underline{T}^{\alpha\beta}{}_{ab0} \quad\longrightarrow\quad 2\alpha\,\phi_I\,t^I{}_{ab}.$$

Since the components $\underline{H}_{ab0}$ reduce to the field strength $F_{\mu\nu}$ of A_μ,

$$\begin{aligned}
&\underline{H}_{\mu\nu t} = \partial_\mu B_{\nu t} + \partial_\nu B_{t\mu} + \partial_t B_{\mu\nu} \longrightarrow \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu},\\
&\underline{H}_{ab0} \to \alpha\theta^{\mu}{}_{a}\theta^{\nu}{}_{b}F_{\mu\nu} = \alpha F_{ab},
\end{aligned}$$

one can see that the components $\underline{H}_{abc}$ are reduced as

$$\begin{aligned}
\underline{H}_{abc} &= \underline{H}^{+}{}_{abc} + \underline{H}^{-}{}_{abc} = \frac{1}{2}\varepsilon_{abc}{}^{de}\left(\underline{H}^{+}{}_{de0} - \underline{H}^{-}{}_{de0}\right) = \frac{1}{2}\varepsilon_{abc}{}^{de}\left(\underline{H}_{de0} - 2\underline{H}^{-}{}_{de0}\right)\\
&\longrightarrow \quad \alpha\,\frac{1}{2}\,\varepsilon_{abc}{}^{de}\left(F_{de} - 4\,\phi_i\,t^i{}_{de}\right).
\end{aligned}$$

We have previously seen that a six-dimensional supersymmetry transformation with a transformation parameter $\underline{\epsilon}^\alpha$ combined with the superconformal transformation with $\underline{\eta}^\alpha$ in (6) is reduced to a five-dimensional supersymmetry transformation. Substituting the parameter $\underline{\eta}^\alpha$ in (6) into the fermionic transformation rules in (10) of the tensor multiplet, one can see that their reduction gives the supersymmetry transformation of the abelian gauge multiplet,

$$\begin{aligned}
&\delta_\epsilon A_\mu = -\frac{i}{4}\Omega_{\alpha\beta}\left(\epsilon^\alpha\right)^T C\gamma_\mu\chi^\alpha, \qquad \delta_\epsilon\phi^I = \frac{i}{4}\left(\Omega\rho^I\right)_{\alpha\beta}\left(\epsilon^\alpha\right)^T C\chi^\beta,\\
&\delta_\epsilon\chi^\alpha = -\frac{1}{2}F_{ab}\gamma^{ab}\epsilon^\alpha - \gamma^\mu\mathcal{D}_\mu\phi^I\left(\rho_I\right)^{\alpha}{}_{\beta}\epsilon^\beta + \frac{1}{2\alpha}G_{ab}\phi^I\left(\rho_I\right)^{\alpha}{}_{\beta}\gamma^{ab}\epsilon^\beta\\
&\qquad + S^I{}_J\phi^J\left(\rho_I\right)^{\alpha}{}_{\beta}\epsilon^\beta + \frac{1}{2}\varepsilon_{IJKLM}S^{IJ}\phi^M\left(\rho^{KL}\right)^{\alpha}{}_{\beta}\epsilon^\beta\\
&\qquad + t^I{}_{ab}\phi^J\left(\rho_{IJ}\right)^{\alpha}{}_{\beta}\gamma^{ab}\epsilon^\beta,
\end{aligned} \tag{14}$$

with the covariant derivative of ϕ^I,

$$\mathcal{D}_\mu \phi^I = \partial_\mu \phi^I - b_\mu \phi^I - A_\mu{}^I{}_J \phi^J .$$

The reduction of the external derivative of the equation (11)

$$d\left(\underline{*}\,\underline{H} + \underline{\phi}_{\alpha\beta}\,\underline{T}^{\alpha\beta}\right) = 0,$$

yields the equation of motion of the gauge field A_μ,

$$\boxed{d\left[\alpha * \left(F - 4\phi_I t^I\right)\right] + F \wedge G = 0,} \tag{15}$$

where

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \qquad t^I = \frac{1}{2} t^I{}_{\mu\nu} dx^\mu \wedge dx^\nu, \qquad G = \frac{1}{2} G_{\mu\nu} dx^\mu \wedge dx^\nu .$$

The equations (12), (13) of motion are reduced into

$$\boxed{\begin{aligned} &\gamma^\mu \mathcal{D}_\mu \chi^\alpha - \frac{1}{8\alpha} G_{ab} \gamma^{ab} \chi^\alpha - \frac{1}{4} S_{IJ} (\rho^{IJ})^\alpha{}_\beta \chi^\beta + \frac{1}{2} t^I{}_{ab} (\rho_I)^\alpha{}_\beta \gamma^{ab} \chi^\beta = 0, \\ &\mathcal{D}^a \mathcal{D}_a \phi^I - S^I{}_J\, S^J{}_K\, \phi^K - \frac{1}{5} R(\Omega)\, \phi^I - \frac{4}{15} M^I{}_J \phi^J \\ &\quad - \frac{1}{20\alpha^2} G_{ab} G^{ab} \phi^I + 4 t^I{}_{ab}\, t_J{}^{ab}\, \phi^J - 2 t^I{}_{ab}\, F^{ab} = 0, \end{aligned}} \tag{16}$$

with

$$\mathcal{D}^a \mathcal{D}_a \phi^I = \theta^{\mu a} \left[\left(\partial_\mu - 2b_\mu\right) \mathcal{D}_a \phi^I + \Omega_{\mu a}{}^b \mathcal{D}_b \phi^I - \frac{1}{4} A_\mu{}^I{}_J \mathcal{D}_a \phi^J \right],$$

where the covariant derivative of χ^α

$$\mathcal{D}_\mu \chi^\alpha = \partial_\mu \chi^\alpha - \frac{3}{2} b_\mu \chi^\alpha + \frac{1}{4} \Omega_\mu{}^{bc} \gamma_{bc} \chi^\alpha - \frac{1}{4} A_\mu{}^{IJ} \left(\rho_{IJ}\right)^\alpha{}_\beta \chi^\beta,$$

with the spin connection $\Omega_\mu{}^{ab} = \omega_\mu{}^{ab} + (e^a{}_\mu \theta^{\nu b} - e^b{}_\mu \theta^{\nu a}) b_\nu$, and the scalar curvature $R(\Omega)$ of $\Omega_\mu{}^{ab}$ is defined by $R(\Omega) = \theta^\mu{}_a \theta^\nu{}_b (\partial_{[\mu} \Omega_{\nu]}{}^{ab} + \Omega_{[\mu}{}^{ae} \Omega_{\nu]e}{}^b)$, which comes from

$$\underline{R}\left(\underline{\Omega}\right) = R(\Omega) + \frac{1}{4\alpha^2} G_{ab} G^{ab} .$$

From the equations of motion (15), (16), one obtains the bosonic part of the action of the abelian gauge multiplet

$$\begin{aligned} L_B = &-\frac{1}{2} \int \left[\alpha \left(F - 4\phi^I t_I\right) \wedge * \left(F - 4\phi^J t_J\right) + C \wedge F \wedge F \right] \\ &+ \frac{1}{2} \int dx^5 \sqrt{g}\, \alpha \left[\mathcal{D}_a \phi^I \mathcal{D}^a \phi_I + \mathcal{M}_{BIJ} \phi^I \phi^J \right], \end{aligned} \tag{17}$$

with

$$\mathcal{M}_{BIJ} = \frac{1}{5} \delta_{IJ} \left(R(\Omega) + \frac{1}{4\alpha^2} G_{ab} G^{ab} \right) + \frac{4}{15} M_{IJ} + 4 t_I{}^{ab} t_{Jab} - S_I{}^K S_{JK},$$

and the fermionic part

$$L_F = \frac{i}{8}\int dx^5\sqrt{g}\,\alpha\left(\chi^\alpha\right)^T C\Big[\gamma^a\mathcal{D}_a\chi^\beta\Omega_{\alpha\beta} - \frac{1}{8\alpha}G_{ab}\gamma^{ab}\chi^\beta\Omega_{\alpha\beta} - \frac{1}{4}S_{IJ}\chi^\beta\left(\Omega\rho^{IJ}\right)_{\alpha\beta} + \frac{1}{2}t^I{}_{ab}\gamma^{ab}\chi^\beta\left(\Omega\rho_I\right)_{\alpha\beta}\Big]. \tag{18}$$

One can verify that the total action $L = L_F + L_B$ is left invariant under the supersymmetry transformation (14). However, it is a lengthy calculation to verify the supersymmetry invariance of the action L. Although we do not intend to pause for a detailed demonstration of it, we will discuss a supersymmetry transformation of the mass term of the scalar fields ϕ^I in the action in Appendix F, which we think is one of the keys to verify the supersymmetry invariance of the action.

4. The generalization for a non-abelian gauge group

The reduced theory of the six-dimensional tensor multiplet gives rise to the abelian gauge theory in five dimensions. We will extend the abelian gauge multiplet $(A_\mu, \phi^I, \chi^\alpha)$ to the adjoint representation of a non-abelian gauge group G and replace the partial derivatives by covariant ones:

$$\partial_\mu\phi^I \quad\longrightarrow\quad \partial_\mu\phi^I + ig\left[A_\mu, \phi^I\right], \qquad \partial_\mu\chi^\alpha \quad\longrightarrow\quad \partial_\mu\chi^\alpha + ig\left[A_\mu, \chi^\alpha\right].$$

We will henceforth denote the covariant derivatives as

$$\mathcal{D}_\mu\phi^I = \partial_\mu\phi^I - b_\mu\phi^I - A_\mu{}^I{}_J\phi^J + ig\left[A_\mu, \phi^I\right],$$
$$\mathcal{D}_\mu\chi^\alpha = \partial_\mu\chi^\alpha - \frac{3}{2}b_\mu\chi^\alpha + \frac{1}{4}\Omega_\mu{}^{bc}\gamma_{bc}\chi^\alpha - \frac{1}{4}A_\mu{}^{IJ}\left(\rho_{IJ}\right)^\alpha{}_\beta\chi^\beta + ig\left[A_\mu, \chi^\alpha\right].$$

For the non-abelian extension of the supersymmetry transformations (14) and the equations of motion (15), (16), there are two conditions to be satisfied. In the flat limit where all the backgrounds go to zero, they should be reduced to the ones in the $\mathcal{N} = 2$ supersymmetric Yang–Mills theory on a flat space, and in the abelian limit $g \to 0$, the extension has to go back to (14), (15), (16). Our ansatz for the non-abelian extension of the supersymmetry transformations is

$$\boxed{\begin{aligned} &\delta_\epsilon A_\mu = -\frac{i}{4}\Omega_{\alpha\beta}\left(\epsilon^\alpha\right)^T C\gamma_\mu\chi^\alpha, \qquad \delta_\epsilon\phi^I = \frac{i}{4}\left(\Omega\rho^I\right)_{\alpha\beta}\left(\epsilon^\alpha\right)^T C\chi^\beta, \\ &\delta_\epsilon\chi^\alpha = -\frac{1}{2}F_{ab}\gamma^{ab}\epsilon^\alpha - \gamma^\mu\mathcal{D}_\mu\phi^I\left(\rho_I\right)^\alpha{}_\beta\epsilon^\beta + \frac{1}{2\alpha}G_{ab}\phi^I\left(\rho_I\right)^\alpha{}_\beta\gamma^{ab}\epsilon^\beta \\ &\qquad + S^I{}_J\phi^J\left(\rho_I\right)^\alpha{}_\beta\epsilon^\beta + \frac{1}{2}\varepsilon_{IJKLM}S^{IJ}\phi^M\left(\rho^{KL}\right)^\alpha{}_\beta\epsilon^\beta \\ &\qquad + t^I{}_{ab}\phi^J\left(\rho_{IJ}\right)^\alpha{}_\beta\gamma^{ab}\epsilon^\beta + \frac{i}{2}g\left[\phi^I, \phi^J\right]\left(\rho_{IJ}\right)^\alpha{}_\beta\epsilon^\beta, \end{aligned}} \tag{19}$$

with the field strength of the non-abelian gauge field A_μ

$$F_{ab} = \theta^\mu{}_a\theta^\nu{}_b F_{\mu\nu} = \theta^\mu{}_a\theta^\nu{}_b\Big(\partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right]\Big).$$

In the abelian gauge theory, the algebra of the supersymmetry transformations (14) is closed on-shell, and in the flat limit of the non-abelian gauge theory, it is also closed on-shell. Therefore, in order to see the closure of the algebra of the supersymmetry transformations (19), we make an ansatz for the equation of motion of the spinor χ^α,

$$\boxed{\begin{aligned}&\gamma^\mu \mathcal{D}_\mu \chi^\alpha + ig\,(\rho_I)^\alpha{}_\beta \left[\phi^I,\,\chi^\beta\right]\\ &\quad -\frac{1}{8\alpha}G_{ab}\gamma^{ab}\chi^\alpha - \frac{1}{4}S_{IJ}(\rho^{IJ})^\alpha{}_\beta\chi^\beta + \frac{1}{2}t^I{}_{ab}(\rho_I)^\alpha{}_\beta\gamma^{ab}\chi^\beta = 0.\end{aligned}} \tag{20}$$

The supersymmetry transforms of $\mathcal{D}_\mu\phi^I$ and $(F_{ab} - 2\phi_I\, t^I{}_{ab})$ may be useful to see that the algebra of the supersymmetry transformations is closed on-shell;

$$\begin{aligned}\delta_\epsilon \mathcal{D}_\mu \phi^I &= \frac{i}{4}\Big[\bar{\epsilon}\cdot\rho^I\mathcal{D}_\mu\chi + \frac{1}{8\alpha}G_{bc}(\bar{\epsilon}\cdot\rho^I\gamma_\mu{}^{bc}\chi) - \frac{1}{2\alpha}G_{\mu b}(\bar{\epsilon}\cdot\rho^I\gamma^b\chi)\\ &\qquad - \frac{1}{4}S_{KL}(\bar{\epsilon}\cdot\rho^{KL}\rho^I\gamma_\mu\chi) - \frac{1}{2}t^J{}_{bc}(\bar{\epsilon}\cdot\rho_J\rho^I\gamma_\mu{}^{bc}\chi)\Big],\\ \delta_\epsilon &\left(F_{ab} - 2\phi_I\, t^I{}_{ab}\right)\\ &= \frac{i}{4}\Big[\bar{\epsilon}\cdot\gamma_{[a}\mathcal{D}_{b]}\chi - \frac{1}{4\alpha}G_{cd}\bar{\epsilon}\cdot\left(\gamma^{cd}{}_{ab} - 3\delta^c{}_{[a}\gamma^d{}_{b]} - 4\delta^c{}_a\delta^d{}_b\right)\chi\\ &\quad + \frac{1}{2}S_{IJ}(\bar{\epsilon}\cdot\rho^{IJ}\gamma_{ab}\chi) + t^I{}_{cd}\bar{\epsilon}\cdot\rho_I\left(\gamma^{cd}{}_{ab} - \delta^c{}_{[a}\gamma^d{}_{b]} - 2\delta^c{}_a\delta^d{}_b\right)\chi\Big].\end{aligned}$$

Using the equation of motion (20) and the Killing spinor equation (7), one can verify that the algebra of the supersymmetry transformations (19) is closed on-shell.

$$\begin{aligned}\left[\delta_\epsilon,\,\delta_\eta\right]A_\mu &= \frac{i}{2}\left[F_{\mu\nu}\xi^\nu + \mathcal{D}_\mu\left(\phi^I\,(\bar{\eta}\cdot\rho_I\epsilon)\right)\right] = -\frac{i}{2}\left[\xi^\nu\partial_\nu A_\mu + \partial_\mu\xi^\nu\cdot A_\nu\right] - \mathcal{D}_\mu\Lambda_G,\\ \left[\delta_\epsilon,\,\delta_\eta\right]\phi^I &= -\frac{i}{2}\left(\xi^\mu\partial_\mu\phi^I - \xi^a b_a\phi^I\right) + ig\left[\Lambda_G,\,\phi^I\right] - \Lambda^I{}_J\phi^J,\\ \left[\delta_\epsilon,\,\delta_\eta\right]\chi^\alpha &= -\frac{i}{2}\left[\xi^\mu\partial_\mu\chi^\alpha - \frac{3}{2}\xi^a b_a\chi^\alpha\right] + ig\left[\Lambda_G,\,\chi^\alpha\right]\\ &\qquad + \frac{1}{4}\Lambda^{ab}\gamma_{ab}\chi^\alpha - \frac{1}{4}\Lambda_{IJ}\left(\rho^{IJ}\right)^\alpha{}_\beta\chi^\beta,\end{aligned} \tag{21}$$

with the Killing vector $\xi^a = (\bar{\eta}\cdot\gamma^a\epsilon)$, where the parameters are given by

$$\begin{aligned}\Lambda_{IJ} &= -\frac{i}{2}\Big[A_{aIJ}\xi^a + S_{IJ}(\bar{\eta}\cdot\epsilon) - \varepsilon_{IJKLM}S^{KL}(\bar{\eta}\cdot\rho^M\epsilon)\\ &\qquad + \frac{1}{2\alpha}G_{ab}(\bar{\eta}\cdot\rho_{IJ}\gamma^{ab}\epsilon) - t^K{}_{ab}(\bar{\eta}\cdot\rho_{IJK}\gamma^{ab}\epsilon)\Big],\\ \Lambda^{ab} &= -\frac{i}{2}\left[\mathcal{D}^a\xi^b + \xi^c\Omega_c{}^{ab}\right], \qquad \Lambda_G = -\frac{i}{2}\left[\xi^a A_a + \phi^I\,(\bar{\eta}\cdot\rho_I\epsilon)\right],\end{aligned}$$

(see Appendix D for the abbreviation $(\bar{\eta}\cdot\rho_{I_1\cdots I_n}\gamma^{a_1\cdots a_m}\epsilon)$), and the covariant derivative of $\phi^I\,(\bar{\eta}\cdot\rho_I\epsilon)$ is

$$\mathcal{D}_\mu\left(\phi^I\,(\bar{\eta}\cdot\rho_I\epsilon)\right) = \partial_\mu\left(\phi^I\,(\bar{\eta}\cdot\rho_I\epsilon)\right) + ig\left[A_\mu,\,\phi^I\,(\bar{\eta}\cdot\rho_I\epsilon)\right].$$

Since we have seen that the supersymmetry transformation (19) gives an on-shell closed algebra with the equation of motion (20), we will proceed with (19) and (20) to obtain the non-abelian extension of the action (17), (18).

A simple calculation shows that the equation of motion (20) may be derived from the fermionic part of the non-abelian action

$$S_F = \frac{i}{8}\int dx^5\sqrt{g}\,\alpha\,\mathrm{tr}\Big[\bar{\chi}\cdot\gamma^a\mathcal{D}_a\chi - \frac{1}{8\alpha}G_{ab}\bar{\chi}\cdot\gamma^{ab}\chi - \frac{1}{4}S_{IJ}\bar{\chi}\cdot\rho^{IJ}\chi \\ + \frac{1}{2}t^I{}_{ab}\bar{\chi}\cdot\rho_I\gamma^{ab}\chi + (ig)\bar{\chi}\cdot\rho_I\Big[\phi^I, \chi\Big]\Big], \tag{22}$$

where the symbol tr denotes a trace in the adjoint representation of the gauge group G.

In the abelian limit $g\to 0$, the non-abelian action should go to (17) – more precisely, the abelian action of the $|G|$ abelian gauge multiplets with $|G|$ denoting the dimension of the adjoint representation of G – and in the flat limit, we must regain the familiar non-abelian action in the $\mathcal{N}=2$ supersymmetric Yang–Mills theory. It therefore seems natural to take the ansatz

$$S_B^{(0)} = -\frac{1}{2}\int \mathrm{tr}\Big[\alpha\left(F-4\phi^I t_I\right)\wedge *\left(F-4\phi^J t_J\right) + C\wedge F\wedge F\Big] \\ + \frac{1}{2}\int dx^5\sqrt{g}\,\alpha\,\mathrm{tr}\Big[\mathcal{D}_a\phi^I\mathcal{D}^a\phi_I + \mathcal{M}_{BIJ}\phi^I\phi^J - \frac{1}{2}(ig)^2[\phi^I,\phi^J][\phi_I,\phi_J]\Big], \tag{23}$$

where

$$\mathcal{M}_{BIJ} = \frac{1}{5}\delta_{IJ}\left(R(\Omega) + \frac{1}{4\alpha^2}G_{ab}G^{ab}\right) + \frac{4}{15}M_{IJ} + 4t_I{}^{ab}t_{Jab} - S_I{}^K S_{JK}. \tag{24}$$

In order to examine the supersymmetry invariance of the sum $S_F + S_B^{(0)}$, one needs to perform a similar calculation to what is done for the abelian action L. The calculation may be painful, especially in the mass term of the scalar fields ϕ^i, of which the details is shown in Appendix F.

However, it turns out that the variation of the sum $S_F + S_B^{(0)}$ under the supersymmetry transformation (19) doesn't vanish at the order $\mathcal{O}(g)$. Therefore, in order to obtain a supersymmetric action, as discussed in [24], one needs the additional term

$$S_B^{(1)} = -\frac{1}{6}\int dx^5\sqrt{g}\,\alpha\,\mathrm{tr}\Big[(ig)\,\varepsilon^{IJKLM}S_{IJ}\phi_K\,[\phi_L,\phi_M]\Big], \tag{25}$$

to cancel the supersymmetry variation of $S_B^{(0)} + S_F$. Thus, one may see that $S = S_B + S_F = S_B^{(0)} + S_B^{(1)} + S_F$ yields a supersymmetric non-abelian action.

5. Supersymmetric backgrounds

In this section, we will discuss the supersymmetric solutions to the Killing spinor equation (7) and the condition (92) from the spinor variation $\delta\chi^{\alpha\beta}{}_{\gamma}$, which gives rise to supersymmetric backgrounds for the five-dimensional supersymmetric Yang–Mills theory.

In this paper, we will make an assumption

$$\boxed{b_\mu = 0,\quad t^i{}_{ab} = 0,\quad S_{i5} = -S_{5i} = 0,\quad A_\mu{}^{i5} = -A_\mu{}^{5i} = 0,\quad (i=1,\cdots,4)} \tag{26}$$

which is satisfied by the background in the previous papers [1,2], as will be seen below. In [1,2], we have considered the product space of a round S^3 and a Riemann surface Σ. In this paper, we are especially interested in supersymmetric backgrounds for deformed 3-spheres – a squashed and an ellipsoid S^3. We will find supersymmetric backgrounds on the product spaces of those 3-spheres and Σ, which turn out to satisfy the assumption (26).

It is convenient under the assumption (26) to decompose the supersymmetry parameter ϵ^α as

$$\rho^5\epsilon^\alpha = \epsilon^\alpha \quad \longrightarrow \quad \epsilon^\alpha = \begin{pmatrix} \epsilon^{\tilde{\alpha}} \\ 0 \end{pmatrix}, \qquad \rho^5\epsilon^\alpha = -\epsilon^\alpha \quad \longrightarrow \quad \epsilon^\alpha = \begin{pmatrix} 0 \\ \varepsilon^{\dot{\alpha}} \end{pmatrix},$$

in the representation with $\rho^5 = \mathrm{diag}.(+\mathbf{1}_2, -\mathbf{1}_2)$. While the Killing spinor equation (7) in a generic background gives a differential equation of $\epsilon^{\tilde{\alpha}}$ and $\varepsilon^{\dot{\alpha}}$ coupled to each other, the assumption (26) splits them into

$$\mathcal{D}_\mu\epsilon^{\tilde{\alpha}} = \frac{1}{4}S_{ij}\left(\sigma^{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\beta}}\,\gamma_\mu\epsilon^{\tilde{\beta}} - \frac{1}{2\alpha}G_{\mu\nu}\gamma^\nu\epsilon^{\tilde{\alpha}} - \frac{1}{8\alpha}G_{bc}\gamma_\mu{}^{bc}\epsilon^{\tilde{\alpha}} + \frac{1}{2}t_{bc}\,\gamma_\mu{}^{bc}\epsilon^{\tilde{\alpha}}, \tag{27}$$

$$\mathcal{D}_\mu\varepsilon^{\dot{\alpha}} = \frac{1}{4}S_{ij}\left(\bar{\sigma}^{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\,\gamma_\mu\varepsilon^{\dot{\beta}} - \frac{1}{2\alpha}G_{\mu\nu}\gamma^\nu\varepsilon^{\dot{\alpha}} - \frac{1}{8\alpha}G_{bc}\gamma_\mu{}^{bc}\varepsilon^{\dot{\alpha}} - \frac{1}{2}t_{bc}\,\gamma_\mu{}^{bc}\varepsilon^{\dot{\alpha}}, \tag{28}$$

with $t_{ab} \equiv t^5{}_{ab}$, where the covariant derivatives are defined by

$$\mathcal{D}_\mu\epsilon^{\tilde{\alpha}} \equiv \partial_\mu\epsilon^{\tilde{\alpha}} + \frac{1}{2}b_\mu\epsilon^{\tilde{\alpha}} + \frac{1}{4}\Omega_\mu{}^{bc}\gamma_{bc}\epsilon^{\tilde{\alpha}} - \frac{1}{4}A_\mu{}^{ij}\left(\sigma_{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\beta}}\epsilon^{\tilde{\beta}},$$

$$\mathcal{D}_\mu\varepsilon^{\dot{\alpha}} \equiv \partial_\mu\varepsilon^{\dot{\alpha}} + \frac{1}{2}b_\mu\varepsilon^{\dot{\alpha}} + \frac{1}{4}\Omega_\mu{}^{bc}\gamma_{bc}\varepsilon^{\dot{\alpha}} - \frac{1}{4}A_\mu{}^{ij}\left(\bar{\sigma}_{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\varepsilon^{\dot{\beta}}.$$

We will further make an ansatz for the Killing spinors,

$$\boxed{\varepsilon^{\dot{\alpha}=1} = \epsilon\otimes\zeta_+, \quad \varepsilon^{\dot{\alpha}=2} = C_3\epsilon^*\otimes\zeta_-; \qquad \epsilon^{\tilde{\alpha}=1} = \tilde{\epsilon}\otimes\zeta_\pm, \quad \epsilon^{\tilde{\alpha}=2} = C_3\tilde{\epsilon}^*\otimes\zeta_\mp,} \tag{29}$$

with two-dimensional spinors ϵ, $\tilde{\epsilon}$ on the S^3 and constant two-dimensional spinors

$$\zeta_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix},$$

on Σ, obeying that $\tau_2\zeta_\pm = \pm\zeta_\pm$, with the Pauli matrix τ_2. Note that they satisfy

$$\gamma_{45}\,\varepsilon^{\dot{\alpha}} = -i\,(\tau_3)^{\dot{\alpha}}{}_{\dot{\beta}}\,\varepsilon^{\dot{\beta}}, \qquad \gamma_{45}\,\epsilon^{\tilde{\alpha}} = \mp i\,(\tau_3)^{\tilde{\alpha}}{}_{\tilde{\beta}}\,\epsilon^{\tilde{\beta}}. \tag{30}$$

For later convenience, let us consider the commutation relation of the covariant derivatives acting on $\varepsilon^{\dot{\alpha}}$, which by definition gives

$$[\mathcal{D}_a, \mathcal{D}_b]\,\varepsilon^{\dot{\alpha}} = \frac{1}{4}\,R_{ab}{}^{cd}(\Omega)\gamma_{cd}\varepsilon^{\dot{\alpha}} - \frac{1}{4}\,F_{ab}{}^{ij}\left(\bar{\sigma}_{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\varepsilon^{\dot{\beta}},$$

and acting γ^{ab} on this, one obtains

$$\gamma^{ab}\mathcal{D}_a\mathcal{D}_b\,\varepsilon^{\dot{\alpha}} = \frac{1}{2}\,\gamma^{ab}\,[\mathcal{D}_a, \mathcal{D}_b]\,\varepsilon^{\dot{\alpha}} = -\frac{1}{4}\,R\,(\Omega)\,\varepsilon^{\dot{\alpha}} - \frac{1}{8}F_{ab}{}^{ij}\left(\bar{\sigma}_{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\gamma^{ab}\varepsilon^{\dot{\beta}}. \tag{31}$$

On the other hand, using the Killing spinor equation (28) twice for $\gamma^{ab}\mathcal{D}_a\mathcal{D}_b\,\varepsilon^{\dot{\alpha}}$ and equating it and the right hand side of (31), one finds that

$$
\begin{aligned}
&-\frac{1}{4}R(\Omega)\varepsilon^{\dot\alpha}-\frac{1}{8}F_{ab}{}^{ij}\left(\bar\sigma_{ij}\right)^{\dot\alpha}{}_{\dot\beta}\gamma^{ab}\varepsilon^{\dot\beta}\\
&=\left(\bar\sigma^{ij}\right)^{\dot\alpha}{}_{\dot\beta}\mathcal{D}_aS_{ij}\gamma^a\varepsilon^{\dot\beta}-\mathcal{D}_at_{bc}\gamma^{abc}\varepsilon^{\dot\alpha}-3\cdot\mathcal{D}^bt_{ba}\gamma^a\varepsilon^{\dot\alpha}-5\cdot\frac{1}{4\alpha}\mathcal{D}^bG_{ba}\gamma^a\varepsilon^{\dot\alpha}\\
&\quad+\frac{5}{4}\cdot S_{ij}S_{kl}\left(\bar\sigma^{ij}\bar\sigma^{kl}\right)^{\dot\alpha}{}_{\dot\beta}\varepsilon^{\dot\beta}\\
&\quad+\left[4\cdot\left(\frac{1}{4\alpha}\right)^2G_{ab}G^{ab}-3\cdot\left(t_{ab}+\frac{1}{4\alpha}G_{ab}\right)\left(t^{ab}+\frac{1}{4\alpha}G^{ab}\right)\right]\varepsilon^{\dot\alpha}\\
&\quad-\frac{9}{2}\cdot\left(\bar\sigma^{ij}\right)^{\dot\alpha}{}_{\dot\beta}\frac{1}{4\alpha}G_{ab}S_{ij}\gamma^{ab}\varepsilon^{\dot\beta}-\frac{3}{2}\cdot\left(\bar\sigma^{ij}\right)^{\dot\alpha}{}_{\dot\beta}t_{ab}S_{ij}\gamma^{ab}\varepsilon^{\dot\beta}-8\cdot\frac{1}{4\alpha}G_a{}^ct_{bc}\gamma^{ab}\varepsilon^{\dot\alpha}\\
&\quad-\left[t_{ab}t_{cd}-5\cdot\left(\frac{1}{4\alpha}\right)^2G_{ab}G_{cd}\right]\gamma^{abcd}\varepsilon^{\dot\alpha}.
\end{aligned}\tag{32}
$$

Decomposing the fields χ^α, ϕ^I in the representation with $\rho^5=\mathrm{diag}.(+\mathbf{1}_2,\,-\mathbf{1}_2)$ as

$$
\chi^\alpha\to\begin{pmatrix}\psi^{\tilde\alpha}\\ \lambda^{\dot\alpha}\end{pmatrix},\qquad \phi^I\to(\phi^{i=1,\cdots,4},\,\phi^5=\sigma),
$$

one can see that the supersymmetry transformation under the assumption (26) becomes

$$
\begin{aligned}
\delta_\epsilon A_\mu&=-\frac{i}{4}\varepsilon_{\tilde\alpha\tilde\beta}\left(\epsilon^{\tilde\alpha}\right)^TC\gamma_\mu\psi^{\tilde\beta}-\frac{i}{4}\varepsilon_{\dot\alpha\dot\beta}\left(\varepsilon^{\dot\alpha}\right)^TC\gamma_\mu\lambda^{\dot\beta},\\
\delta_\epsilon\sigma&=\frac{i}{4}\varepsilon_{\tilde\alpha\tilde\beta}\left(\epsilon^{\tilde\alpha}\right)^TC\psi^{\tilde\beta}-\frac{i}{4}\varepsilon_{\dot\alpha\dot\beta}\left(\varepsilon^{\dot\alpha}\right)^TC\lambda^{\dot\beta},\\
\delta_\epsilon\phi^i&=\frac{i}{4}\left(\varepsilon\sigma^i\right)_{\tilde\alpha\dot\beta}\left(\epsilon^{\tilde\alpha}\right)^TC\lambda^{\dot\beta}+\frac{i}{4}\left(\varepsilon\bar\sigma^i\right)_{\dot\alpha\tilde\beta}\left(\varepsilon^{\dot\alpha}\right)^TC\psi^{\tilde\beta},\\
\delta_\epsilon\psi^{\tilde\alpha}&=-\left(\frac{1}{2}F_{ab}\gamma^{ab}+\gamma^a\mathcal{D}_a\sigma-\frac{1}{2\alpha}G_{ab}\sigma\gamma^{ab}\right)\epsilon^{\tilde\alpha}\\
&\qquad-\left(S^{ij}\sigma-\frac{i}{2}g\left[\phi^i,\,\phi^j\right]\right)\left(\sigma_{ij}\right)^{\tilde\alpha}{}_{\tilde\beta}\epsilon^{\tilde\beta}-\left(\gamma^a\mathcal{D}_a\phi^i-\frac{1}{2\alpha}G_{ab}\phi^i\gamma^{ab}\right.\\
&\qquad\left.-\left(S^{ij}+\varepsilon^{ijkl}S_{kl}\right)\phi_j-t_{ab}\phi^i\gamma^{ab}-ig\left[\sigma,\,\phi^i\right]\right)\left(\sigma_i\right)^{\tilde\alpha}{}_{\dot\beta}\varepsilon^{\dot\beta},\\
\delta_\epsilon\lambda^{\dot\alpha}&=-\left(\frac{1}{2}F_{ab}\gamma^{ab}-\gamma^a\mathcal{D}_a\sigma+\frac{1}{2\alpha}G_{ab}\sigma\gamma^{ab}\right)\varepsilon^{\dot\alpha}\\
&\qquad+\left(S^{ij}\sigma+\frac{i}{2}g\left[\phi^i,\,\phi^j\right]\right)\left(\bar\sigma_{ij}\right)^{\dot\alpha}{}_{\dot\beta}\varepsilon^{\dot\beta}-\left(\gamma^a\mathcal{D}_a\phi^i-\frac{1}{2\alpha}G_{ab}\phi^i\gamma^{ab}\right.\\
&\qquad\left.-\left(S^{ij}-\varepsilon^{ijkl}S_{kl}\right)\phi_j+t_{ab}\phi^i\gamma^{ab}+ig\left[\sigma,\,\phi^i\right]\right)\left(\bar\sigma_i\right)^{\dot\alpha}{}_{\tilde\beta}\epsilon^{\tilde\beta}.
\end{aligned}\tag{33}
$$

The equations of motion of the spinors $\psi^{\tilde\alpha}$, $\lambda^{\dot\alpha}$ under the assumption (26) give

$$
\begin{aligned}
&\gamma^\mu\mathcal{D}_\mu\psi^{\tilde\alpha}+ig\left[\sigma,\,\psi^{\tilde\alpha}\right]+ig\left(\sigma_i\right)^{\tilde\alpha}{}_{\dot\beta}\left[\phi^i,\,\lambda^{\dot\beta}\right]\\
&=\frac{1}{8\alpha}G_{ab}\gamma^{ab}\psi^{\tilde\alpha}+\frac{1}{4}S_{ij}\left(\sigma^{ij}\right)^{\tilde\alpha}{}_{\tilde\beta}\psi^{\tilde\beta}-\frac{1}{2}t_{ab}\gamma^{ab}\psi^{\tilde\alpha},
\end{aligned}
$$

$$\gamma^\mu \mathcal{D}_\mu \lambda^{\dot\alpha} - ig\left[\sigma,\, \lambda^{\dot\alpha}\right] + ig\,(\bar\sigma_i)^{\dot\alpha}{}_{\tilde\beta}\left[\phi^i,\, \psi^{\tilde\beta}\right]$$
$$= \frac{1}{8\alpha} G_{ab}\gamma^{ab}\lambda^{\dot\alpha} + \frac{1}{4} S_{ij}\left(\bar\sigma^{ij}\right)^{\dot\alpha}{}_{\dot\beta}\lambda^{\dot\beta} + \frac{1}{2} t_{ab}\gamma^{ab}\lambda^{\dot\alpha},$$

with the covariant derivatives

$$\mathcal{D}_\mu \psi^{\tilde\alpha} = \partial_\mu \psi^{\tilde\alpha} - \frac{3}{2} b_\mu \psi^{\tilde\alpha} + \frac{1}{4}\Omega_\mu{}^{ab}\gamma_{ab}\psi^{\tilde\alpha} - \frac{1}{4} A_\mu{}^{ij}\left(\sigma_{ij}\right)^{\tilde\alpha}{}_{\tilde\beta}\psi^{\tilde\beta} + ig\left[A_\mu,\, \psi^{\tilde\alpha}\right],$$
$$\mathcal{D}_\mu \lambda^{\dot\alpha} = \partial_\mu \lambda^{\dot\alpha} - \frac{3}{2} b_\mu \lambda^{\dot\alpha} + \frac{1}{4}\Omega_\mu{}^{ab}\gamma_{ab}\lambda^{\dot\alpha} - \frac{1}{4} A_\mu{}^{ij}\left(\bar\sigma_{ij}\right)^{\dot\alpha}{}_{\dot\beta}\lambda^{\dot\beta} + ig\left[A_\mu,\, \lambda^{\dot\alpha}\right].$$

5.1. The $\mathcal{N}=1$ SUSY background in the previous paper

We start with the background in the previous paper [1,2], where the compactification on the product space of a unit round S^3 and a Riemann surface, $S^3 \times \Sigma$ was considered, and we will reinterpret it as a supersymmetric background in terms of $A_\mu{}^i{}_j$, S_{ij}, G_{ab}, $t_{ab} \equiv t^5{}_{ab}$. See Appendix H.1 for the differences of the old notations used in [1] from the ones in this paper.

The background in [1] can be read in the notations of this paper as

$$t_{45} = \frac{1}{4r}, \qquad S_{12} = S_{34} = \frac{1}{2r}, \qquad \frac{1}{4\alpha} G_{45} = -\frac{1}{4r}, \tag{34}$$

in the Lorentz frame t_{ab}, G_{ab}, where we have replaced the unit radius of the S^3 by r.

On the Riemann surface Σ with local coordinates (x^4, x^5), the twisting is required to preserve supersymmetries by turning on the background gauge field $A^i{}_j$ as

$$A^{12} = A^{34} = -\frac{1}{2}\omega^{45}, \tag{35}$$

with the spin connection ω^{45} on the surface Σ. This together with $S_{12} = S_{34}$ break the $\mathrm{Spin}(5)_R$ R-symmetry group to $SU(2)_l \times U(1)_r \subset SU(2)_l \times SU(2)_r$, when regarding the subgroup Spin(4) of the $\mathrm{Spin}(5)_R$ as $SU(2)_l \times SU(2)_r$. We refer to it as the $\mathcal{N}=1$ twisting, following [16].

The supersymmetry condition (92) determines the background M_{IJ}

$$\frac{4}{15} M_{55} = \frac{1}{5}\left[\frac{1}{r^2} - R(\Sigma)\right], \qquad \frac{4}{15} M_{ij} = -\frac{1}{20}\left[\frac{1}{r^2} - R(\Sigma)\right]\delta_{ij} \quad (i, j = 1, \cdots, 4)$$

where the scalar curvature $R(\Sigma)$ is derived from the spin connection ω^{45},

$$\frac{1}{2} R(\Sigma)\, e^4 \wedge e^5 = d\omega^{45},$$

and substituting these into (24) gives[6]

$$\mathcal{M}_{B\,55} = \frac{2}{r^2}, \qquad \mathcal{M}_{B\,ij} = \frac{1}{4}\left[R(\Sigma) + \frac{4}{r^2}\right]\delta_{ij} \quad (i, j = 1, \cdots, 4).$$

The Killing spinor equation (28) in the background (34) is identical to the one in [1],

$$\mathcal{D}_\mu \varepsilon^{\dot\alpha} = -\frac{1}{2r}\gamma_\mu{}^{45}\varepsilon^{\dot\alpha},$$

with the ansatz (29).

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[6] In the previous paper [1] (v3 on the arXiv), the scalar curvature $R(\Sigma)$ was dropped from the mass terms $\mathcal{M}_{B\,ij}$ of the $\mathcal{N}=1$ hypermultiplet scalars.

The scalar curvature $R(\Omega)$ on the $S^3 \times \Sigma$ is given by

$$R(\Omega) = R(S^3) + R(\Sigma) = \frac{6}{r^2} + R(\Sigma),$$

for the round S^3 of radius r. Since the gauge field $A^i{}_j$ is minus the half of the spin connection ω^{45} on the surface Σ, the field strength of A^{ij} results in

$$F_{45}{}^{12} = F_{45}{}^{34} = -\frac{1}{4} R(\Sigma).$$

The equation (32) identically holds for the curvatures and the background fields, and it is consistent with the existence of the Killing spinor $\varepsilon^{\dot{\alpha}}$. In fact, as explained in [1,2] and in Appendix G, the Killing spinor is given by

$$\varepsilon^{\dot{\alpha}=1} = \epsilon_0 \otimes \zeta_+, \qquad \varepsilon^{\dot{\alpha}=2} = C_3^{-1} \epsilon_0{}^* \otimes \zeta_-,$$

with ϵ_0 a constant spinor on the S^3, which is consistent with our ansatz (29).

For the other supersymmetry parameter $\epsilon^{\tilde{\alpha}}$, the Killing spinor equation (27) in the same background gives

$$\mathcal{D}_a \epsilon^{\tilde{\alpha}} = -\frac{1}{2\alpha} G_{ab} \gamma^b \epsilon^{\tilde{\alpha}} + \frac{1}{2r} \gamma_a{}^{45} \epsilon^{\tilde{\alpha}}.$$

Note that $S^{12} = S^{34}$ obeys $S^{ij}\sigma_{ij} = 0$. With $A^{12} = A^{34}$, we have $A^{ij}\sigma_{ij} = 0$, and the twisting of the background A^{ij} have no effects inside the covariant derivative $\mathcal{D}_a \epsilon^{\tilde{\alpha}}$. In a generic Riemann surface Σ, we don't have a solution to the above Killing spinor equation. In fact, the calculation of $\gamma^{ab}\mathcal{D}_a\mathcal{D}_b\epsilon^{\tilde{\alpha}}$ shows that the scalar curvature $R(\Sigma)$ is an obstacle to the existence of a Killing spinor for $\epsilon^{\tilde{\alpha}}$.

We can see from (34), (35) that the background breaks the $\mathrm{Spin}(5)_R$ group of the R-symmetry into $SU(2)_l \times U(1)_r$, which is a subgroup of $SU(2)_l \times SU(2)_r \simeq \mathrm{Spin}(4)_R \subset \mathrm{Spin}(5)_R$. The symmetry breaking is caused by the twisting $A^{12} = A^{34}$ (and also $S^{12} = S^{34}$). As we have seen just above, the twisting only retains the half of the supersymmetries. Therefore, it is consistent with the fact that the $SU(2)_l$ symmetry doesn't give rise to the $SU(2)_R$ R-symmetry in four-dimensional $\mathcal{N} = 2$ supersymmetric theories [6,16].

The background (34) is not a unique solution[7] to yield an $\mathcal{N} = 1$ supersymmetric background on the round S^3. Even under the ansatz

$$S_{12} = S_{34} = \frac{1}{2} S,$$

with only non-zero components G_{45} and t_{45}, there exists a Killing spinor for $\varepsilon^{\dot{\alpha}}$, if

$$\frac{1}{2} S + 2 \cdot \frac{1}{4\alpha} G_{45} = 0, \qquad \frac{1}{2} S + \frac{1}{4\alpha} G_{45} + t_{45} = \frac{1}{2r},$$

which can be read from the Killing spinor equation (28). They may therefore be parametrized by S;

$$\frac{1}{4\alpha} G_{45} = -\frac{1}{4} S, \qquad t_{45} = -\frac{1}{4} S + \frac{1}{2r}.$$

[7] It has been pointed out in [28] in the context of five-dimensional $\mathcal{N} = 1$ supersymmetric theories.

The other supersymmetric condition (92) gives one more constraint – the backgrounds are constant on Σ,

$$\mathcal{D}_4 S = \mathcal{D}_5 S = 0,$$

and determines the remaining background M_{IJ},

$$\frac{4}{15} M_{55} = -\frac{1}{5} R(\Sigma) + \frac{4}{5r^2} - \frac{3}{5} S^2, \qquad \frac{4}{15} M_{ij} = \left[\frac{1}{20} R(\Sigma) - \frac{1}{5r^2} + \frac{3}{20} S^2\right] \delta_{ij},$$

for $i, j = 1, \cdots, 4$. The scalar mass parameters $\mathcal{M}_{B\,IJ}$ are given by

$$\mathcal{M}_{B\,55} = \frac{2}{r}\left(\frac{2}{r} - S\right), \qquad \mathcal{M}_{B\,11} = \cdots = \mathcal{M}_{B\,44} = \frac{1}{4} R(\Sigma) + \frac{1}{r^2}.$$

When $S = 1/r$, it certainly retains the mass term of the scalar σ in the previous papers [1,2].

5.2. $\mathcal{N} = 2$ SUSY backgrounds on the round $S^3 \times \Sigma$

While the background in [1,2] preserves half of the supersymmetries, we will find a new supersymmetric background preserving both of $\varepsilon^{\dot\alpha}$ and $\epsilon^{\tilde\alpha}$ on the $S^3 \times \Sigma$.

Taking the breaking of the R-symmetry group $\mathrm{Spin}(5)_R$ into account, we will turn on A^{12} and $S^{12} = S$ only, and it would break the $\mathrm{Spin}(5)_R$ group down to $U(1)_R \times SU(2)_R$. We could instead turn on A^{34} or S^{34} only, but it is just a matter of convention. We refer to this partial twisting as the $\mathcal{N} = 2$ twisting.

Since we have the covariant derivatives with the ansatz (29),

$$\mathcal{D}_\mu \epsilon^{\tilde\alpha} = \partial_\mu \epsilon^{\tilde\alpha} - \frac{i}{2}\left(A_\mu{}^{12} \pm \omega_\mu{}^{45}\right)(\tau_3)^{\tilde\alpha}{}_{\tilde\beta}\epsilon^{\tilde\beta},$$

$$\mathcal{D}_\mu \varepsilon^{\dot\alpha} = \partial_\mu \varepsilon^{\dot\alpha} - \frac{i}{2}\left(A_\mu{}^{12} + \omega_\mu{}^{45}\right)(\tau_3)^{\dot\alpha}{}_{\dot\beta}\varepsilon^{\dot\beta},$$

in order to cancel the spin connection ω^{45} by A_{12} in both of the covariant derivatives, the chirality of $\epsilon^{\tilde\alpha}$ on the surface Σ should be the same as the one of $\varepsilon^{\dot\alpha}$; $i\gamma_{45}\varepsilon^{\tilde\alpha} = (\tau_3)^{\tilde\alpha}{}_{\tilde\beta}\epsilon^{\tilde\beta}$. Therefore, the twisting

$$A^{12} = -\omega^{45}$$

works for both of $\varepsilon^{\dot\alpha}$ and $\epsilon^{\tilde\alpha}$. When we turn on the components G_{45} and t_{45} only, the Killing spinor equations (27), (28) become

$$\mathcal{D}_a \epsilon^{\tilde\alpha} = \frac{i}{2} S\,(\tau_3)^{\tilde\alpha}{}_{\tilde\beta}\gamma_a\epsilon^{\tilde\beta} - \frac{1}{2\alpha} G_{ab}\gamma^b\epsilon^{\tilde\alpha} - \left(\frac{1}{4\alpha} G_{45} - t_{45}\right)\gamma_a{}^{45}\epsilon^{\tilde\alpha},$$

$$\mathcal{D}_a \varepsilon^{\dot\alpha} = \frac{i}{2} S\,(\tau_3)^{\dot\alpha}{}_{\dot\beta}\gamma_a\varepsilon^{\dot\beta} - \frac{1}{2\alpha} G_{ab}\gamma^b\varepsilon^{\dot\alpha} - \left(\frac{1}{4\alpha} G_{45} + t_{45}\right)\gamma_a{}^{45}\varepsilon^{\dot\alpha}.$$

For $a = 4, 5$, the Killing spinor equation is satisfied with $\epsilon^{\tilde\alpha}$ and $\varepsilon^{\dot\alpha}$ constant on Σ, if

$$S + 4 \cdot \frac{1}{4\alpha} G_{45} = 0.$$

With the ansatz (29), the Killing spinors on the round S^3 (see Appendix G.1) are lifted to

$$\mathcal{D}_a \varepsilon^{\dot\alpha} = -\frac{1}{2r}\gamma_a{}^{45}\varepsilon^{\dot\alpha}, \qquad \mathcal{D}_a \epsilon^{\tilde\alpha} = \mp\frac{1}{2r}\gamma_a{}^{45}\epsilon^{\tilde\alpha},$$

and the comparison of this with the above Killing spinor equations for $a = 1, 2, 3$ leads to

$$\frac{1}{2}S + \frac{1}{4\alpha}G_{45} + t_{45} = \frac{1}{2r}, \qquad \frac{1}{2}S + \frac{1}{4\alpha}G_{45} - t_{45} = \pm\frac{1}{2r}.$$

Depending upon the sign, there are two solutions:

$$S = -4 \cdot \frac{1}{4\alpha}G_{45} = \frac{2}{r}, \quad \frac{1}{4\alpha}G_{45} = -\frac{1}{2r}, \qquad t_{45} = 0,$$

and

$$S = \frac{1}{4\alpha}G_{45} = 0, \qquad t_{45} = \frac{1}{2r}.$$

We will call the former background type B and the latter type A, respectively.

Let us begin with the type A background:

$$S_{ij} = 0, \qquad \frac{1}{4\alpha}G_{ab} = 0, \quad t_{45} = \frac{1}{2r}.$$

In the background, since the Killing spinor equation (7) is reduced into

$$\mathcal{D}_\mu \epsilon^{\tilde{\alpha}} = \frac{1}{2r}\gamma_\mu{}^{45}\epsilon^{\tilde{\alpha}}, \qquad \mathcal{D}_\mu \varepsilon^{\dot{\alpha}} = -\frac{1}{2r}\gamma_\mu{}^{45}\varepsilon^{\dot{\alpha}},$$

one obtains the solution to them,

$$\left(\epsilon^{\tilde{\alpha}=1}, \epsilon^{\tilde{\alpha}=2}\right) = \left(U^{-1}\tilde{\epsilon}_0 \otimes \zeta_+,\, C_3^{-1}U^T\tilde{\epsilon}_0^* \otimes \zeta_-\right),$$
$$\left(\varepsilon^{\dot{\alpha}=1}, \varepsilon^{\dot{\alpha}=2}\right) = \left(\epsilon_0 \otimes \zeta_+,\, C_3^{-1}\epsilon_0^* \otimes \zeta_-\right),$$

with ϵ_0 and $\tilde{\epsilon}_0$ constant spinors and U the mapping of the 3-sphere to the $SU(2)$ group given in Appendix G and with C_3 the three-dimensional charge conjugation matrix explained in Appendix B.

The supersymmetry condition (92) determines the background M_{IJ}:

$$\frac{4}{15}M_{55} = \frac{4}{5}\frac{1}{r^2} - \frac{1}{5}R(\Sigma),$$
$$\frac{4}{15}M_{11} = \frac{4}{15}M_{22} = -\frac{1}{5}\frac{1}{r^2} + \frac{3}{10}R(\Sigma), \qquad \frac{4}{15}M_{33} = \frac{4}{15}M_{44} = -\frac{1}{5}\frac{1}{r^2} - \frac{1}{5}R(\Sigma),$$

which gives rise to the masses $\mathcal{M}_{B\,IJ}$ of the scalar fields ϕ^I,

$$\mathcal{M}_{B\,55} = \frac{4}{r^2}, \quad \mathcal{M}_{B\,11} = \mathcal{M}_{B\,22} = \frac{1}{r^2} + \frac{1}{2}R(\Sigma), \quad \mathcal{M}_{B\,33} = \mathcal{M}_{B\,44} = \frac{1}{r^2}.$$

Turning on the field $t_{45} = t_{45}^{I=5}$ breaks the Spin$(5)_R$ symmetry group into Spin$(4)_R$ and with the twisting by $A^{12} = -\omega^{45}$ into $U(1) \times U(1)$. Thus, the background doesn't respect the R-symmetry of the four-dimensional $\mathcal{N} = 2$ conformal algebra, but it retains the $\mathcal{N} = 2$ supersymmetry.

Let us move on to the type B background:

$$S = \frac{2}{r}, \qquad \frac{1}{4\alpha}G_{45} = -\frac{1}{2r}, \qquad t_{45} = 0.$$

It gives rise to the Killing spinor equation

$$\mathcal{D}_\mu \epsilon^{\tilde{\alpha}} = -\frac{1}{2r}\gamma_\mu{}^{45}\epsilon^{\tilde{\alpha}}, \qquad \mathcal{D}_\mu \varepsilon^{\dot{\alpha}} = -\frac{1}{2r}\gamma_\mu{}^{45}\varepsilon^{\dot{\alpha}},$$

and one gives the same constant solution for the both $\epsilon^{\tilde{\alpha}}$ and $\varepsilon^{\dot{\alpha}}$:

$$\left(\epsilon^{\tilde{\alpha}=1}, \epsilon^{\tilde{\alpha}=2}\right) = \left(\tilde{\epsilon}_0 \otimes \zeta_+,\, \tilde{\epsilon}_0^* \otimes \zeta_-\right), \qquad \left(\varepsilon^{\dot{\alpha}=1}, \varepsilon^{\dot{\alpha}=2}\right) = \left(\epsilon_0 \otimes \zeta_+,\, C_3^{-1}\epsilon_0^* \otimes \zeta_-\right),$$

with ϵ_0 and $\tilde{\epsilon}_0$ constant spinors as above.

The supersymmetric condition (92) is obeyed by the background, if the background fields M_{IJ} satisfy

$$\frac{4}{15}M_{55} = \frac{4}{15}M_{33} = \frac{4}{15}M_{44} = -\frac{8}{5}\frac{1}{r^2} - \frac{1}{5}R(\Sigma),$$
$$\frac{4}{15}M_{11} = \frac{4}{15}M_{22} = \frac{12}{5}\frac{1}{r^2} + \frac{3}{10}R(\Sigma),$$

which surely respects the R-symmetry group $U(1)_R \times SU(2)_R$. The scalars σ, ϕ^3, ϕ^4 remain massless, while the remaining ϕ^1, ϕ^2, are lifted by a half of the scalar curvature $R(\Sigma)$:

$$\mathcal{M}_{B\,55} = \mathcal{M}_{B\,33} = \mathcal{M}_{B\,44} = 0, \qquad \mathcal{M}_{B\,11} = \mathcal{M}_{B\,22} = \frac{1}{2}R(\Sigma).$$

Thus, they respect the remaining R-symmetry group $U(1)_R \times SU(2)_R$.

Turning on either of A^{12} or A^{34} without $t_{45} \neq 0$ breaks the R-symmetry group $SO(5)_R$ into $SO(2) \times SO(3) \simeq U(1)_R \times SU(2)_R$, which can be identified with the R-symmetry group of the $\mathcal{N} = 2$ superconformal group, if the theory flows into an infrared fixed point. On the other hand, as in the previous papers [1,2], turning on both A^{12} and A^{34} such that $A^{12} = A^{34}$, the $SO(5)_R$ group is broken to $SU(2)_l \times U(1)_r$, which is the subgroup of $SU(2)_l \times SU(2)_r \simeq SO(4) \subset SO(5)_R$. The subgroup $SU(2)_l$ cannot be identified to the R-symmetry group $SU(2)_R$, because the above results shows that such a background preserves only half of the supersymmetries.[8] This is consistent with the result in [6,16].

5.3. A squashed 3-sphere with constant Killing spinors

A squashed 3-sphere is a deformation of a round S^3, and regarding it as a circle fibration over a round 2-sphere, *i.e.*, the Hopf fibration, the radius of the fiber differs from the radius of the base. See Appendix G for more details. In [21], three-dimensional supersymmetric field theories on the squashed 3-sphere has been discussed, and we will make use of their construction for the five-dimensional theory.

The constant solution $\varepsilon^{\dot{\alpha}}$ on the round S^3: $\left(\varepsilon^1, \varepsilon^2\right) = \left(\epsilon_0 \otimes \zeta_+,\, C_3^{-1}\epsilon_0^* \otimes \zeta_-\right)$ with

$$\epsilon_0 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad C_3^{-1}\epsilon_0^* = \begin{pmatrix}0\\1\end{pmatrix}, \qquad \gamma^3\varepsilon^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}}, \tag{36}$$

solves the differential equation

$$\left(d + \frac{1}{4}\omega^{ab}\gamma_{ab}\right)\varepsilon^{\dot{\alpha}} - \frac{i}{\tilde{r}}\left(1 - \frac{\tilde{r}^2}{r^2}\right)(\tau_3)^{\dot{\alpha}}{}_{\dot{\beta}}e_3\varepsilon^{\dot{\beta}} = -\frac{1}{2}\frac{\tilde{r}}{r^2}e^a\gamma_a{}^{45}\varepsilon^{\dot{\alpha}}, \tag{37}$$

[8] We thank Yuji Tachikawa for clarification on this point.

where ω^{ab} is the spin connection of the squashed S^3 with the fiber radius $\tilde{r}$ and the base radius r. See Appendix G for the squashed S^3.

We will begin with the $\mathcal{N}=2$ twisting by turning on A^{12} only. A comparison of (37) with the Killing spinor equation (28) suggests that

$$A^{12}=\frac{2}{\tilde{r}}\left(1-\frac{\tilde{r}^2}{r^2}\right)e^3-\omega^{45}, \qquad \frac{1}{4\alpha}G_{45}=-\frac{1}{2}\frac{\tilde{r}}{r^2}, \qquad S_{12}=\frac{2\tilde{r}}{r^2}. \tag{38}$$

For the other supersymmetry parameter $\epsilon^{\tilde{\alpha}}$, it is easy to find a Killing spinor on the squashed S^3, if we make the same ansatz as for $\varepsilon^{\dot{\alpha}}$; it is a constant spinor $\left(\epsilon^1,\epsilon^2\right)=\left(\tilde{\epsilon}_0\otimes\zeta_+,\, C_3^{-1}\tilde{\epsilon}_0^*\otimes\zeta_-\right)$ obeying $\tilde{\epsilon}_0=(1,0)^T$. One then see that it obeys the same differential equation (37), and thus the background (38) preserves the both Killing spinors $\varepsilon^{\dot{\alpha}}$, $\epsilon^{\tilde{\alpha}}$.

The other supersymmetry condition (92) determines the background fields M_{IJ},

$$\frac{4}{15}M_{11}=\frac{4}{15}M_{22}=\frac{3}{10}R(\Sigma)+\frac{12}{5}\frac{1}{r^2},$$
$$\frac{4}{15}M_{33}=\frac{4}{15}M_{44}=\frac{4}{15}M_{55}=-\frac{1}{5}R(\Sigma)-\frac{8}{5}\frac{1}{r^2},$$

and plugging them into (24), one obtains the scalar masses $\mathcal{M}_{B\,IJ}$,

$$\mathcal{M}_{B\,11}=\mathcal{M}_{B\,22}=\frac{1}{2}R(\Sigma)+\frac{4}{r^2}\left(1-\frac{\tilde{r}^2}{r^2}\right), \qquad \mathcal{M}_{B\,33}=\mathcal{M}_{B\,44}=\mathcal{M}_{B\,55}=0.$$

Let us proceed to the $\mathcal{N}=1$ twisting so that we will turn on the gauge field A^{ij} of only one $SU(2)$ subgroup of the Spin$(5)_R$ group by requiring that $A^{12}=A^{34}$, and then a comparison with the Killing spinor equation (7) identifies the background R-symmetry gauge field

$$A^{12}=A^{34}=\frac{1}{\tilde{r}}\left(1-\frac{\tilde{r}^2}{r^2}\right)e^3-\frac{1}{2}\omega^{45},$$

and for the other background fields, taking account of (92), one finds that

$$S_{12}=S_{34}=\frac{1}{2}S, \qquad \frac{1}{4\alpha}G_{45}=-\frac{1}{4}S, \qquad t_{45}=-\frac{1}{4}S+\frac{1}{2}\frac{\tilde{r}}{r^2},$$
$$\frac{4}{15}M_{55}=\frac{4}{5}\frac{\tilde{r}^2}{r^4}-\frac{1}{5}R(\Sigma)-\frac{3}{5}S^2-\frac{8}{5}\frac{1}{r^2}\left(1-\frac{\tilde{r}^2}{r^2}\right),$$
$$\frac{4}{15}M_{ij}=\left[-\frac{1}{5}\frac{\tilde{r}^2}{r^4}+\frac{1}{20}R(\Sigma)+\frac{3}{20}S^2+\frac{2}{5}\frac{1}{r^2}\left(1-\frac{\tilde{r}^2}{r^2}\right)\right]\delta_{ij}, \tag{39}$$

for $i,j=1,\cdots,4$. In the limit $\tilde{r}\to r$, one regains the $\mathcal{N}=1$ supersymmetric background on the round S^3 in the previous subsection. It follows from (24) that

$$\mathcal{M}_{B\,55}=\frac{2\tilde{r}}{r^2}\left(\frac{2\tilde{r}}{r^2}-S\right),$$

$$\mathcal{M}_{B\,11}=\mathcal{M}_{B\,22}=\mathcal{M}_{B\,33}=\mathcal{M}_{B\,44}=\frac{1}{4}R(\Sigma)+\frac{\tilde{r}^2}{r^4}+\frac{2}{r^2}\left(1-\frac{\tilde{r}^2}{r^2}\right).$$

5.4. A squashed 3-sphere with non-constant Killing spinors

Upon the Kaluza–Klein compactification on the time circle to the round $S^3 \times \Sigma$, the periodic boundary condition $t \to t + 2\pi$ was assumed in the previous papers [1,2]. The partition function is supposed to give the index of the six-dimensional theory. Let us generalize this by considering a slant boundary condition

$$t \quad \sim \quad t + 2\pi, \qquad \psi \quad \sim \quad \psi + \frac{4u\pi}{r\alpha},$$

where ψ is the fiber coordinate in the Hopf fibration of the 3-sphere. See Appendix G for more details.

It has been explained in [22] that the Kaluza–Klein reduction along this slant circle gives rise to a squashed S^3. Changing the local coordinates $(t,\ \psi)$ into $(\tilde{t},\ \tilde{\psi})$ by

$$\tilde{t} = t \quad \sim \quad \tilde{t} + 2\pi, \qquad \tilde{\psi} = \psi - \frac{2u}{r\alpha} t \quad \sim \quad \tilde{\psi},$$

the ordinary reduction in the $\tilde{t}$ direction will be carried out. Then, the mapping $U(\psi, \theta, \phi)$ in (95) from the 3-sphere to the $SU(2)$ group is given in terms of the new coordinates by

$$U(\psi, \theta, \phi) = e^{\frac{i}{2}\phi\tau_3} e^{\frac{i}{2}\theta\tau_3} e^{\frac{i}{2}\psi\tau_3} = U(\tilde{\psi}, \theta, \phi) e^{i\frac{u\tilde{t}}{r\alpha}\tau_3},$$

and the vielbein $\tilde{\mu}^{(0)}$ in the new coordinates of the 3-sphere,

$$\tilde{\mu}^{(0)} = \left(\frac{1}{i}\right) U^{-1}(\tilde{\psi}, \theta, \phi) dU(\tilde{\psi}, \theta, \phi)$$

is related to $\mu^{(0)}$ in the original coordinates by

$$\tilde{\mu}_1^{(0)} = \cos\left(\frac{2u}{r\alpha}t\right)\mu_1^{(0)} + \sin\left(\frac{2u}{r\alpha}t\right)\mu_2^{(0)}, \quad \tilde{\mu}_2^{(0)} = \cos\left(\frac{2u}{r\alpha}t\right)\mu_2^{(0)} - \sin\left(\frac{2u}{r\alpha}t\right)\mu_1^{(0)},$$
$$\tilde{\mu}_3^{(0)} = \mu_3^{(0)} - \frac{u}{r\alpha}dt.$$

Under the change of coordinates, in the six-dimensional metric

$$ds_6^2 = ds_\Sigma^2 + r^2\left[\left(\mu_1^{(0)}\right)^2 + \left(\mu_2^{(0)}\right)^2 + \left(\mu_3^{(0)}\right)^2\right] - \frac{1}{\alpha^2}dt^2,$$

in addition to the trivial change of the base part in the Hopf fibration

$$r^2\left[\left(\mu_1^{(0)}\right)^2 + \left(\mu_2^{(0)}\right)^2\right] = r^2\left[\left(\tilde{\mu}_1^{(0)}\right)^2 + \left(\tilde{\mu}_2^{(0)}\right)^2\right],$$

the last two terms are changed into

$$-\frac{1}{\alpha^2}dt^2 + r^2\left(\mu_3^{(0)}\right)^2 \to -\frac{1-u^2}{\alpha^2}\left(d\tilde{t} - \alpha\frac{ru}{1-u^2}\tilde{\mu}_3^{(0)}\right)^2 + \frac{r^2}{1-u^2}\left(\tilde{\mu}_3^{(0)}\right)^2$$
$$= -\frac{1}{\tilde{\alpha}^2}\left(d\tilde{t} + C\right)^2 + e_3^2,$$

where

$$\tilde{\alpha}=\frac{1}{\sqrt{1-u^2}}\alpha, \qquad C=-\alpha\frac{ru}{1-u^2}\tilde{\mu}_3^{(0)}=-\tilde{\alpha}ue_3,$$
$$e_3=\frac{r}{\sqrt{1-u^2}}\tilde{\mu}_3^{(0)}=\tilde{r}\left(\frac{d\tilde{\psi}+\cos\theta d\phi}{2}\right),$$

with $\tilde{r}=r/\sqrt{1-u^2}$. Therefore, the slant boundary condition turns on the graviphoton field C and deforms the radius of the circle fiber, which results in a squashed 3-sphere.

Upon the reduction to five dimensions, one has the metric

$$ds_5^2=ds_\Sigma^2+\left(e_1^2+e_2^2+e_3^2\right), \qquad e_1=r\tilde{\mu}_1^{(0)}, \quad e_2=r\tilde{\mu}_2^{(0)},$$

and the field strength of the graviphoton,

$$\frac{1}{\tilde{\alpha}}G=\frac{1}{\tilde{\alpha}}dC=-\frac{2}{r}\frac{u}{\sqrt{1-u^2}}e_1\wedge e_2.$$

Note that $\tilde{\alpha}$ is the radius of the circle in the $\tilde{t}$ direction, while $\tilde{r}$ is the radius of the fiber in the Hopf fibration of the squashed S^3.

If we started with a non-vanishing graviphoton field C in the round $S^3\times\Sigma$, we wouldn't gain a simple squashed 3-sphere. Therefore, let us consider the supersymmetric backgrounds in Subsection 5.2, where we have $C=0$ on a round $S^3\times\Sigma$. One can now see that the above change of coordinates leads the backgrounds in Subsection 5.2 to supersymmetric backgrounds on the squashed $S^3\times\Sigma$. We will begin with the $\mathcal{N}=1$ supersymmetric background in 5.2 by turning on

$$A^{12}=A^{34}=-\frac{1}{2}\omega^{45},$$

breaking the $\mathrm{Spin}(5)_R$ symmetry down to $SU(2)_l\times U(1)_r$. Besides the R-symmetry gauge field $A^i{}_j$, the only auxiliary field t_{45} is turned on in Subsection 5.2.

Returning to six dimensions, the sechsbein

$$\left(e^1,e^2,e^3,e^0=\frac{1}{\tilde{\alpha}}\left(d\tilde{t}+C\right)\right)$$

is related to the sechsbein $(\mu^1,\mu^2,\mu^3,\mu^0=(1/\alpha)dt)$, where $\mu_a=r\mu_a^{(0)}$ $(a=1,2,3)$, by a local Lorentz transformation,

$$e^a=\sum_{b=0}^{3}\Lambda^a{}_b\,\mu^b \quad (a=0,\cdots,3), \tag{40}$$

where

$$\left(\Lambda^a{}_b\right)=\begin{pmatrix}\cos(2ut/r\alpha) & \sin(2ut/r\alpha) & 0 & 0\\ -\sin(2ut/r\alpha) & \cos(2ut/r\alpha) & 0 & 0\\ 0 & 0 & \cosh\xi & -\sinh\xi\\ 0 & 0 & -\sinh\xi & \cosh\xi\end{pmatrix},$$

with $\cosh\xi=1/\sqrt{1-u^2}$ and $\sinh\xi=u/\sqrt{1-u^2}$.

From the six-dimensional view point, the background t_{45} may be regarded as

$$\underline{T}^{\alpha\beta}{}_{045}=-\underline{T}^{\alpha\beta}{}_{123}=t_{45}\left(\rho^5\Omega^{-1}\right)^{\alpha\beta}.$$

Recall that $\underline{T}^{\alpha\beta}{}_{\underline{abc}}$ is anti-self under the Hodge duality. The field $\underline{T}^{\alpha\beta}{}_{\underline{abc}}$ is transformed under the Lorentz transformation (40), and one obtains

$$\tilde{\underline{T}}^{\alpha\beta}{}_{045} = t_{45}\cosh\xi, \qquad \tilde{\underline{T}}^{\alpha\beta}{}_{012} = -t_{45}\sinh\xi.$$

Therefore, on the squashed $S^3 \times \Sigma$, besides the R-symmetry gauge field $A^{12} = A^{34}$ and the graviphoton field G, the auxiliary fields

$$\tilde{t}_{45} = t_{45}\cosh\xi, \qquad \tilde{t}_{12} = -t_{45}\sinh\xi$$

are turned on.

Under the Lorentz transformation (40), a six-dimensional supersymmetry parameter $\underline{\epsilon}^\alpha$ transforms as

$$\underline{\epsilon}^\alpha \quad \to \quad \tilde{\underline{\epsilon}}^\alpha = \exp\left(\frac{ut}{r\alpha}\underline{\Gamma}^{12}\right)\exp\left(\frac{1}{2}\xi\,\underline{\Gamma}^{03}\right)\underline{\epsilon}^\alpha, \tag{41}$$

and recalling that it is of positive chirality,

$$\underline{\epsilon}^\alpha = \begin{pmatrix} \epsilon^\alpha \\ 0 \end{pmatrix},$$

one can see that the five-dimensional spinor ϵ^α transforms as

$$\epsilon^\alpha \quad \to \quad \tilde{\epsilon}^\alpha = \exp\left(\frac{ut}{r\alpha}\gamma^{12}\right)\exp\left(\frac{1}{2}\xi\,\gamma^{3}\right)\epsilon^\alpha.$$

The Lorentz transform (41) of the Killing spinors in Subsection 5.2 also gives the Killing spinors in the background on the squashed $S^3 \times \Sigma$. In fact in Subsection 5.2, depending on the sign of the background $t_{45} = \pm 1/2r$, one has the Killing spinors

$$\left(\varepsilon^{\dot{\alpha}=1}, \varepsilon^{\dot{\alpha}=2}\right) = \left(\epsilon_0 \otimes \zeta_+,\, C_3^{-1}\epsilon_0^* \otimes \zeta_-\right), \quad \text{for } t_{45} = 1/2r,$$
$$\left(\varepsilon^{\dot{\alpha}=1}, \varepsilon^{\dot{\alpha}=2}\right) = \left(U^{-1}\epsilon_0 \otimes \zeta_+,\, C_3^{-1}U^T\epsilon_0^* \otimes \zeta_-\right), \quad \text{for } t_{45} = -1/2r,$$

and they are transformed under the Lorentz transformation into

$$\varepsilon^1 \to \exp\left(i\frac{ut}{r\alpha}\tau_3\right)\exp\left(\frac{1}{2}\xi\,\tau_3\right)\epsilon_0 \otimes \zeta_+, \quad \text{for } t_{45} = 1/2r,$$
$$\begin{aligned}\varepsilon^1 &\to \exp\left(i\frac{ut}{r\alpha}\tau_3\right)\exp\left(\frac{1}{2}\xi\,\tau_3\right)\cdot\exp\left(-i\frac{ut}{r\alpha}\tau_3\right)U^{-1}(\tilde{\psi},\theta,\phi)\epsilon_0 \otimes \zeta_+ \\ &= \exp\left(\frac{1}{2}\xi\,\tau_3\right)U^{-1}(\tilde{\psi},\theta,\phi)\epsilon_0 \otimes \zeta_+, \quad \text{for } t_{45} = -1/2r.\end{aligned}$$

One can thus see that the former never survive the Kaluza–Klein reduction, and the latter yields the Killing spinor on the squashed $S^3 \times \Sigma$,

$$\tilde{\varepsilon}^{\dot{\alpha}=1} = \exp\left(\frac{1}{2}\xi\,\tau_3\right)U^{-1}(\tilde{\psi},\theta,\phi)\epsilon_0 \otimes \zeta_+,$$
$$\tilde{\varepsilon}^{\dot{\alpha}=2} = \exp\left(-\frac{1}{2}\xi\,\tau_3\right)C_3^{-1}U^T(\tilde{\psi},\theta,\phi)\epsilon_0^* \otimes \zeta_-,$$

which is the solution to

$$
\begin{aligned}
\mathcal{D}_a \varepsilon^{\dot{\alpha}} &= -\frac{1}{2\tilde{\alpha}} G_{12} \left(\delta_a{}^1 \delta_b{}^2 - \delta_a{}^2 \delta_b{}^1 \right) \gamma^b \varepsilon^{\dot{\alpha}} - \left[\frac{1}{4\tilde{\alpha}} G_{12} + \tilde{t}_{12} \right] \gamma_a{}^{12} \varepsilon^{\dot{\alpha}} - \tilde{t}_{45}\, \gamma_a{}^{45} \varepsilon^{\dot{\alpha}} \\
&= -\frac{1}{r} \sinh \xi \left(\delta_a{}^1 \delta_b{}^2 - \delta_a{}^2 \delta_b{}^1 \right) \gamma^b \varepsilon^{\dot{\alpha}} + \frac{1}{2r} \cosh \xi\, \gamma_a{}^{45} \varepsilon^{\dot{\alpha}},
\end{aligned}
$$

which agrees with the Killing spinor equation (28) with the background obtained in this subsection.

Let us turn to the remaining supersymmetry condition (92) from $\delta_\epsilon \chi^{\alpha\beta}{}_\gamma = 0$, which determines the auxiliary field M_{IJ}. Substituting the background fields into (92) and noticing that[9]

$$
\mathcal{D}_2 t_{23} = \mathcal{D}_1 t_{13} = 2 t_{12} \cdot t_{45}, \qquad \frac{1}{\alpha} \mathcal{D}_2 G_{23} = \frac{1}{\alpha} \mathcal{D}_1 G_{13} = 4 \frac{\tilde{r}}{r^2} t_{12},
$$

one obtains

$$
\begin{aligned}
&\frac{4}{15} M_{55} = \frac{4}{5} \frac{1}{r^2} - \frac{1}{5} R(\Sigma), \\
&\frac{4}{15} M_{11} = \frac{4}{15} M_{22} = -\frac{1}{5} \frac{1}{r^2} - \frac{1}{5} R(\Sigma) - F_{45}{}^{12} = -\frac{1}{5} \frac{1}{r^2} + \frac{1}{20} R(\Sigma), \\
&\frac{4}{15} M_{33} = \frac{4}{15} M_{44} = -\frac{1}{5} \frac{1}{r^2} - \frac{1}{5} R(\Sigma) - F_{45}{}^{34} = -\frac{1}{5} \frac{1}{r^2} + \frac{1}{20} R(\Sigma),
\end{aligned}
$$

and the scalar masses $\mathcal{M}_{B\,IJ}$

$$
\mathcal{M}_{B\,55} = 4 \frac{\tilde{r}^2}{r^4}, \qquad \mathcal{M}_{B\,11} = \mathcal{M}_{B\,22} = \mathcal{M}_{B\,33} = \mathcal{M}_{B\,44} = \frac{1}{r^2} + \frac{1}{4} R(\Sigma).
$$

In summary, we have found the supersymmetric background on the squashed $S^3 \times \Sigma$,

$$
\tilde{t}_{12} = -\frac{1}{4\tilde{\alpha}} G_{12} = \frac{1}{2r} \sinh \xi, \qquad \tilde{t}_{45} = -\frac{1}{2r} \cosh \xi, \qquad A^{12} = A^{34} = -\frac{1}{2} \omega^{12},
$$

with the above scalar masses $\mathcal{M}_{B\,IJ}$.

5.5. An ellipsoid 3-sphere

As explained in [21], an ellipsoid 3-sphere is defined by the set of solutions $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$ to

$$
\frac{x_1^2 + x_2^2}{r^2} + \frac{x_3^2 + x_4^2}{\tilde{r}^2} = 1,
$$

which is solved by polar coordinates (ϕ, χ, θ)

$$
x_1 = r \cos\theta \cos\varphi, \quad x_2 = r \cos\theta \sin\varphi, \qquad x_3 = \tilde{r} \sin\theta \cos\chi, \quad x_4 = \tilde{r} \sin\theta \sin\chi,
$$

with $0 \leq \phi \leq 2\pi$, $0 \leq \chi \leq 2\pi$, $0 \leq \theta \leq \pi/2$. The metric is induced by embedding it into a flat $\mathbf{R}^4$,

$$
ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.
$$

For more details, see [21] or Appendix G.5.

[9] The latter formula is written for later use.

The Killing spinors ϵ and $\epsilon^c = C_3^{-1}\epsilon^*$ on the ellipsoid S^3 are given[10] in Appendix G.5. Using them, we form five-dimensional Killing spinors, $\varepsilon^1 = \epsilon \otimes \zeta_+$, $\varepsilon^2 = \epsilon^c \otimes \zeta_-$, with the same $\zeta_\pm$ as before, and see that they obey

$$\mathcal{D}_\mu \varepsilon^{\dot{\alpha}} = -\frac{1}{2f}\gamma_\mu{}^{45}\varepsilon^{\dot{\alpha}}, \tag{42}$$

where the R-symmetry gauge field $A^i{}_j$ in the covariant derivative $\mathcal{D}\varepsilon^{\dot{\alpha}}$ is given by

$$\frac{1}{2}\left(A^{12} + A^{34}\right) = -V - \frac{1}{2}\omega^{45},$$

with V the background $U(1)$ gauge field on the ellipsoid S^3 given in (103) of Appendix G.5, and with ω^{45} the spin connection on the Riemann surface Σ.

We will consider an $\mathcal{N} = 1$ supersymmetric background by taking $A^{12} = A^{34}$, and break the $\mathrm{Spin}(5)_R$ symmetry group to $SU(2)_l \times U(1)_r$.

For the other background fields S_{ij}, G_{ab}, and t_{ab}, the Killing spinor equation (28) in the background satisfying

$$S + 2\cdot\frac{1}{4\alpha}G_{45} = 0, \qquad S + \frac{1}{4\alpha}G_{45} + t_{45} = \frac{1}{2f}, \tag{43}$$

where we have assumed that $S = S_{12} = S_{34}$, is reduced into (42). However, substituting them into the other supersymmetry condition (92), we find that the background

$$S = 0, \qquad \frac{1}{4\alpha}G_{45} = 0, \qquad t_{45} = \frac{1}{2f}, \tag{44}$$

is the only solution to (92), and that the background fields M_{IJ} are given by

$$\frac{4}{15}M_{55} = \frac{2}{f^2} - \frac{1}{5}R(\Omega) = \frac{2}{5}\left[\frac{4f^2 - r^2 - \tilde{r}^2}{f^4} - \frac{1}{2}R(\Sigma)\right],$$
$$\frac{4}{15}M_{ij} = -\delta_{ij}\frac{1}{15}M_{55}, \quad (i, j = 1, \cdots, 4).$$

It means that the scalar masses $\mathcal{M}_{B\,IJ}$ are

$$\mathcal{M}_{B\,55} = \frac{4}{f^2}, \qquad \mathcal{M}_{B\,ij} = \frac{1}{2}\left[\frac{r^2 + \tilde{r}^2}{f^2} + \frac{1}{2}R(\Sigma)\right]\delta_{ij}, \quad (i, j = 1, \cdots, 4).$$

6. The off-shell formulation of the reduced theory

In order to implement the localization technique in calculating the partition function of the five-dimensional theory in a supergravity background, a supersymmetry transformation for the localization must be defined off-shell. To this end, the half of the supersymmetry transformations (19) will be extended off-shell by introducing auxiliary fields. The supersymmetry parameters $\epsilon^{\tilde{\alpha}}$, $\varepsilon^{\dot{\alpha}}$ are decoupled in the supersymmetry transformations, if the background fields obey the conditions

$$t^i{}_{ab} = 0, \qquad S_{i5} = 0, \qquad A_\mu{}^{i5} = 0 \qquad (i = 1, \cdots, 4).$$

[10] The Killing spinors are the same as in [21].

The backgrounds we would like to consider in this paper, obey these conditions. Therefore, we will content ourselves with the construction of an off-shell formulation of the theory in this restricted type of backgrounds. In addition, the backgrounds in this paper also satisfy the condition $b_\mu = 0$, and we will add it to the above conditions.

We would like to use one of the supersymmetry transformations for the localization. Since the supersymmetry parameters $\epsilon^{\tilde{\alpha}}$, $\varepsilon^{\dot{\alpha}}$ are decoupled in the supersymmetry transformations in the restricted type of background, we will turn off the parameter $\epsilon^{\tilde{\alpha}}$ and focus only on $\varepsilon^{\dot{\alpha}}$. It is then convenient to regard the $\mathcal{N} = 2$ gauge multiplet as the sum of an $\mathcal{N} = 1$ gauge multiplet (σ, A_μ, $\lambda^{\dot{\alpha}}$) and $\mathcal{N} = 1$ hypermultiplets (ϕ^i, $\psi^{\tilde{\alpha}}$).

We will introduce an auxiliary field $D^{\dot{\alpha}}{}_{\dot{\beta}}$ in the adjoint representation of the $SU(2)_r$ subgroup of the $\mathrm{Spin}(5)_R$ R-symmetry group:

$$D^{\dot{\alpha}\dot{\beta}} = \varepsilon^{\dot{\beta}\dot{\gamma}} D^{\dot{\alpha}}{}_{\dot{\gamma}} = D^{\dot{\beta}\dot{\alpha}},$$

and replace

$$\left(\bar{\sigma}^{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}} S_{ij}\sigma + \frac{i}{2} g \left[\phi^i, \phi^j\right] \left(\bar{\sigma}_{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \quad \rightarrow \quad D^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{45}$$

in (33), to which the supersymmetry transformation (19) is reduced in the restricted type of backgrounds.

In a consistent way to the above replacement, the supersymmetry transformation of $D^{\dot{\alpha}}{}_{\dot{\beta}}$ is determined by using the equation of motion of $\lambda^{\dot{\alpha}}$, and one obtains

$$\begin{aligned}
&\delta_\epsilon A_\mu = -\frac{i}{4}(\bar{\varepsilon} \cdot \gamma_\mu \lambda), \qquad \delta_\epsilon \sigma = -\frac{i}{4}(\bar{\varepsilon} \cdot \lambda), \\
&\delta_\epsilon \lambda^{\dot{\alpha}} = -\frac{1}{2} F_{ab}\gamma^{ab}\varepsilon^{\dot{\alpha}} + \gamma^a \mathcal{D}_a \sigma \cdot \varepsilon^{\dot{\alpha}} + D^{\dot{\alpha}}{}_{\dot{\beta}}\varepsilon^{\dot{\beta}} - \frac{1}{2\alpha} G_{ab}\sigma\gamma^{ab}\varepsilon^{\dot{\alpha}}, \\
&\delta_\epsilon D^{\dot{\alpha}\dot{\beta}} = -\frac{i}{4}\bar{\varepsilon}^{(\dot{\alpha}} \left[\gamma^a \mathcal{D}_a \lambda^{\dot{\beta})} - ig\left[\sigma, \lambda^{\dot{\beta})}\right] - \frac{1}{8\alpha} G_{ab}\gamma^{ab}\lambda^{\dot{\beta})} - \frac{1}{4} S_{ij}\left(\bar{\sigma}^{ij}\right)^{\dot{\beta})}{}_{\dot{\gamma}}\lambda^{\dot{\gamma}} \right. \\
&\qquad\qquad \left. - \frac{1}{2} t_{ab}\gamma^{ab}\lambda^{\dot{\beta})}\right] + \frac{i}{4} S_{ij}\left(\bar{\sigma}^{ij}\right)^{\dot{\alpha}}{}_{\dot{\gamma}}\varepsilon^{\dot{\gamma}\dot{\beta}}\left(\bar{\eta} \cdot \lambda\right).
\end{aligned} \tag{46}$$

The off-shell supersymmetry transformations (46) are closed into the other bosonic transformations. Using the Killing spinor equation (28) in this type of backgrounds, one obtains

$$\begin{aligned}
&\left[\delta_\epsilon, \delta_\eta\right] A_\mu = \frac{i}{2}\xi^\nu F_{\mu\nu} - \frac{i}{2}\left(\bar{\eta} \cdot \varepsilon\right) \mathcal{D}_\mu \sigma + \frac{i}{2\alpha}\xi^\nu G_{\mu\nu}\sigma \\
&\qquad\qquad = -\frac{i}{2}\left(\xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu \cdot A_\nu\right) - \partial_\mu \Lambda_G - ig\left[A_\mu, \Lambda_G\right], \\
&\left[\delta_\epsilon, \delta_\eta\right]\sigma = -\frac{i}{2}\xi^a \mathcal{D}_a \sigma = -\frac{i}{2}\xi^a \partial_a \sigma + \Lambda_D \sigma + ig\left[\Lambda_G, \sigma\right], \\
&\left[\delta_\epsilon, \delta_\eta\right]\lambda^{\dot{\alpha}} \\
&\quad = -\frac{i}{2}\xi^a \mathcal{D}_a \lambda^{\dot{\alpha}} + \frac{i}{2}\cdot(ig)\left[\left(\bar{\eta} \cdot \varepsilon\right)\sigma, \lambda^{\dot{\alpha}}\right] - \frac{i}{8}\left(\mathcal{D}_a \xi_b\right)\gamma^{ab}\lambda^{\dot{\alpha}} - \frac{1}{4}\Lambda^{(0)}_{ij}\left(\bar{\sigma}^{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\lambda^{\dot{\beta}} \\
&\quad = -\frac{i}{2}\xi^\mu \partial_\mu \lambda^{\dot{\alpha}} + \frac{3}{2}\Lambda_D \lambda^{\dot{\alpha}} + ig\left[\Lambda_G, \lambda^{\dot{\alpha}}\right] + \frac{1}{4}\Lambda_{ab}\gamma^{ab}\lambda^{\dot{\alpha}} - \frac{1}{4}\Lambda^{ij}\left(\bar{\sigma}_{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\lambda^{\dot{\beta}},
\end{aligned}$$

$$\begin{aligned}
&\left[\delta_\epsilon, \delta_\eta\right] D^{\dot\alpha\dot\beta} \\
&= -\frac{i}{2}\xi^a \mathcal{D}_a D^{\dot\alpha\dot\beta} + \frac{i}{2}\cdot(ig)\left[(\bar\eta\cdot\epsilon)\,\sigma,\ D^{\dot\alpha\dot\beta}\right] - \frac{1}{4}\Lambda^{(0)}_{ij}\left[\left(\bar\sigma^{ij}\right)^{\dot\alpha}{}_{\dot\gamma} D^{\dot\gamma\dot\beta} + \left(\bar\sigma^{ij}\right)^{\dot\beta}{}_{\dot\gamma} D^{\dot\alpha\dot\gamma}\right] \\
&= -\frac{i}{2}\xi^\mu\partial_\mu D^{\dot\alpha\dot\beta} + 2\Lambda_D D^{\dot\alpha\dot\beta} + ig\left[\Lambda_G,\ D^{\dot\alpha\dot\beta}\right] \\
&\quad -\frac{1}{4}\Lambda_{ij}\left[\left(\bar\sigma^{ij}\right)^{\dot\alpha}{}_{\dot\gamma} D^{\dot\gamma\dot\beta} + \left(\bar\sigma^{ij}\right)^{\dot\beta}{}_{\dot\gamma} D^{\dot\alpha\dot\gamma}\right],
\end{aligned} \tag{47}$$

with the transformation parameters

$$\begin{aligned}
&\xi^a = \bar\eta\cdot\gamma^a\varepsilon, \qquad \Lambda_D = \frac{i}{2}\xi^a b_a, \qquad \Lambda_{ab} = -\frac{i}{2}\left[\mathcal{D}_a\xi_b + \xi^\mu\left(\Omega_\mu\right)_{ab}\right], \\
&\Lambda_G = -\frac{i}{2}\left[\xi^a A_a - (\bar\eta\cdot\varepsilon)\,\sigma\right], \\
&\Lambda^{(0)}_{ij} = -\frac{i}{2}\left[3S_{ij}\left(\bar\eta\cdot\epsilon\right) + \left(t_{ab} + \frac{1}{2\alpha}G_{ab}\right)\left(\bar\eta\cdot\bar\sigma_{ij}\gamma^{ab}\epsilon\right)\right], \\
&\Lambda_{ij} = \Lambda^{(0)}_{ij} - \frac{i}{2}\left(\xi^a A_a\right)_{ij}.
\end{aligned} \tag{48}$$

Let us proceed to the hypermultiplets $(\phi^i,\ \psi^{\tilde\alpha})$. In writing the off-shell transformation for them, it will turn out that the spinor index notation of the scalars $\phi^{\tilde\alpha}{}_{\dot\beta}$,

$$\phi^{\tilde\alpha}{}_{\dot\beta} = \phi^i\left(\sigma_i\right)^{\tilde\alpha}{}_{\dot\beta}, \qquad \bar\phi^{\dot\alpha}{}_{\tilde\beta} = \phi^i\left(\bar\sigma_i\right)^{\dot\alpha}{}_{\tilde\beta},$$

will be convenient. In order to formulate an off-shell supersymmetry transformation[11] of the hypermultiplets, we will introduce auxiliary fields $F^{\tilde\alpha}{}_{\check\beta}$ with the index $\check\beta$ labeling a doublet of a new $SU(2)$ flavor group, which is not a subgroup of the $\mathrm{Spin}(5)_R$ R-symmetry group, following [29].

Further, we will also introduce different supersymmetry parameters $\varepsilon^{\check\alpha}$ from $\varepsilon^{\dot\alpha}$ and $\epsilon^{\tilde\alpha}$, the former of which span the whole four-dimensional spinor space with $\varepsilon^{\dot\alpha}$.

The auxiliary fields $F^{\tilde\alpha}{}_{\check\beta}$ and the new parameters $\varepsilon^{\check\alpha}$ are expected to play the role to impose the equation of motion of the spinors in the off-shell supersymmetry formulation. To this end, requiring that $\delta_\epsilon F^{\tilde\alpha}{}_{\check\beta}$ be proportional to the equation of motion of the spinor $\psi^{\tilde\alpha}$, one obtains an off-shell supersymmetry transformation

$$\begin{aligned}
\delta_\epsilon\phi^{\tilde\alpha}{}_{\dot\beta} &= \frac{i}{2}\left(\bar\varepsilon_{\dot\beta}\cdot\psi^{\tilde\alpha}\right), \\
\delta_\epsilon\psi^{\tilde\alpha} &= -\gamma^a\mathcal{D}_a\phi^{\tilde\alpha}{}_{\dot\beta}\cdot\varepsilon^{\dot\beta} + ig\left[\sigma,\ \phi^{\tilde\alpha}{}_{\dot\beta}\right]\varepsilon^{\dot\beta} + \left(t_{ab} + \frac{1}{2\alpha}G_{ab}\right)\phi^{\tilde\alpha}{}_{\dot\beta}\gamma^{ab}\varepsilon^{\dot\beta} \\
&\quad - \frac{1}{4}S_{ij}\left[3\phi^{\tilde\alpha}{}_{\dot\gamma}\left(\bar\sigma^{ij}\right)^{\dot\gamma}{}_{\dot\beta} + \left(\sigma^{ij}\right)^{\tilde\alpha}{}_{\tilde\gamma}\phi^{\tilde\gamma}{}_{\dot\beta}\right]\varepsilon^{\dot\beta} + F^{\tilde\alpha}{}_{\check\beta}\varepsilon^{\check\beta}, \\
\delta_\epsilon F^{\tilde\alpha}{}_{\check\beta} &= \frac{i}{2}\bar\varepsilon_{\check\beta}\left[\gamma^a\mathcal{D}_a\psi^{\tilde\alpha} + ig\left[\sigma,\ \psi^{\tilde\alpha}\right] + ig\left[\phi^{\tilde\alpha}{}_{\dot\gamma},\ \lambda^{\dot\gamma}\right]\right. \\
&\quad \left. + \frac{1}{2}\left(t_{ab} - \frac{1}{4\alpha}G_{ab}\right)\gamma^{ab}\psi^{\tilde\alpha} - \frac{1}{4}S_{ij}\left(\sigma^{ij}\right)^{\tilde\alpha}{}_{\tilde\gamma}\psi^{\tilde\gamma}\right].
\end{aligned} \tag{49}$$

---

[11] We will follow the prescription in [29,1,2], where we won't expect to have a full-fledged off-shell formulation. However, it will be sufficient to carry out the localization.

In terms of the vector notation of the scalars ϕ^i, it gives

$$\begin{aligned}
\delta_\epsilon \phi^i &= \frac{i}{4}(\bar{\sigma}^i)^{\dot{\alpha}}{}_{\tilde{\beta}}(\bar{\varepsilon}_{\dot{\alpha}} \cdot \psi^{\tilde{\beta}}), \\
\delta_\epsilon \psi^{\tilde{\alpha}} &= (\sigma_i)^{\tilde{\alpha}}{}_{\dot{\beta}}\Big[-\gamma^a \mathcal{D}_a \phi^i \cdot \varepsilon^{\dot{\beta}} + ig\left[\sigma, \phi^i\right]\varepsilon^{\dot{\beta}} + \left(t_{ab} + \frac{1}{2\alpha}G_{ab}\right)\phi^i \gamma^{ab}\varepsilon^{\dot{\beta}} \\
&\qquad + \left(S^{ij} + \varepsilon^{ijkl}S_{kl}\right)\phi_j \varepsilon^{\dot{\beta}}\Big] + F^{\tilde{\alpha}}{}_{\check{\beta}}\varepsilon^{\check{\beta}}, \\
\delta_\epsilon F^{\tilde{\alpha}}{}_{\check{\beta}} &= \frac{i}{2}\bar{\varepsilon}_{\check{\beta}}\Big[\gamma^a \mathcal{D}_a \psi^{\tilde{\alpha}} + ig\left[\sigma, \psi^{\tilde{\alpha}}\right] + ig(\sigma_i)^{\tilde{\alpha}}{}_{\dot{\gamma}}\left[\phi^i, \lambda^{\dot{\gamma}}\right] \\
&\qquad + \frac{1}{2}\left(t_{ab} - \frac{1}{4\alpha}G_{ab}\right)\gamma^{ab}\psi^{\tilde{\alpha}} - \frac{1}{4}S_{ij}\left(\sigma^{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\gamma}}\psi^{\tilde{\gamma}}\Big],
\end{aligned}$$

where the conditions are assumed:

$$\begin{aligned}
&\left(\varepsilon^{\check{\alpha}}\right)^T C\eta^{\dot{\beta}} - \left(\eta^{\check{\alpha}}\right)^T C\varepsilon^{\dot{\beta}} = 0, \qquad (\bar{\eta}_{\dot{\beta}}\,\varepsilon^{\dot{\beta}}) = (\bar{\eta}_{\check{\beta}}\,\varepsilon^{\check{\beta}}), \\
&(\bar{\eta}_{\dot{\beta}}\gamma^a \varepsilon^{\dot{\beta}}) = -(\bar{\eta}_{\check{\beta}}\gamma^a \varepsilon^{\check{\beta}}),
\end{aligned} \tag{50}$$

which will be necessary for the off-shell closure of the supersymmetry transformations on $\phi^{\tilde{\alpha}}{}_{\dot{\beta}}$ and $\psi^{\tilde{\alpha}}$. Note that in terms of the spinor index notation, the covariant derivative $\mathcal{D}_\mu \phi^{\tilde{\alpha}}{}_{\dot{\beta}}$ can be read as

$$\mathcal{D}_\mu \phi^{\tilde{\alpha}}{}_{\dot{\beta}} = \partial_\mu \phi^{\tilde{\alpha}}{}_{\dot{\beta}} - b_\mu \phi^{\tilde{\alpha}}{}_{\dot{\beta}} + ig\left[A_\mu, \phi^{\tilde{\alpha}}{}_{\dot{\beta}}\right] - \frac{1}{4}A_\mu{}^{ij}\left[\left(\sigma_{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\gamma}}\phi^{\tilde{\gamma}}{}_{\dot{\beta}} - \phi^{\tilde{\alpha}}{}_{\dot{\gamma}}\left(\bar{\sigma}_{ij}\right)^{\dot{\gamma}}{}_{\dot{\beta}}\right].$$

Making use of (50), one can verify that

$$\begin{aligned}
\left[\delta_\epsilon, \delta_\eta\right]\phi^{\tilde{\alpha}}{}_{\dot{\beta}} &= -\frac{i}{2}\xi^a \mathcal{D}_a \phi^{\tilde{\alpha}}{}_{\dot{\beta}} + \frac{i}{2}\cdot(ig)\left[(\bar{\eta}\cdot\varepsilon)\sigma, \phi^{\tilde{\alpha}}{}_{\dot{\beta}}\right] + \frac{1}{4}\Lambda^{(0)}_{ij}\phi^{\tilde{\alpha}}{}_{\dot{\gamma}}\left(\bar{\sigma}^{ij}\right)^{\dot{\gamma}}{}_{\dot{\beta}} \\
&\qquad - \frac{1}{4}\tilde{\Lambda}^{(0)}_{ij}\left(\sigma^{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\gamma}}\phi^{\tilde{\gamma}}{}_{\dot{\beta}} \\
&= -\frac{i}{2}\xi^\mu \partial_\mu \phi^{\tilde{\alpha}}{}_{\dot{\beta}} + \Lambda_D\,\phi^{\tilde{\alpha}}{}_{\dot{\beta}} + ig\left[\Lambda_G, \phi^{\tilde{\alpha}}{}_{\dot{\beta}}\right] \\
&\qquad + \frac{1}{4}\Lambda_{ij}\phi^{\tilde{\alpha}}{}_{\dot{\gamma}}\left(\bar{\sigma}^{ij}\right)^{\dot{\gamma}}{}_{\dot{\beta}} - \frac{1}{4}\tilde{\Lambda}_{ij}\left(\sigma^{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\gamma}}\phi^{\tilde{\gamma}}{}_{\dot{\beta}}, \\
\left[\delta_\epsilon, \delta_\eta\right]\psi^{\tilde{\alpha}} & \\
&= -\frac{i}{2}\xi^a \mathcal{D}_a \psi^{\tilde{\alpha}} - \frac{i}{8}\left(\mathcal{D}_a \xi_b\right)\gamma^{ab}\psi^{\tilde{\alpha}} + \frac{i}{2}\cdot(ig)\left[(\bar{\eta}\cdot\varepsilon)\sigma, \phi^{\tilde{\alpha}}{}_{\dot{\beta}}\right] - \frac{1}{4}\tilde{\Lambda}^{(0)}_{ij}\left(\sigma^{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\gamma}}\psi^{\tilde{\gamma}} \\
&= -\frac{i}{2}\xi^\mu \partial_\mu \psi^{\tilde{\alpha}} + \frac{3}{2}\Lambda_D\,\psi^{\tilde{\alpha}} + \frac{1}{4}\Lambda_{ab}\gamma^{ab}\psi^{\tilde{\alpha}} + ig\left[\Lambda_G, \phi^{\tilde{\alpha}}{}_{\dot{\beta}}\right] - \frac{1}{4}\tilde{\Lambda}_{ij}\left(\sigma^{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\gamma}}\psi^{\tilde{\gamma}},
\end{aligned} \tag{51}$$

with the parameters in (48), where the transformation parameters of the other $SU(2)$ subgroup of the $\mathrm{Spin}(5)_R$ are given by

$$\tilde{\Lambda}^{(0)}_{ij} = \frac{i}{2}(\bar{\eta}\cdot\varepsilon)S_{ij}, \qquad \tilde{\Lambda}_{ij} = \tilde{\Lambda}^{(0)}_{ij} - \frac{i}{2}\xi^\mu A_{\mu ij}.$$

So far, we have seen that the supersymmetry transformations (49) on $\phi^{\tilde{\alpha}}{}_{\dot{\beta}}$, $\psi^{\tilde{\alpha}}$ are closed off-shell, if we require the condition (50) on the supersymmetry parameters. However, we will see that the supersymmetry transformations (49) on the auxiliary fields $F^{\tilde{\alpha}}{}_{\breve{\beta}}$ are not automatically closed. In order to achieve an off-shell supersymmetry transformation, it seems that one has to require the supergravity backgrounds to obey additional conditions.

Let us look at the supersymmetry transformations on $F^{\tilde{\alpha}}{}_{\breve{\beta}}$. Using the condition (50), one obtains

$$\begin{aligned}\left[\delta_\epsilon, \delta_\eta\right] F^{\tilde{\alpha}}{}_{\breve{\beta}} &= -\frac{i}{2}\xi^\mu \partial_\mu F^{\tilde{\alpha}}{}_{\breve{\gamma}} + \Lambda_D F^{\tilde{\alpha}}{}_{\breve{\gamma}} + ig\left[\Lambda_G, F^{\tilde{\alpha}}{}_{\breve{\gamma}}\right] + \frac{1}{4}\tilde{\Lambda}_{ij}\left(\sigma_{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\gamma}} F^{\tilde{\gamma}}{}_{\breve{\beta}} \\ &\quad + \breve{\Lambda}_{\breve{\beta}}{}^{\breve{\gamma}} F^{\tilde{\alpha}}{}_{\breve{\gamma}} + \frac{1}{4}\left(\Theta^{ij}\right)_{\breve{\beta}}{}^{\dot{\gamma}}\left(\sigma_{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\delta}}\phi^{\tilde{\delta}}{}_{\dot{\gamma}} + \Delta_{\breve{\beta}}{}^{\dot{\gamma}}\phi^{\tilde{\alpha}}{}_{\dot{\gamma}},\end{aligned} \tag{52}$$

where the parameter $\breve{\Lambda}_{\breve{\alpha}}{}^{\breve{\beta}}$ of the new $SU(2)$ transformation is given by

$$\begin{aligned}\breve{\Lambda}_{\breve{\alpha}}{}^{\breve{\beta}} &= \frac{i}{2}\bigg[\left(\bar{\eta}_{\breve{\beta}}\gamma^a\mathcal{D}_a\varepsilon^{\breve{\gamma}} - \bar{\varepsilon}_{\breve{\beta}}\gamma^a\mathcal{D}_a\eta^{\breve{\gamma}}\right) \\ &\quad + \frac{1}{2}\left(t_{ab} - \frac{1}{4\alpha}G_{ab}\right)\left(\bar{\eta}_{\breve{\beta}}\gamma^{ab}\varepsilon^{\breve{\gamma}} - \bar{\varepsilon}_{\breve{\beta}}\gamma^{ab}\eta^{\breve{\gamma}}\right)\bigg],\end{aligned}$$

and the last two terms on the right hand side suggest that the supersymmetry transformations fail to be closed off-shell, where the parameters $\left(\Theta^{ij}\right)_{\breve{\beta}}{}^{\dot{\gamma}}$, and $\Delta_{\breve{\beta}}{}^{\dot{\gamma}}$ are given by

$$\begin{aligned}\left(\Theta^{ij}\right)_{\breve{\beta}}{}^{\dot{\gamma}} &= i\left(\bar{\eta}_{\breve{\beta}}\gamma^{ab}\varepsilon^{\dot{\gamma}} - \bar{\varepsilon}_{\breve{\beta}}\gamma^{ab}\eta^{\dot{\gamma}}\right)\left(\frac{1}{4}F_{ab}{}^{ij} + \frac{1}{4\alpha}G_{ab}S^{ij}\right) \\ &\quad - \frac{i}{2}\left(\bar{\eta}_{\breve{\beta}}\gamma^a\varepsilon^{\dot{\gamma}} - \bar{\varepsilon}_{\breve{\beta}}\gamma^a\eta^{\dot{\gamma}}\right)\mathcal{D}_a S^{ij},\end{aligned}$$

$$\begin{aligned}\Delta_{\breve{\beta}}{}^{\dot{\gamma}} &= \frac{i}{2}\bigg[\left(\frac{1}{4\alpha}\right)^2 G_{ab}G_{cd}\left(\bar{\eta}_{\breve{\beta}}\gamma^{abcd}\varepsilon^{\dot{\gamma}} - \bar{\varepsilon}_\beta\gamma^{abcd}\eta^{\dot{\gamma}}\right) \\ &\quad - \left(\mathcal{D}^a t_{ab} + \frac{1}{4\alpha}\mathcal{D}^a G_{ab}\right)\left(\bar{\eta}_{\breve{\beta}}\gamma^b\varepsilon^{\dot{\gamma}} - \bar{\varepsilon}_\beta\gamma^b\eta^{\dot{\gamma}}\right)\bigg] \\ &\quad + \frac{i}{2}\left(\bar{\sigma}^{ij}\right)^{\dot{\gamma}}{}_{\dot{\delta}}\bigg[\frac{1}{4}\mathcal{D}_a S_{ij}\left(\bar{\eta}_{\breve{\beta}}\gamma^a\varepsilon^{\dot{\delta}} - \bar{\varepsilon}_\beta\gamma^a\eta^{\dot{\delta}}\right) \\ &\quad - \frac{1}{2}\left(t_{ab} + 2\frac{1}{4\alpha}G_{ab}\right)S_{ij}\left(\bar{\eta}_{\breve{\beta}}\gamma^{ab}\varepsilon^{\dot{\delta}} - \bar{\varepsilon}_\beta\gamma^{ab}\eta^{\dot{\delta}}\right)\bigg].\end{aligned}$$

Therefore, in order to gain an off-shell closed algebra of the supersymmetry transformations, the conditions

$$\left(\Theta^{ij}\right)_{\breve{\beta}}{}^{\dot{\gamma}}\left(\sigma_{ij}\right)^{\tilde{\alpha}}{}_{\tilde{\delta}} = 0, \qquad \Delta_{\breve{\beta}}{}^{\dot{\gamma}} = 0, \tag{53}$$

are required. Although the implications of the condition (53) have been unclear for the authors, the backgrounds in Section 5 satisfy the condition (53). So henceforth, we will assume that the backgrounds satisfy the condition (53).

Let us make sure that the supersymmetric backgrounds in Section 5 obey the conditions (50) and (53). With the ansatz (29) and as explained in Appendix G, the supersymmetry parameters $\varepsilon^{\dot{\alpha}}$, $\varepsilon^{\breve{\alpha}}$ are given by

$$\varepsilon^{\dot{\alpha}=1} = \epsilon\otimes\zeta_+, \quad \varepsilon^{\dot{\alpha}=2} = C_3^{-1}\epsilon^*\otimes\zeta_-; \qquad \varepsilon^{\breve{\alpha}=1} = \epsilon\otimes\zeta_-, \quad \varepsilon^{\breve{\alpha}=2} = C_3^{-1}\epsilon^*\otimes\zeta_+,$$

obeying that $\varepsilon^{\check{\alpha}} = \gamma_5 \varepsilon^{\check{\alpha}}$. It follows from this that

$$\left(\varepsilon^{\check{\alpha}}\right)^T C\eta^{\dot{\beta}} - \left(\eta^{\check{\alpha}}\right)^T C\varepsilon^{\dot{\beta}} = 0, \qquad \bar{\eta}_{\check{\alpha}}\varepsilon^{\check{\alpha}} = \bar{\eta}_{\dot{\alpha}}\varepsilon^{\dot{\alpha}}, \qquad \bar{\eta}_{\check{\alpha}}\gamma^a\varepsilon^{\check{\alpha}} = -\bar{\eta}_{\dot{\alpha}}\gamma^a\varepsilon^{\dot{\alpha}},$$

and the condition (50) is satisfied by the supersymmetry parameters $\varepsilon^{\dot{\alpha}}, \varepsilon^{\check{\alpha}}$.

Since the field S_{ij} in all the $\mathcal{N} = 1$ supersymmetric backgrounds in Section 5 satisfies $S^{ij}\sigma_{ij} = 0$, and the field strength $F_{ab}{}^{ij}$ satisfies $F_{ab}{}^{ij}\sigma_{ij} = 0$, it is easy to see that they satisfy the former condition in (53). In the $\mathcal{N} = 2$ background in Subsection 5.3, the non-zero components of $F_{ab}{}^{ij}$ are $F_{45}{}^{ij}$ and $F_{12}{}^{ij}$, and G_{ab} has only nonzero component G_{45}. At the first sight, the nonzero $F_{45}{}^{ij}$ and G_{45} seems to give the contributions to the former condition of (53), but, since we have the formula

$$\bar{\eta}_{\check{\alpha}}\gamma^{45}\varepsilon^{\dot{\beta}} - \bar{\varepsilon}_{\check{\alpha}}\gamma^{45}\eta^{\dot{\beta}} = 0, \tag{54}$$

they yield no contributions to the condition. Furthermore, the nonzero $F_{12}{}^{ij}$ appears on the left hand side of the condition with the term

$$\bar{\eta}_{\check{\alpha}}\gamma^{12}\varepsilon^{\dot{\beta}} - \bar{\varepsilon}_{\check{\alpha}}\gamma^{12}\eta^{\dot{\beta}} \quad \propto \quad \bar{\eta}_{\check{\alpha}}\gamma^{3}\varepsilon^{\dot{\beta}} - \bar{\varepsilon}_{\check{\alpha}}\gamma^{3}\eta^{\dot{\beta}}, \tag{55}$$

but, the conditions $\gamma^3\varepsilon^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}}$ and $\gamma^3\eta^{\dot{\alpha}} = \eta^{\dot{\alpha}}$ reduce $\bar{\eta}_{\check{\alpha}}\gamma^{3}\varepsilon^{\dot{\beta}} - \bar{\varepsilon}_{\check{\alpha}}\gamma^{3}\eta^{\dot{\beta}}$ to the left hand side of the first condition in (50). Therefore, the former condition in (53) is obeyed also for the $\mathcal{N} = 2$ supersymmetric background.

The covariant derivatives $\mathcal{D}_a t_{bc}$, $\mathcal{D}_a G_{bc}$, and $\mathcal{D}_a S_{ij}$ are vanishing except for the background in Subsection 5.4. However, ever for the background, $\mathcal{D}^a t_{ab} + \mathcal{D}^a G_{ab}/(4\alpha) = 0$. Further, it is obvious that $G_{ab}G_{cd}\varepsilon^{abcde} = 0$. Therefore, taking (54) into account, one can see that all the backgrounds in Section 5 also satisfy the latter condition in (53). Now we can see that all the supersymmetry backgrounds in (53) allow the off-shell supersymmetry.

Let us proceed to the construction of an off-shell supersymmetric action. In order to perform the replacement (45) with $D^{\dot{\alpha}}{}_{\dot{\beta}}$ within the on-shell invariant action $S = S_F + S_B^{(0)} + S_B^{(1)}$ in (22)–(25), we will add the term

$$\begin{aligned}
&\frac{1}{2}\int dx^5\sqrt{g}\,\alpha\,\mathrm{tr}\Big[-\frac{1}{2}\hat{D}^{\dot{\alpha}}{}_{\dot{\beta}}\hat{D}^{\dot{\beta}}{}_{\dot{\alpha}}\Big] \\
&\quad\text{with}\quad \hat{D}^{\dot{\alpha}}{}_{\dot{\beta}} \equiv D^{\dot{\alpha}}{}_{\dot{\beta}} - \left(\bar{\sigma}^{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\left(S_{ij}\sigma + \frac{1}{2}(ig)\left[\phi_i, \phi_j\right]\right)
\end{aligned}$$

to the on-shell action S.

For the hypermultiplets, the off-shell supersymmetry transformation of $\psi^{\tilde{\alpha}}$ has an additional term $F'^{a}{}_{\check{\gamma}}\varepsilon^{\check{\gamma}}$, compared to the on-shell supersymmetry transformations. Therefore, under the off-shell supersymmetry transformation, the on-shell fermionic action S_F in (22) gains an additional term

$$\begin{aligned}
&-\frac{1}{2}\varepsilon_{\tilde{\alpha}\tilde{\beta}}F^{\tilde{\alpha}}{}_{\check{\gamma}}\left(\varepsilon^{\check{\gamma}}\right)^T C\left(\gamma^a\mathcal{D}_a\psi^{\tilde{\beta}} + \cdots\right) \\
&\quad = -\frac{1}{2}\varepsilon_{\tilde{\alpha}\tilde{\beta}}F^{\tilde{\alpha}}{}_{\check{\gamma}}\cdot\varepsilon^{\check{\gamma}\check{\delta}}\delta_\epsilon F^{\tilde{\beta}}{}_{\check{\delta}} = -\frac{1}{4}\delta_\epsilon\left(\varepsilon_{\tilde{\alpha}\tilde{\beta}}\varepsilon^{\check{\gamma}\check{\delta}}F^{\tilde{\alpha}}{}_{\check{\gamma}}F^{\tilde{\beta}}{}_{\check{\delta}}\right)
\end{aligned}$$

with the ellipsis denoting the other terms of the equation of motion for $\psi^{\tilde{\alpha}}$, from the term

$$-\frac{i}{4}\varepsilon_{\tilde{\alpha}\tilde{\beta}}\left(\delta_\epsilon\psi^{\tilde{\alpha}}\right)^T C\left(\gamma^a\mathcal{D}_a\psi^{\tilde{\beta}} + \cdots\right) + \cdots$$

in $\delta_\epsilon S_F$. Thus, it is necessary to add the term

$$-\int \sqrt{g}\, d^5x\, \frac{\alpha}{4}\, \mathrm{tr}\left[\varepsilon_{\tilde{\alpha}\tilde{\beta}}\varepsilon^{\check{\gamma}\check{\delta}} F^{\tilde{\alpha}}{}_{\check{\gamma}} F^{\tilde{\beta}}{}_{\check{\delta}}\right]$$

to cancel the additional term from $\delta_\epsilon S_F$.

Finally, the construction of an off-shell action is achieved as

$$\mathcal{S} = -\int \mathrm{tr}\left[\frac{1}{2} C \wedge F \wedge F + \frac{\alpha}{2}(F - 4t\sigma) \wedge *(F - 4t\sigma)\right] - \int \sqrt{g}\, d^5x\, \alpha\, \mathrm{tr}[\mathcal{L}], \tag{56}$$

where the ‘matter’ Lagrangian $\mathcal{L}$ is given by

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\mathcal{D}_a\sigma\mathcal{D}^a\sigma - \frac{1}{2}\mathcal{D}_a\phi^i\mathcal{D}^a\phi^i - \frac{1}{2}\mathcal{M}_\sigma\sigma^2 - \frac{1}{2}\mathcal{M}_{ij}\phi^i\phi^j + \frac{1}{4}D^{\dot{\alpha}}{}_{\dot{\beta}}D^{\dot{\beta}}{}_{\dot{\alpha}} \\
&+ \frac{1}{4}\varepsilon_{\tilde{\alpha}\tilde{\beta}}\varepsilon^{\check{\gamma}\check{\delta}} F^{\tilde{\alpha}}{}_{\check{\gamma}} F^{\tilde{\beta}}{}_{\check{\delta}} - \frac{1}{2}D^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{\sigma}^{ij})^{\dot{\beta}}{}_{\dot{\alpha}}\left(S_{ij}\sigma + \frac{1}{2}(ig)\left[\phi_i, \phi_j\right]\right) \\
&+ \frac{1}{2}(ig)^2\left[\sigma, \phi^i\right]\left[\sigma, \phi^i\right] - igS_{ij}\sigma\left[\phi^i, \phi^j\right] - \frac{i}{8}\bar{\lambda}\cdot\gamma^a\mathcal{D}_a\lambda - \frac{i}{8}\bar{\psi}\cdot\gamma^a\mathcal{D}_a\psi \\
&- \frac{i}{16}\left(t_{ab} - \frac{1}{4\alpha}G_{ab}\right)\bar{\psi}\cdot\gamma^{ab}\psi + \frac{i}{16}\left(t_{ab} + \frac{1}{4\alpha}G_{ab}\right)\bar{\lambda}\cdot\gamma^{ab}\lambda \\
&+ \frac{i}{32}S_{ij}\bar{\psi}\cdot\sigma^{ij}\psi + \frac{i}{32}S_{ij}\bar{\lambda}\cdot\bar{\sigma}^{ij}\lambda - \frac{i}{8}(ig)\bar{\psi}_{\tilde{\alpha}}\cdot\left[\sigma, \psi^{\tilde{\alpha}}\right] + \frac{i}{8}(ig)\bar{\lambda}_{\dot{\alpha}}\cdot\left[\sigma, \lambda^{\dot{\alpha}}\right] \\
&- \frac{i}{8}(ig)\bar{\psi}_{\tilde{\alpha}}\cdot(\sigma_i)^{\tilde{\alpha}}{}_{\dot{\beta}}\left[\phi^i, \lambda^{\dot{\beta}}\right] - \frac{i}{8}(ig)\bar{\lambda}_{\dot{\alpha}}\cdot(\bar{\sigma}_i)^{\dot{\alpha}}{}_{\tilde{\beta}}\left[\phi^i, \psi^{\tilde{\beta}}\right],
\end{aligned} \tag{57}$$

with the ‘mass’ parameters

$$\begin{aligned}
\mathcal{M}_\sigma &= \left(\frac{4}{15}M_{55} + \frac{1}{5}R(\Omega) + \frac{1}{20\alpha^2}G_{ab}G^{ab} + 4t_{ab}t^{ab} - \frac{1}{2}\mathrm{tr}(\bar{\sigma}^{ij}\bar{\sigma}^{kl})S_{ij}S_{kl}\right), \\
\mathcal{M}_{ij} &= \left[\frac{4}{15}M_{ij} + \frac{1}{5}\left(R(\Omega) + \frac{1}{4\alpha^2}G_{ab}G^{ab}\right)\delta_{ij} - S_i{}^k S_{jk}\right].
\end{aligned}$$

7. Localization and twistings

In this section, let us proceed to compute the partition function of the theory by using the localization. Before going on, there is a subtle point that the kinetic terms of the fields σ, $D^{\dot{\alpha}}{}_{\dot{\beta}}$, ϕ^i, and $F^{\tilde{\alpha}}{}_{\check{\beta}}$ have the negative sign in the Lagrangian (57). In order to circumvent it, we would like to follow the same strategy for σ, $D^{\dot{\alpha}}{}_{\dot{\beta}}$, and $F^{\tilde{\alpha}}{}_{\check{\beta}}$ as in [1,2]. Recall that the scalars ϕ^i had the positive kinetic terms in [1,2], where the five-dimensional theory was obtained by the dimensional reduction from the six-dimensional maximally supersymmetric Yang–Mills theory.

To this end, we will perform the ‘analytic continuation’ for the scalars,

$$\sigma \quad \to \quad i\sigma, \qquad \phi^i \quad \to \quad -i\phi^i.$$

For the auxiliary fields $D^{\dot{\alpha}}{}_{\dot{\beta}}$ and $F^{\tilde{\alpha}}{}_{\check{\beta}}$, let us carefully recall what we have done in the previous papers [1,2]. First, we have shifted $D^{\mathrm{i}}{}_{\mathrm{i}}$ as[12]

$$D^{\mathrm{i}}{}_{\mathrm{i}} \quad \to \quad D^{\mathrm{i}}{}_{\mathrm{i}} - iF_{45} - \frac{i}{\alpha}G_{45}\sigma, \tag{58}$$

12 In the previous papers [1,2], although we haven’t considered the supergravity backgrounds such as G_{ab}, we can interpret what was done as the shift (58) in terms of supergravity backgrounds.

and we then impose the reality condition

$$\left(D^{\dot\alpha}{}_{\dot\beta}\right)^* = D^{\dot\beta}{}_{\dot\alpha}.$$

In previous papers, we implicitly left the sign of the kinetic term of $F^{\tilde\alpha}{}_{\check\beta}$ negative. Since the integration over the auxiliary fields $F^{\tilde\alpha}{}_{\check\beta}$ is trivial; there is no dependence on the vacuum expectation value of σ, we have just ignored the divergence from it. Therefore, we assumed that

$$\left(F^{\tilde\alpha}{}_{\check\beta}\right)^* = -F^{\tilde\gamma}{}_{\check\delta}\,\varepsilon^{\check\delta\check\beta}\,\varepsilon_{\tilde\gamma\tilde\alpha}.$$

Although we don't have any rationale for the prescriptions, it seems to work well, and we will also follow the same prescriptions in this paper.

In order to carry out the localization, we will define a BRST transformation out of the supersymmetry transformation by setting both of $\varepsilon^{\dot 2}$ and $\varepsilon^{\check 2}$ to be zero, following the strategy in [1,2]. Note that this is possible, because $\varepsilon^{\dot 2}$ decouples from $\varepsilon^{\dot 1}$ in the Killing spinor equation. This is also the case for $\varepsilon^{\check\alpha}$. Furthermore, introducing bosonic Killing spinors ε and $\check\varepsilon$, we take

$$\varepsilon^{\dot 1} = \Upsilon\,\varepsilon, \qquad \varepsilon^{\check 1} = \Upsilon\,\check\varepsilon,$$

where Υ is a Grassmann odd number. For a generic field Φ, then we define the BRST transformation of Φ by

$$\delta_\epsilon\Phi = \Upsilon\,\delta_Q\Phi.$$

Before the shift (58), it follows from (46) that the BRST transformation on the gauge multiplet is given by

$$\begin{aligned}
&\delta_Q A_\mu = \frac{i}{4}\bar\lambda^{\dot 2}\gamma_\mu\varepsilon, \qquad \delta_Q\sigma = \frac{1}{4}\bar\lambda^{\dot 2}\varepsilon\\
&\delta_Q\lambda^{\dot 1} = -\frac{1}{2}F_{ab}\gamma^{ab}\varepsilon + i\mathcal{D}_a\sigma\,\gamma^a\varepsilon + D^{\dot 1}{}_{\dot 1}\varepsilon - \frac{i}{2\alpha}G_{ab}\,\sigma\gamma^{ab}\varepsilon, \qquad \delta_Q\lambda^{\dot 2} = D^{\dot 2}{}_{\dot 1}\,\varepsilon,\\
&\delta_Q D^{\dot 1}{}_{\dot 2} = -\frac{i}{2}\Big[\bar\varepsilon\gamma^a\mathcal{D}_a\lambda^{\dot 1} + g\left[\sigma,\,\bar\varepsilon\lambda^{\dot 1}\right] - \left(\frac{1}{8\alpha}G_{ab} + \frac{1}{2}t_{ab}\right)\bar\varepsilon\gamma^{ab}\lambda^{\dot 1}\\
&\qquad - \frac{1}{4}S_{ij}\left(\bar\sigma^{ij}\right)^1{}_1\bar\varepsilon\lambda^{\dot 1}\Big],\\
&\delta_Q D^{\dot 1}{}_{\dot 1} = -\frac{i}{4}\Big[\mathcal{D}_a\bar\lambda^{\dot 2}\gamma^a\varepsilon + g\left[\sigma,\,\bar\lambda^{\dot 2}\varepsilon\right] + \frac{1}{8\alpha}G_{ab}\bar\lambda^{\dot 2}\gamma^{ab}\varepsilon + \frac{1}{2}t_{ab}\bar\lambda^{\dot 2}\gamma^{ab}\varepsilon\\
&\qquad - \frac{3}{4}S_{ij}\left(\bar\sigma^{ij}\right)^1{}_1\bar\lambda^{\dot 2}\varepsilon\Big],
\end{aligned}\tag{59}$$

and from (49) that for the hypermultiplets,

$$\begin{aligned}
&\delta_Q\phi^i = \frac{1}{4}\left(\bar\sigma^i\right)^2{}_{\tilde\alpha}\,\bar\psi^{\tilde\alpha}\varepsilon, \qquad \delta_Q F^{\tilde\alpha}{}_1 = 0,\\
&\delta_Q F^{\tilde\alpha}{}_2 = \frac{i}{2}\check\varepsilon^T C\Big[\gamma^a\mathcal{D}_a\psi^{\tilde\alpha} - g\left[\sigma,\,\psi^{\tilde\alpha}\right] + g\left(\sigma_i\right)^{\tilde\alpha}{}_{\dot\gamma}\left[\phi^i,\,\lambda^{\dot\gamma}\right]\\
&\qquad + \frac{1}{2}\left(t_{ab} - \frac{1}{4\alpha}G_{ab}\right)\gamma^{ab}\psi^{\tilde\alpha}\Big],\\
&\delta_Q\psi^{\tilde\alpha} = i\left(\sigma_i\right)^{\tilde\alpha}{}_1\Big[\mathcal{D}_a\phi^i\gamma^a + g\left[\sigma,\,\phi^i\right] - \frac{1}{2\alpha}G_{ab}\phi^i\gamma^{ab} - t_{ab}\phi^i\gamma^{ab}\\
&\qquad - \left(S_{ij} + \varepsilon_{ijkl}S^{kl}\right)\phi^j\Big]\varepsilon + F^{\tilde\alpha}{}_1\check\varepsilon,
\end{aligned}\tag{60}$$

where $\bar{\lambda}^{\dot{\alpha}}$ is an abbreviation for $(\lambda^{\dot{\alpha}})^T C$, $\bar{\varepsilon}$ is for $\varepsilon^T C$.

The algebra of the supersymmetry transformations in (47), (51) and (52) may be used to check the nilpotency of the BRST transformation, assuming that (50) and (53) are satisfied. Substituting

$$\delta_\epsilon \Phi = \Upsilon\, \delta_Q \Phi, \qquad \delta_\eta \Phi = \Upsilon'\, \delta_Q \Phi,$$

into (47)–(52), through the relation

$$\left[\delta_\epsilon,\, \delta_\eta\right] \Phi = 2\Upsilon\Upsilon' (\delta_Q)^2 \Phi,$$

one can compute $(\delta_Q)^2\Phi$. Since $\varepsilon^{\dot{2}} = \eta^{\dot{2}} = 0$,

$$\bar{\eta}\cdot\varepsilon = (\eta^{\dot{1}})^T C \varepsilon^{\dot{2}} - (\eta^{\dot{2}})^T C \varepsilon^{\dot{1}} = 0, \qquad \xi^a = \bar{\eta}\cdot\gamma^a\varepsilon = (\eta^{\dot{1}})^T C\gamma^a \varepsilon^{\dot{2}} - (\eta^{\dot{2}})^T C\gamma^a \varepsilon^{\dot{1}} = 0,$$

so that $\bar{\eta}\cdot\varepsilon$ and ξ^a on the right hand sides of (47)–(52) are zero. Recalling that ε is chiral – $i\gamma^{45}\varepsilon = \varepsilon$ – on Σ, one can see that

$$\bar{\eta}\cdot\bar{\sigma}_{ij}\gamma^{ab}\epsilon = -\Upsilon\Upsilon' \left(\bar{\sigma}_{ij}\right)^2{}_1\, \varepsilon^T C\gamma^{ab}\varepsilon \quad (a, b = 1, 2, 3),$$
$$\bar{\eta}\cdot\bar{\sigma}_{ij}\gamma^{45}\epsilon = -\Upsilon\Upsilon' \left(\bar{\sigma}_{ij}\right)^2{}_1\, \varepsilon^T C\gamma^{45}\varepsilon = 0.$$

However, since $\bar{\eta}\cdot\bar{\sigma}_{ij}\gamma^{a4}\epsilon$ and $\bar{\eta}\cdot\bar{\sigma}_{ij}\gamma^{a5}\epsilon$ are not necessarily zero for a generic background, the BRST transformation isn't always nilpotent. But, for the backgrounds of our interest in this paper, since there are no fields carrying the mixed components tangent to the 3-sphere and to the Riemann surface at the same time, one can find that it is nilpontent.

Let us now take the shift (58) into account. Although it never affects the nilpotency of the BRST transformation δ_Q, it does affect $\delta_Q\lambda^{\dot{1}}$ and $\delta_Q D^{\dot{1}}{}_{\dot{1}}$,

$$\delta_Q\lambda^{\dot{1}} = -\frac{1}{2}\sum_{(a,b)\neq(4,5),(5,4)}\left(F_{ab} + \frac{i}{\alpha}G_{ab}\sigma\right)\gamma^{ab}\varepsilon + i\mathcal{D}_a\sigma\,\gamma^a\varepsilon + D^{\dot{1}}{}_{\dot{1}}\varepsilon,$$

$$\delta_Q D^{\dot{1}}{}_{\dot{1}} = -\frac{i}{4}\left[\sum_{m=1}^{3}\mathcal{D}_m\bar{\lambda}^{\dot{2}}\gamma^m\varepsilon + g\left[\sigma,\, \bar{\lambda}^{\dot{2}}\varepsilon\right] - \frac{i}{\alpha}G_{45}\bar{\lambda}^{\dot{2}}\varepsilon \right.$$
$$\left. + \frac{1}{2}\left(t_{ab} + \frac{1}{4\alpha}G_{ab}\right)\bar{\lambda}^{\dot{2}}\gamma^{ab}\varepsilon - \frac{3}{4}S_{ij}\left(\bar{\sigma}^{ij}\right)^1{}_1\bar{\lambda}^{\dot{2}}\varepsilon\right],$$

where we have assumed that the Killing spinor ε obeys $i\gamma_{45}\varepsilon = \varepsilon$ and $\mathcal{D}_4\varepsilon = \mathcal{D}_5\varepsilon = 0$. These conditions are satisfied by the Killing spinors in Section 5.

The partition function of the theory with the action $\mathcal{S}$ in (56)

$$Z = \int [d\Phi]\, \exp(\mathcal{S})$$

is invariant under a deformation of the action

$$Z = \int [d\Phi]\, \exp\left(\mathcal{S} + t\mathcal{S}_Q\right)$$

by the 'regulator' action

$$\mathcal{S}_Q = \delta_Q\Psi,$$

which is the BRST transform of a functional Ψ of the fields, with a parameter t. More explicitly, we will choose the regulator action to be

$$\mathcal{S}_Q = -\int d^5x\sqrt{g}\,\delta_Q\,\mathrm{tr}\left[\left(\delta_Q\lambda^{\dot{\alpha}}\right)^\dagger\cdot\lambda^{\dot{\alpha}} + \left(\delta_Q\psi^{\tilde{\alpha}}\right)^\dagger\cdot\psi^{\tilde{\alpha}}\right].$$

Since the partition function Z never depends on the parameter t, one can take a large t limit, while leaving the value of Z intact. In the large t limit, the main contributions to Z comes from the fixed points of the fields given by $\delta_Q\lambda^{\dot{\alpha}} = 0$ and $\delta_Q\psi^{\tilde{\alpha}} = 0$. Then, writing the fields Φ in terms of quantum fluctuations $\tilde{\Phi}$ about one of the fixed points Φ_0 as

$$\Phi = \Phi_0 + \frac{1}{\sqrt{t}}\tilde{\Phi},$$

and interacting over the fluctuations to carry out the one-loop computation, one may compute the partition function Z exactly.

In order to carry out the localization, it is convenient to convert spinor and vector fields to scalar fields[13] on the 3-spheres, and then there is no need to introduce spinor or vector spherical harmonics on the three-spheres.

For the $\mathcal{N} = 1$ hypermultiplet,

$$\begin{aligned}
&\psi^{\tilde{1}} = \left(\chi\otimes\epsilon + \xi\otimes\epsilon^c\right)\otimes\zeta_+ + \left(\eta\otimes\epsilon + \kappa\otimes\epsilon^c\right)\otimes\zeta_-,\\
&\psi^{\tilde{2}} = \left(\tilde{\chi}\otimes\epsilon + \tilde{\xi}\otimes\epsilon^c\right)\otimes\zeta_+ + \left(\tilde{\eta}\otimes\epsilon + \tilde{\kappa}\otimes\epsilon^c\right)\otimes\zeta_-,\\
&\tilde{H} = \frac{1}{\sqrt{2}}(\sigma_i)^1{}_1\phi^i = -\frac{i}{\sqrt{2}}\left(\phi^3 + i\phi^4\right), \qquad H = \frac{1}{\sqrt{2}}(\bar{\sigma}_i)^2{}_1\phi^i = \frac{i}{\sqrt{2}}\left(\phi^1 + i\phi^2\right),\\
&\tilde{H}^* = \frac{1}{\sqrt{2}}(\bar{\sigma}_i)^1{}_1\phi^i = \frac{i}{\sqrt{2}}\left(\phi^3 - i\phi^4\right), \qquad H^* = \frac{1}{\sqrt{2}}(\sigma_i)^1{}_2\phi^i = -\frac{i}{\sqrt{2}}\left(\phi^1 - i\phi^2\right).
\end{aligned}$$

Note that ϵ and ϵ^c are the Killing spinors on each of the S^3s and that they are linearly independent as two-component vectors, as discussed in Appendix G. The fields $(\chi, \xi, \eta, \kappa)$ and $(\tilde{\chi}, \tilde{\xi}, \tilde{\eta}, \tilde{\kappa})$ are scalar fields on the three-spheres.

For the $\mathcal{N} = 1$ vector multiplet,

$$\begin{aligned}
&\lambda^1 = \left(\xi\,\epsilon + \eta\,\epsilon^c\right)\otimes\zeta_+ + \left(\varphi\,\epsilon + \chi\,\epsilon^c\right)\otimes\zeta_-,\\
&\left(\lambda^2\right)^T = -\left[\left(\tilde{\varphi}\,\epsilon^\dagger + \tilde{\chi}\,\epsilon^{c\dagger}\right)\otimes\zeta_+ + \left(\tilde{\xi}\,\epsilon^\dagger + \tilde{\eta}\,\epsilon^{c\dagger}\right)\otimes\zeta_-\right]C^{-1},\\
&A_m = \left(\epsilon^\dagger\tau_m\epsilon\right)V_0 + \left(\epsilon^{c\dagger}\tau_m\epsilon\right)V_- + \left(\epsilon^\dagger\tau_m\epsilon^c\right)V_+,
\end{aligned}$$

for $m = 1, 2, 3$. The three-component vectors $\left(\epsilon^\dagger\tau_m\epsilon\right)$, $\left(\epsilon^{c\dagger}\tau_m\epsilon\right)$, and $\left(\epsilon^\dagger\tau_m\epsilon^c\right)$ are orthogonal among them, and are normalized by $\epsilon^\dagger\epsilon = \epsilon^{c\dagger}\epsilon^c = 1$. See Appendix G in more detail. Let us recall that the background on the squashed S^3 in Subsection 5.4 is not up to our mind here, because we will leave the calculation of the partition function for the background undone in this paper, as explained in Introduction.

[13] They are scalar fields on the 3-spheres, but not necessarily scalar fields on Σ. We will see the spin content of them on Σ, below.

In order to denote the scalar fields in the gauge multiplet, we use the same Greek letters χ, ξ, η, as for the ones in the hypermultiplet. But, we never mean that they are the same fields. What it really means is the shortage of the Greek letters we can assign to each of the fields. We will compute the one-loop contributions from the gauge multiplet and the hypermultiplet, separately. Therefore, we believe and hope that it wouldn't cause any confusion.

7.1. The $\mathcal{N}=2$ twisting and the $\mathcal{N}=1$ twisting

As we have seen in Section 5, the $\mathcal{N}=2$ twisting by turning on A^{12} only gives rise to the $\mathcal{N}=2$ supersymmetric backgrounds on a round and a squashed S^3, and on the other hand, the $\mathcal{N}=1$ twisting by turning on A^{12} and A^{34} with $A^{12}=A^{34}$ gives rise to the $\mathcal{N}=1$ supersymmetric backgrounds on a round, a squashed and an ellipsoid S^3. The difference between the $\mathcal{N}=2$ twisting and the $\mathcal{N}=1$ twisting has no effects on the BRST transformation of the $\mathcal{N}=1$ gauge multiplet, but affects the transformation of the $\mathcal{N}=1$ hypermultiplet. Therefore, the one-loop contributions from the $\mathcal{N}=1$ gauge multiplet don't depend on which twisting is done and yield the same results on an identical sphere.

Therefore, before proceeding to the one-loop calculations, let us see how the spin content of the two-dimensional fields in the hypermultiplet is changed upon each of the twistings. Then, we will see the spin content of the $\mathcal{N}=1$ gauge multiplet after the twistings, too.

The spin content of the two-dimensional fields on Σ from the hypermultiplet can be read from the covariant derivatives of the component fields of the hypermultiplet along the surface Σ with the local coordinates (x^4, x^5).

For the $\mathcal{N}=2$ twisting,

$$\begin{aligned}
\mathcal{D}_z\psi^{\tilde{\alpha}} &= \frac{1}{2}\left(\mathcal{D}_4\psi^{\tilde{\alpha}} - i\mathcal{D}_5\psi^{\tilde{\alpha}}\right)\\
&= \partial_z\psi^{\tilde{\alpha}} + \frac{1}{2}\omega_z{}^{45}\gamma_{45}\psi^{\tilde{\alpha}} + \frac{1}{2}\omega_z{}^{45}\left(\sigma_{12}\right)^{\tilde{\alpha}}{}_{\tilde{\beta}}\psi^{\tilde{\beta}} + ig\left[A_z,\ \psi^{\tilde{\alpha}}\right],\\
\mathcal{D}_z\phi^1 &= \partial_z\phi^1 + \omega_z{}^{45}\phi^2 + ig\left[A_z,\ \phi^1\right], \qquad \mathcal{D}_z\phi^2 = \partial_z\phi^2 - \omega_z{}^{45}\phi^1 + ig\left[A_z,\ \phi^2\right],\\
\mathcal{D}_z\phi^3 &= \partial_z\phi^3 + ig\left[A_z,\ \phi^3\right], \qquad \mathcal{D}_z\phi^4 = \partial_z\phi^4 + ig\left[A_z,\ \phi^4\right],
\end{aligned}$$

where the complex coordinate z is defined by $z=x^4+ix^5$ and $\partial_z=(\partial_4-i\partial_5)/2$.

Therefore, we can see that the bosonic field $\tilde{H}\sim(\phi^3+i\phi^4)$ remains a scalar under the $\mathcal{N}=2$ twisting, but $H\sim(\phi^1+i\phi^2)$ becomes a $(0,1)$-form; $H\to H_{\bar{z}}$. The fermionic fields (χ,ξ) are changed to be a scalar, and (η,κ) to be $(1,0)$-forms; $(\eta,\kappa)\to(\eta_z,\kappa_z)$. On the other hand, the fermionic fields $(\tilde{\chi},\tilde{\xi})$ give $(0,1)$-forms; $(\tilde{\chi},\tilde{\xi})\to(\tilde{\chi}_{\bar{z}},\tilde{\xi}_{\bar{z}})$, and (η,κ) become scalars.

For the $\mathcal{N}=1$ twisting,

$$\begin{aligned}
\mathcal{D}_z\psi^{\tilde{\alpha}} &= \frac{1}{2}\left(\mathcal{D}_4\psi^{\tilde{\alpha}} - i\mathcal{D}_5\psi^{\tilde{\alpha}}\right) = \partial_z\psi^{\tilde{\alpha}} + \frac{1}{2}\omega_z{}^{45}\gamma_{45}\psi^{\tilde{\alpha}} + ig\left[A_z,\ \psi^{\tilde{\alpha}}\right],\\
\mathcal{D}_z\phi^i &= \partial_z\phi^i + \frac{1}{2}\omega_z{}^{45}\epsilon^{ij}\phi^i + ig\left[A_z,\ \phi^i\right],
\end{aligned}$$

for $i,j=1,\cdots,4$, where ϵ^{ij} is an antisymmetric tensor with non-zero components $\epsilon^{12}=-\epsilon^{21}=\epsilon^{34}=-\epsilon^{43}=1$ only.

Table 4
The scalar fields on the S^3's of the hypermultiplet upon the twistings.

5D fields	Scalars	The $\mathcal{N}=2$ twisting		The $\mathcal{N}=1$ twisting	
		Spin (k,l)	Charge q	Spin (k,l)	Charge q
ϕ^i	$\tilde{H}$	$(0,0)$	0	$(\frac{1}{2},0)$	-1
	H	$(0,1)$	-2	$(\frac{1}{2},0)$	-1
	$\tilde{H}^*$	$(0,0)$	0	$(0,\frac{1}{2})$	1
	H^*	$(1,0)$	2	$(0,\frac{1}{2})$	1
$\psi^{\tilde{1}}$	χ	$(0,0)$	0	$(\frac{1}{2},0)$	-1
	ξ	$(0,0)$	2	$(\frac{1}{2},0)$	1
	η	$(1,0)$	0	$(0,\frac{1}{2})$	-1
	κ	$(1,0)$	2	$(0,\frac{1}{2})$	1
$\psi^{\tilde{2}}$	$\tilde{\chi}$	$(0,1)$	-2	$(\frac{1}{2},0)$	-1
	$\tilde{\xi}$	$(0,1)$	0	$(\frac{1}{2},0)$	1
	$\tilde{\eta}$	$(0,0)$	-2	$(0,\frac{1}{2})$	-1
	$\tilde{\kappa}$	$(0,0)$	0	$(0,\frac{1}{2})$	1

The bosonic fields $(\tilde{H},\, H)$ and the fermionic fields $(\chi,\, \xi)$, $(\tilde{\chi},\, \tilde{\xi})$ become two-dimensional Weyl spinors of positive chirality, and $(\eta,\, \kappa)$ and $(\tilde{\eta},\, \tilde{\kappa})$ are Weyl spinors of negative chirality.[14]

The results are summarized in Table 4. The notation (k,l) in the table denotes a (k,l)-form for integers k, l. For half integers k, l, $(\frac{1}{2},0)$ denotes a Weyl spinor of positive chirality, and $(0,\frac{1}{2})$ of negative chirality. Whichever k and l are integer or half-integer, the covariant derivative of a field Φ of (k,l) carries the spin connection ω^{45} of Σ as

$$\mathcal{D}_z\Phi = \partial_z\Phi + i(k-l){\omega_z}^{45}\Phi + ig\left[A_z,\, \Phi\right].$$

On a squashed and an ellipsoid S^3, as we have seen in Subsections 5.3 and 5.5, we have also turned on the background field ${A^i}_j$ along the S^3. When the Killing spinors are reduced to the three-dimensional ones ϵ and ϵ^c on the spheres, we refer to the background R-symmetry field as V so that the covariant derivatives of ϵ and ϵ^c are given by

$$\mathcal{D}\epsilon = \left(d + \frac{1}{4}\omega_{mn}\tau^{mn} + iV\right)\epsilon, \qquad \mathcal{D}\epsilon^c = \left(d + \frac{1}{4}\omega_{mn}\tau^{mn} - iV\right)\epsilon^c.$$

Then, on the squashed S^3 in Subsection 5.3, we have for the $\mathcal{N}=2$ twisting,

$$A^{12}|_{S^3} = \frac{2}{\tilde{r}}\left(1 - \frac{\tilde{r}^2}{r^2}\right)e^3 = -2V,$$

and for the $\mathcal{N}-1$ twisting,

$$A^{12}|_{S^3} = A^{34}|_{S^3} = \frac{1}{\tilde{r}}\left(1 - \frac{\tilde{r}^2}{r^2}\right)e^3 = -V,$$

where $|_{S^3}$ denotes the components along the S^3.

[14] In a two-dimensional Euclidean space, the complex conjugate of a Weyl spinor of negative chirality is of positive chirality. Therefore, essentially, after the $\mathcal{N}=1$ twisting, all the fields in the hypermultiplet carry the same spin on Σ.

Table 5
The charges q of the two-dimensional fields from the gauge multiplet under the background V.

2D fields	φ	χ	$\tilde{\varphi}$	$\tilde{\chi}$	ξ	η	$\tilde{\xi}$	$\tilde{\eta}$	V_0	V_+	V_-
spin (k, l)	$(1, 0)$		$(0, 1)$		$(0, 0)$		$(0, 0)$			$(0, 0)$	
charge q	0	2	0	-2	0	2	0	-2	0	2	-2

On an ellipsoid S^3 in Subsection 5.5, the $\mathcal{N}=1$ twisting causes along the S^3 the background field

$$A^{12}|_{S^3} = A^{34}|_{S^3} = -V,$$

which is identical to V given in (103).

When a two-dimensional field Φ has the covariant derivative

$$\mathcal{D}\Phi|_{S^3} = d\Phi|_{S^3} + iqV\Phi,$$

we will say that the field Φ carries charge q under the background field V. The charges of the two-dimensional fields from the hypermultiplet are listed in Table 4.

Let us turn to the two-dimensional fields on the three-spheres in the $\mathcal{N}=1$ gauge multiplet and see how the spin content of them is changed under the twisting. As mentioned before, both of the $\mathcal{N}=2$ and $\mathcal{N}=1$ twistings affect the spin content of them in the same way.

The two-dimensional fields from the $\mathcal{N}=1$ gauge multiplet are also charged under the gauge field V. But, the charges of them don't depend upon which twisting we perform. The charges under V which two-dimensional fields from the $\mathcal{N}=1$ gauge multiplet carry are also listed in Table 5.

8. Localization on the round and squashed S^3

In this section, we will compute the partition function by localization for the backgrounds on the squashed S^3 discussed in Subsection 5.3. In the round limit $\tilde{r} \to r$, we will see that the previous results in [1,2] are regained for the $\mathcal{N}=1$ twisting, and we will obtain new results for the $\mathcal{N}=2$ twisting on the round S^3 in Subsection 5.2 and on the squashed S^3 in Subsection 5.3.

As mentioned before, in order to carry out localization, we need to find fixed points of the regulator action $\mathcal{S}_Q$, which are given by $\delta_Q\lambda^{\dot{\alpha}} = 0$ and $\delta_Q\psi^{\tilde{\alpha}} = 0$. In the squashed S^3 background in Subsection 5.3, the former conditions gives

$$\delta_Q\lambda^{\dot{1}} = -\frac{1}{2}\sum_{(a,b)\neq(4,5),(5,4)} F_{ab}\gamma^{ab}\varepsilon + i\mathcal{D}_a\sigma\,\gamma^a\varepsilon + D^{\dot{1}}{}_{\dot{1}}\,\varepsilon = 0, \qquad \delta_Q\lambda^{\dot{2}} = D^{\dot{2}}{}_{\dot{1}}\,\varepsilon = 0,$$

which are reduced to

$$\frac{1}{2}\epsilon_{mkl}F_{kl} = \mathcal{D}_m\sigma, \qquad \left[F_{mz}\tau^m - i\mathcal{D}_z\sigma\right]\epsilon = 0, \qquad D^{\dot{\alpha}}{}_{\dot{\beta}} = 0,$$

with the complex coordinate $z = x^4 + ix^5$ combining the local coordinates (x^4, x^5) of Σ.

The first equation means that

$$A_m = 0, \qquad e^m\partial_m\sigma = 0 \quad \to \quad \sigma = \sigma(z, \bar{z}),$$

and the second equation in turn implies that

$$\mathcal{D}_z\sigma = \partial_z\sigma + ig\left[A_z,\, \sigma\right] = 0, \qquad e^m\partial_m A_z = 0 \quad \to \quad A_z = A_z(z, \bar{z}).$$

We will 'diagonalize' the scalar field σ at the fixed point by partial gauge fixing,

$$\sigma(z,\bar{z}) = \sum_{i=1}^{r} \sigma^i(z,\bar{z})\, H_i\,,$$

where H_i $(i = 1, \cdots, r)$ are the generators of the Cartan subalgebra of the gauge group G with r the rank of G. It then follows from $\mathcal{D}_z\sigma = 0$ that the gauge field A_z takes values in the Cartan subalgebra, too,

$$A_z(z,\bar{z}) = \sum_{i=1}^{r} A_z^i(z,\bar{z})\, H_i\,,$$

and that $\partial_z\sigma^i = 0$, *i.e.*, the solution σ^i is a constant.

As for the latter conditions $\delta_Q\psi^{\tilde{\alpha}} = 0$, a simple examination shows that the solution to $\delta_Q\psi^{\tilde{\alpha}} = 0$ is given by

$$\tilde{H} = H = 0, \qquad F^1{}_1 = 2\sqrt{2}\mathcal{D}_z\tilde{H} = 0, \qquad F^2{}_1 = -2\sqrt{2}\mathcal{D}_z H = 0,$$

for both of the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ twistings.

We will proceed to calculate the one-loop contributions about the fixed points of the regulator action $\mathcal{S}_Q$ in the next two subsections.

8.1. One-loop contributions from the $\mathcal{N} = 1$ gauge multiplet

The BRST transformations of the $\mathcal{N} = 1$ gauge multiplet are the same for both of the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ twistings. As discussed in Section 7, we would like to reduce all the component fields in the gauge multiplet into scalar fields on the S^3.

In particular, when we will convert the gauge field A_m to V_0 and $V_\pm$, the field strength $F_{mn} = \mathcal{D}_m A_n - \mathcal{D}_n A_m + ig\,[A_m,\, A_n]$ may be rewritten in terms of them as

$$\begin{aligned}
\frac{1}{2}\epsilon_{mkl}\left(\epsilon^\dagger\tau_m\epsilon\right)F_{kl} &= \epsilon_{mkl}\left(\epsilon^\dagger\tau_m\epsilon\right)\mathcal{D}_k A_l + \cdots \\
&= \frac{2\tilde{r}}{r^2}V_0 + i\left(\epsilon^{c\dagger}\tau_m\epsilon\right)\mathcal{D}_m^{(-2)}V_- - i\left(\epsilon^\dagger\tau_m\epsilon^c\right)\mathcal{D}_m^{(2)}V_+ + \cdots\,, \\
\frac{1}{2}\epsilon_{mkl}\left(\epsilon^{c\dagger}\tau_m\epsilon\right)F_{kl} &= \epsilon_{mkl}\left(\epsilon^{c\dagger}\tau_m\epsilon\right)\mathcal{D}_k A_l + \cdots \\
&= \frac{4\tilde{r}}{r^2}V_+ + 2i\left(\epsilon^\dagger\tau_m\epsilon\right)\mathcal{D}_m^{(2)}V_+ - i\left(\epsilon^{c\dagger}\tau_m\epsilon\right)\mathcal{D}_m^{(0)}V_0 + \cdots\,,
\end{aligned}$$

where the ellipsis stands for the gauge interaction terms, which gives no contributions to the partition function in the large t limit, and we will omit them. Note that the formulas (101) were used to derive these. Also omitting the gauge interaction terms, the field strength F_{mz} is given in terms of this language by

$$\begin{aligned}
F_{mz} = &\left(\epsilon^{c\dagger}\tau_m\epsilon\right)\left[\frac{1}{2}\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}_n^{(0)}A_z - \mathcal{D}_z V_-\right] \\
&+ \left(\epsilon^\dagger\tau_m\epsilon^c\right)\left[\frac{1}{2}\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}_n^{(0)}A_z - \mathcal{D}_z V_+\right] \\
&+ \left(\epsilon^\dagger\tau_m\epsilon\right)\left[\left(\epsilon^\dagger\tau^n\epsilon\right)\mathcal{D}_n^{(0)}A_z - \mathcal{D}_z V_0\right].
\end{aligned}$$

After the conversion, we can see that the BRST transformation of the bosonic fields is given by

$$
\begin{aligned}
&\delta_Q \tilde{\sigma} = -\frac{1}{4}\tilde{\xi}, \qquad \delta_Q V_0 = -\frac{i}{4}\tilde{\xi}, \quad \delta_Q V_- = -\frac{i}{4}\tilde{\eta}, \quad \delta_Q V_+ = 0, \qquad \delta_Q A_{\bar{z}} = \frac{1}{4}\tilde{\varphi}, \\
&\delta_Q A_z = 0, \\
&\delta_Q D^1{}_1 = \frac{i}{4}\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}_m^{(0)}\tilde{\xi} - 2i\frac{\tilde{r}}{r^2}\tilde{\xi} + g\left[\sigma,\ \tilde{\xi}\right] + \left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}_m^{(-2)}\tilde{\eta}\right], \\
&\delta_Q D^1{}_2 = \frac{i}{4}\left[-\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}_m^{(2)}\chi + 2i\frac{\tilde{r}}{r^2}\chi - g\left[\sigma,\ \chi\right] + \left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}_m^{(0)}\varphi - 2i\mathcal{D}_z\eta\right],
\end{aligned}
$$

where we denote a fixed point of the scalar field σ as the same letter σ, and the fluctuation about this fixed point σ as $\tilde{\sigma}$. Henceforth, we will keep this notation until the end of this section.

The BRST transformation of the fermionic fields is given by

$$
\begin{aligned}
&\delta_Q \tilde{\xi} = 0, \qquad \delta_Q \tilde{\eta} = 0, \qquad \delta_Q \tilde{\varphi} = 0, \qquad \delta_Q \tilde{\chi} = -D^2{}_1, \\
&\delta_Q \xi = -2i\frac{\tilde{r}}{r^2}V_0 + g\left[\sigma,\ V_0\right] + i\left(\epsilon^\dagger \tau^m \epsilon\right)\partial_m \tilde{\sigma} + \left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}_m^{(-2)}V_- \\
&\qquad - \left(\epsilon^\dagger \tau^m \epsilon^c\right)\mathcal{D}_m^{(2)}V_+ + D^1{}_1, \\
&\delta_Q \eta = -4i\frac{\tilde{r}}{r^2}V_+ + 2g\left[\sigma,\ V_+\right] + 2\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}_m^{(2)}V_+ + i\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\partial_m \tilde{\sigma} \\
&\qquad - \left(\epsilon^{c\dagger}\tau^m \epsilon\right)\partial_m V_0, \\
&\delta_Q \varphi = 2i\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\partial_m A_z + g\left[\sigma,\ A_z\right] - \mathcal{D}_z V_0 + i\mathcal{D}_z\tilde{\sigma}\right], \\
&\delta_Q \chi = 2i\left[\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\partial_m A_z - 2\mathcal{D}_z V_+\right].
\end{aligned}
$$

Taking account of $(V_\pm)^\dagger = V_\mp$, we deduce that

$$
\begin{aligned}
&\delta_Q\left(\delta_Q \xi\right)^\dagger = \frac{1}{2}\left[2\frac{\tilde{r}}{r^2}\tilde{\xi} + ig\left[\sigma,\ \tilde{\xi}\right] + i\left(\epsilon^\dagger \tau^m \epsilon\right)\partial_m \tilde{\xi} + i\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}_m^{(-2)}\tilde{\eta}\right], \\
&\delta_Q\left(\delta_Q \eta\right)^\dagger = -\frac{i}{2}\left[2i\frac{\tilde{r}}{r^2}\tilde{\eta} - g\left[\sigma,\ \tilde{\eta}\right] + \left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}_m^{(-2)}\tilde{\eta} - \left(\epsilon^\dagger \tau^m \epsilon^c\right)\mathcal{D}_m \tilde{\xi}\right], \\
&\delta_Q\left(\delta_Q \varphi\right)^\dagger = -\frac{i}{2}\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\partial_m \tilde{\varphi} - g\left[\sigma,\ \tilde{\varphi}\right] + 2i\mathcal{D}_{\bar{z}}\tilde{\xi}\right], \\
&\delta_Q\left(\delta_Q \chi\right)^\dagger = -\frac{i}{2}\left[\left(\epsilon^\dagger \tau^m \epsilon^c\right)\partial_m \tilde{\varphi} + 2i\mathcal{D}_{\bar{z}}\tilde{\eta}\right], \\
&\delta_Q\left(\delta_Q \tilde{\chi}\right)^\dagger = -\frac{i}{2}\left[-\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}_m \chi + 2i\frac{\tilde{r}}{r^2}\chi - g\left[\sigma,\ \chi\right] + \left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}_m \varphi - 2i\mathcal{D}_z \eta\right].
\end{aligned}
\tag{61}
$$

Each of these fluctuations is in the adjoint representation of the gauge group G, whose Cartan generators we denote as H_i $(i = 1, \cdots, r)$ with r the rank of G, and the remaining generators as E_α with α a root of G. We assume that they obey

$$
\left[H_i,\ E_\alpha\right] = \alpha_i E_\alpha, \qquad \left[E_\alpha,\ E_{-\alpha}\right] = \sum_{i=1}^{r}\alpha_i H_i \equiv \alpha \cdot H,
$$

and are normalized as

$$\mathrm{tr}\left[H_i H_j\right] = \delta_{i,j}, \qquad \mathrm{tr}\left[E_{-\alpha} E_\alpha\right] = 1.$$

Since the fluctuations have no interactions in the large t limit, the fluctuations taking values in the Cartan subalgebra are decoupled from the remaining sector, and they yield an overall constant to the partition function. We are interested in the dependence of the partition function on the value σ at one of the fixed points, and therefore we will focus on the remaining sector, where the fluctuations are expanded in terms of the basis $\{E_\alpha\}_{\alpha\in\Lambda}$ with Λ the set of all the roots of G.

We then assume that $(\sigma\cdot\alpha) = \sum_{i=1}^r \sigma_i\alpha^i$ is non-zero for a generic $(\sigma^1, \cdots, \sigma^r)$. It implies that the operator $[\sigma, \cdot]$ acting on the sector we are interested in is invertible, and the following shifts are allowed to be done:

$$\begin{aligned} V_0 \quad &\to \quad V_0 - i\frac{1}{g\,[\sigma,\,\cdot]}\left(\epsilon^\dagger\tau^m\epsilon\right)\partial_m\tilde{\sigma}, \qquad & V_+ \quad &\to \quad V_+ - \frac{i}{2}\frac{1}{g\,[\sigma,\,\cdot]}\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\partial_m\tilde{\sigma}, \\ V_- \quad &\to \quad V_- - \frac{i}{2}\frac{1}{g\,[\sigma,\,\cdot]}\left(\epsilon^\dagger\tau^m\epsilon^c\right)\partial_m\tilde{\sigma}, \qquad & A_z \quad &\to \quad A_z - i\frac{1}{g\,[\sigma,\,\cdot]}\partial_z\tilde{\sigma}, \end{aligned}$$

which enables us to 'gauge away' the fluctuation $\tilde{\sigma}$ in the above BRST transformation. In order to ensure this, we need to use (102) in Appendix G.3.

We would now like to contemplate the relation of the scalar σ with a parameter θ of the gauge transformation. In order to elucidate the discussion, we will refer to the value σ at one of the fixed points as σ_0. Before 'diagonalizing' σ_0, the scalar field σ is given by the sum

$$\sigma = \sigma_0(z,\bar{z}) + \tilde{\sigma},$$

where the fluctuation $\tilde{\sigma}$ is defined as the non-zero modes on the S^3 so that

$$\tilde{\sigma} = \sum_{l=1/2}^{\infty}\sum_{m,\tilde{m}=-l}^{l}\tilde{\sigma}_{l,m,\tilde{m}}(z,\bar{z})\varphi_{l,m,\tilde{m}},$$

with the scalar spherical harmonics $\varphi_{l,m,\tilde{m}}$ ($l = 0, 1/2, 1, 3/2, \cdots; -l \le m, \tilde{m} \le l$) on the S^3. The fixed point $\sigma_0(z,\bar{z})$ therefore corresponds to the zero mode $\varphi_{0,0,0}$ on the S^3. With the parameter θ of the gauge transformation, the scalar field σ is transformed infinitesimally as

$$\sigma \quad \to \quad \sigma + ig\,[\theta,\sigma].$$

The parameter θ may be expanded in terms of the harmonics $\varphi_{l,m,\tilde{m}}$,

$$\theta = \sum_{l=0}^{\infty}\sum_{m,\tilde{m}=-l}^{l}\theta_{l,m,\tilde{m}}(z,\bar{z})\varphi_{l,m,\tilde{m}} = \theta_0(z,\bar{z}) + \sum_{l=1/2}^{\infty}\sum_{m,\tilde{m}=-l}^{l}\theta_{l,m,\tilde{m}}(z,\bar{z})\varphi_{l,m,\tilde{m}}.$$

Since there is the correspondence between $\tilde{\sigma}_{l,m,\tilde{m}}$ and $\theta_{l,m,\tilde{m}}$ for $l \neq 0$, the measure of the non-zero modes $\prod_{l=1/2}\prod_{-l\le m,\tilde{m}\le l}[d\tilde{\sigma}_{l,m,\tilde{m}}]$ in the path integral can be canceled by the gauge degrees of freedom, $\prod_{l=1/2}\prod_{-l\le m,\tilde{m}\le l}[d\theta_{l,m,\tilde{m}}]$, if the fluctuation $\tilde{\sigma}$ never appear in the integrand of the path integral. This is indeed the case, as we have seen above in the large t limit.

The remaining gauge degrees of freedom $\theta_0(z,\bar{z})$ are used to 'diagonalize' $\sigma_0(z,\bar{z})$, as explained before. As we have elucidated in the previous paper [1], the ratio of the measures $[d\sigma_0]/[d\theta_0]$ gives rise to the Faddeev–Popov determinant

$$
\begin{aligned}
Z_{\mathrm{FP}} &= \prod_{\alpha\in\Lambda} \mathrm{Det}_{(0,0)}\left[ig\left(\sigma\cdot\alpha\right)\right] \\
&= \int [d\bar{c}(z,\bar{z})dc(z,\bar{z})]\exp\left[-ig\sum_{\alpha\in\Lambda}\int_{\Sigma} d^2z\sqrt{g_\Sigma}\left(\sigma_0\cdot\alpha\right)\bar{c}_{-\alpha}c_\alpha\right],
\end{aligned}
\tag{62}
$$

with the Faddeev–Popov ghost $c_\alpha(z,\bar{z})$, $\bar{c}_\alpha(z,\bar{z})$ $(\alpha\in\Lambda)$, which scalar fields on Σ.

Thus, we will set $\tilde{\sigma}$ to zero in the BRST transformations, and let us proceed to the evaluation of the one-loop determinants in the partition function.

From the bosonic part of the gauge multiplet in the regulator action $\mathcal{S}_Q$,

$$
\begin{aligned}
&-\int d^5x\sqrt{g}\,\mathrm{tr}\Big[\left(\delta_Q\xi\right)^\dagger\cdot\delta_Q\xi+\left(\delta_Q\eta\right)^\dagger\cdot\delta_Q\eta+\left(\delta_Q\varphi\right)^\dagger\cdot\delta_Q\varphi+\left(\delta_Q\chi\right)^\dagger\cdot\delta_Q\chi \\
&\quad+\left(\delta_Q\tilde{\chi}\right)^\dagger\cdot\delta_Q\tilde{\chi}\Big],
\end{aligned}
\tag{63}
$$

we can see that the auxiliary fields ${D^1}_2$ and ${D^2}_1$ show up in the last term $\left(\delta_Q\tilde{\chi}\right)^\dagger\cdot\delta_Q\tilde{\chi}\sim\left|{D^2}_1\right|^2$, and we will immediately integrated them out in the path integral. Furthermore, we will also integrate out the auxiliary field ${D^1}_1$, since it appears only in the first term $\left(\delta_Q\xi\right)^\dagger\cdot\delta_Q\xi\sim\left|{D^1}_1\right|^2+\cdots$, with no ${D^1}_1$ in the ellipsis. Then, the sum of the first term and the second term is reduced to

$$
\begin{aligned}
&-\int d^5x\sqrt{g}\,\mathrm{tr}\Big[\Big|-2i\frac{\tilde{r}}{r^2}V_0+g\left[\sigma,\ V_0\right]+\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(-2)}V_--\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}_m^{(2)}V_+\Big|^2 \\
&\quad+\Big|-4i\frac{\tilde{r}}{r^2}V_++2g\left[\sigma,\ V_+\right]+2\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}_m^{(2)}V_+-\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\partial_mV_0\Big|^2\Big],
\end{aligned}
\tag{64}
$$

and the third and fourth terms are summed to yield

$$
-4\int d^5x\sqrt{g}\,\mathrm{tr}\Big[\Big|\left(\epsilon^\dagger\tau^m\epsilon\right)\partial_mA_z+g\left[\sigma,\ A_z\right]-\mathcal{D}_zV_0\Big|^2+\Big|\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\partial_mA_z-2\mathcal{D}_zV_+\Big|^2\Big],
$$

which we will integrate by parts to obtain

$$
\begin{aligned}
&-4\int d^5x\sqrt{g}\,\mathrm{tr}\Big[A_z\Delta_0A_{\bar{z}}+A_z\mathcal{D}_{\bar{z}}\left(\left(\epsilon^\dagger\tau^m\epsilon\right)\partial_mV_0+g\left[\sigma,\ V_0\right]+2\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(-2)}V_-\right) \\
&\quad+\mathcal{D}_z\left(\left(\epsilon^\dagger\tau^m\epsilon\right)\partial_mV_0-g\left[\sigma,\ V_0\right]+2\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}_m^{(2)}V_+\right)\cdot A_{\bar{z}} \\
&\quad+\mathcal{D}_zV_0\mathcal{D}_{\bar{z}}V_0+4\mathcal{D}_zV_+\mathcal{D}_{\bar{z}}V_-\Big],
\end{aligned}
$$

where Δ_0 denotes the differential operator

$$
-\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\partial_m+g\left[\sigma,\ \cdot\right]\right]\left[\left(\epsilon^\dagger\tau^n\epsilon\right)\partial_n-g\left[\sigma,\ \cdot\right]\right]-\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(-2)}\left(\epsilon^\dagger\tau^n\epsilon^c\right)\partial_n.
$$

Since the three differential operators

$$
\left(\epsilon^\dagger\tau^m\epsilon\right)\partial_m=\frac{2i}{\tilde{r}}L_3,\quad\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\partial_m=\frac{2i}{r}L_+,\quad\left(\epsilon^\dagger\tau^m\epsilon^c\right)\partial_m=\frac{2i}{r}L_-,
\tag{65}
$$

with $d=e^m\partial_m$ obey the Lie algebra of $SU(2)$,

$$
\left[L_3,\ L_\pm\right]=\pm L_\pm,\qquad\left[L_+,\ L_-\right]=2L_3,
$$

we can regard Δ_0 as

$$\frac{4}{\tilde{r}^2} L_3^2 + \frac{4}{r^2} L_+ L_- + g^2 [\sigma, [\sigma, \cdot]],$$

which is positive in the root sector expanded in the basis $\{E_\alpha\}$, and the operator Δ_0 is invertible in the sector. Using the inverse of it, we will shift A_z and $A_{\bar{z}}$ in the above integrand to give

$$-4 \int d^5 x \sqrt{g}\, \mathrm{tr} \left[A_z \Delta_0 A_{\bar{z}} + \mathcal{D}_z J_+ \cdot \frac{1}{\Delta_{-2}} (\mathcal{D}_z J_+)^\dagger \right], \tag{66}$$

after integrations by parts, with

$$J_+ = 2 \left(\left(\epsilon^\dagger \tau^m \epsilon \right) \partial_m V_+ + g \left[\sigma, V_+ \right] - 2i \frac{\tilde{r}}{r^2} V_+ \right) - \left(\epsilon^{c\dagger} \tau^m \epsilon \right) \partial_m V_0,$$

where we have defined the operator Δ_{-2} by

$$\begin{aligned} &- \left[\left(\epsilon^\dagger \tau^m \epsilon \right) \mathcal{D}_m^{(-2)} + 2i \frac{\tilde{r}}{r^2} + g [\sigma, \cdot] \right] \\ &\quad \times \left[\left(\epsilon^\dagger \tau^n \epsilon \right) \mathcal{D}_n^{(-2)} + 2i \frac{\tilde{r}}{r^2} - g [\sigma, \cdot] \right] - \left(\epsilon^\dagger \tau^n \epsilon^c \right) \mathcal{D}_n^{(0)} \left(\epsilon^{c\dagger} \tau^m \epsilon \right) \mathcal{D}_m^{(-2)}. \end{aligned}$$

From the definition, it is obvious that

$$\begin{aligned} &\left(\epsilon^{c\dagger} \tau^m \epsilon \right) \mathcal{D}_m^{(-2)} = \left(\epsilon^{c\dagger} \tau^m \epsilon \right) \partial_m = \frac{2i}{r} L_+, \\ &\left(\epsilon^\dagger \tau^m \epsilon \right) \mathcal{D}_m^{(-2)} + 2i \frac{\tilde{r}}{r^2} = \left(\epsilon^\dagger \tau^m \epsilon \right) \left(\partial_m + 2i \frac{1}{\tilde{r}} \left(1 - \frac{\tilde{r}^2}{r^2} \right) e_m^3 \right) + 2i \frac{\tilde{r}}{r^2} = \frac{2i}{\tilde{r}} (L_3 + 1), \end{aligned} \tag{67}$$

and so the operator Δ_{-2} may be rewritten as

$$\frac{4}{\tilde{r}^2} (L_3 + 1)^2 + g^2 [\sigma, [\sigma, \cdot]] + \frac{4}{r^2} L_- L_+.$$

Therefore, with the same reason as for Δ_0, we can see that the operator Δ_{-2} is invertible.

To achieve the above expression (66) of the integrand, we have used the formulas (102), repeatedly. In particular, from (102), we can deduce more customized formulas for this purpose,

$$\begin{aligned} &D_0^{(q+2)} D_+^{(q)} = D_+^{(q)} \left[D_0^{(q)} + 2i \frac{\tilde{r}}{r^2} \right], \qquad D_-^{(q)} D_0^{(q)} = \left[D_0^{(q-2)} + 2i \frac{\tilde{r}}{r^2} \right] D_-^{(q)}, \\ &D_0^{(0)} \Delta_0 = \Delta_0 D_0^{(0)}, \qquad D_-^{(0)} \Delta_0 = \Delta_{-2} D_-^{(0)}, \qquad \Delta_0 D_+^{(-2)} = D_+^{(-2)} \Delta_{-2}, \end{aligned} \tag{68}$$

with the abbreviations,

$$D_0^{(q)} = \left(\epsilon^\dagger \tau^m \epsilon \right) \mathcal{D}_m^{(q)}, \qquad D_+^{(q)} = \left(\epsilon^{c\dagger} \tau^m \epsilon \right) \mathcal{D}_m^{(q)}, \qquad D_-^{(q)} = \left(\epsilon^\dagger \tau^m \epsilon^c \right) \mathcal{D}_m^{(q)}.$$

In the sum of (64) and (66), we will shift $V_\pm$ as

$$V_\pm \quad \to \quad V_\pm + \frac{1}{2} \frac{1}{D_0^{(\pm 2)} \mp 2i \frac{\tilde{r}}{r^2} \pm g [\sigma, \cdot]} D_\pm^{(0)} V_0.$$

This shift is possible, because the operators acting on the root E_α

$$D_0^{(\pm 2)} \mp 2i \frac{\tilde{r}}{r^2} \pm g [\sigma, \cdot] = \frac{2i}{\tilde{r}} (L_3 \mp 1) \pm g (\sigma \cdot \alpha)$$

have no zero-modes for a generic $(\sigma \cdot \alpha)$. Since the term in $\delta_Q \xi$,

$$D_-^{(2)} V_+ - D_+^{(-2)} V_- + \left(2i \frac{\tilde{r}}{r^2} - g\,[\sigma, \cdot]\right) V_0$$

is shifted to become $D_-^{(2)} V_+ - D_+^{(-2)} V_- - \mathcal{K}_0 V_0$, with $\mathcal{K}_0 V_0$ denoting

$$\frac{g\,[\sigma, \cdot]}{\left(D_0^{(0)} + g\,[\sigma, \cdot]\right)\left(D_0^{(0)} - g\,[\sigma, \cdot]\right)} \times \left[\left(D_0^{(0)} - 2i\frac{\tilde{r}}{r^2} + g\,[\sigma, \cdot]\right)\left(D_0^{(0)} - g\,[\sigma, \cdot]\right) + D_+^{(-2)} D_-^{(0)}\right] V_0.$$

We obtain the integrand of the resulting sum of (64) and (66), after integrations by parts,

$$A_z \Delta_0 A_{\bar{z}} - V_+ \frac{1}{\Delta_{-2}} \left[\Delta_{-2} - 4\mathcal{D}_z \mathcal{D}_{\bar{z}}\right] \left(D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g\,[\sigma, \cdot]\right) \times \left(D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} - g\,[\sigma, \cdot]\right) V_- + \frac{1}{4}\left|D_-^{(2)} V_+ - D_+^{(-2)} V_- - \mathcal{K}_0 V_0\right|^2,$$

where we will shift V_0 appropriately to eliminate the term $D_-^{(2)} V_+ - D_+^{(-2)} V_-$. This is possible, since the operator $\mathcal{K}_0$ is invertible in the same sense as explained above. The last term in the above integrand then gives

$$-\frac{1}{4}\left(\mathcal{K}_0 V_0\right)^2.$$

Note that $\mathcal{K}_0 V_0$ is pure imaginary, which may be ensured by using

$$\left(D_0^{(0)} - 2i\frac{\tilde{r}}{r^2} \mp g\,[\sigma, \cdot]\right)\left(D_0^{(0)} \pm g\,[\sigma, \cdot]\right) + D_+^{(-2)} D_-^{(0)} = \left(D_0^{(0)} + 2i\frac{\tilde{r}}{r^2} \pm g\,[\sigma, \cdot]\right)\left(D_0^{(0)} \mp g\,[\sigma, \cdot]\right) + D_-^{(2)} D_+^{(0)}. \tag{69}$$

We now see that the resulting integrand is ‘diagonalized’, and it is a simple of matter to compute the one-loop determinants from the bosonic fields of the gauge multiplet,

$$Z_{\mathrm{V,B}}^{\text{1-loop}} = Z_{\mathrm{V,0}}^{\text{1-loop}} \times \frac{\mathcal{D}\mathrm{et}_{(0,0)}\left[\Delta_{-2}\right]}{\mathcal{D}\mathrm{et}_{(0,1)}\left[\Delta_0\right] \mathcal{D}\mathrm{et}_{(0,0)}\left[\left[\Delta_{-2} - 4\mathcal{D}_z\mathcal{D}_{\bar{z}}\right]\left(D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g\,[\sigma, \cdot]\right)\left(D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} - g\,[\sigma, \cdot]\right)\right]},$$

where the determinant $\mathcal{D}\mathrm{et}_{(k,l)}[D]$ for an operator D is defined by

$$\frac{1}{\mathcal{D}\mathrm{et}_{(k,l)}\,[D]} = \int [d\varphi^\dagger][d\varphi] \exp\left[-\int d^5x \sqrt{g}\, \mathrm{tr}[\varphi^\dagger D \varphi]\right],$$

for a bosonic (k, l)-form field φ on Σ and its partner $\varphi^\dagger$, both of which are also scalar fields on the S^3. We denote the one-loop contribution from V_0 as $Z_{\mathrm{V},0}$. Since V_0 is a real field; $V_0^\dagger = V_0$, some care is required to integrate over it. Upon expanding it in terms of the basis $\{E_\alpha\}_{\alpha\in\Lambda}$, we have

$$V_0 = \sum_{i=1}^{r} V_0^i H_i + \sum_{\alpha\in\Lambda} V_0^\alpha E_\alpha,$$

and the reality condition implies that $\left(V_0^\alpha\right)^\dagger = V_0^{-\alpha}$. Therefore, $Z_{V,0}$ is given by the path integral

$$\int \prod_{\alpha\in\Lambda} \left[dV_0^\alpha\right] \exp\left[\frac{1}{4}\int d^5x\sqrt{g}\,\mathrm{tr}\left[(\mathcal{K}_0 V_0)^2\right]\right]$$

with the exponent $\mathrm{tr}\left[(\mathcal{K}_0 V_0)^2\right]$ expanded as

$$-2\sum_{\alpha\in\Lambda_+}\left|\frac{g\,(\sigma\cdot\alpha)}{\left(D_0^{(0)}+g\,(\sigma\cdot\alpha)\right)\left(D_0^{(0)}-g\,(\sigma\cdot\alpha)\right)}\right.$$
$$\left.\times\left[\left(D_0^{(0)}-2i\frac{\tilde{r}}{r^2}+g\,(\sigma\cdot\alpha)\right)\left(D_0^{(0)}-g\,(\sigma\cdot\alpha)\right)+D_+^{(-2)}D_-^{(0)}\right]V_0^\alpha\right|^2,$$

up to the Cartan part, with Λ_+ the set of all the positive roots of Λ. Taking account of (69), we will integrate it to obtain

$$Z_{V,0}^{\text{1-loop}} = \prod_{\alpha\in\Lambda_+}\mathrm{Det}_{(0,0)}\left[\frac{g\,(\sigma\cdot\alpha)}{\left(D_0^{(0)}+g\,(\sigma\cdot\alpha)\right)\left(D_0^{(0)}-g\,(\sigma\cdot\alpha)\right)}\right]^{-2}$$
$$\times\,\mathrm{Det}_{(0,0)}\left[\left(D_0^{(0)}-2i\frac{\tilde{r}}{r^2}+g\,(\sigma\cdot\alpha)\right)\left(D_0^{(0)}-g\,(\sigma\cdot\alpha)\right)+D_+^{(-2)}D_-^{(0)}\right]^{-2},$$

where we defined the determinant $\mathrm{Det}_{(k,l)}[D]$ for an operator D as

$$\frac{1}{\mathrm{Det}_{(k,l)}[D]} = \int [d\,(\varphi_\alpha)^\dagger][d\varphi_\alpha]\exp\left[-\int d^5x\sqrt{g}\,[(\varphi_\alpha)^\dagger\,D\varphi_\alpha]\right],$$

for a bosonic (k,l)-form field φ_α on Σ and its partner $(\varphi_\alpha)^\dagger$, both of which are also scalar fields on the S^3. They are just one components of an adjoint φ in the expansion

$$\varphi = \sum_{i=1}^{r}\varphi_i H_i + \sum_{\alpha\in\Lambda}\varphi_\alpha E_\alpha.$$

Therefore, up to an overall constant including the Cartan part,

$$Z_{\text{V},0}^{\text{1-loop}} = \frac{1}{Z_{\text{FP}}}\cdot\frac{\left(\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)}-g\,[\sigma,\,\cdot]\right]\right)^2}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\left(D_0^{(0)}-2i\frac{\tilde{r}}{r^2}+g\,[\sigma,\,\cdot]\right)\left(D_0^{(0)}-g\,[\sigma,\,\cdot]\right)+D_+^{(-2)}D_-^{(0)}\right]}.$$

Let us proceed to the one-loop contributions from the fermionic part of the gauge multiplet, of which the part in the regulator action $\mathcal{S}_Q$ is

$$-\int d^5x\sqrt{g}\left(\delta_Q\left(\delta_Q\xi\right)^\dagger\cdot\xi+\delta_Q\left(\delta_Q\eta\right)^\dagger\cdot\eta+\delta_Q\left(\delta_Q\varphi\right)^\dagger\cdot\varphi+\delta_Q\left(\delta_Q\chi\right)^\dagger\cdot\chi\right.$$
$$\left.+\,\delta_Q\left(\delta_Q\tilde{\chi}\right)^\dagger\cdot\tilde{\chi}\right).$$

Substituting (61) into this and integrating by parts, the first two terms become

$$-\left(\frac{i}{2}\right)\left[\left(\tilde{\xi},\tilde{\eta}\right)\begin{pmatrix} D_0^{(0)}+2i\frac{\tilde{r}}{r^2}+g\,[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)}+2i\frac{\tilde{r}}{r^2}+g\,[\sigma,\cdot]\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right],$$

and the remaining terms yield

$$\left(\frac{i}{2}\right)\left[\begin{pmatrix}\tilde{\varphi}, \tilde{\chi}\end{pmatrix}\begin{pmatrix} D_0^{(0)} - g[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + 2i\frac{\tilde{r}}{r^2} - g[\sigma,\cdot]\end{pmatrix}\begin{pmatrix}\varphi \\ \chi\end{pmatrix}\right] + \begin{pmatrix}\mathcal{D}_{\bar{z}}\tilde{\xi}, \mathcal{D}_{\bar{z}}\tilde{\eta}\end{pmatrix}\begin{pmatrix}\varphi \\ \chi\end{pmatrix} + \begin{pmatrix}\tilde{\varphi}, & \tilde{\chi}\end{pmatrix}\begin{pmatrix}0 \\ \mathcal{D}_z\chi\end{pmatrix}.$$

Integrating over φ, χ, $\tilde{\varphi}$, and $\tilde{\chi}$ to give the one-loop determinant

$$\mathcal{D}\mathrm{et}_{(1,0)}\left[\begin{pmatrix} D_0^{(0)} - g[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + 2i\frac{\tilde{r}}{r^2} - g[\sigma,\cdot]\end{pmatrix}\right],$$

the latter terms in the integrand are reduced to

$$-2i\left[\begin{pmatrix}\tilde{\xi}, \tilde{\eta}\end{pmatrix}\begin{pmatrix} D_0^{(0)} - g[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + 2i\frac{\tilde{r}}{r^2} - g[\sigma,\cdot]\end{pmatrix}^{-1}\begin{pmatrix}0 \\ \mathcal{D}_{\bar{z}}\mathcal{D}_z\eta\end{pmatrix}\right],$$

after integration by parts. Summing this and the first two terms in the integrand results in

$$-\frac{i}{2}\left[\begin{pmatrix}\tilde{\xi}, \tilde{\eta}\end{pmatrix}\begin{pmatrix} D_0^{(0)} - g[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + 2i\frac{\tilde{r}}{r^2} - g[\sigma,\cdot]\end{pmatrix}^{-1}\begin{pmatrix}D_1 & 0 \\ D_3 & D_4\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right],$$

where the operators D_1, D_3, and D_4 denote

$$\begin{aligned}
D_1 &= \left(D_0^{(0)} - g[\sigma,\cdot]\right)\left(D_0^{(0)} + 2i\frac{\tilde{r}}{r^2} + g[\sigma,\cdot]\right) + D_-^{(2)}D_+^{(0)}, \\
D_3 &= D_+^{(0)}\left(D_0^{(0)} + 2i\frac{\tilde{r}}{r^2} + g[\sigma,\cdot]\right) - \left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} + g[\sigma,\cdot]\right)D_+^{(0)}, \\
D_4 &= 4\mathcal{D}_{\bar{z}}\mathcal{D}_z + \left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} + g[\sigma,\cdot]\right)\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} - g[\sigma,\cdot]\right) + D_+^{(0)}D_-^{(2)},
\end{aligned}$$

and the zero in the top right component of the matrix is seen from the calculation

$$\left(D_0^{(0)} - g[\sigma,\cdot]\right)D_-^{(2)} - D_-^{(2)}\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} - g[\sigma,\cdot]\right) = 0,$$

with help of (68). Integrating over ξ, η, $\tilde{\xi}$, and $\tilde{\eta}$, we obtain the one-loop determinants

$$\frac{\mathcal{D}\mathrm{et}_{(0,0)}\left[\begin{pmatrix}D_1 & 0 \\ D_2 & D_4\end{pmatrix}\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\begin{pmatrix} D_0^{(0)} - g[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + 2i\frac{\tilde{r}}{r^2} - g[\sigma,\cdot]\end{pmatrix}\right]}.$$

Thus, we compute the one-loop contributions from the fermionic fields of the gauge multiplet,

$$Z_{V,F}^{\text{1-loop}} = \frac{\mathcal{D}\mathrm{et}_{(1,0)}\left[\begin{pmatrix} D_0^{(0)} - g[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + 2i\frac{\tilde{r}}{r^2} - g[\sigma,\cdot]\end{pmatrix}\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\begin{pmatrix} D_0^{(0)} - g[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + 2i\frac{\tilde{r}}{r^2} - g[\sigma,\cdot]\end{pmatrix}\right]}\mathcal{D}\mathrm{et}_{(0,0)}\left[\begin{pmatrix}D_1 & 0 \\ D_2 & D_4\end{pmatrix}\right],$$

where the last factor is easily evaluated as

$$\mathcal{D}\mathrm{et}_{(0,0)}\left[\begin{pmatrix} D_1 & 0 \\ D_2 & D_4 \end{pmatrix}\right] = \mathcal{D}\mathrm{et}_{(0,0)}\left[D_1\right]\mathcal{D}\mathrm{et}_{(0,0)}\left[D_4\right]$$

$$= \mathcal{D}\mathrm{et}_{(0,0)}\left[\left(D_0^{(0)} - g\left[\sigma,\,\cdot\right]\right)\left(D_0^{(0)} + 2i\frac{\tilde{r}}{r^2} + g\left[\sigma,\,\cdot\right]\right) + D_-^{(2)}D_+^{(0)}\right]$$

$$\times\,\mathcal{D}\mathrm{et}_{(0,0)}\left[4\mathcal{D}_{\bar{z}}\mathcal{D}_z + \left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} + g\left[\sigma,\cdot\right]\right)\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} - g\left[\sigma,\cdot\right]\right)\right.$$

$$\left. + D_+^{(0)}D_-^{(2)}\right].$$

Let us evaluate the determinant

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[\begin{pmatrix} D_0^{(0)} - g\left[\sigma,\cdot\right] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + 2i\frac{\tilde{r}}{r^2} - g\left[\sigma,\cdot\right] \end{pmatrix}\right].$$

For four operators A, B, C, and D, we have the formula (see, for example, [30])

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] = \mathcal{D}\mathrm{et}_{(k,l)}\left[A - B\frac{1}{D}C\right]\mathcal{D}\mathrm{et}_{(k,l)}\left[D\right],$$

for an invertible D. If there is another differential operator D' satisfying the relation

$$B\frac{1}{D} = \frac{1}{D'}B,$$

we then obtain the formula

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] = \frac{\mathcal{D}\mathrm{et}_{(k,l)}\left[D\right]}{\mathcal{D}\mathrm{et}_{(k,l)}\left[D'\right]}\mathcal{D}\mathrm{et}_{(k,l)}\left[D'A - BC\right].$$

When we regard

$$B = D_-^{(2)}, \qquad D = -D_0^{(2)} + 2i\frac{\tilde{r}}{r^2} - g\left[\sigma,\cdot\right],$$

using (68), we find the operator

$$D' = -D_0^{(0)} - g\left[\sigma,\cdot\right],$$

and the determinant gives

$$\frac{\mathcal{D}\mathrm{et}_{(k,l)}\left[D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} + g\left[\sigma,\cdot\right]\right]}{\mathcal{D}\mathrm{et}_{(k,l)}\left[D_0^{(0)} + g\left[\sigma,\cdot\right]\right]}\mathcal{D}\mathrm{et}_{(k,l)}$$

$$\times\left[-\left(D_0^{(0)} + g\left[\sigma,\cdot\right]\right)\left(D_0^{(0)} - g\left[\sigma,\cdot\right]\right) - D_-^{(2)}D_+^{(0)}\right].$$

On the other hand, we also have the formula (see, for example, [30])

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] = \mathcal{D}\mathrm{et}_{(k,l)}\left[A\right]\mathcal{D}\mathrm{et}_{(k,l)}\left[D - C\frac{1}{A}B\right],$$

for an invertible A. If there is another differential operator A' satisfying the relation

$$C\frac{1}{A} = \frac{1}{A'}C,$$

we then obtain the formula

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] = \frac{\mathcal{D}\mathrm{et}_{(k,l)}\left[A\right]}{\mathcal{D}\mathrm{et}_{(k,l)}\left[A'\right]}\mathcal{D}\mathrm{et}_{(k,l)}\left[A'D - CB\right].$$

If we then identify

$$A = D_0^{(0)} - g\left[\sigma, \cdot\right], \qquad C = D_+^{(0)},$$

using (68), we can find the operator

$$A' = D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} - g\left[\sigma, \cdot\right],$$

and therefore, the same determinant have another expression

$$\frac{\mathcal{D}\mathrm{et}_{(k,l)}\left[D_0^{(0)} - g\left[\sigma, \cdot\right]\right]}{\mathcal{D}\mathrm{et}_{(k,l)}\left[D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} - g\left[\sigma, \cdot\right]\right]}\mathcal{D}\mathrm{et}_{(k,l)}$$
$$\times\left[-\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} - g\left[\sigma, \cdot\right]\right)\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} + g\left[\sigma, \cdot\right]\right) - D_+^{(0)}D_-^{(2)}\right].$$

For a (k, l)-form fermionic field υ on Σ of charge 2, which is also a scalar on the S^3, and its hermitian conjugate υ^*, integration by part is used to deduce

$$\int d^5x\sqrt{g}\,\mathrm{tr}\left[\upsilon^*\left(\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} - g\left[\sigma, \cdot\right]\right)\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} + g\left[\sigma, \cdot\right]\right) + D_+^{(0)}D_-^{(2)}\right)\upsilon\right]$$
$$= \int d^5x\sqrt{g}\,\mathrm{tr}\left[\upsilon\left(\left(D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} - g\left[\sigma, \cdot\right]\right)\left(D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g\left[\sigma, \cdot\right]\right)\right.\right.$$
$$\left.\left. + D_-^{(0)}D_+^{(-2)}\right)\upsilon^*\right],$$

which implies that

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[-\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} - g\left[\sigma, \cdot\right]\right)\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} + g\left[\sigma, \cdot\right]\right) - D_+^{(0)}D_-^{(2)}\right]$$
$$= \mathcal{D}\mathrm{et}_{(l,k)}\left[-\left(D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g\left[\sigma, \cdot\right]\right)\left(D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} - g\left[\sigma, \cdot\right]\right) - D_-^{(0)}D_+^{(-2)}\right]$$
$$= \mathcal{D}\mathrm{et}_{(l,k)}\left[\Delta_{-2}\right].$$

Similarly, we can see that

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[-\left(D_0^{(0)} + g\left[\sigma, \cdot\right]\right)\left(D_0^{(0)} - g\left[\sigma, \cdot\right]\right) - D_-^{(2)}D_+^{(0)}\right]$$
$$= \mathcal{D}\mathrm{et}_{(l,k)}\left[-\left(D_0^{(0)} + g\left[\sigma, \cdot\right]\right)\left(D_0^{(0)} - g\left[\sigma, \cdot\right]\right) - D_+^{(-2)}D_-^{(0)}\right] = \mathcal{D}\mathrm{et}_{(l,k)}\left[\Delta_0\right],$$

and that

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} \pm g\left[\sigma, \cdot\right]\right] = \mathcal{D}\mathrm{et}_{(l,k)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} \pm g\left[\sigma, \cdot\right]\right].$$

Using these, we may rewrite the determinant

$$
\begin{aligned}
&\mathcal{D}\mathrm{et}_{(k,l)}\left[\begin{pmatrix} D_0^{(0)} - g\,[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + 2i\frac{\tilde{r}}{r^2} - g\,[\sigma,\cdot] \end{pmatrix}\right] \\
&= \frac{\mathcal{D}\mathrm{et}_{(l,k)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(l,k)}\left[D_0^{(0)} + g\,[\sigma,\cdot]\right]} \mathcal{D}\mathrm{et}_{(l,k)}\left[\Delta_0\right] \\
&= \frac{\mathcal{D}\mathrm{et}_{(l,k)}\left[D_0^{(0)} - g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(l,k)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} - g\,[\sigma,\cdot]\right]} \mathcal{D}\mathrm{et}_{(l,k)}\left[\Delta_{-2}\right],
\end{aligned}
\tag{70}
$$

which also implies the formula

$$
\frac{\mathcal{D}\mathrm{et}_{(l,k)}\left[\Delta_0\right]}{\mathcal{D}\mathrm{et}_{(l,k)}\left[\Delta_{-2}\right]} = \frac{\mathcal{D}\mathrm{et}_{(l,k)}\left[D_0^{(0)} - g\,[\sigma,\cdot]\right]\mathcal{D}\mathrm{et}_{(l,k)}\left[D_0^{(0)} + g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(l,k)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} - g\,[\sigma,\cdot]\right]\mathcal{D}\mathrm{et}_{(l,k)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g\,[\sigma,\cdot]\right]}.
$$

Using (70) twice in a bit tricky way, we obtain $Z_{\mathrm{V,F}}^{\text{1-loop}}$

$$
\begin{aligned}
&\frac{\mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g\,[\sigma,\cdot]\right]\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} - g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(0)} + g\,[\sigma,\cdot]\right]\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)} - g\,[\sigma,\cdot]\right]} \frac{\mathcal{D}\mathrm{et}_{(0,1)}\left[\Delta_0\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\Delta_{-2}\right]} \\
&\times \mathcal{D}\mathrm{et}_{(0,0)}\left[D_1\right]\mathcal{D}\mathrm{et}_{(0,0)}\left[D_4\right].
\end{aligned}
$$

With the same reasoning as the argument about integration by parts in the integrand, it is easy to see that

$$
\begin{aligned}
\mathcal{D}\mathrm{et}_{(0,0)}\left[D_1\right] &= \mathcal{D}\mathrm{et}_{(0,0)}\left[\left(D_0^{(0)} - 2i\frac{\tilde{r}}{r^2} + g\,[\sigma,\cdot]\right)\left(D_0^{(0)} - g\,[\sigma,\cdot]\right) + D_+^{(-2)}D_-^{(0)}\right], \\
\mathcal{D}\mathrm{et}_{(0,0)}\left[D_4\right] &= \mathcal{D}\mathrm{et}_{(0,0)}\left[4\mathcal{D}_z\mathcal{D}_{\bar{z}} - \Delta_{-2}\right].
\end{aligned}
$$

Using this, we will combine the one-loop contributions $Z_{\mathrm{V,B}}^{\text{1-loop}}$, $Z_{\mathrm{V,F}}^{\text{1-loop}}$, and Z_{FP} from the gauge multiplet to yield

$$
\boxed{
\begin{aligned}
Z_{\mathrm{V}}^{\text{1-loop}} &= Z_{\mathrm{FP}} Z_{\mathrm{V,B}}^{\text{1-loop}} Z_{\mathrm{V,F}}^{\text{1-loop}} \\
&= \frac{\mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g\,[\sigma,\cdot]\right]} \frac{\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)} - g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(0)} - g\,[\sigma,\cdot]\right]},
\end{aligned}
}
\tag{71}
$$

where we have made use of the invariance

$$
\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)} - g\,[\sigma,\cdot]\right] = \mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)} + g\,[\sigma,\cdot]\right],
$$

under $\alpha \to -\alpha$ for all the roots $\alpha \in \Lambda$.

The determinant $\mathcal{D}\mathrm{et}_{(k,l)}$ can be evaluated by using the basis $\{\varphi_{l,m,\tilde{m}} \otimes v \otimes E_\alpha, \varphi_{l,m,\tilde{m}} \otimes v \otimes H_i\}$, for v running over all the basis vectors of $\Omega^{(k,l)}(\Sigma)$, the set of all (k,l)-forms on Σ, upon regarding $\Omega^{(k,l)}(\Sigma)$ as a linear space. Here, $\varphi_{l,m,\tilde{m}}$ $(l = 0, 1/2, 1, 3/2, \cdots; -l \le m, \tilde{m} \le l)$ denote the scalar spherical harmonics on the S^3, and through the relations (65) of the differential operators with the generators of the Lie algebra of $SU(2)$, they provide the representations of the $SU(2)$ algebra;

$$L_3\varphi_{l,m,\tilde{m}} = m\varphi_{l,m,\tilde{m}}, \qquad L_\pm\varphi_{l,m,\tilde{m}} = \sqrt{(l\mp m)(l\pm m+1)}\varphi_{l,m\pm 1,\tilde{m}}.$$

On the basis[15] $\{\varphi_{l,m,\tilde{m}}\otimes v\otimes E_\alpha\}$, using (67), we deduce

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[D_0^{(0)} - g\,[\sigma,\cdot]\right] = \prod_{\alpha\in\Lambda}\prod_{l\in\frac{1}{2}\mathbf{Z}_{\geq 0}}\prod_{m,\tilde{m}=-l}^{l}\det{}_{(k,l)}\left[\frac{2i}{\tilde{r}}m - g\,(\sigma\cdot\alpha)\right],$$

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g\,[\sigma,\cdot]\right] = \prod_{\alpha\in\Lambda}\prod_{l\in\frac{1}{2}\mathbf{Z}_{\geq 0}}\prod_{m,\tilde{m}=-l}^{l}\det{}_{(k,l)}\left[\frac{2i}{\tilde{r}}(m+1) + g\,(\sigma\cdot\alpha)\right],$$

with $\frac{1}{2}\mathbf{Z}_{\geq 0}$ the set of non-negative half integers, where the determinant $\det_{(k,l)}$ is defined over the space $\Omega^{(k,l)}(\Sigma)$.

As explained in [31], the Hodge decomposition implies that for the space $\Omega^{k,l}(\Sigma)$ of all the (k,l)-forms on the Riemann surface Σ,

$$\Omega^{1,0}(\Sigma)\oplus\Omega^{0,1}(\Sigma) = \left(\Omega^{0,0}(\Sigma)\ominus H^0(\Sigma)\right)\oplus\left(\Omega^{0,0}(\Sigma)\ominus H^0(\Sigma)\right)\oplus H^1(\Sigma),$$

where $H^p(\Sigma)$ is the space of all the harmonic p-forms on Σ. It follows from this that for a constant D,

$$\frac{\det_{(0,0)}[D]}{\det_{(0,1)}[D]} = D^{b_0(\Sigma)-\frac{1}{2}b_1(\Sigma)} = D^{\frac{1}{2}\chi(\Sigma)} \tag{72}$$

with $b_i(\Sigma) = \dim H_i(\Sigma)$ the i-th Betti number, and with the Euler number $\chi(\Sigma)$ of the surface Σ:

$$\chi(\Sigma) = b_0(\Sigma) - b_1(\Sigma) + b_2(\Sigma) = 2b_0(\Sigma) - b_1(\Sigma),$$

where we have used the Hodge duality; $b_0(\Sigma) = b_2(\Sigma)$.

Taking account of this, we can reduce $Z_{\mathrm{V}}^{\text{1-loop}}$ to

$$\begin{aligned}Z_{\mathrm{V}}^{\text{1-loop}} &= \prod_{\alpha\in\Lambda}\prod_{l\in\frac{1}{2}\mathbf{Z}_{\geq 0}}\prod_{m,\tilde{m}=-l}^{l}\left(\frac{\frac{2i}{\tilde{r}}m - g\,(\sigma\cdot\alpha)}{\frac{2i}{\tilde{r}}(m+1) + g\,(\sigma\cdot\alpha)}\right)^{\frac{1}{2}\chi(\Sigma)}\\ &= \prod_{\alpha\in\Lambda_+}\left[\tilde{r}g\,(\sigma\cdot\alpha)\prod_{n=1}\left(n^2 + \tilde{r}^2g^2\,(\sigma\cdot\alpha)^2\right)\right]^{\chi(\Sigma)},\end{aligned}$$

where we have replaced l by $n = 2l$ ($n = 0, 1, 2, \cdots$).

From the formula

$$\frac{1}{\pi}\sinh\pi x = x\prod_{m=1}^{\infty}\left(1 + \frac{x^2}{m^2}\right),$$

15 Since the contributions from the basis vectors $\varphi_{l,m,\tilde{m}}\otimes v\otimes H_i$ to the determinants are constant factors to the partition function, we will omit them.

together with the zeta regularization,

$$\prod_{m=1}^{\infty} m = e^{\sum_{m=1}^{\infty} \log m} \quad \rightarrow \quad e^{-\zeta'(0)} = \sqrt{2\pi},$$

it follows that

$$\boxed{Z_{\mathrm{V}}^{\text{1-loop}} = \prod_{\alpha \in \Lambda_+} \left[2 \sinh\left(\pi \tilde{r} g\,(\sigma \cdot \alpha)\right)\right]^{\chi(\Sigma)}.} \tag{73}$$

In the round limit $\tilde{r} \to r$, Z_{V} is in agreement with the previous result in [1].

8.2. *One-loop contributions from the $\mathcal{N} = 1$ hypermultiplet*

Let us proceed to compute the one-loop contributions from the hypermultiplet by localization. Since the BRST transformation in the $\mathcal{N} = 1$ twisting differs from the one in the $\mathcal{N} = 2$ twisting, we will discuss them separately in the next two sub-subsections.

However, the BRST transformations of the scalar fields H, $H^\dagger$ of the hypermultiplet are common in both the $\mathcal{N} = 1$ and the $\mathcal{N} = 2$ twistings. From (60), the BRST transformation of the scalar fields is reduced to

$$\delta_Q \tilde{H} = 0, \quad \delta_Q H = 0, \qquad \delta_Q \tilde{H}^\dagger = -\frac{1}{2\sqrt{2}} \tilde{\kappa}, \quad \delta_Q H^\dagger = -\frac{1}{2\sqrt{2}} \kappa.$$

8.2.1. *The $\mathcal{N} = 1$ twisting*

Let us begin with the $\mathcal{N} = 1$ twisting to calculate the one-loop contributions from the hypermultiplet to the partition function by localization.

The BRST transformation of the fermions in the hypermultiplet is given by

$$\begin{aligned}
\delta_Q \chi &= \sqrt{2} i \left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-1)}{}_m \tilde{H} + g \left[\sigma,\, \tilde{H}\right] + i \frac{\tilde{r}}{r^2} \tilde{H} \right], \\
\delta_Q \xi &= \sqrt{2} i \left(\epsilon^{c\dagger} \tau^m \epsilon\right) \mathcal{D}^{(-1)}{}_m \tilde{H}, \\
\delta_Q \eta &= F^1{}_1 - 2\sqrt{2} \mathcal{D}_z \tilde{H}, \qquad \delta_Q \kappa = 0, \\
\delta_Q \tilde{\chi} &= -\sqrt{2} i \left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-1)}{}_m H + g\,[\sigma,\, H] + i \frac{\tilde{r}}{r^2} H \right], \\
\delta_Q \tilde{\xi} &= -\sqrt{2} i \left(\epsilon^{c\dagger} \tau^m \epsilon\right) \mathcal{D}^{(-1)}{}_m H, \\
\delta_Q \tilde{\eta} &= F^2{}_1 + 2\sqrt{2} \mathcal{D}_z H, \qquad \delta_Q \tilde{\kappa} = 0,
\end{aligned}$$

and its hermitian conjugate by

$$\begin{aligned}
\left(\delta_Q \chi\right)^\dagger &= -\sqrt{2} i \left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(1)}{}_m \tilde{H}^\dagger - g \left[\sigma,\, \tilde{H}^\dagger\right] - i \frac{\tilde{r}}{r^2} \tilde{H}^\dagger \right], \\
\left(\delta_Q \xi\right)^\dagger &= -\sqrt{2} i \left(\epsilon^\dagger \tau^m \epsilon^c\right) \mathcal{D}^{(1)}{}_m \tilde{H}^\dagger, \\
\left(\delta_Q \eta\right)^\dagger &= -F^2{}_2 - 2\sqrt{2} \mathcal{D}_{\bar{z}} \tilde{H}^\dagger, \qquad \left(\delta_Q \kappa\right)^\dagger = 0,
\end{aligned}$$

$$\left(\delta_Q \tilde{\chi}\right)^\dagger = \sqrt{2} i \left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(1)}{}_m H^\dagger - g\left[\sigma,\, H^\dagger\right] - i \frac{\tilde{r}}{r^2} H^\dagger\right],$$
$$\left(\delta_Q \tilde{\xi}\right)^\dagger = \sqrt{2} i \left(\epsilon^\dagger \tau^m \epsilon^c\right) \mathcal{D}^{(1)}{}_m H^\dagger,$$
$$\left(\delta_Q \tilde{\eta}\right)^\dagger = F^1{}_2 + 2\sqrt{2} \mathcal{D}_{\bar{z}} H^\dagger, \qquad \left(\delta_Q \tilde{\kappa}\right)^\dagger = 0.$$

Using the BRST transformation of the auxiliary fields $F^1{}_2$, $F^2{}_2$,

$$\delta_Q F^1{}_2 = \frac{i}{2}\left[\left(\epsilon^{c\dagger} \tau^m \epsilon\right) \mathcal{D}^{(-1)}{}_m \chi - \left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(1)}{}_m \eta - g\left[\sigma,\, \eta\right] + i \frac{\tilde{r}}{r^2} \eta - 2i \mathcal{D}_{\bar{z}} \kappa\right],$$
$$\delta_Q F^2{}_2 = \frac{i}{2}\left[\left(\epsilon^{c\dagger} \tau^m \epsilon\right) \mathcal{D}^{(-1)}{}_m \tilde{\chi} - \left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(1)}{}_m \tilde{\eta} - g\left[\sigma,\, \tilde{\eta}\right] + i \frac{\tilde{r}}{r^2} \tilde{\eta} - 2i \mathcal{D}_{\bar{z}} \tilde{\kappa}\right],$$

where we have omitted the terms $g\left(\sigma^i\right)^{\tilde{\alpha}}{}_{\dot{\gamma}}\left[\phi^i,\, \lambda^{\dot{\gamma}}\right]$ on the right hand sides of both the equations, because they vanish in the large t limit, we find that

$$\delta_Q\left(\delta_Q \chi\right)^\dagger = \frac{i}{2}\left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(1)}{}_m \tilde{\kappa} - g\left[\sigma,\, \tilde{\kappa}\right] - i \frac{\tilde{r}}{r^2} \tilde{\kappa}\right],$$
$$\delta_Q\left(\delta_Q \xi\right)^\dagger = \frac{i}{2}\left(\epsilon^\dagger \tau^m \epsilon^c\right) \mathcal{D}^{(1)}{}_m \tilde{\kappa},$$
$$\delta_Q\left(\delta_Q \eta\right)^\dagger = -\frac{i}{2}\left\{\left(\epsilon^{c\dagger} \tau^m \epsilon\right) \mathcal{D}^{(-1)}{}_m \tilde{\chi} - \left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(1)}{}_m \tilde{\xi} + g\left[\sigma,\, \tilde{\xi}\right] - i \frac{\tilde{r}}{r^2} \tilde{\xi}\right]\right\},$$
$$\delta_Q\left(\delta_Q \kappa\right)^\dagger = 0,$$
$$\delta_Q\left(\delta_Q \tilde{\chi}\right)^\dagger = -\frac{i}{2}\left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(1)}{}_m \kappa - g\left[\sigma,\, \kappa\right] - i \frac{\tilde{r}}{r^2} \kappa\right],$$
$$\delta_Q\left(\delta_Q \tilde{\xi}\right)^\dagger = -\frac{i}{2}\left(\epsilon^\dagger \tau^m \epsilon^c\right) \mathcal{D}^{(1)}{}_m \kappa,$$
$$\delta_Q\left(\delta_Q \tilde{\eta}\right)^\dagger = \frac{i}{2}\left\{\left(\epsilon^{c\dagger} \tau^m \epsilon\right) \mathcal{D}^{(-1)}{}_m \chi - \left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(1)}{}_m \xi + g\left[\sigma,\, \xi\right] - i \frac{\tilde{r}}{r^2} \xi\right]\right\},$$
$$\delta_Q\left(\delta_Q \tilde{\kappa}\right)^\dagger = 0.$$

The system of $(\tilde{H}, \tilde{H}^\dagger, \chi, \xi, \tilde{\eta}, \tilde{\kappa}, F^1{}_2)$ is identical to the one of $(H, H^\dagger, \tilde{\chi}, \tilde{\xi}, \eta, \kappa, F^2{}_2)$. If the former contributes the one-loop determinant $Z_H^{\text{1-loop}}$ to the partition function, both of the systems contribute $(Z_H^{\text{1-loop}})^2$. Therefore, we will focus on the former system only.

From the fermionic part of the system $(\tilde{H}, \tilde{H}^\dagger, \chi, \xi, \tilde{\eta}, \tilde{\kappa}, F^1{}_2)$ of the regulator action $\mathcal{S}_Q$,

$$-\int \sqrt{g} d^5 x\left[\delta_Q\left(\delta_Q \chi\right)^\dagger \cdot \chi + \delta_Q\left(\delta_Q \xi\right)^\dagger \cdot \xi + \delta_Q\left(\delta_Q \tilde{\eta}\right)^\dagger \cdot \tilde{\eta} + \delta_Q\left(\delta_Q \tilde{\kappa}\right)^\dagger \cdot \tilde{\kappa}\right]$$
$$= \frac{i}{2} \int \sqrt{g} d^5 x$$
$$\times\left[\begin{pmatrix}\chi, & \xi\end{pmatrix}\begin{pmatrix}\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(1)}{}_m - g\left[\sigma, \cdot\right] - i \frac{\tilde{r}}{r^2} & \left(\epsilon^{c\dagger} \tau^m \epsilon\right) \mathcal{D}^{(-1)}{}_m \\ \left(\epsilon^\dagger \tau^m \epsilon^c\right) \mathcal{D}^{(1)}{}_m & -\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-1)}{}_m - g\left[\sigma, \cdot\right] - i \frac{\tilde{r}}{r^2}\end{pmatrix}\begin{pmatrix}\tilde{\kappa} \\ \tilde{\eta}\end{pmatrix}\right],$$

where we have performed integration by parts, the one-loop determinant from the fermions of the system $(\tilde{H}, \tilde{H}^\dagger, \chi, \xi, \tilde{\eta}, \tilde{\kappa}, F^1{}_2)$ can be read as

$$Z^{\text{1-loop}}_{\text{H,F}}$$
$$= \mathcal{D}\text{et}_{(0,\frac{1}{2})}\left[\begin{pmatrix} \left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m - g\,[\sigma,\cdot] - i\frac{\tilde{r}}{r^2} & \left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m \\ \left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(1)}{}_m & -\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m - g\,[\sigma,\cdot] - i\frac{\tilde{r}}{r^2}\end{pmatrix}\right],$$

where the determinant $\mathcal{D}\text{et}_{(0,\frac{1}{2})}$ is defined such that the path integral over a fermionic (k,l)-form ψ on Σ and its partner λ with a differential operator D yields

$$\int [d\lambda]\,[d\psi]\exp\left[\int \sqrt{g}d^5x\,\lambda D\psi\right] = \mathcal{D}\text{et}_{(k,l)}\,[D]\,.$$

For four differential operators $D_1,\cdots,D_4$, we have the formula

$$\mathcal{D}\text{et}_{(k,l)}\left[\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4\end{pmatrix}\right] = \mathcal{D}\text{et}_{(k,l)}\,[D_1]\,\mathcal{D}\text{et}_{(k,l)}\left[D_4 - D_3\frac{1}{D_1}D_2\right],$$

for an invertible D_1. If there is another differential operator $D'{}_1$ satisfying the relation

$$D_3\frac{1}{D_1} = \frac{1}{D'{}_1}D_3,$$

we obtain the formula

$$\mathcal{D}\text{et}_{(k,l)}\left[\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4\end{pmatrix}\right] = \frac{\mathcal{D}\text{et}_{(k,l)}\,[D_1]}{\mathcal{D}\text{et}_{(k,l)}\,[D'_1]}\mathcal{D}\text{et}_{(k,l)}\left[D'{}_1D_4 - D_3D_2\right].$$

In our case, we have

$$D_1 = \left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m - g\,[\sigma,\cdot] - i\frac{\tilde{r}}{r^2}, \qquad D_3 = \left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(1)}{}_m,$$

which both act on the spinor $\tilde{\kappa}$ of negative chirality on Σ and of charge $q = 1$. Using (99) in Appendix G.3, we can find the operator $D'{}_1$,

$$\begin{aligned} D_3D_1 &= \left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(1)}{}_n\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m - g\,[\sigma,\cdot] - i\frac{\tilde{r}}{r^2}\right] \\ &= \left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m - g\,[\sigma,\cdot] + i\frac{\tilde{r}}{r^2}\right]\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(1)}{}_n = D'{}_1D_3.\end{aligned}$$

Therefore, we find that

$$Z^{\text{1-loop}}_{\text{H,F}} = \mathcal{D}\text{et}_{(0,\frac{1}{2})}\left[\Delta^{\mathcal{N}=1}_{\text{H,B}}\right]\frac{\mathcal{D}\text{et}_{(0,\frac{1}{2})}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m - g\,[\sigma,\cdot] - i\frac{\tilde{r}}{r^2}\right]}{\mathcal{D}\text{et}_{(0,\frac{1}{2})}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m - g\,[\sigma,\cdot] + i\frac{\tilde{r}}{r^2}\right]},$$

where the differential operator $\Delta^{\mathcal{N}=1}_{\text{H,B}}$ denotes

$$\begin{aligned}\Delta^{\mathcal{N}=1}_{\text{H,B}} = &-\left(\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m - g\,[\sigma,\cdot] + i\frac{\tilde{r}}{r^2}\right)\left(\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m + g\,[\sigma,\cdot] + i\frac{\tilde{r}}{r^2}\right) \\ &-\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(1)}{}_m\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m.\end{aligned}$$

Note that

$$\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\left(\epsilon^{\dagger}\tau^{m}\epsilon\right)\mathcal{D}^{(1)}{}_{m}-g\,[\sigma,\cdot]-i\frac{\tilde{r}}{r^{2}}\right]=\int\left[d\chi\right]\left[d\tilde{\kappa}\right]\exp\left[\frac{i}{2}\int\sqrt{g}d^{5}x\,\chi\,D_{1}\tilde{\kappa}\right]$$
$$=\int\left[d\chi\right]\left[d\tilde{\kappa}\right]\exp\left[\frac{i}{2}\int\sqrt{g}d^{5}x\,\tilde{\kappa}\,D'{}_{1}\chi\right]$$
$$=\mathcal{D}\mathrm{et}_{(\frac{1}{2},0)}\left[\left(\epsilon^{\dagger}\tau^{m}\epsilon\right)\mathcal{D}^{(-1)}{}_{m}-g\,[\sigma,\cdot]+i\frac{\tilde{r}}{r^{2}}\right].$$

The differential operator $D'{}_1$ in the determinant $\mathcal{D}\mathrm{et}_{(\frac{1}{2},0)}\left[D'{}_1\right]$ on the most right hand side doesn't depend on the chirality of χ, and therefore, $\mathcal{D}\mathrm{et}_{(\frac{1}{2},0)}\left[D'{}_1\right]=\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[D'{}_1\right]$. It means that the ratio of the determinants is unity;

$$\frac{\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\left(\epsilon^{\dagger}\tau^{m}\epsilon\right)\mathcal{D}^{(1)}{}_{m}-g\,[\sigma,\cdot]-i\frac{\tilde{r}}{r^{2}}\right]}{\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\left(\epsilon^{\dagger}\tau^{m}\epsilon\right)\mathcal{D}^{(-1)}{}_{m}-g\,[\sigma,\cdot]+i\frac{\tilde{r}}{r^{2}}\right]}=1,$$

and we obtain

$$Z^{\text{1-loop}}_{\mathrm{H,F}}=\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\Delta^{\mathcal{N}=1}_{\mathrm{H,B}}\right].$$

In the bosonic part of the system $(\tilde{H},\tilde{H}^{\dagger},\chi,\xi,\tilde{\eta},\tilde{\kappa},F^{1}{}_{2})$ of the regulator action $\mathcal{S}_{Q}$,

$$-\int d^{5}x\sqrt{g}\left[\left(\delta_{Q}\chi\right)^{\dagger}\cdot\delta_{Q}\chi+\left(\delta_{Q}\xi\right)^{\dagger}\cdot\delta_{Q}\xi+\left(\delta_{Q}\tilde{\eta}\right)^{\dagger}\cdot\delta_{Q}\tilde{\eta}+\left(\delta_{Q}\tilde{\kappa}\right)^{\dagger}\cdot\delta_{Q}\tilde{\kappa}\right],$$

we will shift the auxiliary fields $F^{\tilde{\alpha}}{}_{\check{\beta}}$ so that we can trivially integrate them out. We will integrate the remaining part of the action by parts to obtain

$$-2\int d^{5}x\sqrt{g}\tilde{H}^{\dagger}\Delta^{\mathcal{N}=1}_{\mathrm{H,B}}\tilde{H},$$

and see that the one-loop determinant from the bosonic fields of the system $(\tilde{H},\tilde{H}^{\dagger},\chi,\xi,\tilde{\eta},\tilde{\kappa})$ is given by

$$Z^{\text{1-loop}}_{\mathrm{H,B}}=\frac{1}{\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\Delta^{\mathcal{N}=1}_{\mathrm{H,B}}\right]}.$$

Therefore, the contributions from the hypermultiplet to the partition function are trivial;

$$\left(Z^{\text{1-loop}}_{\mathrm{H}}\right)^{2}=\left(Z^{\text{1-loop}}_{\mathrm{H,F}}Z^{\text{1-loop}}_{\mathrm{H,B}}\right)^{2}=1.$$

In the round limit $\tilde{r}\to r$, the contributions from the hypermultiplet reproduce the previous results about the hypermultiplet in [1].

8.2.2. The $\mathcal{N}=2$ twisting

Let us proceed to the $\mathcal{N}=2$ twisting. In contrast to the $\mathcal{N}=1$ twisting, the system of the fluctuations $(\tilde{H},\tilde{H}^{\dagger},\chi,\xi,\tilde{\eta},\tilde{\kappa},F^{1}{}_{2})$ yields the different contribution to the partition function from the one from $(H,H^{\dagger},\tilde{\chi},\tilde{\xi},\eta,\kappa,F^{2}{}_{2})$, and we will treat them separately below.

The BRST transformation of the fermions in the hypermultiplet is given by

$$\delta_Q \chi = \sqrt{2}i\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m\tilde{H} + g\left[\sigma,\ \tilde{H}\right] + 2i\frac{\tilde{r}}{r^2}\tilde{H}\right],$$
$$\delta_Q \xi = \sqrt{2}i\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m\tilde{H},$$
$$\delta_Q \eta = F^1{}_1 - 2\sqrt{2}\mathcal{D}_z\tilde{H}, \qquad \delta_Q\kappa = 0,$$
$$\delta_Q \tilde{\chi} = -\sqrt{2}i\Big[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(-2)}{}_m H + g\left[\sigma,\ H\right]\Big], \quad \delta_Q\tilde{\xi} = -\sqrt{2}i\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(-2)}{}_m H,$$
$$\delta_Q\tilde{\eta} = F^2{}_1 + 2\sqrt{2}\mathcal{D}_z H, \qquad \delta_Q\tilde{\kappa} = 0,$$

and its hermitian conjugate by

$$\left(\delta_Q\chi\right)^\dagger = -\sqrt{2}i\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m\tilde{H}^\dagger - g\left[\sigma,\ \tilde{H}^\dagger\right] - 2i\frac{\tilde{r}}{r^2}\tilde{H}^\dagger\right],$$
$$\left(\delta_Q\xi\right)^\dagger = -\sqrt{2}i\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(0)}{}_m\tilde{H}^\dagger,$$
$$\left(\delta_Q\eta\right)^\dagger = -F^2{}_2 - 2\sqrt{2}\mathcal{D}_{\bar{z}}\tilde{H}^\dagger, \qquad \left(\delta_Q\kappa\right)^\dagger = 0,$$
$$\left(\delta_Q\tilde{\chi}\right)^\dagger = \sqrt{2}i\Big[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(2)}{}_m H^\dagger - g\left[\sigma,\ H^\dagger\right]\Big],$$
$$\left(\delta_Q\tilde{\xi}\right)^\dagger = \sqrt{2}i\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(2)}{}_m H^\dagger,$$
$$\left(\delta_Q\tilde{\eta}\right)^\dagger = F^1{}_2 + 2\sqrt{2}\mathcal{D}_{\bar{z}}H^\dagger, \qquad \left(\delta_Q\tilde{\kappa}\right)^\dagger = 0.$$

Using the BRST transformation of the auxiliary fields $F^1{}_2$, $F^2{}_2$,

$$\delta_Q F^1{}_2 = \frac{i}{2}\left[\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m\chi - \left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m\eta - g\left[\sigma,\ \eta\right] - 2i\mathcal{D}_{\bar{z}}\kappa\right],$$
$$\delta_Q F^2{}_2 = \frac{i}{2}\left[\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m\tilde{\chi} - \left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m\tilde{\eta} - g\left[\sigma,\ \tilde{\eta}\right] + 2i\frac{\tilde{r}}{r^2}\tilde{\eta} - 2i\mathcal{D}_{\bar{z}}\tilde{\kappa}\right],$$

where we have omitted the terms $g\left(\sigma^i\right)^{\tilde{\alpha}}{}_{\dot{\gamma}}\left[\phi^i,\ \lambda^{\dot{\gamma}}\right]$ on the right hand sides of both the equations, because they vanish in the large t limit, we find that

$$\delta_Q\left(\delta_Q\chi\right)^\dagger = \frac{i}{2}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m\tilde{\kappa} - g\left[\sigma,\ \tilde{\kappa}\right] - 2i\frac{\tilde{r}}{r^2}\tilde{\kappa}\right],$$
$$\delta_Q\left(\delta_Q\xi\right)^\dagger = \frac{i}{2}\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(0)}{}_m\tilde{\kappa},$$
$$\delta_Q\left(\delta_Q\eta\right)^\dagger = -\frac{i}{2}\left\{\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(-2)}{}_m\tilde{\chi} - \left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m\tilde{\xi} + g\left[\sigma,\ \tilde{\xi}\right] - 2i\frac{\tilde{r}}{r^2}\tilde{\xi}\right]\right\},$$
$$\delta_Q\left(\delta_Q\kappa\right)^\dagger = 0,$$
$$\delta_Q\left(\delta_Q\tilde{\chi}\right)^\dagger = -\frac{i}{2}\Big[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(2)}{}_m\kappa - g\left[\sigma,\ \kappa\right]\Big],$$
$$\delta_Q\left(\delta_Q\tilde{\xi}\right)^\dagger = -\frac{i}{2}\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(2)}{}_m\kappa,$$
$$\delta_Q\left(\delta_Q\tilde{\eta}\right)^\dagger = \frac{i}{2}\left\{\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m\chi - \Big[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(2)}{}_m\xi + g\left[\sigma,\ \xi\right]\Big]\right\},$$
$$\delta_Q\left(\delta_Q\tilde{\kappa}\right)^\dagger = 0.$$

From the fermionic part of the system $(\tilde{H}, \tilde{H}^\dagger, \chi, \xi, \tilde{\eta}, \tilde{\kappa}, F^1{}_2)$ of the regulator action $\mathcal{S}_Q$,

$$\begin{aligned}
&-\int \sqrt{g} d^5x \left[\delta_Q \left(\delta_Q \chi\right)^\dagger \cdot \chi + \delta_Q \left(\delta_Q \xi\right)^\dagger \cdot \xi + \delta_Q \left(\delta_Q \tilde{\eta}\right)^\dagger \cdot \tilde{\eta} + \delta_Q \left(\delta_Q \tilde{\kappa}\right)^\dagger \cdot \tilde{\kappa}\right] \\
&= \frac{i}{2} \int \sqrt{g} d^5x \\
&\quad \times \left[\begin{pmatrix} \chi, & \xi \end{pmatrix} \begin{pmatrix} \left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(0)}{}_m - g\,[\sigma, \cdot] - 2i \frac{\tilde{r}}{r^2} & \left(\epsilon^{c\dagger} \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m \\ \left(\epsilon^\dagger \tau^m \epsilon^c\right) \mathcal{D}^{(0)}{}_m & -\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m - g\,[\sigma, \cdot] \end{pmatrix} \begin{pmatrix} \tilde{\kappa} \\ \tilde{\eta} \end{pmatrix} \right],
\end{aligned}$$

where we have performed integration by parts, the one-loop determinant from the fermions of the system $(\tilde{H}, \tilde{H}^\dagger, \chi, \xi, \tilde{\eta}, \tilde{\kappa}, F^1{}_2)$ can be read as

$$Z^{\text{1-loop}}_{\text{H,F}} = \mathcal{D}\text{et}_{(0,0)} \left[\begin{pmatrix} \left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(0)}{}_m - g\,[\sigma, \cdot] - 2i \frac{\tilde{r}}{r^2} & \left(\epsilon^{c\dagger} \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m \\ \left(\epsilon^\dagger \tau^m \epsilon^c\right) \mathcal{D}^{(0)}{}_m & -\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m - g\,[\sigma, \cdot] \end{pmatrix} \right], \tag{74}$$

with the determinant $\mathcal{D}\text{et}_{(0,0)}$ defined in the previous Subsection 8.2.1.

As in the $\mathcal{N} = 1$ twisting in Subsection 8.2.1, upon computing the one-loop determinant (74), we may identify the differential operators D_1 and D_3 with

$$D_1 = \left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(0)}{}_m - g\,[\sigma, \cdot] - 2i \frac{\tilde{r}}{r^2}, \qquad D_3 = \left(\epsilon^\dagger \tau^m \epsilon^c\right) \mathcal{D}^{(0)}{}_m,$$

respectively, and using (99) in Appendix G.3, we obtain the operator D'_1,

$$\begin{aligned}
D_3 D_1 &= \left(\epsilon^\dagger \tau^n \epsilon^c\right) \mathcal{D}^{(0)}{}_n \left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(0)}{}_m - g\,[\sigma, \cdot] - 2i \frac{\tilde{r}}{r^2}\right] \\
&= \left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m - g\,[\sigma, \cdot]\right] \left(\epsilon^\dagger \tau^n \epsilon^c\right) \mathcal{D}^{(0)}{}_n = D'_1 D_3.
\end{aligned}$$

Using this relation, we can compute the one-loop determinant

$$Z^{\text{1-loop}}_{\text{H,F}} = \mathcal{D}\text{et}_{(0,0)} \left[\Delta^{\mathcal{N}=2}_{\text{H,B}}\right] \frac{\mathcal{D}\text{et}_{(0,0)}\left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(0)}{}_m - g\,[\sigma, \cdot] - 2i \frac{\tilde{r}}{r^2}\right]}{\mathcal{D}\text{et}_{(0,0)}\left[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m - g\,[\sigma, \cdot]\right]},$$

where the differential operator $\Delta^{\mathcal{N}=2}_{\text{H,B}}$ denotes

$$\begin{aligned}
\Delta^{\mathcal{N}=2}_{\text{H,B}} = &-\Big(\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m - g\,[\sigma, \cdot]\Big)\Big(\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m + g\,[\sigma, \cdot]\Big) \\
&- \left(\epsilon^\dagger \tau^m \epsilon^c\right) \mathcal{D}^{(0)}{}_m \left(\epsilon^{c\dagger} \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m.
\end{aligned}$$

In the bosonic part of the system $(\tilde{H}, \tilde{H}^\dagger, \chi, \xi, \tilde{\eta}, \tilde{\kappa})$ of the regulator action $\mathcal{S}_Q$,

$$-\int d^5x \sqrt{g} \left[\left(\delta_Q \chi\right)^\dagger \cdot \delta_Q \chi + \left(\delta_Q \xi\right)^\dagger \cdot \delta_Q \xi + \left(\delta_Q \tilde{\eta}\right)^\dagger \cdot \delta_Q \tilde{\eta} + \left(\delta_Q \tilde{\kappa}\right)^\dagger \cdot \delta_Q \tilde{\kappa}\right],$$

we will shift the auxiliary fields $F^{\tilde{\alpha}}{}_{\check{\beta}}$ so that we can trivially integrate them out. We will integrate the remaining part of the action by parts to obtain

$$-2 \int d^5x \sqrt{g} \tilde{H}^\dagger \tilde{\Delta}^{\mathcal{N}=2}_{\text{H,B}} \tilde{H},$$

with the differential operator $\tilde{\Delta}^{\mathcal{N}=2}_{\mathrm{H,B}}$ given by

$$-\left(\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] + 2i\frac{\tilde{r}}{r^2}\right)\left(\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m + g\,[\sigma,\cdot] + 2i\frac{\tilde{r}}{r^2}\right)$$
$$-\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(2)}{}_m\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m.$$

Therefore, we can read the one-loop determinant from the bosonic fields of the system $(\tilde{H},\tilde{H}^\dagger,\chi,\xi,\tilde{\eta},\tilde{\kappa},F^1{}_2)$ as

$$Z^{\text{1-loop}}_{\mathrm{H,B}} = \frac{1}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\tilde{\Delta}^{\mathcal{N}=2}_{\mathrm{H,B}}\right]}.$$

Let us move onto the fermionic part of the system $(H,H^\dagger,\tilde{\chi},\tilde{\xi},\eta,\kappa,F^2{}_2)$ of the regulator action $\mathcal{S}_Q$,

$$-\int\sqrt{g}d^5x\left[\delta_Q\left(\delta_Q\tilde{\chi}\right)^\dagger\cdot\tilde{\chi} + \delta_Q\left(\delta_Q\tilde{\xi}\right)^\dagger\cdot\tilde{\xi} + \delta_Q\left(\delta_Q\eta\right)^\dagger\cdot\eta + \delta_Q\left(\delta_Q\kappa\right)^\dagger\cdot\kappa\right]$$
$$=\frac{i}{2}\int\sqrt{g}d^5x$$
$$\times\left[\begin{pmatrix}\tilde{\chi}, & \tilde{\xi}\end{pmatrix}\begin{pmatrix}\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(2)}{}_m - g\,[\sigma,\cdot] & \left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m \\ \left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(2)}{}_m & -\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] - 2i\frac{\tilde{r}}{r^2}\end{pmatrix}\begin{pmatrix}\kappa\\ \eta\end{pmatrix}\right],$$

up to an integration by parts. It gives rise to the one-loop determinant

$$\tilde{Z}^{\text{1-loop}}_{\mathrm{H,F}} = \mathcal{D}\mathrm{et}_{(1,0)}\left[\begin{pmatrix}\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(2)}{}_m - g\,[\sigma,\cdot] & \left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m \\ \left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(2)}{}_m & -\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] - 2i\frac{\tilde{r}}{r^2}\end{pmatrix}\right], \tag{75}$$

with the determinant $\mathcal{D}\mathrm{et}_{(1,0)}$ defined in the previous Subsection 8.2.1.

As we have done just above, identifying the differential operators D_1 and D_3 with

$$D_1 = \left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(2)}{}_m - g\,[\sigma,\cdot], \qquad D_3 = \left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(2)}{}_m,$$

and using (99) in Appendix G.3, the operator D'_1 is found to be

$$D_3D_1 = \left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(2)}{}_n\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(2)}{}_m - g\,[\sigma,\cdot]\right]$$
$$=\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] + 2i\frac{\tilde{r}}{r^2}\right]\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(2)}{}_n = D'_1D_3.$$

It follows from this relation that the one-loop determinant is computed to give

$$\tilde{Z}^{\text{1-loop}}_{\mathrm{H,F}} = \mathcal{D}\mathrm{et}_{(1,0)}\left[\tilde{\Delta}^{\mathcal{N}=2}_{\mathrm{H,B}}\right]\frac{\mathcal{D}\mathrm{et}_{(1,0)}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(2)}{}_m - g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(1,0)}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] + 2i\frac{\tilde{r}}{r^2}\right]},$$

with $\tilde{\Delta}^{\mathcal{N}=2}_{\mathrm{H,B}}$ given just above.

In the bosonic part of the system $(H,H^\dagger,\tilde{\chi},\tilde{\xi},\eta,\kappa,F^2{}_2)$ of the regulator action $\mathcal{S}_Q$,

$$-\int d^5x\sqrt{g}\left[\left(\delta_Q\tilde{\chi}\right)^\dagger\cdot\delta_Q\tilde{\chi} + \left(\delta_Q\tilde{\xi}\right)^\dagger\cdot\delta_Q\tilde{\xi} + \left(\delta_Q\eta\right)^\dagger\cdot\delta_Q\eta + \left(\delta_Q\kappa\right)^\dagger\cdot\delta_Q\kappa\right],$$

we can trivially integrate $F^{\tilde{\alpha}}{}_{\check{\beta}}$ out in the same way as above. We will integrate the remaining part of the action by parts to obtain

$$-2\int d^5x\sqrt{g}H^\dagger \Delta^{\mathcal{N}=2}_{\mathrm{H,B}}H,$$

with the same $\tilde{\Delta}^{\mathcal{N}=2}_{\mathrm{H,B}}$ as above. It immediately gives the one-loop determinant from the bosonic fields of the system $(H, H^\dagger, \tilde{\chi}, \tilde{\xi}, \eta, \kappa, F^2{}_2)$,

$$\tilde{Z}^{\text{1-loop}}_{\mathrm{H,B}} = \frac{1}{\mathcal{D}\mathrm{et}_{(0,1)}\left[\Delta^{\mathcal{N}=2}_{\mathrm{H,B}}\right]}.$$

Combining the one-loop determinants from both the systems, we obtain

$$\begin{aligned}
Z^{\text{1-loop}}_{\mathrm{H}} &= Z^{\text{1-loop}}_{\mathrm{H,F}}\tilde{Z}^{\text{1-loop}}_{\mathrm{H,F}}Z^{\text{1-loop}}_{\mathrm{H,B}}\tilde{Z}^{\text{1-loop}}_{\mathrm{H,B}} \\
&= \frac{\mathcal{D}\mathrm{et}_{(0,0)}\left[\Delta^{\mathcal{N}=2}_{\mathrm{H,B}}\right]}{\mathcal{D}\mathrm{et}_{(0,1)}\left[\Delta^{\mathcal{N}=2}_{\mathrm{H,B}}\right]}\frac{\mathcal{D}\mathrm{et}_{(1,0)}\left[\tilde{\Delta}^{\mathcal{N}=2}_{\mathrm{H,B}}\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\tilde{\Delta}^{\mathcal{N}=2}_{\mathrm{H,B}}\right]}\frac{\mathcal{D}\mathrm{et}_{(0,0)}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] - 2i\frac{\tilde{r}}{r^2}\right]}{\mathcal{D}\mathrm{et}_{(1,0)}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] + 2i\frac{\tilde{r}}{r^2}\right]} \\
&\quad\times\frac{\mathcal{D}\mathrm{et}_{(1,0)}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(2)}{}_m - g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(-2)}{}_m - g\,[\sigma,\cdot]\right]}.
\end{aligned}$$

We may regard the differential operators

$$\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}_m = \frac{2i}{\tilde{r}}L_3, \quad \left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(0)}_m = \frac{2i}{r}L_+, \quad \left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(0)}_m = \frac{2i}{r}L_-,$$

as the generators of the Lie algebra of $SU(2)$ satisfying that

$$\left[L_3,\, L_\pm\right] = \pm L_\pm, \qquad \left[L_+,\, L_-\right] = 2L_3.$$

Then, the scalar spherical harmonics $\varphi_{l,m,\tilde{m}}$ $(l = 0, 1/2, 1, 3/2, \cdots; -l \le m, \tilde{m} \le l)$ on the S^3 obey

$$L_3\varphi_{l,m,\tilde{m}} = m\varphi_{l,m,\tilde{m}}, \qquad L_\pm\varphi_{l,m,\tilde{m}} = \sqrt{(l\mp m)(l\pm m+1)}\varphi_{l,m\pm1,\tilde{m}}.$$

Each of the fluctuations is in the adjoint representation of the gauge group G, whose Cartan generators we denote as H_i $(i = 1, \cdots, r)$ with r the rank of G, and the remaining generators as E_α with α a root of G. We assume that they obey

$$[H_i,\, E_\alpha] = \alpha_i E_\alpha, \qquad \left[E_\alpha,\, E_{-\alpha}\right] = \sum_{i=1}^{r}\alpha_i H_i \equiv \alpha\cdot H,$$

and are normalized as

$$\mathrm{tr}\left[H_i H_j\right] = \delta_{i,j}, \qquad \mathrm{tr}\left[E_{-\alpha}E_\alpha\right] = 1.$$

As explained in [31], the Hodge decomposition implies that for the space $\Omega^{k,l}(\Sigma)$ of all the (k,l)-forms on the Riemann surface Σ,

$$\Omega^{1,0}(\Sigma)\oplus\Omega^{0,1}(\Sigma) = \left(\Omega^{0,0}(\Sigma)\ominus H^0(\Sigma)\right)\oplus\left(\Omega^{0,0}(\Sigma)\ominus H^0(\Sigma)\right)\oplus H^1(\Sigma),$$

where $H^p(\Sigma)$ is the space of all the harmonic p-forms on Σ. It follows from this that for a constant D,

$$\frac{\mathcal{D}\mathrm{et}_{(0,0)}\,[D]}{\mathcal{D}\mathrm{et}_{(1,0)}\,[D]} = D^{b_0(\Sigma)-\frac{1}{2}b_1(\Sigma)} = D^{\frac{1}{2}\chi(\Sigma)}$$

with $b_i(\Sigma) = \dim H_i(\Sigma)$ the i-th Betti number, and with the Euler number $\chi(\Sigma)$ of the surface Σ:

$$\chi(\Sigma) = b_0(\Sigma) - b_1(\Sigma) + b_2(\Sigma) = 2b_0(\Sigma) - b_1(\Sigma),$$

where we have used the Hodge duality; $b_0(\Sigma) = b_2(\Sigma)$.

The one-loop determinant $\mathcal{D}\mathrm{et}_{(k,l)}$ in $Z_{\mathrm{H}}^{\text{1-loop}}$ is defined over the space with the basis[16] $\{\varphi_{l,m,\tilde{m}} \otimes E_\alpha \otimes v\}$, where $v \in \Omega^{k,l}(\Sigma)$.

In terms of the basis $\{\varphi_{l,m,\tilde{m}} \otimes E_\alpha \otimes v\}$, we obtain

$$\frac{\mathcal{D}\mathrm{et}_{(0,0)}\Big[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] - 2i\,\frac{\tilde{r}}{r^2}\Big]}{\mathcal{D}\mathrm{et}_{(1,0)}\Big[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] + 2i\,\frac{\tilde{r}}{r^2}\Big]}$$

$$= \left[\prod_{\alpha\in\Lambda}\prod_{l\in\frac{1}{2}\mathbf{Z}_{\geq 0}}\prod_{m=-l}^{l}\prod_{\tilde{m}=-l}^{l}\left(\frac{2i}{\tilde{r}}m - g\,(\sigma\cdot\alpha) + 2i\,\frac{\tilde{r}}{r^2}\right)\right]^{\frac{1}{2}\chi(\Sigma)},$$

up to an overall constant, where α is a root of the Lie algebra of the gauge group, and Λ is the set of all the roots of it.

By the hermitian conjugation, we can see that

$$\mathcal{D}\mathrm{et}_{(0,0)}\Big[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m - g\,[\sigma,\cdot]\Big] = \mathcal{D}\mathrm{et}_{(0,0)}\Big[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(2)}{}_m - g\,[\sigma,\cdot]\Big],$$

and in a similar way to above, we can compute

$$\frac{\mathcal{D}\mathrm{et}_{(1,0)}\Big[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(2)}{}_m - g\,[\sigma,\cdot]\Big]}{\mathcal{D}\mathrm{et}_{(0,0)}\Big[\left(\epsilon^\dagger \tau^m \epsilon\right) \mathcal{D}^{(-2)}{}_m - g\,[\sigma,\cdot]\Big]}$$

$$= \frac{1}{\left[\prod_{\alpha\in\Lambda}\prod_{l\in\frac{1}{2}\mathbf{Z}_{\geq 0}}\prod_{m=-l}^{l}\prod_{\tilde{m}=-l}^{l}\left(\frac{2i}{\tilde{r}}m + \frac{2i}{\tilde{r}}\left(1-\frac{\tilde{r}^2}{r^2}\right) - g\,(\sigma\cdot\alpha)\right)\right]^{\frac{1}{2}\chi(\Sigma)}},$$

up to a constant factor.

[16] More precisely, the basis consists of $\{\varphi_{l,m,\tilde{m}} \otimes E_\alpha \otimes v\}$ and $\{\varphi_{l,m,\tilde{m}} \otimes H_i \otimes v\}$, with $v \in \Omega^{k,l}(\Sigma)$. However, the Cartan part of the Lie algebra of G contributes a constant to the determinant, and we will omit them in computing the partition function.

After replacing spin l by $n = 2l = 0, 1, 2, \cdots$, and shifting $n \to n-1$, we find that

$$\frac{\mathcal{D}\mathrm{et}_{(0,0)}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] - 2i\frac{\tilde{r}}{r^2}\right]}{\mathcal{D}\mathrm{et}_{(1,0)}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(0)}{}_m - g\,[\sigma,\cdot] + 2i\frac{\tilde{r}}{r^2}\right]}\frac{\mathcal{D}\mathrm{et}_{(1,0)}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(2)}{}_m - g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(-2)}{}_m - g\,[\sigma,\cdot]\right]}$$

$$= \prod_{\alpha\in\Lambda}\left[\prod_{n=1}^{\infty}\left(\frac{n-1+2\frac{\tilde{r}^2}{r^2} - i\tilde{r}g\,(\sigma\cdot\alpha)}{n+1-2\frac{\tilde{r}^2}{r^2} + i\tilde{r}g\,(\sigma\cdot\alpha)}\right)^n\right]^{\frac{1}{2}\chi(\Sigma)}$$

$$= \prod_{\alpha\in\Lambda}\left[\frac{1}{s_{b=1}\left(i - 2i\frac{\tilde{r}^2}{r^2} - \tilde{r}g\,(\sigma\cdot\alpha)\right)}\right]^{\frac{1}{2}\chi(\Sigma)},$$

where $s_b(x)$ is a double sine function:

$$s_b(x) = \prod_{m,n=0}^{\infty}\frac{mb + nb^{-1} + \frac{1}{2}Q - ix}{mb + nb^{-1} + \frac{1}{2}Q + ix},$$

with $Q = b + b^{-1}$. In particular, when $b = 1$, it is reduced to

$$s_{b=1}(x) = \prod_{n=1}^{\infty}\left(\frac{n - ix}{n + ix}\right)^n. \tag{76}$$

For more details on double sine functions, see [32,21,33,34].

The remaining factor in the one-loop contribution $Z_{\mathrm{H}}^{\text{1-loop}}$ is computed in a similar way to yield

$$\frac{\mathcal{D}\mathrm{et}_{(0,0)}\left[\Delta_{\mathrm{H,B}}^{\mathcal{N}=2}\right]}{\mathcal{D}\mathrm{et}_{(0,1)}\left[\Delta_{\mathrm{H,B}}^{\mathcal{N}=2}\right]}\frac{\mathcal{D}\mathrm{et}_{(1,0)}\left[\tilde{\Delta}_{\mathrm{H,B}}^{\mathcal{N}=2}\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\tilde{\Delta}_{\mathrm{H,B}}^{\mathcal{N}=2}\right]}$$

$$= \prod_{\alpha\in\Lambda}\prod_{l\in\frac{1}{2}\mathbf{Z}_{\geq 0}}\prod_{m,\tilde{m}=-l}^{l}\left(\frac{\frac{4}{\tilde{r}^2}(m+1-\frac{\tilde{r}^2}{r^2})^2 + \frac{4}{r^2}(l-m)(l+m+1) + g^2\,(\sigma\cdot\alpha)^2}{\frac{4}{\tilde{r}^2}(m+\frac{\tilde{r}^2}{r^2})^2 + \frac{4}{r^2}(l-m)(l+m+1) + g^2\,(\sigma\cdot\alpha)^2}\right)^{\frac{1}{2}\chi(\Sigma)}$$

$$= \prod_{\alpha\in\Lambda}\prod_{l\in\frac{1}{2}\mathbf{Z}_{\geq 0}}\prod_{\tilde{m}=-l}^{l}\left(\frac{\frac{4}{\tilde{r}^2}(l+1-\frac{\tilde{r}^2}{r^2})^2 + g^2\,(\sigma\cdot\alpha)^2}{\frac{4}{\tilde{r}^2}(-l-\frac{\tilde{r}^2}{r^2})^2 + g^2\,(\sigma\cdot\alpha)^2}\right)^{\frac{1}{2}\chi(\Sigma)}$$

$$= \prod_{\alpha\in\Lambda}\prod_{n=1}^{\infty}\left(\frac{(n+1-\frac{2\tilde{r}^2}{r^2})^2 + \tilde{r}^2g^2\,(\sigma\cdot\alpha)^2}{(n-1+\frac{2\tilde{r}^2}{r^2})^2 + \tilde{r}^2g^2\,(\sigma\cdot\alpha)^2}\right)^{\frac{n}{2}\chi(\Sigma)}$$

$$= \prod_{\alpha\in\Lambda}\left[s_{b=1}\left(i - 2i\frac{\tilde{r}^2}{r^2} - \tilde{r}g\,(\sigma\cdot\alpha)\right)s_{b=1}\left(i - 2i\frac{\tilde{r}^2}{r^2} + \tilde{r}g\,(\sigma\cdot\alpha)\right)\right]^{\frac{1}{2}\chi(\Sigma)}.$$

Therefore, wrapping up all the factors, we obtain

$$\boxed{Z_{\mathrm{H}}^{\text{1-loop}} = \prod_{\alpha\in\Lambda_+}\left[s_{b=1}\left(i-2i\frac{\tilde{r}^2}{r^2}-\tilde{r}g\,(\sigma\cdot\alpha)\right)s_{b=1}\left(i-2i\frac{\tilde{r}^2}{r^2}+\tilde{r}g\,(\sigma\cdot\alpha)\right)\right]^{\frac{1}{2}\chi(\Sigma)},}$$

with Λ_+ the set of all the positive roots in Λ.

Let us consider the round limit $\tilde{r}\to r$ of $Z_{\mathrm{H}}^{\text{1-loop}}$. To this end, we will derive the formula

$$s_{b=1}(-i+x)s_{b=1}(-i-x) = \big(2\sinh(\pi x)\big)^2, \tag{77}$$

by using the zeta regularization,

$$\prod_{m=1}^{\infty} m = e^{\sum_{m=1}^{\infty}\log m} \quad\to\quad e^{-\zeta'(0)} = \sqrt{2\pi}.$$

In this regularization, we can prove the above formula as follows:

$$\begin{aligned} s_{b=1}(-i+x)s_{b=1}(-i-x) &= \prod_{n=1}^{\infty}\left(\frac{n-1-ix}{n+1+ix}\right)^n\left(\frac{n-1+ix}{n+1-ix}\right)^n = x^2\prod_{m=1}^{\infty}\left(m^2+x^2\right)^2 \\ &= \left(\prod_{m=1}^{\infty} m^4\right)\left[x\prod_{m=1}^{\infty}\left(1+\frac{x^2}{m^2}\right)\right]^2 = \big(2\sinh(\pi x)\big)^2, \end{aligned}$$

where we have used the formula

$$\frac{1}{\pi}\sinh\pi x = x\prod_{m=1}^{\infty}\left(1+\frac{x^2}{m^2}\right).$$

We can make use of (77) to see the round limit $\tilde{r}\to r$ of $Z_{\mathrm{H}}^{\text{1-loop}}$,

$$\begin{aligned} Z_{\mathrm{H}}^{\text{1-loop}} \quad &\to \quad \prod_{\alpha\in\Lambda_+}\Big[s_{b=1}\left(-i-\tilde{r}g\,(\sigma\cdot\alpha)\right)s_{b=1}\left(-i+\tilde{r}g\,(\sigma\cdot\alpha)\right)\Big]^{\frac{1}{2}\chi(\Sigma)} \\ &= \prod_{\alpha\in\Lambda_+}\Big[2\sinh\left(\pi r g\,(\sigma\cdot\alpha)\right)\Big]^{\chi(\Sigma)}. \end{aligned}$$

In summary, we have seen that the one-loop contribution in the $\mathcal{N}=2$ twisting on the squashed S^3 is given by

$$\boxed{\begin{aligned} Z^{\text{1-loop}} &= Z_{\mathrm{V}}^{\text{1-loop}} Z_{\mathrm{H}}^{\text{1-loop}} \\ &= \prod_{\alpha\in\Lambda_+}\Big[2\sinh\left(\pi r g\,(\sigma\cdot\alpha)\right)\Big]^{\chi(\Sigma)}\left[s_{b=1}\left(i-2i\frac{\tilde{r}^2}{r^2}-\tilde{r}g\,(\sigma\cdot\alpha)\right)s_{b=1}\right. \\ &\qquad \left.\times\left(i-2i\frac{\tilde{r}^2}{r^2}+\tilde{r}g\,(\sigma\cdot\alpha)\right)\right]^{\frac{1}{2}\chi(\Sigma)}. \end{aligned}}$$

9. Localization on the ellipsoid S^3

We will calculate the partition function by localization on the ellipsoid S^3 in the background discussed in Subsection 5.5. The calculations we will carry out are quite parallel to what we have done for the round and squashed S^3's in the previous section. All we have to do is to replace the background gauge field V by the one in (103), and $\tilde{r}/r^2$ by $1/f$. The fixed points discussed in the beginning of Section 8 are the same as for the background on the ellipsoid S^3.

Therefore, we will briefly explain the calculations of the one-loop contributions from the $\mathcal{N}=1$ gauge multiplet and the $\mathcal{N}=1$ hypermultiplet, separately in the next two subsections.

9.1. One-loop contributions from the $\mathcal{N}=1$ gauge multiplet

For the BRST transformations of the $\mathcal{N}=1$ gauge multiplet, as discussed in Section 7 and done in previous Section 8, we will reduce all the component fields in the gauge multiplet into scalar fields on the S^3.

As seen in Section 8, upon converting the gauge field A_m to V_0 and $V_\pm$, the field strength F_{mn} and F_{mz} are given, up to the gauge interactions, by

$$\begin{aligned}
&\frac{1}{2}\epsilon_{mkl}\left(\epsilon^\dagger\tau_m\epsilon\right)F_{kl}=\frac{2\tilde{r}}{r^2}V_0+i\left(\epsilon^{c\dagger}\tau_m\epsilon\right)\mathcal{D}_m^{(-2)}V_- - i\left(\epsilon^\dagger\tau_m\epsilon^c\right)\mathcal{D}_m^{(2)}V_+,\\
&\frac{1}{2}\epsilon_{mkl}\left(\epsilon^{c\dagger}\tau_m\epsilon\right)F_{kl}=\frac{4\tilde{r}}{r^2}V_+ + 2i\left(\epsilon^\dagger\tau_m\epsilon\right)\mathcal{D}_m^{(2)}V_+ - i\left(\epsilon^{c\dagger}\tau_m\epsilon\right)\mathcal{D}_m^{(0)}V_0,\\
&F_{mz}=\left(\epsilon^{c\dagger}\tau_m\epsilon\right)\left[\frac{1}{2}\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}_n^{(0)}A_z-\mathcal{D}_zV_-\right]\\
&\qquad+\left(\epsilon^\dagger\tau_m\epsilon^c\right)\left[\frac{1}{2}\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}_n^{(0)}A_z-\mathcal{D}_zV_+\right]\\
&\qquad+\left(\epsilon^\dagger\tau_m\epsilon\right)\left[\left(\epsilon^\dagger\tau^n\epsilon\right)\mathcal{D}_n^{(0)}A_z-\mathcal{D}_zV_0\right],
\end{aligned}$$

where we have used (105), and we will omit the gauge interactions, as before, since they have no effects on the partition function in the large t limit.

The BRST transformation of the bosonic fields is given by

$$\begin{aligned}
&\delta_Q\tilde{\sigma}=-\frac{1}{4}\tilde{\xi},\qquad \delta_QV_0=-\frac{i}{4}\tilde{\xi},\quad \delta_QV_-=-\frac{i}{4}\tilde{\eta},\quad \delta_QV_+=0,\\
&\delta_QA_{\bar{z}}=\frac{1}{4}\tilde{\varphi},\quad \delta_QA_z=0,\\
&\delta_QD^1{}_1=\frac{i}{4}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}_m^{(0)}\tilde{\xi}-\frac{2i}{f}\tilde{\xi}+g\left[\sigma,\,\tilde{\xi}\right]+\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(-2)}\tilde{\eta}\right],\\
&\delta_QD^1{}_2=\frac{i}{4}\left[-\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}_m^{(2)}\chi+\frac{2i}{f}\chi-g\left[\sigma,\,\chi\right]+\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(0)}\varphi-2i\mathcal{D}_z\eta\right],
\end{aligned}$$

where we denote a fixed point of the scalar field σ as the same letter σ, and the fluctuation about this fixed point σ as $\tilde{\sigma}$, as we have done in the previous section. Henceforth, we will keep this notation until the end of this section.

The BRST transformation of the fermionic fields is given by

$$\begin{aligned}
&\delta_Q \tilde{\xi} = 0, \qquad \delta_Q \tilde{\eta} = 0, \qquad \delta_Q \tilde{\varphi} = 0, \qquad \delta_Q \tilde{\chi} = -D^2{}_1, \\
&\delta_Q \xi = -\frac{2i}{f} V_0 + g\,[\sigma,\ V_0] + i\left(\epsilon^\dagger \tau^m \epsilon\right)\partial_m \tilde{\sigma} + \left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}_m^{(-2)} V_- \\
&\qquad\quad - \left(\epsilon^\dagger \tau^m \epsilon^c\right)\mathcal{D}_m^{(2)} V_+ + D^1{}_1, \\
&\delta_Q \eta = -\frac{4i}{f} V_+ + 2g\left[\sigma,\ V_+\right] + 2\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}_m^{(2)} V_+ + i\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\partial_m \tilde{\sigma} - \left(\epsilon^{c\dagger}\tau^m \epsilon\right)\partial_m V_0, \\
&\delta_Q \varphi = 2i\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\partial_m A_z + g\left[\sigma, A_z\right] - \mathcal{D}_z V_0 + i\mathcal{D}_z \tilde{\sigma}\right], \\
&\delta_Q \chi = 2i\left[\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\partial_m A_z - 2\mathcal{D}_z V_+\right],
\end{aligned}$$

and furthermore, we find that

$$\begin{aligned}
&\delta_Q\left(\delta_Q \xi\right)^\dagger = \frac{1}{2}\left[\frac{2}{f}\tilde{\xi} + ig\left[\sigma,\ \tilde{\xi}\right] + i\left(\epsilon^\dagger \tau^m \epsilon\right)\partial_m \tilde{\xi} + i\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}_m^{(-2)}\tilde{\eta}\right], \\
&\delta_Q\left(\delta_Q \eta\right)^\dagger = -\frac{i}{2}\left[\frac{2i}{f}\tilde{\eta} - g\left[\sigma,\ \tilde{\eta}\right] + \left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}_m^{(-2)}\tilde{\eta} - \left(\epsilon^\dagger \tau^m \epsilon^c\right)\mathcal{D}_m \tilde{\xi}\right], \\
&\delta_Q\left(\delta_Q \varphi\right)^\dagger = -\frac{i}{2}\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\partial_m \tilde{\varphi} - g\left[\sigma,\ \tilde{\varphi}\right] + 2i\mathcal{D}_{\bar{z}}\tilde{\xi}\right], \\
&\delta_Q\left(\delta_Q \chi\right)^\dagger = -\frac{i}{2}\left[\left(\epsilon^\dagger \tau^m \epsilon^c\right)\partial_m \tilde{\varphi} + 2i\mathcal{D}_{\bar{z}}\tilde{\eta}\right], \\
&\delta_Q\left(\delta_Q \tilde{\chi}\right)^\dagger = -\frac{i}{2}\left[-\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}_m \chi + \frac{2i}{f}\chi - g\left[\sigma,\ \chi\right] + \left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}_m \varphi - 2i\mathcal{D}_z \eta\right].
\end{aligned}$$

As we have discussed in the previous Section 8, assuming that $(\sigma\cdot\alpha) = \sum_{i=1}^r \sigma_i \alpha^i$ is non-zero for a generic $(\sigma^1, \cdots, \sigma^r)$, we can see that the operator $[\sigma,\ \cdot]$ acting on the sector with the basis $\{E_\alpha\}$ we are interested in is invertible, and we will 'gauge away' the fluctuation $\tilde{\sigma}$ by the shifts

$$\begin{aligned}
&V_0 \quad\to\quad V_0 - i\frac{1}{g\,[\sigma,\ \cdot]}\left(\epsilon^\dagger \tau^m \epsilon\right)\partial_m \tilde{\sigma}, \qquad V_+ \quad\to\quad V_+ - \frac{i}{2}\frac{1}{g\,[\sigma,\ \cdot]}\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\partial_m \tilde{\sigma}, \\
&V_- \quad\to\quad V_- - \frac{i}{2}\frac{1}{g\,[\sigma,\ \cdot]}\left(\epsilon^\dagger \tau^m \epsilon^c\right)\partial_m \tilde{\sigma}, \qquad A_z \quad\to\quad A_z - i\frac{1}{g\,[\sigma,\ \cdot]}\partial_z \tilde{\sigma},
\end{aligned}$$

in the BRST transformation in the large t limit, where we used (107) in Appendix G.5.

Using the remaining gauge transformations, we will 'diagonalize' the value of the scalar σ at one of the fixed points. The latter results in the Faddeev–Popov determinant

$$\begin{aligned}
Z_{\mathrm{FP}} &= \prod_{\alpha\in\Lambda} \mathrm{Det}_{(0,0)}\left[ig\,(\sigma\cdot\alpha)\right] \\
&= \int [d\bar{c}(z,\bar{z})dc(z,\bar{z})]\exp\left[-ig\sum_{\alpha\in\Lambda}\int_\Sigma d^2z\sqrt{g_\Sigma}\,(\sigma\cdot\alpha)\,\bar{c}_{-\alpha}c_\alpha\right],
\end{aligned} \tag{78}$$

with the Faddeev–Popov ghost $c_\alpha(z,\bar{z})$, $\bar{c}_\alpha(z,\bar{z})$ $(\alpha\in\Lambda)$, which are scalar fields on Σ.

This gauge-fixing procedure is quite the same as for the squash S^3 in Section 8, and we will set $\tilde{\sigma}$ to zero in the BRST transformations.

The bosonic part (63) of the gauge multiplet in the regulator action $\mathcal{S}_Q$, after integrating out the auxiliary fields $D^{\dot\alpha}{}_{\dot\beta}$ and integrating by parts, is reduced to the sum of

$$-\int d^5x\sqrt{g}\,\mathrm{tr}\Bigg[\Big|-\frac{2i}{f}V_0+g\,[\sigma,\ V_0]+\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(-2)}V_- -\left(\epsilon^{\dagger}\tau^m\epsilon^c\right)\mathcal{D}_m^{(2)}V_+\Big|^2$$
$$+\Big|-\frac{4i}{f}V_+ +2g\,[\sigma,\ V_+]+2\left(\epsilon^{\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(2)}V_+ -\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\partial_m V_0\Big|^2\Bigg], \tag{79}$$

and

$$-4\int d^5x\sqrt{g}\,\mathrm{tr}\Big[A_z\Delta_0A_{\bar z}+A_z\mathcal{D}_{\bar z}\Big(\left(\epsilon^{\dagger}\tau^m\epsilon\right)\partial_m V_0+g\,[\sigma,\ V_0]+2\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(-2)}V_-\Big)$$
$$+\mathcal{D}_z\Big(\left(\epsilon^{\dagger}\tau^m\epsilon\right)\partial_m V_0-g\,[\sigma,\ V_0]+2\left(\epsilon^{\dagger}\tau^m\epsilon^c\right)\mathcal{D}_m^{(2)}V_+\Big)\cdot A_{\bar z}$$
$$+\mathcal{D}_zV_0\mathcal{D}_{\bar z}V_0+4\mathcal{D}_zV_+\mathcal{D}_{\bar z}V_-\Big],$$

where Δ_0 denotes the differential operator

$$-\Big[\left(\epsilon^{\dagger}\tau^m\epsilon\right)\partial_m+g\,[\sigma,\ \cdot]\Big]\Big[\left(\epsilon^{\dagger}\tau^n\epsilon\right)\partial_n-g\,[\sigma,\ \cdot]\Big]-\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(-2)}\left(\epsilon^{\dagger}\tau^n\epsilon^c\right)\partial_n,$$

which is positive and so invertible in the root sector expanded in the basis $\{E_\alpha\}$.

As we have done for the squashed S^3 in the previous Section 8.1, using

$$D_0^{(q+2)}D_+^{(q)}=D_+^{(q)}\left(D_0^{(q)}+\frac{2i}{f}\right)-i\left(\frac{q+2}{2}\right)\left(\epsilon^{c\dagger}\tau^{mn}\epsilon\right)V_{mn},$$
$$D_-^{(q)}D_0^{(q)}=\left(D_0^{(q-2)}+\frac{2i}{f}\right)D_-^{(q)}+i\left(\frac{q}{2}\right)\left(\epsilon^{\dagger}\tau^{mn}\epsilon^c\right)V_{mn}, \tag{80}$$

with the abbreviations,

$$D_0^{(q)}=\left(\epsilon^{\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(q)},\qquad D_+^{(q)}=\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(q)},\qquad D_-^{(q)}=\left(\epsilon^{\dagger}\tau^m\epsilon^c\right)\mathcal{D}_m^{(q)},$$

derived from (107) in Appendix G.5, we will shift A_z and $A_{\bar z}$ in the latter integrand to give

$$-4\int d^5x\sqrt{g}\,\mathrm{tr}\Big[A_z\Delta_0A_{\bar z}+\mathcal{D}_zJ_+\cdot\frac{1}{\Delta_{-2}}\left(\mathcal{D}_zJ_+\right)^\dagger\Big], \tag{81}$$

after integrations by parts, with

$$J_+=2\left(\left(\epsilon^{\dagger}\tau^m\epsilon\right)\partial_mV_+ +g\,[\sigma,\ V_+]-\frac{2i}{f}V_+\right)-\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\partial_mV_0,$$

where we have defined the operator Δ_{-2} by

$$-\Big[\left(\epsilon^{\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(-2)}+\frac{2i}{f}+g\,[\sigma,\ \cdot]\Big]\Big[\left(\epsilon^{\dagger}\tau^n\epsilon\right)\mathcal{D}_n^{(-2)}+\frac{2i}{f}-g\,[\sigma,\ \cdot]\Big]$$
$$-\left(\epsilon^{\dagger}\tau^n\epsilon^c\right)\mathcal{D}_n^{(0)}\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m^{(-2)},$$

which is also invertible in the sector we are interested in. Here, we have made use of

$$\frac{1}{\Delta_0}D_+^{(0)}=D_+^{(-2)}\frac{1}{\Delta_{-2}},$$

which is also deduced from (107).

When we shift $V_\pm$ as

$$V_\pm \quad \to \quad V_\pm + \frac{1}{2}\frac{1}{D_0^{(\pm 2)} \mp \frac{2i}{f} \pm g\,[\sigma,\,\cdot]} D_\pm^{(0)} V_0,$$

for a generic $(\sigma \cdot \alpha)$, the term in $\delta_Q \xi$,

$$D_-^{(2)} V_+ - D_+^{(-2)} V_- + \left(\frac{2i}{f} - g\,[\sigma,\,\cdot]\right) V_0$$

is shifted to become $D_-^{(2)} V_+ - D_+^{(-2)} V_- - \mathcal{K}_0 V_0$, with $\mathcal{K}_0 V_0$ denoting

$$\frac{g\,[\sigma,\,\cdot]}{\left(D_0^{(0)} + g\,[\sigma,\,\cdot]\right)\left(D_0^{(0)} - g\,[\sigma,\,\cdot]\right)} \\ \times \left[\left(D_0^{(0)} - \frac{2i}{f} + g\,[\sigma,\,\cdot]\right)\left(D_0^{(0)} - g\,[\sigma,\,\cdot]\right) + D_+^{(-2)} D_-^{(0)}\right] V_0,$$

where we have used the formula

$$\mathcal{D}_\pm^{(2)} \frac{1}{\mathcal{D}_0^{(2)} \pm \frac{2i}{f} \mp g\,[\sigma,\,\cdot]} = \frac{1}{\mathcal{D}_0^{(0)} \mp g\,[\sigma,\,\cdot]} \mathcal{D}_\pm^{(2)},$$

which follow from (80) and

$$\mathcal{D}_0^{(q-2)} \mathcal{D}_-^{(q)} = \mathcal{D}_-^{(q)} \left(\mathcal{D}_0^{(q)} - \frac{2i}{f}\right) + i\left(\frac{q-2}{2}\right)\left(\epsilon^\dagger \tau^{mn} \epsilon^c\right) V_{mn},$$

and the formula

$$\mathcal{D}_+^{(-2)} \mathcal{D}_-^{(0)} - \mathcal{D}_-^{(2)} \mathcal{D}_+^{(0)} = \frac{4i}{f} \mathcal{D}_0^{(0)},$$

of (107), together with (106) in Appendix G.5.

Therefore, the integrand of the sum of (79) and (81), after integrations by parts, becomes

$$A_z \Delta_0 A_{\bar{z}} - V_+ \frac{1}{\Delta_{-2}} \left[\Delta_{-2} - 4\mathcal{D}_z \mathcal{D}_{\bar{z}}\right] \\ \times \left(D_0^{(-2)} + \frac{2i}{f} + g\,[\sigma,\,\cdot]\right)\left(D_0^{(-2)} + \frac{2i}{f} - g\,[\sigma,\,\cdot]\right) V_- + \frac{1}{4}\left|\mathcal{K}_0 V_0\right|^2,$$

where we have shifted V_0 appropriately to eliminate the term $D_-^{(2)} V_+ - D_+^{(-2)} V_-$, as before.

Integrating over the remaining fluctuations, we obtain the one-loop determinants from the bosonic fields of the gauge multiplet,

$$Z_{\mathrm{V,B}}^{\text{1-loop}} = Z_{\mathrm{V},0} \\ \times \frac{\mathcal{D}\mathrm{et}_{(0,0)}\left[\Delta_{-2}\right]}{\mathcal{D}\mathrm{et}_{(0,1)}\left[\Delta_0\right] \mathcal{D}\mathrm{et}_{(0,0)}\left[\left[\Delta_{-2} - 4\mathcal{D}_z \mathcal{D}_{\bar{z}}\right]\left(D_0^{(-2)} + \frac{2i}{f} + g\,[\sigma,\,\cdot]\right)\left(D_0^{(-2)} + \frac{2i}{f} - g\,[\sigma,\,\cdot]\right)\right]},$$

where $Z_{\mathrm{V},0}$ denotes the one-loop contribution from V_0.

Taking account of the fact that V_0 is a real field; $V_0^\dagger = V_0$, we find that

$$Z_{V,0} = \prod_{\alpha\in\Lambda_+} \mathrm{Det}_{(0,0)}\left[\frac{g\,(\sigma\cdot\alpha)}{\left(D_0^{(0)} + g\,(\sigma\cdot\alpha)\right)\left(D_0^{(0)} - g\,(\sigma\cdot\alpha)\right)}\right]^{-2}$$
$$\times \mathrm{Det}_{(0,0)}\left[\left(D_0^{(0)} - \frac{2i}{f} + g\,(\sigma\cdot\alpha)\right)\left(D_0^{(0)} - g\,(\sigma\cdot\alpha)\right) + D_+^{(-2)} D_-^{(0)}\right]^{-2}.$$

Therefore, up to an overall constant including the Cartan part,

$$Z_{\mathrm{V},0} = \frac{1}{Z_{\mathrm{FP}}}\cdot\frac{\left(\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)} - g\,[\sigma,\,\cdot]\right]\right)^2}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\left(D_0^{(0)} - \frac{2i}{f} + g\,[\sigma,\,\cdot]\right)\left(D_0^{(0)} - g\,[\sigma,\,\cdot]\right) + D_+^{(-2)} D_-^{(0)}\right]}.$$

The computation of the one-loop contributions from the fermionic part of the gauge multiplet in the regulator action $\mathcal{S}_Q$ is also parallel to that for the squashed S^3.

Integrating by parts, the fermionic part is reduced to the sum of

$$-\left(\frac{i}{2}\right)\left[\left(\tilde{\xi},\tilde{\eta}\right)\begin{pmatrix} D_0^{(0)} + \frac{2i}{f} + g\,[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + \frac{2i}{f} + g\,[\sigma,\cdot]\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right],$$

and

$$\left(\frac{i}{2}\right)\left[\left(\tilde{\varphi},\tilde{\chi}\right)\begin{pmatrix} D_0^{(0)} - g\,[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + \frac{2i}{f} - g\,[\sigma,\cdot]\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\right]$$
$$+\left(\mathcal{D}_{\bar{z}}\tilde{\xi},\mathcal{D}_{\bar{z}}\tilde{\eta}\right)\begin{pmatrix}\varphi\\ \chi\end{pmatrix} + \left(\tilde{\varphi},\quad \tilde{\chi}\right)\begin{pmatrix}0\\ \mathcal{D}_z\chi\end{pmatrix}.$$

Integrating over φ, χ, $\tilde{\varphi}$, and $\tilde{\chi}$ gives the one-loop determinant

$$\mathcal{D}\mathrm{et}_{(1,0)}\left[\begin{pmatrix} D_0^{(0)} - g\,[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + \frac{2i}{f} - g\,[\sigma,\cdot]\end{pmatrix}\right], \tag{82}$$

and leaves the integrand

$$-\frac{i}{2}\left[\left(\tilde{\xi},\tilde{\eta}\right)\begin{pmatrix} D_0^{(0)} - g\,[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + \frac{2i}{f} - g\,[\sigma,\cdot]\end{pmatrix}^{-1}\begin{pmatrix} D_1 & 0\\ D_3 & D_4\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right],$$

after integration by parts, where the operators D_1, D_3, and D_4 denote

$$D_1 = \left(D_0^{(0)} - g\,[\sigma,\,\cdot]\right)\left(D_0^{(0)} + \frac{2i}{f} + g\,[\sigma,\,\cdot]\right) + D_-^{(2)} D_+^{(0)},$$

$$D_3 = D_+^{(0)}\left(D_0^{(0)} + \frac{2i}{f} + g\,[\sigma,\,\cdot]\right) - \left(D_0^{(2)} - \frac{2i}{f} + g\,[\sigma,\,\cdot]\right) D_+^{(0)},$$

$$D_4 = 4\mathcal{D}_{\bar{z}}\mathcal{D}_z + \left(D_0^{(2)} - \frac{2i}{f} + g\,[\sigma,\,\cdot]\right)\left(D_0^{(2)} - \frac{2i}{f} - g\,[\sigma,\,\cdot]\right) + D_+^{(0)} D_-^{(2)}.$$

Integrating over the remaining ξ, η, $\tilde{\xi}$, and $\tilde{\eta}$, and combining the resulting determinant with (82), we obtain the one-loop contributions from the fermionic fields of the gauge multiplet,

$$Z_{V,F}^{\text{1-loop}} = \frac{\mathcal{D}\mathrm{et}_{(1,0)}\left[\begin{pmatrix} D_0^{(0)} - g\,[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + \frac{2i}{f} - g\,[\sigma,\cdot] \end{pmatrix}\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\begin{pmatrix} D_0^{(0)} - g\,[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + \frac{2i}{f} - g\,[\sigma,\cdot] \end{pmatrix}\right]} \mathcal{D}\mathrm{et}_{(0,0)}\left[\begin{pmatrix} D_1 & 0 \\ D_2 & D_4 \end{pmatrix}\right],$$

with $\mathcal{D}\mathrm{et}_{(0,0)}\left[\begin{pmatrix} D_1 & 0 \\ D_2 & D_4 \end{pmatrix}\right]$ evaluated to give

$$\mathcal{D}\mathrm{et}_{(0,0)}\left[\left(D_0^{(0)} - g\,[\sigma,\,\cdot]\right)\left(D_0^{(0)} + \frac{2i}{f} + g\,[\sigma,\,\cdot]\right) + D_-^{(2)} D_+^{(0)}\right]$$
$$\times\, \mathcal{D}\mathrm{et}_{(0,0)}\left[4\mathcal{D}_{\bar{z}}\mathcal{D}_z + \left(D_0^{(2)} - \frac{2i}{f} + g\,[\sigma,\,\cdot]\right)\left(D_0^{(2)} - \frac{2i}{f} - g\,[\sigma,\,\cdot]\right) + D_+^{(0)} D_-^{(2)}\right].$$

The determinant

$$\mathcal{D}\mathrm{et}_{(1,0)}\left[\begin{pmatrix} D_0^{(0)} - g\,[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + \frac{2i}{f} - g\,[\sigma,\cdot] \end{pmatrix}\right],$$

noting the relation

$$D_-^{(2)} \frac{1}{-D_0^{(2)} + \frac{2i}{f} - g\,[\sigma,\cdot]} = \frac{1}{-D_0^{(0)} - g\,[\sigma,\cdot]} D_-^{(2)},$$

is evaluated to yield

$$\frac{\mathcal{D}\mathrm{et}_{(1,0)}\left[D_0^{(2)} - \frac{2i}{f} + g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(1,0)}\left[D_0^{(0)} + g\,[\sigma,\cdot]\right]}$$
$$\times\, \mathcal{D}\mathrm{et}_{(1,0)}\left[-\left(D_0^{(0)} + g\,[\sigma,\cdot]\right)\left(D_0^{(0)} - g\,[\sigma,\cdot]\right) - D_-^{(2)} D_+^{(0)}\right]. \tag{83}$$

Since the determinant of an operator is the same as the one of its adjoint operator, it follows that

$$\mathcal{D}\mathrm{et}_{(1,0)}\left[-\left(D_0^{(0)} + g\,[\sigma,\cdot]\right)\left(D_0^{(0)} - g\,[\sigma,\cdot]\right) - D_-^{(2)} D_+^{(0)}\right] = \mathcal{D}\mathrm{et}_{(0,1)}\left[\Delta_0\right],$$
$$\mathcal{D}\mathrm{et}_{(1,0)}\left[D_0^{(2)} - \frac{2i}{f} \pm g\,[\sigma,\cdot]\right] = \mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(-2)} + \frac{2i}{f} \pm g\,[\sigma,\cdot]\right],$$
$$\mathcal{D}\mathrm{et}_{(1,0)}\left[D_0^{(0)} \pm g\,[\sigma,\cdot]\right] = \mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(0)} \pm g\,[\sigma,\cdot]\right].$$

Using them, (83) may be rewritten as

$$\frac{\mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(2)} + \frac{2i}{f} + g\,[\sigma,\cdot]\right]}{\mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(0)} + g\,[\sigma,\cdot]\right]} \mathcal{D}\mathrm{et}_{(0,1)}\left[\Delta_0\right]. \tag{84}$$

For the determinant

$$\mathcal{D}\mathrm{et}_{(0,0)}\left[\begin{pmatrix} D_0^{(0)} - g\,[\sigma,\cdot] & D_-^{(2)} \\ D_+^{(0)} & -D_0^{(2)} + \frac{2i}{f} - g\,[\sigma,\cdot] \end{pmatrix}\right],$$

using the formula

$$D_+^{(0)} \frac{1}{D_0^{(0)} - g[\sigma, \cdot]} = \frac{1}{D_0^{(2)} - \frac{2i}{f} - g[\sigma, \cdot]} D_+^{(0)},$$

and the relation of the determinant of an operator with that of the adjoint operator,

$$\begin{aligned}
&\mathcal{D}\mathrm{et}_{(0,0)}\left[-\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} - g[\sigma, \cdot]\right)\left(D_0^{(2)} - 2i\frac{\tilde{r}}{r^2} + g[\sigma, \cdot]\right) - D_+^{(0)} D_-^{(2)}\right] \\
&\quad = \mathcal{D}\mathrm{et}_{(0,0)}\left[\Delta_{-2}\right]. \\
&\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(2)} - \frac{2i}{f} \pm g[\sigma, \cdot]\right] = \mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(-2)} + \frac{2i}{f} \pm g[\sigma, \cdot]\right], \\
&\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)} \pm g[\sigma, \cdot]\right] = \mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)} \pm g[\sigma, \cdot]\right],
\end{aligned}$$

we can see that it is reduced to

$$\frac{\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)} - g[\sigma, \cdot]\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(-2)} + \frac{2i}{f} - g[\sigma, \cdot]\right]} \mathcal{D}\mathrm{et}_{(0,0)}\left[\Delta_{-2}\right]. \tag{85}$$

Substituting (84) and (85) into $Z_{\mathrm{V,F}}^{\text{1-loop}}$, we obtain

$$\begin{aligned}
&\frac{\mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(-2)} + \frac{2i}{f} + g[\sigma, \cdot]\right] \mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(-2)} + \frac{2i}{f} - g[\sigma, \cdot]\right]}{\mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(0)} + g[\sigma, \cdot]\right] \mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)} - g[\sigma, \cdot]\right]} \frac{\mathcal{D}\mathrm{et}_{(0,1)}\left[\Delta_0\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[\Delta_{-2}\right]} \\
&\quad \times \mathcal{D}\mathrm{et}_{(0,0)}\left[D_1\right] \mathcal{D}\mathrm{et}_{(0,0)}\left[D_4\right].
\end{aligned}$$

With the same argument about the adjoint operators, we can show that

$$\begin{aligned}
&\mathcal{D}\mathrm{et}_{(0,0)}\left[D_1\right] = \mathcal{D}\mathrm{et}_{(0,0)}\left[\left(D_0^{(0)} - \frac{2i}{f} + g[\sigma, \cdot]\right)\left(D_0^{(0)} - g[\sigma, \cdot]\right) + D_+^{(-2)} D_-^{(0)}\right], \\
&\mathcal{D}\mathrm{et}_{(0,0)}\left[D_4\right] = \mathcal{D}\mathrm{et}_{(0,0)}\left[4\mathcal{D}_z \mathcal{D}_{\bar{z}} - \Delta_{-2}\right],
\end{aligned}$$

and using this, the one-loop contributions $Z_{\mathrm{V,B}}^{\text{1-loop}}$, $Z_{\mathrm{V,F}}^{\text{1-loop}}$, and Z_{FP} from the gauge multiplet are summarized to give

$$\boxed{\begin{aligned}
Z_{\mathrm{V}}^{\text{1-loop}} &= Z_{\mathrm{FP}} Z_{\mathrm{V,B}}^{\text{1-loop}} Z_{\mathrm{V,F}}^{\text{1-loop}} \\
&= \frac{\mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(-2)} + \frac{2i}{f} + g[\sigma, \cdot]\right]}{\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(-2)} + \frac{2i}{f} + g[\sigma, \cdot]\right]} \frac{\mathcal{D}\mathrm{et}_{(0,0)}\left[D_0^{(0)} - g[\sigma, \cdot]\right]}{\mathcal{D}\mathrm{et}_{(0,1)}\left[D_0^{(0)} - g[\sigma, \cdot]\right]}.
\end{aligned}} \tag{86}$$

The determinant $\mathcal{D}\mathrm{et}_{(k,l)}$ can be evaluated by using the basis $\{h_{n,m,k} \otimes v \otimes E_\alpha, h_{n,m,k} \otimes v \otimes H_i\}$, for v running over all the basis vectors of $\Omega^{(k,l)}(\Sigma)$, the set of all (k,l)-forms on Σ, upon regarding $\Omega^{(k,l)}(\Sigma)$ as a linear space. Here, $h_{n,m,k}$ $(n, m = 0, 1, 2, \cdots; k = 0, 1, \cdots, n+m)$ denote scalar spherical harmonics on the S^3 (See Appendix G.5 for more details) and obey

$$\mathcal{D}_0^{(0)} h_{n,m,k} = \left[\frac{i}{r}\left(i\frac{\partial}{\partial\varphi}\right) + \frac{i}{\tilde{r}}\left(-i\frac{\partial}{\partial\chi}\right)\right] h_{n,m,k} = \left[\frac{i(n-k)}{r} + \frac{i(m-k)}{\tilde{r}}\right] h_{n,m,k},$$

$$\left(\mathcal{D}_0^{(-2)} + \frac{2i}{f}\right) h_{n,m,k} = \left[\frac{i}{r}\left(i\frac{\partial}{\partial\varphi} + 1\right) + \frac{i}{\tilde{r}}\left(-i\frac{\partial}{\partial\chi} + 1\right)\right] h_{n,m,k}$$
$$= \left[\frac{i(n-k+1)}{r} + \frac{i(m-k+1)}{\tilde{r}}\right] h_{n,m,k}.$$

On the basis[17] $\{\varphi_{l,m,\tilde{m}} \otimes v \otimes E_\alpha\}$, the determinants in $Z_{\mathrm{V}}^{\text{1-loop}}$ are computed to give

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[D_0^{(0)} - g[\sigma,\cdot]\right] = \prod_{\alpha\in\Lambda}\prod_{n,m=0}^{\infty}\prod_{k=0}^{m+n} \det{}_{(k,l)}\left[\frac{i(n-k)}{r} + \frac{i(m-k)}{\tilde{r}} - g(\sigma\cdot\alpha)\right],$$

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[D_0^{(-2)} + 2i\frac{\tilde{r}}{r^2} + g[\sigma,\cdot]\right]$$
$$= \prod_{\alpha\in\Lambda}\prod_{n,m=0}^{\infty}\prod_{k=0}^{m+n} \det{}_{(k,l)}\left[\frac{i(n-k+1)}{r} + \frac{i(m-k+1)}{\tilde{r}} + g(\sigma\cdot\alpha)\right],$$

where the determinant $\det_{(k,l)}$ is defined over the space $\Omega^{(k,l)}(\Sigma)$.

Taking account of (72), we can simplify Z_{V},

$$Z_{\mathrm{V}} = \prod_{\alpha\in\Lambda}\prod_{n,m=0}^{\infty}\prod_{k=0}^{m+n}\left(\frac{\frac{i(n-k)}{r} + \frac{i(m-k)}{\tilde{r}} + g(\sigma\cdot\alpha)}{\frac{i(n-k+1)}{r} + \frac{i(m-k+1)}{\tilde{r}} + g(\sigma\cdot\alpha)}\right)^{\frac{1}{2}\chi(\Sigma)}$$
$$= \prod_{\alpha\in\Lambda}\prod_{n,m=0}^{\infty}\left(\frac{-\frac{im}{r} - \frac{in}{\tilde{r}} + g(\sigma\cdot\alpha)}{\frac{i(n+1)}{r} + \frac{i(m+1)}{\tilde{r}} + g(\sigma\cdot\alpha)}\right)^{\frac{1}{2}\chi(\Sigma)}$$
$$= \prod_{\alpha\in\Lambda_+}\left[(ig(\sigma\cdot\alpha))^2\prod_{n=1}^{\infty}\left(\frac{n^2}{r^2} + g^2(\sigma\cdot\alpha)^2\right)\left(\frac{n^2}{\tilde{r}^2} + g^2(\sigma\cdot\alpha)^2\right)\right]^{\frac{1}{2}\chi(\Sigma)}.$$

Furthermore, from the formula

$$\frac{1}{\pi}\sinh\pi x = x\prod_{m=1}^{\infty}\left(1 + \frac{x^2}{m^2}\right),$$

it follows that

$$\boxed{Z_{\mathrm{V}}^{\text{1-loop}} = \prod_{\alpha\in\Lambda_+}\left[2\sinh\left(\pi r g(\sigma\cdot\alpha)\right)\cdot 2\sinh\left(\pi\tilde{r}g(\sigma\cdot\alpha)\right)\right]^{\frac{1}{2}\chi(\Sigma)}.} \tag{87}$$

In the round limit $\tilde{r} \to r$, $Z_{\mathrm{V}}^{\text{1-loop}}$ recovers the result for the round S^3 in Section 8.

[17] We will again omit the contributions from the basis vectors $\varphi_{l,m,\tilde{m}} \otimes v \otimes H_i$ to the determinants, as done for the squashed S^3.

9.2. One-loop contributions from the $\mathcal{N}=1$ hypermultiplet

Let us turn to compute the one-loop contributions from the hypermultiplet by localization.

Since the BRST transformations of the scalar fields H, $H^\dagger$ of the hypermultiplet are independent of the background fields, they are the same as for the round and squashed S^3's;

$$\delta_Q \tilde{H} = 0, \quad \delta_Q H = 0, \qquad \delta_Q \tilde{H}^\dagger = -\frac{1}{2\sqrt{2}}\tilde{\kappa}, \quad \delta_Q H^\dagger = -\frac{1}{2\sqrt{2}}\kappa.$$

The BRST transformation of the fermions in the hypermultiplet is given by

$$\begin{aligned}
\delta_Q \chi &= \sqrt{2}i\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m \tilde{H} + g\left[\sigma,\ \tilde{H}\right] + \frac{i}{f}\tilde{H}\right],\\
\delta_Q \xi &= \sqrt{2}i\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m \tilde{H},\\
\delta_Q \eta &= F^1{}_1 - 2\sqrt{2}\mathcal{D}_z \tilde{H}, \qquad \delta_Q \kappa = 0,\\
\delta_Q \tilde{\chi} &= -\sqrt{2}i\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m H + g\left[\sigma,\ H\right] + \frac{i}{f}H\right],\\
\delta_Q \tilde{\xi} &= -\sqrt{2}i\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m H,\\
\delta_Q \tilde{\eta} &= F^2{}_1 + 2\sqrt{2}\mathcal{D}_z H, \qquad \delta_Q \tilde{\kappa} = 0,
\end{aligned}$$

and its hermitian conjugate by

$$\begin{aligned}
\left(\delta_Q \chi\right)^\dagger &= -\sqrt{2}i\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(1)}{}_m \tilde{H}^\dagger - g\left[\sigma,\ \tilde{H}^\dagger\right] - \frac{i}{f}\tilde{H}^\dagger\right],\\
\left(\delta_Q \xi\right)^\dagger &= -\sqrt{2}i\left(\epsilon^\dagger \tau^m \epsilon^c\right)\mathcal{D}^{(1)}{}_m \tilde{H}^\dagger,\\
\left(\delta_Q \eta\right)^\dagger &= -F^2{}_2 - 2\sqrt{2}\mathcal{D}_{\bar{z}} \tilde{H}^\dagger, \qquad \left(\delta_Q \kappa\right)^\dagger = 0,\\
\left(\delta_Q \tilde{\chi}\right)^\dagger &= \sqrt{2}i\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(1)}{}_m H^\dagger - g\left[\sigma,\ H^\dagger\right] - \frac{i}{f}H^\dagger\right],\\
\left(\delta_Q \tilde{\xi}\right)^\dagger &= \sqrt{2}i\left(\epsilon^\dagger \tau^m \epsilon^c\right)\mathcal{D}^{(1)}{}_m H^\dagger,\\
\left(\delta_Q \tilde{\eta}\right)^\dagger &= F^1{}_2 + 2\sqrt{2}\mathcal{D}_{\bar{z}} H^\dagger, \qquad \left(\delta_Q \tilde{\kappa}\right)^\dagger = 0.
\end{aligned}$$

Using the BRST transformation of the auxiliary fields $F^1{}_2$, $F^2{}_2$,

$$\begin{aligned}
\delta_Q F^1{}_2 &= \frac{i}{2}\left[\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m \chi - \left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(1)}{}_m \eta - g\left[\sigma,\ \eta\right] + \frac{i}{f}\eta - 2i\mathcal{D}_{\bar{z}}\kappa\right],\\
\delta_Q F^2{}_2 &= \frac{i}{2}\left[\left(\epsilon^{c\dagger}\tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m \tilde{\chi} - \left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(1)}{}_m \tilde{\eta} - g\left[\sigma,\ \tilde{\eta}\right] + \frac{i}{f}\tilde{\eta} - 2i\mathcal{D}_{\bar{z}}\tilde{\kappa}\right],
\end{aligned}$$

where we have omitted the terms $g\left(\sigma^i\right)^{\tilde{\alpha}}{}_{\dot{\gamma}}\left[\phi^i,\ \lambda^{\dot{\gamma}}\right]$ on the right hand sides of both the equations, because their contributions vanish in the large t limit, we find that

$$\delta_Q\left(\delta_Q\chi\right)^\dagger = \frac{i}{2}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m\tilde{\kappa} - g\left[\sigma,\,\tilde{\kappa}\right] - \frac{i}{f}\tilde{\kappa}\right],$$
$$\delta_Q\left(\delta_Q\xi\right)^\dagger = \frac{i}{2}\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(1)}{}_m\tilde{\kappa},$$
$$\delta_Q\left(\delta_Q\eta\right)^\dagger = -\frac{i}{2}\left\{\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m\tilde{\chi} - \left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m\tilde{\xi} + g\left[\sigma,\,\tilde{\xi}\right] - \frac{i}{f}\tilde{\xi}\right]\right\},$$
$$\delta_Q\left(\delta_Q\kappa\right)^\dagger = 0,$$
$$\delta_Q\left(\delta_Q\tilde{\chi}\right)^\dagger = -\frac{i}{2}\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m\kappa - g\left[\sigma,\,\kappa\right] - \frac{i}{f}\kappa\right],$$
$$\delta_Q\left(\delta_Q\tilde{\xi}\right)^\dagger = -\frac{i}{2}\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(1)}{}_m\kappa,$$
$$\delta_Q\left(\delta_Q\tilde{\eta}\right)^\dagger = \frac{i}{2}\left\{\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m\chi - \left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m\xi + g\left[\sigma,\,\xi\right] - \frac{i}{f}\xi\right]\right\},$$
$$\delta_Q\left(\delta_Q\tilde{\kappa}\right)^\dagger = 0.$$

The system of $(\tilde{H}, \tilde{H}^\dagger, \chi, \xi, \tilde{\eta}, \tilde{\kappa}, F^1{}_2)$ is identical to the one of $(H, H^\dagger, \tilde{\chi}, \tilde{\xi}, \eta, \kappa, F^2{}_2)$, as we have seen for the squashed S^3 case in Subsection 8.2.1. If the former contributes the one-loop determinant $Z_H^{\text{1-loop}}$ to the partition function, both of the systems contribute $(Z_H^{\text{1-loop}})^2$. Therefore, we will focus on the former system only.

From the fermionic part of the system $(\tilde{H}, \tilde{H}^\dagger, \chi, \xi, \tilde{\eta}, \tilde{\kappa}, F^1{}_2)$ of the regulator action $\mathcal{S}_Q$,

$$\begin{aligned}
&-\int\sqrt{g}d^5x\left[\delta_Q\left(\delta_Q\chi\right)^\dagger\cdot\chi + \delta_Q\left(\delta_Q\xi\right)^\dagger\cdot\xi + \delta_Q\left(\delta_Q\tilde{\eta}\right)^\dagger\cdot\tilde{\eta} + \delta_Q\left(\delta_Q\tilde{\kappa}\right)^\dagger\cdot\tilde{\kappa}\right] \\
&= \frac{i}{2}\int\sqrt{g}d^5x \\
&\quad\times\left[\begin{pmatrix}\chi, & \xi\end{pmatrix}\begin{pmatrix}\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m - g\left[\sigma,\cdot\right] - \frac{i}{f} & \left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m \\ \left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(1)}{}_m & -\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m - g\left[\sigma,\cdot\right] - \frac{i}{f}\end{pmatrix}\begin{pmatrix}\tilde{\kappa} \\ \tilde{\eta}\end{pmatrix}\right].
\end{aligned}$$

As we have done in Subsection 8.2.1, for four differential operators $D_1, \cdots, D_4$, we have the formula

$$\mathcal{D}\text{et}_{(k,l)}\left[\begin{pmatrix}D_1 & D_2 \\ D_3 & D_4\end{pmatrix}\right] = \mathcal{D}\text{et}_{(k,l)}\left[D_1\right]\mathcal{D}\text{et}_{(k,l)}\left[D_4 - D_3\frac{1}{D_1}D_2\right],$$

for an invertible D_1. In the above case, we have

$$D_1 = \left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m - g\left[\sigma,\cdot\right] - \frac{i}{f}, \qquad D_3 = \left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(1)}{}_m,$$

which both act on the spinor $\tilde{\kappa}$ of negative chirality on Σ and of charge $q = 1$. Using (107) in Appendix G.5, we can identify the operator D'_1,

$$\begin{aligned}
D_3D_1 &= \left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(1)}{}_n\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(1)}{}_m - g\left[\sigma,\cdot\right] - \frac{i}{f}\right] \\
&= \left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(-1)}{}_m - g\left[\sigma,\cdot\right] + \frac{i}{f}\right]\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(1)}{}_n = D'_1D_3,
\end{aligned}$$

and obtain

$$\mathcal{D}\mathrm{et}_{(k,l)}\left[\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}\right] = \frac{\mathcal{D}\mathrm{et}_{(k,l)}\left[D_1\right]}{\mathcal{D}\mathrm{et}_{(k,l)}\left[D'_1\right]}\mathcal{D}\mathrm{et}_{(k,l)}\left[D'{}_1 D_4 - D_3 D_2\right].$$

We thus find that

$$Z_{\mathrm{H,F}}^{\text{1-loop}} = \mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\Delta_{\mathrm{H,B}}^{\mathcal{N}=1}\right] \frac{\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(1)}{}_m - g\left[\sigma,\cdot\right] - \frac{i}{f}\right]}{\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m - g\left[\sigma,\cdot\right] + \frac{i}{f}\right]},$$

where the differential operator $\Delta_{\mathrm{H,B}}^{\mathcal{N}=1}$ denotes

$$\begin{aligned}\Delta_{\mathrm{H,B}}^{\mathcal{N}=1} = &-\left(\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m - g\left[\sigma,\cdot\right] + \frac{i}{f}\right)\left(\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m + g\left[\sigma,\cdot\right] + \frac{i}{f}\right) \\ &- \left(\epsilon^\dagger \tau^m \epsilon^c\right)\mathcal{D}^{(1)}{}_m \left(\epsilon^{c\dagger} \tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m.\end{aligned}$$

With the same reason as in Subsection 8.2.1, we can show that

$$\frac{\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(1)}{}_m - g\left[\sigma,\cdot\right] - \frac{i}{f}\right]}{\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}^{(-1)}{}_m - g\left[\sigma,\cdot\right] + \frac{i}{f}\right]} = 1,$$

and it follows from this that

$$Z_{\mathrm{H,F}}^{\text{1-loop}} = \mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\Delta_{\mathrm{H,B}}^{\mathcal{N}=1}\right].$$

In the bosonic part of the system $(\tilde{H}, \tilde{H}^\dagger, \chi, \xi, \tilde{\eta}, \tilde{\kappa}, F^1{}_2)$ of the regulator action $\mathcal{S}_Q$,

$$-\int d^5x\sqrt{g}\left[\left(\delta_Q\chi\right)^\dagger \cdot \delta_Q\chi + \left(\delta_Q\xi\right)^\dagger \cdot \delta_Q\xi + \left(\delta_Q\tilde{\eta}\right)^\dagger \cdot \delta_Q\tilde{\eta} + \left(\delta_Q\tilde{\kappa}\right)^\dagger \cdot \delta_Q\tilde{\kappa}\right],$$

we will immediately integrate the auxiliary fields $F^{\tilde{\alpha}}{}_{\check{\beta}}$ out, and integrate the remaining part of the action by parts to obtain

$$-2\int d^5x\sqrt{g}\tilde{H}^\dagger \Delta_{\mathrm{H,B}}^{\mathcal{N}=1}\tilde{H},$$

and see that the one-loop determinant from the bosonic fields of the system $(\tilde{H}, \tilde{H}^\dagger, \chi, \xi, \tilde{\eta}, \tilde{\kappa})$ is given by

$$Z_{\mathrm{H,B}}^{\text{1-loop}} = \frac{1}{\mathcal{D}\mathrm{et}_{(0,\frac{1}{2})}\left[\Delta_{\mathrm{H,B}}^{\mathcal{N}=1}\right]}.$$

Therefore, the contributions from the hypermultiplet to the partition function are trivial;

$$\left(Z_{\mathrm{H}}^{\text{1-loop}}\right)^2 = \left(Z_{\mathrm{H,F}}^{\text{1-loop}} Z_{\mathrm{H,B}}^{\text{1-loop}}\right)^2 = 1.$$

In the round limit $\tilde{r} \to r$, the contributions from the hypermultiplet reproduce the previous results on the round S^3 about the hypermultiplet in Section 8.

10. Summary and discussions

In this paper, we have seen the effects caused by changing the twisting and by deforming a round 3-sphere to a squashed and an ellipsoid 3-spheres on the partition function on the round S^3, which was computed in the previous paper [1,2].

We have discussed the two kinds of twistings – the $\mathcal{N}=1$ twisting and the $\mathcal{N}=2$ twisting, the former of which breaks the $\mathrm{Spin}(5)_R$ symmetry group to $U(1)_r \times SU(2)_l$, which is the subgroup of $SU(2)_r \times SU(2)_l \simeq \mathrm{Spin}(4) \subset \mathrm{Spin}(5)_R$, while the latter breaks the $\mathrm{Spin}(5)_R$ to $U(1)_R \times SU(2)_R$, which is the subgroup $\mathrm{Spin}(2)_R \times \mathrm{Spin}(3)_R$ of the $\mathrm{Spin}(5)_R$. In the $\mathcal{N}=1$ twisting, the only supersymmetry transformation with the parameter $\varepsilon^{\dot{\alpha}}$ is preserved, and in the $\mathcal{N}=2$ twisting, the ones with both $\varepsilon^{\dot{\alpha}}$ and $\epsilon^{\tilde{\alpha}}$ are available.

The change of the twisting affects on the spin content of the $\mathcal{N}=1$ hypermultiplet, when the $\mathcal{N}=2$ gauge multiplet is viewed as the sum of the $\mathcal{N}=1$ gauge multiplet and the $\mathcal{N}=1$ hypermultiplet.

In all the cases we discussed in this paper, the classical action, $\mathcal{S}_{\mathrm{cl}}$ which is the value of the off-shell action (56) at one σ^i of the fixed-points, can be compactly written down[18] as

$$\mathcal{S}_{\mathrm{cl}} = \left(\int_{S^3} \frac{2}{f} e^1 \wedge e^2 \wedge e^3 \right) i \sum_{i=1}^{r} \int_{\Sigma} F^i_{45} \sigma^i \, dx^4 \wedge dx^5,$$

in the zero-area limit of the Riemann surface Σ, where $1/f$ is replaced by $\tilde{r}/^2 r$ for the squashed S^3 and by $1/r$ for the round S^3. Recall that σ^i is a constant at the fixed point, and notice that the scalar curvature $R(\Sigma)$ of Σ disappears in the mass parameter $\mathcal{M}_\sigma$. The integration in the prefactor can be easily done to give

$$\int_{S^3} \frac{2}{f} e^1 \wedge e^2 \wedge e^3 = \begin{cases} (2\pi r)^2 & \text{for the round } S^3, \\ (2\pi \tilde{r})^2 & \text{for the squashed } S^3, \\ (2\pi r)(2\pi \tilde{r}) & \text{for the ellipsoid } S^3. \end{cases}$$

Therefore, combining the classical action $\mathcal{S}_{\mathrm{cl}}$ with the one-loop contributions $Z^{\text{1-loop}} = Z_{\mathrm{V}}^{\text{1-loop}} Z_{\mathrm{H}}^{\text{1-loop}}$, we obtain the partition function

$$Z^{\mathcal{N}}_{S^3} = \sum_m \prod_{i=1}^{r} \int d\sigma^i \exp[\mathcal{S}_{\mathrm{cl}}] \, Z^{\text{1-loop}},$$

where the integers m_i are the 'monopole' charges, which will be explained below for, just for brevity, the gauge group $G = SU(2)$.

In the $\mathcal{N}=1$ twisting, we have seen that the one-loop contributions from the hypermultiplet are trivial to the partition functions.

$$Z_{\mathrm{H}}^{\text{1-loop}} = 1.$$

Furthermore, on the squashed S^3, the one-loop determinants from the $\mathcal{N}=1$ gauge multiplet remains the same as on the round S^3, when we replace r by $\tilde{r}$. In fact, the partition function $Z^{\mathcal{N}=1}_{\text{squashed}}$ for the squashed S^3,

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18 Recall that upon the localization, we rotated $\sigma^i \to i\sigma^i$ and shifted $D^{\dot{1}}{}_{\dot{1}} - iF_{45} - \frac{i}{\alpha} G_{45}\sigma$, as discussed at the beginning of Section 7.

$$\sum_m \prod_{i=1}^{r} \int d\sigma^i \exp[\mathcal{S}_{\text{cl}}]\, Z_{\text{V}}^{\text{1-loop}} Z_{\text{H}}^{\text{1-loop}}$$
$$= \sum_m \prod_{i=1}^{r} \int d\sigma^i \exp[\mathcal{S}_{\text{cl}}] \prod_{\alpha\in\Lambda_+} \left[2\sinh\left(\pi\tilde{r}g\,(\sigma\cdot\alpha)\right)\right]^{\chi(\Sigma)}$$

is reduced to the one for the round S^3, in the round limit $\tilde{r} \to r$.

More specifically, let us take the gauge group G to be $SU(2)$, and then the generators $\{H,\ E_\pm\}$ obey

$$[H,\ E_\pm] = \pm\sqrt{2}E_\pm, \qquad [E_+,\ E_-] = \sqrt{2}H,$$

with our normalization, implying that the positive root $\alpha = \sqrt{2}$. In our convention, we have

$$\int_\Sigma F_{45}\, dx^4 \wedge dx^5 = \frac{2\pi}{g}\sqrt{2}m,$$

where m, which was referred to above as the 'monopole charge', runs over all the integers. Substituting it to the classical action $\mathcal{S}_{\text{cl}}$, we see that the path integral gains the contributions only from the configurations

$$\sigma = \frac{1}{\sqrt{2}}\frac{g}{(2\pi\tilde{r})^2}n, \qquad \text{for } n \in \mathbf{Z}.$$

Therefore, the partition function on the squashed S^3 is turned to

$$Z_{\text{squashed}}^{\mathcal{N}=1} = 2\sum_{n=1}^{\infty}\left(e^{-\frac{g^2}{4\pi\tilde{r}}n} - e^{\frac{g^2}{4\pi\tilde{r}}n}\right)^{\chi(\Sigma)} = 2\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{\chi(\Sigma)} \sum_{n=1}^{\infty}\left[\chi_n(q)\right]^{\chi(\Sigma)},$$

with $q = e^{-g^2/(2\pi\tilde{r})}$, where the character $\chi_n(q)$ is defined by

$$\chi_n(q) = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

Especially when we consider the round S^3 and replace $\tilde{r}$ by r in the above $Z_{\text{squashed}}^{\mathcal{N}=1}$, the result in the previous paper [1] is recovered;

$$Z_{\text{round}}^{\mathcal{N}=1} = 2\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{\chi(\Sigma)} \sum_{n=1}^{\infty}\left[\chi_n(q)\right]^{\chi(\Sigma)}, \tag{88}$$

with $q = e^{-g^2/(2\pi r)}$, and we can see that it is consistent with the superconformal index computed in [3]. Using the $2(g-1)$ structure constants and $3(g-1)$ propagators in [3] to compute the index for the surface Σ of genus[19] g (therefore, $\chi(\Sigma) = 2 - 2g$) with no punctures, one obtains the above $Z_{\text{round}}^{\mathcal{N}=1}$ up to an factor.[20] In the review article [35], it has been elucidated[21] that the discrepancy is attributed to the difference of the renormalization prescriptions used here and there, and that it can be improved by the requirement of the S-duality (a.k.a. the bootstrap).

[19] This g is *not* the gauge coupling constant g.

[20] In the terminology of [3], the factor is given by $(\mathcal{N}_{222})^{-\chi(\Sigma)}$.

[21] We thank Yuji Tachikawa for elucidating this point.

From the point of view of the number of supersymmetries, this result seems puzzling. The partition function $Z^{\mathcal{N}=1}_{\text{round}}$ computed under the $\mathcal{N}=1$ twisting is supposed to be the index of a four-dimensional $\mathcal{N}=1$ supersymmetric theory, while the index in [3] was computed for a four-dimensional $\mathcal{N}=2$ superconformal theory. In [20], the superconformal index of $\mathcal{N}=1$ class $\mathcal{S}$ fixed points has been calculated in four dimensions. Among their results, the mixed Schur index carries two fugacities p and q in their notations. When we take $p=q$, the index takes the same form as the Schur index of the $\mathcal{N}=2$ fixed points given in [3].

The partition function $Z^{\mathcal{N}=1}_{\text{squashed}}$ on the squashed S^3 is essentially the same as $Z^{\mathcal{N}=1}_{\text{round}}$ on the round S^3. However, the partition function $Z^{\mathcal{N}=1}_{\text{ellipsoid}}$ on the ellipsoid S^3 is deformed from the one on the round S^3.

$$\begin{aligned} Z^{\mathcal{N}=1}_{\text{ellipsoid}} &= \sum_m \prod_{i=1}^r \int d\sigma^i \exp[\mathcal{S}_{\text{cl}}]\, Z_{\text{V}}^{\text{1-loop}} Z_{\text{H}}^{\text{1-loop}} \\ &= \sum_m \prod_{i=1}^r \int d\sigma^i \exp[\mathcal{S}_{\text{cl}}] \prod_{\alpha\in\Lambda_+} \left[2\sinh\left(\pi r g\,(\sigma\cdot\alpha)\right)\cdot 2\sinh\left(\pi\tilde{r} g\,(\sigma\cdot\alpha)\right)\right]^{\frac{1}{2}\chi(\Sigma)}. \end{aligned}$$

This is a similar situation to the three-dimensional case in [21]. As we have done just before, taking the gauge group $G=SU(2)$ and summing over the monopole charge m, we can see that the only configurations

$$\sigma = \frac{1}{\sqrt{2}} \frac{g}{(2\pi\tilde{r})(2\pi r)} n, \qquad \text{for } n\in\mathbf{Z}$$

contribute to the partition function, and therefore the summation of n over integers yields

$$Z^{\mathcal{N}=1}_{\text{ellipsoid}} = 2\left[\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)\left(p^{\frac{1}{2}}-p^{-\frac{1}{2}}\right)\right]^{\frac{1}{2}\chi(\Sigma)} \sum_{n=1}^{\infty}\left[\chi_n(q)\chi_n(p)\right]^{\frac{1}{2}\chi(\Sigma)},$$

where $q=e^{-g^2/(2\pi r)}$ and $p=e^{-g^2/(2\pi\tilde{r})}$. It is also consistent with the mixed Schur index[22] of $\mathcal{N}=1$ rank one class $\mathcal{S}$ fixed points in [20], up to an factor from the renormalization mentioned above.

Let us turn to the $\mathcal{N}=2$ twisting. On the round S^3, we have seen that the hypermultiplet contributes the same one-loop determinants to the partition function as the $\mathcal{N}=1$ gauge multiplet does. Deforming the round S^3 to the squashed S^3, we have observed that the one-loop contributions from the hypermultiplet are deformed by the deformation parameter of the S^3.

In fact, in the partition function $Z^{\mathcal{N}=2}_{\text{squashed}}$ on the squashed S^3

$$Z^{\mathcal{N}=2}_{\text{squashed}} = \sum_m \prod_{i=1}^r \int d\sigma^i \exp[\mathcal{S}_{\text{cl}}]\, Z_{\text{V}}^{\text{1-loop}} Z_{\text{H}}^{\text{1-loop}},$$

we have seen that the one-loop contributions $Z_{\text{V}}^{\text{1-loop}} Z_{\text{H}}^{\text{1-loop}}$ are given by

[22] More specifically, our result corresponds to the case with $l_1=l_2$ in their notations of [20], and to the $\mathcal{N}=1$ twist in [16], as may be seen from the background R-symmetry gauge field.

$$\prod_{\alpha\in\Lambda_+}\left[\left(2\sinh\left(\pi\tilde{r}g\left(\sigma\cdot\alpha\right)\right)\right)^2 s_{b=1}\left(i-2i\frac{\tilde{r}^2}{r^2}-\tilde{r}g\left(\sigma\cdot\alpha\right)\right)s_{b=1}\right.$$
$$\left.\times\left(i-2i\frac{\tilde{r}^2}{r^2}+\tilde{r}g\left(\sigma\cdot\alpha\right)\right)\right]^{\frac{1}{2}\chi(\Sigma)}.$$

Therefore, for the gauge group $G=SU(2)$, upon summing the magnetic charge m over integers, we obtain

$$Z^{\mathcal{N}=2}_{\text{squashed}}=2\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)^{\chi(\Sigma)}$$
$$\times\sum_{n=1}^{\infty}\left[\left(\chi_n(q)\right)^2 s_{b=1}\left(i-2i\frac{\tilde{r}^2}{r^2}-\frac{g^2}{4\pi^2\tilde{r}}n\right)s_{b=1}\left(i-2i\frac{\tilde{r}^2}{r^2}+\frac{g^2}{4\pi^2\tilde{r}}n\right)\right]^{\frac{1}{2}\chi(\Sigma)},$$

with $q=e^{-g^2/(2\pi\tilde{r})}$, where the double sine functions may be rewritten as

$$s_{b=1}\left(i-2i\frac{\tilde{r}^2}{r^2}-\frac{g^2}{4\pi^2\tilde{r}}n\right)s_{b=1}\left(i-2i\frac{\tilde{r}^2}{r^2}+\frac{g^2}{4\pi^2\tilde{r}}n\right)$$
$$=\prod_{m=1}^{\infty}\left[\frac{\left(m+1-2\frac{\tilde{r}^2}{r^2}\right)^2+\left(\frac{g^2}{4\pi^2\tilde{r}}n\right)^2}{\left(m-1+2\frac{\tilde{r}^2}{r^2}\right)^2+\left(\frac{g^2}{4\pi^2\tilde{r}}n\right)^2}\right]^m,$$

and in the round limit $\tilde{r}\to r$, they reduce to

$$\left[\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)\cdot\chi_n(q)\right]^2,$$

with $q=e^{-g^2/(2\pi r)}$. When recognizing $\chi_n(q)$ as the q-deformed number[23] $[n]_q$, we may regard the square root of the double sine functions as a deformation of $[n]_q$.

We thus find that the partition function $Z^{\mathcal{N}=2}_{\text{round}}$ on the round S^3 is given for $G=SU(2)$ by

$$Z^{\mathcal{N}=2}_{\text{round}}=\sum_m\int d\sigma\exp\left[\mathcal{S}_{\text{cl}}\right]\left[2\sinh\left(\pi rg\left(\sigma\cdot\alpha\right)\right)\right]^{2\chi(\Sigma)}$$
$$=2\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)^{2\chi(\Sigma)}\sum_{n=1}^{\infty}\left[\chi_n(q)\right]^{2\chi(\Sigma)}.$$

This result suggests that the partition function $Z^{\mathcal{N}=2}_{\text{round}}$ does not corresponds to the Schur limit of the superconformal index discussed in [3]. We expect that it gives another simple limit of the superconformal index of $\mathcal{N}=2$ rank one class $\mathcal{S}$ fixed points, where the index can be calculated by the two-dimensional q-deformed Yang–Mills theory but with the measure $\left[2\sinh\left(\pi rg\left(\sigma\cdot\alpha\right)\right)\right]^{\chi(\Sigma)}$ squared.

23 This definition of the q-deformed number $[n]_q$ slightly differs from the one in [1] by the factor $1/(q^{\frac{1}{2}}-q^{-\frac{1}{2}})$.

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Appendix A. Our conventions of (anti-)symmetrization of indices and differential forms

The convention of the antisymmetrization and symmetrization[24] may be seen from

$$A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu, \qquad X_{(\mu}Y_{\nu)} = X_\mu Y_\nu + X_\nu Y_\mu.$$

However, for the six-dimensional gamma matrices (for the definition of them, see the next appendix), we define

$$\underline{\Gamma}^{\underline{ab}} = \frac{1}{2}\underline{\Gamma}^{[\underline{a}}\underline{\Gamma}^{\underline{b}]}, \qquad \underline{\Gamma}^{\underline{a_1\cdots a_n}} = \frac{1}{n}\underline{\Gamma}^{[\underline{a_1}}\underline{\Gamma}^{\underline{a_2\cdots a_n}]} = \frac{1}{n!}\underline{\Gamma}^{[\underline{a_1}}\underline{\Gamma}^{\underline{a_2}}\cdots\underline{\Gamma}^{\underline{a_n}]}.$$

For the five-dimensional gamma matrices, $\gamma^{a_1\cdots a_n}$ is defined in the same way.

For an n-form $\underline{A}_n$ in six dimensions, we define

$$\underline{A}_n = \frac{1}{n!}\underline{A}_{\underline{\mu_1\cdots\mu_n}}\, d\underline{X}^{\underline{\mu}_1}\wedge\cdots\wedge d\underline{X}^{\underline{\mu}_n} = \frac{1}{n!}\underline{A}_{\underline{a_1\cdots a_n}}\,\underline{E}^{\underline{a}_1}\wedge\cdots\wedge\underline{E}^{\underline{a}_n},$$

where X^μ $(\mu = 0, 1, \cdots, 5)$ denote the local coordinates and $\underline{E}^{\underline{a}}$ $(a = 0, 1, \cdots, 5)$ are the sechsbein one-form. We also define the Hodge dual of the form $\underline{A}_n$ by

$$\underline{*}\,\underline{A}_n = \frac{1}{(6-n)!}\left[\frac{1}{n!}\varepsilon_{\underline{a_1\cdots a_{6-n}b_1\cdots b_n}}\underline{A}^{\underline{b_1\cdots b_n}}\right]\underline{E}^{\underline{a}_1}\wedge\cdots\wedge\underline{E}^{\underline{a}_{6-n}},$$

with $\varepsilon_{01\cdots 5} = 1$. Because of the property $\underline{**}\,\underline{A}_n = (-)^{n+1}\underline{A}_n$, one may define the self-dual and the anti-self dual parts, respectively, of a three-form $\underline{A}_3$ by

$$\underline{A}_3^{\pm} = \frac{1}{2}\left(\underline{A}_3 \pm \underline{*}\,\underline{A}_3\right) = \pm\underline{*}\,\underline{A}_3^{\pm}.$$

We define the external derivative $d\underline{A}_n$ of the n-form

$$\begin{aligned} d\underline{A}_n &= \frac{1}{n!}\,\partial_{\underline{\mu}_1}\underline{A}_{\underline{\mu_2\cdots\mu_{n+1}}}\,d\underline{X}^{\underline{\mu}_1}\wedge\cdots\wedge d\underline{X}^{\underline{\mu}_{n+1}} \\ &= \frac{1}{(n+1)!}\,\partial_{[\underline{\mu}_1}\underline{A}_{\underline{\mu_2\cdots\mu_{n+1}}]}\,d\underline{X}^{\underline{\mu}_1}\wedge\cdots\wedge d\underline{X}^{\underline{\mu}_{n+1}}. \end{aligned}$$

For an n-form a in five dimensions,

$$a = \frac{1}{n!}a_{\mu_1\cdots\mu_n}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} = \frac{1}{n!}a_{a_1\cdots a_n}\,e^{a_1}\wedge\cdots\wedge e^{a_n},$$

similarly, the external derivative da and the Hodge dual $*a$ are

[24] Ref. [25] gives weights to each of the terms on the right hand side; for example, $A_{[\mu}B_{\nu]} = (1/2)(A_\mu B_\nu - A_\nu B_\mu)$, $X_{(\mu}Y_{\nu)} = (1/2)(X_\mu Y_\nu + X_\nu Y_\mu)$.

$$da = \frac{1}{(n+1)!} \partial_{[\mu_1} a_{\mu_2 \cdots \mu_{n+1}]} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{n+1}},$$
$$*a = \frac{1}{(5-n)!} \left[\frac{1}{n!} \varepsilon_{a_1 \cdots a_{5-n} b_1 \cdots b_n} a^{b_1 \cdots b_n} \right] de^{a_1} \wedge \cdots \wedge de^{a_{5-n}}.$$

Appendix B. Gamma matrices of the 6-dimensional Lorentz group

We define the six-dimensional gamma matrices $\underline{\Gamma}^{\underline{a}}$ ($\underline{a} = 0, 1, \cdots, 5$) such that they satisfy

$$\left\{ \underline{\Gamma}^{\underline{a}}, \underline{\Gamma}^{\underline{b}} \right\} = 2\underline{\eta}^{\underline{ab}} \mathbf{1}_8,$$

with $\mathbf{1}_8$ the 8×8 unit matrix and the Lorentz metric $(\eta^{\underline{ab}}) = \mathrm{diag}\,.(-1, +1, \cdots, +1)$. While only $\underline{\Gamma}^0$ is anti-hermitian, the others are hermitian.

The chirality is defined by the matrix

$$\underline{\Gamma}^7 = \underline{\Gamma}^0 \underline{\Gamma}^1 \cdots \underline{\Gamma}^5, \qquad \left(\underline{\Gamma}^7\right)^2 = \mathbf{1}_8,$$

and it enjoys the properties

$$\underline{\Gamma}^{\underline{a_1 \cdots a_6}} = -\varepsilon^{\underline{a_1 \cdots a_6}} \underline{\Gamma}^7, \qquad \underline{\Gamma}^{\underline{abc}} = -\frac{1}{3!} \varepsilon^{\underline{abcdef}} \underline{\Gamma}_{\underline{def}} \underline{\Gamma}^7, \tag{89}$$

with the convections $\varepsilon^{01 \cdots 5} = -1$ ($\varepsilon_{01 \ldots 5} = +1$).

The charge conjugation matrix $\underline{C}$ is a unitary matrix satisfying

$$\underline{C}^T = \underline{C}, \qquad \left(\underline{\Gamma}_{\underline{a}}\right)^T = -\underline{C} \underline{\Gamma}_{\underline{a}} \underline{C}^{-1}, \tag{90}$$

with T denoting the transpose of the matrices, and thus $\left(\underline{\Gamma}^7\right)^T = -\underline{C} \underline{\Gamma}^7 \underline{C}^{-1}$.

On the reduction along the time direction from the six-dimensional Minkowski space to the five-dimensional Euclidean space, where the gamma matrices are five 4×4 hermitian matrices γ^a ($a = 1, \cdots, 5$) satisfying

$$\left\{ \gamma^a, \gamma^b \right\} = 2\delta^{ab} \mathbf{1}_4, \qquad \gamma^1 \cdots \gamma^5 = \mathbf{1}_4,$$

with $\mathbf{1}_4$ the 4×4 unit matrix, we define

$$\underline{\Gamma}^0 = \mathbf{1}_4 \otimes i\tau_2 = \begin{pmatrix} 0 & \mathbf{1}_4 \\ -\mathbf{1}_4 & 0 \end{pmatrix}, \quad \underline{\Gamma}^a = \gamma^a \otimes \tau_1 = \begin{pmatrix} 0 & \gamma^a \\ \gamma^a & 0 \end{pmatrix},$$
$$\underline{\Gamma}^7 = \mathbf{1}_4 \otimes \tau_3 = \begin{pmatrix} \mathbf{1}_4 & 0 \\ 0 & -\mathbf{1}_4 \end{pmatrix},$$

for $a = 1, \cdots, 5$, with the Pauli matrices τ_1, τ_2, τ_3.

The property (89) is reduced to

$$\gamma^{abcde} \equiv \frac{1}{5!} \gamma^{[a} \gamma^b \gamma^c \gamma^d \gamma^{e]} = \varepsilon^{abcde},$$

with $\varepsilon^{12345} = \varepsilon_{12345} = 1$.

The six-dimensional charge conjugate matrix $\underline{C}$ is related to the five-dimensional charge conjugation matrix C by

$$\underline{C} = C \otimes i\tau_2 = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix},$$

and one can see that the charge conjugation matrix C enjoys the properties

$$C^T = -C, \qquad (\gamma_a)^T = C\gamma_a C^{-1}.$$

It follows from them that

$$\left(C\gamma^{a_1\cdots a_n}\right)^T = -(-)^{\frac{n(n-1)}{2}}\left(C\gamma^{a_1\cdots a_n}\right), \qquad \left(\gamma^{a_1\cdots a_n}C^{-1}\right)^T = -(-)^{\frac{n(n-1)}{2}}\left(\gamma^{a_1\cdots a_n}C^{-1}\right).$$

A more explicit form of the five-dimensional gamma matrices γ^μ takes

$$\gamma^a = \tau_a \otimes \tau_2 \quad (a = 1, 2, 3), \qquad \gamma^4 = \mathbf{1}_2 \otimes \tau_1, \qquad \gamma^5 = \mathbf{1}_2 \otimes \tau_3,$$

with the charge conjugation matrix $C = C_3 \otimes \mathbf{1}_2$, where $C_3 = i\tau_2$.

Appendix C. Gamma matrices of the R-symmetry group Spin(5)$_R$

We give the explicit form of the gamma matrices of the R-symmetry group Spin(5)$_R$

$$\rho^1 = \tau^1 \otimes \tau^2, \quad \rho^2 = \tau^2 \otimes \tau^2, \quad \rho^3 = \tau^3 \otimes \tau^2, \quad \rho^4 = \mathbf{1} \otimes \tau^1,$$
$$\rho^5 = \mathbf{1} \otimes \tau^3 = \rho^1 \cdots \rho^4,$$

with the Pauli matrices τ_1, τ_2, τ_3, satisfying that

$$\left\{\rho^I, \rho^J\right\} = 2\delta^{IJ},$$

where I, J run from 1 to 5. We use them to define

$$\rho^{I_1\cdots I_n} = \frac{1}{n!}\rho^{[I_1}\cdots\rho^{I_n]}, \qquad \rho^{I_1\cdots I_5} = \varepsilon^{I_1\cdots I_5},$$

with $\varepsilon^{12345} = 1$.

We also explicitly give the charge conjugation matrix Ω of the Spin(5)$_R$

$$\Omega = i\tau_2 \otimes \mathbf{1} = -\Omega^\dagger = -\Omega^T,$$

where Ω^T is the transpose of the matrix Ω, which satisfies

$$\Omega\left(\rho^I\right)^T\Omega^{-1} = \rho^I \qquad (I = 1, 2, \cdots, 5).$$

It follows from these properties that

$$\left(\Omega\rho^{I_1\cdots I_n}\right)^T = -(-)^{\frac{n(n-1)}{2}}\left(\Omega\rho^{I_1\cdots I_n}\right), \qquad \left(\rho^{I_1\cdots I_n}\Omega^{-1}\right)^T = -(-)^{\frac{n(n-1)}{2}}\left(\rho^{I_1\cdots I_n}\Omega^{-1}\right).$$

Given the components

$$\Omega = \left(\Omega_{\alpha\beta}\right), \quad \Omega^{-1} = -\left(\Omega^{\alpha\beta}\right) \qquad (\alpha, \beta = 1, \cdots, 4),$$

one has

$$\Omega^{\alpha\gamma}\Omega_{\beta\gamma} = \delta^\alpha{}_\beta.$$

The index α of a spinor ϵ^α of the Spin(5)$_R$ is lowered by Ω as

$$\epsilon_\alpha = \epsilon^\beta\Omega_{\beta\alpha}, \qquad \epsilon^\alpha = \Omega^{\alpha\beta}\epsilon_\beta,$$

and this convention is consistent with

$$\Omega^{\alpha\gamma}\,\Omega^{\beta\delta}\,\Omega_{\gamma\delta}=\Omega^{\alpha\beta}.$$

Since the components $\Omega_{\alpha\beta}$ are real,

$$\left(\Omega_{\alpha\beta}\right)^*=\Omega^{\alpha\beta},$$

where $\left(\Omega_{\alpha\beta}\right)^*$ denotes the complex conjugate of $\Omega_{\alpha\beta}$.

The Fierz transformation of two matrices $(M_{\alpha\beta})$, $(N_{\alpha\beta})$

$$\begin{aligned}M_{\alpha\beta}\,N_{\gamma\delta}=\frac{1}{4}\Big[&\left(M\Omega^{-1}N\right)_{\alpha\delta}\Omega_{\gamma\beta}+\left(M\rho^I\Omega^{-1}N\right)_{\alpha\delta}(\Omega\rho_I)_{\gamma\beta}\\&-\frac{1}{2}\left(M\rho^{IJ}\Omega^{-1}N\right)_{\alpha\delta}(\Omega\rho_{IJ})_{\gamma\beta}\Big]\end{aligned}$$

may be useful to verify some of the calculations in the text.

C.1. $\mathrm{Spin}(5)_R\longrightarrow\mathrm{Spin}(4)_R\simeq SU(2)_l\times SU(2)_r$

A vector v^I $(I=1,\cdots,4,5)$ of the $\mathrm{Spin}(5)_R$ group is decomposed into irreducible representations of the subgroup $\mathrm{Spin}(4)_R$ as one vector v^i $(i=1,\cdots,4)$ and one singlet v^5. A spinor ψ^α $(\alpha=1,\cdots,4)$ of the $\mathrm{Spin}(5)_R$ group is decomposed into

$$\psi^\alpha=\begin{pmatrix}\psi^{\tilde\alpha}\\ \psi^{\dot\alpha}\end{pmatrix}\qquad(\tilde\alpha=1,2;\ \dot\alpha=1,2),$$

with $\psi^{\tilde\alpha}$ in $(\mathbf{2},\mathbf{1})$ and $\psi^{\dot\alpha}$ in $(\mathbf{1},\mathbf{2})$ of the $SU(2)_l\times SU(2)_r\simeq\mathrm{Spin}(4)_R$ group.

The gamma matrices of the $\mathrm{Spin}(5)_R$ group are reduced into

$$\rho^i=\begin{pmatrix}&\sigma^i\\ \bar\sigma^i&\end{pmatrix}\quad(i=1,\cdots,4),\qquad\rho^5=\begin{pmatrix}\mathbf{1}_2&\\&-\mathbf{1}_2\end{pmatrix}=\rho^1\cdots\rho^4,$$

where

$$(\sigma^i)=(-i\overrightarrow{\tau},1)=(\sigma^{i\tilde\alpha}{}_{\dot\beta}),\qquad(\bar\sigma^i)=(i\overrightarrow{\tau},1)=(\bar\sigma^{i\dot\alpha}{}_{\tilde\beta}),$$

with τ^a $(a=1,2,3)$ the Pauli matrices. The matrices σ^i, $\bar\sigma^i$ $(i=1,\cdots,4)$ obey the relations

$$\begin{aligned}&\bar\sigma^{i\dot\beta}{}_{\tilde\alpha}=\varepsilon_{\tilde\alpha\tilde\gamma}\varepsilon^{\dot\beta\dot\delta}\sigma^{i\tilde\gamma}{}_{\dot\delta},\qquad\sigma^{i\tilde\gamma}{}_{\dot\delta}=\varepsilon^{\tilde\gamma\tilde\alpha}\varepsilon_{\dot\delta\dot\beta}\bar\sigma^{i\dot\beta}{}_{\tilde\alpha},\\&\sigma^{i\tilde\alpha}{}_{\dot\beta}\,\bar\sigma^{i\dot\gamma}{}_{\tilde\delta}=2\delta^{\tilde\alpha}{}_{\tilde\delta}\,\delta^{\dot\gamma}{}_{\dot\beta},\qquad\sigma^{i\tilde\alpha}{}_{\dot\beta}\,\sigma^{i\tilde\gamma}{}_{\dot\delta}=2\varepsilon^{\tilde\alpha\tilde\gamma}\,\varepsilon_{\dot\beta\dot\delta},\qquad\bar\sigma^{i\dot\alpha}{}_{\tilde\beta}\,\bar\sigma^{i\dot\gamma}{}_{\tilde\delta}=2\varepsilon^{\dot\alpha\dot\gamma}\,\varepsilon_{\tilde\beta\tilde\delta},\\&\mathrm{tr}\left[\sigma^i\bar\sigma^j\right]=\sigma^{i\tilde\alpha}{}_{\dot\beta}\bar\sigma^{j\dot\beta}{}_{\tilde\alpha}=2\delta^{ij}.\end{aligned}$$

The generators of the $\mathrm{Spin}(4)_R$ group in the spinor representation become the direct sum

$$\rho^{ij}=\frac{1}{2}\begin{pmatrix}\sigma^i\bar\sigma^j-\sigma^j\bar\sigma^i&\\&\bar\sigma^i\sigma^j-\bar\sigma^j\sigma^i\end{pmatrix}\equiv\begin{pmatrix}\sigma^{ij}&\\&\bar\sigma^{ij}\end{pmatrix}\quad(i,j=1,\cdots,4)$$

obeying

$$\sigma^{ij}=-\frac{1}{2}\varepsilon^{ijkl}\sigma_{kl},\qquad\bar\sigma^{ij}=\frac{1}{2}\varepsilon^{ijkl}\bar\sigma_{kl}.$$

For example, one has

$$\bar{\sigma}^{12} = \bar{\sigma}^{34} = i\tau^3, \qquad \sigma^{12} = -\sigma^{34} = i\tau^3.$$

The charge conjugation matrix gives

$$\Omega = \begin{pmatrix} \varepsilon_{\tilde{\alpha}\tilde{\beta}} & \\ & \varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} = \begin{pmatrix} i\tau_2 & \\ & i\tau_2 \end{pmatrix}, \qquad \Omega^{-1} = \begin{pmatrix} -\varepsilon^{\tilde{\alpha}\tilde{\beta}} & \\ & -\varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} = \begin{pmatrix} -i\tau_2 & \\ & -i\tau_2 \end{pmatrix},$$

with $\varepsilon_{12} = \varepsilon^{12} = 1$.

The spinor indices $\tilde{\alpha}$, $\dot{\alpha}$ are raised or lowered as

$$\psi_{\tilde{\alpha}} = \psi^{\tilde{\beta}} \varepsilon_{\tilde{\beta}\tilde{\alpha}}, \quad \psi^{\tilde{\alpha}} = \varepsilon^{\tilde{\alpha}\tilde{\beta}} \psi_{\tilde{\beta}}; \qquad \psi_{\dot{\alpha}} = \psi^{\dot{\beta}} \varepsilon_{\dot{\beta}\dot{\alpha}}, \quad \psi^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \psi_{\dot{\beta}}.$$

Appendix D. Symplectic Majorana–Weyl spinors

For a six-dimensional Dirac spinor $\underline{\psi}$, we define $\overline{\underline{\psi}} = \left(\underline{\psi}\right)^\dagger \underline{\Gamma}^0$. In six-dimensional Minkowski space, the symplectic Majorana condition on an even number of spinors can be imposed. In our case, all the spinors in the Weyl multiplet and the tensor multiplet of the supergravity carry the spinor indices of the $\mathrm{Spin}(5)_R$ symmetry group, and the dimension of the spinor representation is four - an even number. Let us take one of such spinors, say $\underline{\psi}^\alpha$, and it obeys the symplectic Majorana condition

$$\left(\underline{\psi}^\alpha\right)^\dagger \underline{\Gamma}^0 = \left(\underline{\psi}^\beta\right)^T \underline{C} \Omega_{\beta\alpha},$$

and the other spinors in the multiplets obey the same condition.

In the Minkowski space, the Weyl condition $\underline{\Gamma}^7 \underline{\psi} = \pm \underline{\psi}$ and the symplectic Majorana condition can be imposed on spinors at the same time. In fact, all the spinors of the multiplets are symplectic Majorana–Weyl spinors, and also so are the parameters of the supersymmetry and the conformal supersymmetry transformations, as explained in the text.

After the dimensional reduction, the spinors in the supergravity multiplets give rise to symplectic Majorana spinors in the five-dimensional Euclidean space. If $\underline{\psi}^\alpha$ is a symplectic Majorana–Weyl spinor of positive chirality, it takes the form

$$\underline{\psi}^\alpha = \begin{pmatrix} \psi^\alpha \\ 0 \end{pmatrix},$$

and is reduced to the symplectic Majorana spinor ψ^α obeying

$$\left(\psi^\alpha\right)^\dagger = \left(\psi^\beta\right)^T C \Omega_{\beta\alpha}, \tag{91}$$

in the five-dimensional Euclidean space. If it is of negative chirality, one can see from

$$\underline{\psi}^\alpha = \begin{pmatrix} 0 \\ \psi^\alpha \end{pmatrix},$$

that ψ^α also obeys the same condition (91).

It is convenient to introduce the notations for the conjugate of a five-dimensional spinor ϵ^α

$$\bar{\epsilon}^\alpha \equiv \left(\epsilon^\alpha\right)^T C,$$

and the abbreviation of the spinor bilinear

$$\left(\bar{\epsilon} \cdot \rho_{I_1 \cdots I_n} \gamma^{a_1 \cdots a_m} \eta\right) \equiv \left(\Omega \rho_{I_1 \cdots I_n}\right)_{\alpha\beta} \cdot \left(\epsilon^{\alpha}\right)^T C \gamma^{a_1 \cdots a_m} \eta^{\beta},$$

of two five-dimensional spinors ϵ^{α}, η^{α}.

The Fierz transformation of five-dimensional spinors ϵ^{α}, η^{α} gives

$$\eta^{\alpha} \bar{\epsilon}^{\beta} = \eta^{\alpha} \left(\epsilon^{\beta}\right)^T C = -\frac{1}{4}\left[\left(\epsilon^{\beta} C \eta^{\alpha}\right) \mathbf{1}_4 + \left(\epsilon^{\beta} C \gamma^{a} \eta^{\alpha}\right) \gamma_a - \frac{1}{2}\left(\epsilon^{\beta} C \gamma^{ab} \eta^{\alpha}\right) \gamma_{ab}\right],$$

and the following formula is repeatedly used in the calculations in the text:

$$\begin{aligned}\eta^{\alpha} \bar{\epsilon}^{\beta} - \epsilon^{\alpha} \bar{\eta}^{\beta} = -\frac{1}{8}\Big[& (\bar{\epsilon} \cdot \eta) \left(\Omega^{-1}\right)^{\alpha\beta} + (\bar{\epsilon} \cdot \rho_I \eta) \left(\rho^I \Omega^{-1}\right)^{\alpha\beta} + \left(\bar{\epsilon} \cdot \gamma^a \eta\right) \left(\Omega^{-1}\right)^{\alpha\beta} \gamma_a \\ & + \left(\bar{\epsilon} \cdot \rho_I \gamma^a \eta\right) \left(\rho^I \Omega^{-1}\right)^{\alpha\beta} \gamma_a + \frac{1}{4}\left(\bar{\epsilon} \cdot \rho_{IJ} \gamma^{ab} \eta\right) \left(\rho^{IJ} \Omega^{-1}\right)^{\alpha\beta} \gamma_{ab}\Big].\end{aligned}$$

The following abbreviation for the bilinears of spinors $\psi^{\tilde{\alpha}}$, $\chi^{\tilde{\alpha}}$ in $(\mathbf{2}, \mathbf{1})$ and $\lambda^{\dot{\alpha}}$, $\epsilon^{\dot{\alpha}}$ in $(\mathbf{1}, \mathbf{2})$ of the $SU(2)_l \times SU(2)_r$ R-symmetry group is used:

$$\bar{\psi} \cdot \sigma_{i_1 \cdots i_{2n}} \gamma^{a_1 \cdots a_m} \chi \equiv \varepsilon_{\tilde{\alpha}\tilde{\gamma}} (\sigma_{i_1 \cdots i_{2n}})^{\tilde{\gamma}}{}_{\tilde{\beta}} \cdot \left(\psi^{\tilde{\alpha}}\right)^T C \gamma^{a_1 \cdots a_m} \chi^{\tilde{\beta}},$$

$$\bar{\psi} \cdot \sigma_{i_1 \cdots i_{2n+1}} \gamma^{a_1 \cdots a_m} \lambda \equiv \varepsilon_{\tilde{\alpha}\tilde{\gamma}} (\sigma_{i_1 \cdots i_{2n+1}})^{\tilde{\gamma}}{}_{\dot{\beta}} \cdot \left(\psi^{\tilde{\alpha}}\right)^T C \gamma^{a_1 \cdots a_m} \lambda^{\dot{\beta}},$$

$$\bar{\epsilon} \cdot \bar{\sigma}_{i_1 \cdots i_{2n+1}} \gamma^{a_1 \cdots a_m} \chi \equiv \varepsilon_{\dot{\alpha}\dot{\gamma}} (\bar{\sigma}_{i_1 \cdots i_{2n+1}})^{\dot{\gamma}}{}_{\tilde{\beta}} \cdot \left(\epsilon^{\dot{\alpha}}\right)^T C \gamma^{a_1 \cdots a_m} \chi^{\tilde{\beta}},$$

$$\bar{\epsilon} \cdot \bar{\sigma}_{i_1 \cdots i_{2n}} \gamma^{a_1 \cdots a_m} \lambda \equiv \varepsilon_{\dot{\alpha}\dot{\gamma}} (\bar{\sigma}_{i_1 \cdots i_{2n}})^{\dot{\gamma}}{}_{\dot{\beta}} \cdot \left(\epsilon^{\dot{\alpha}}\right)^T C \gamma^{a_1 \cdots a_m} \lambda^{\dot{\beta}}.$$

For the spinors $\varepsilon^{\dot{\alpha}}$, $\eta^{\dot{\alpha}}$, the Fierz transformation

$$\varepsilon^{\dot{\alpha}} \bar{\eta}_{\dot{\beta}} = \varepsilon^{\dot{\alpha}} \eta^{\dot{\gamma}} C \varepsilon_{\dot{\gamma}\dot{\beta}} = -\frac{1}{4}\left[\left(\bar{\eta}_{\dot{\beta}}\, \varepsilon^{\dot{\alpha}}\right) \mathbf{1}_4 + \left(\bar{\eta}_{\dot{\beta}} \gamma^a \varepsilon^{\dot{\alpha}}\right) \gamma_a - \frac{1}{2}\left(\bar{\eta}_{\dot{\beta}} \gamma^{ab} \varepsilon^{\dot{\alpha}}\right) \gamma_{ab}\right],$$

is also useful to verify computations in the text such as the algebras of the supersymmetry transformations. In particular, we have often made use of the formula

$$\eta^{\dot{\alpha}} \bar{\varepsilon}_{\dot{\beta}} - \varepsilon^{\dot{\alpha}} \bar{\eta}_{\dot{\beta}} = -\frac{1}{4}\left[(\bar{\varepsilon} \cdot \eta)\, \delta^{\dot{\alpha}}{}_{\dot{\beta}} + \left(\bar{\varepsilon} \cdot \gamma^a \eta\right) \delta^{\dot{\alpha}}{}_{\dot{\beta}}\, \gamma_a + \frac{1}{8}\left(\bar{\varepsilon} \cdot \bar{\sigma}_{ij} \gamma^{ab} \eta\right) \left(\bar{\sigma}^{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\, \gamma_{ab}\right].$$

Appendix E. The supersymmetry Condition from the spinor $\chi^{\alpha\beta}{}_{\gamma}$

One of the supersymmetry conditions which the supersymmetric backgrounds in the five-dimensional supergravity should obey is the requirement that $\delta \chi^{\alpha\beta}{}_{\gamma} \left(\Omega \rho^I\right)_{\beta\alpha}$ should vanish. It yields the condition

$$\begin{aligned}&\frac{4}{5\alpha}\left[G^{ab} t^I{}_{ab} \Omega_{\gamma\delta} - \frac{1}{4} G^{ab} t^J{}_{ab} \left(\Omega \rho^I{}_J\right)_{\gamma\delta}\right] \epsilon^{\delta} \\ &\quad - 2\left[t^I{}_{ab} t^{Jab} \left(\Omega \rho_J\right)_{\gamma\delta} - \frac{1}{5} t^J{}_{ab} t_J{}^{ab} \left(\Omega \rho^I\right)_{\gamma\delta}\right] \epsilon^{\delta} \\ &\quad + \frac{4}{15} M^I{}_J \left(\Omega \rho^J\right)_{\gamma\delta} \epsilon^{\delta} + \varepsilon^{abcde}\left[t^I{}_{ab} t^J{}_{cd} \left(\Omega \rho_J\right)_{\gamma\delta} - \frac{1}{5} t_{Jab} t^J{}_{cd} \left(\Omega \rho^I\right)_{\gamma\delta}\right] \gamma_e \epsilon^{\delta}\end{aligned}$$

$$
\begin{aligned}
&+\frac{8}{5}\left[\mathcal{D}^a t^I{}_{ab}\Omega_{\gamma\delta}-\frac{1}{4}\mathcal{D}^a t^K{}_{ab}\left(\Omega\rho^I{}_K\right)_{\gamma\delta}\right]\gamma^b\epsilon^\delta\\
&-\frac{3}{5}\left[\mathcal{D}_a S^I{}_J\left(\Omega\rho^J\right)_{\gamma\delta}-\frac{1}{3}\mathcal{D}_a S_{KL}\left(\Omega\rho^{IKL}\right)_{\gamma\delta}\right]\gamma^a\epsilon^\delta\\
&+\frac{2}{5}\varepsilon^{abcde}\left[\mathcal{D}_a t^I{}_{bc}\Omega_{\gamma\delta}-\frac{1}{4}\mathcal{D}_a t^J{}_{bc}\left(\Omega\rho^I{}_J\right)_{\gamma\delta}\right]\gamma_{de}\epsilon^\delta\\
&+\frac{3}{10}\left[F_{ab}{}^{IJ}\left(\Omega\rho_J\right)_{\gamma\delta}-\frac{1}{3}F_{ab}{}^{KL}\left(\Omega\rho^I{}_{KL}\right)_{\gamma\delta}\right]\gamma^{ab}\epsilon^\delta\\
&-\frac{8}{5\alpha}\left[G_a{}^c t^I{}_{bc}\Omega_{\gamma\delta}-\frac{1}{4}G_a{}^c t_{Jbc}\left(\Omega\rho^{IJ}\right)_{\gamma\delta}\right]\gamma^{ab}\epsilon^\delta\\
&-\frac{12}{5}\left[t^I{}_{ac}t^J{}_b{}^c\left(\Omega\rho_J\right)_{\gamma\delta}-\frac{1}{3}t^K{}_{ac}t^L{}_b{}^c\left(\Omega\rho^I{}_{KL}\right)_{\gamma\delta}\right]\gamma^{ab}\epsilon^\delta\\
&+\frac{3}{10\alpha}\left[G_{ab}S^I{}_J\left(\Omega\rho^J\right)_{\gamma\delta}-\frac{1}{3}G_{ab}S_{KL}\left(\Omega\rho^{IKL}\right)_{\gamma\delta}\right]\gamma^{ab}\epsilon^\delta\\
&+\frac{4}{5}\left[t^I{}_{ab}S_{KL}\left(\Omega\rho^{KL}\right)_{\gamma\delta}-\frac{1}{2}t^K{}_{ab}S_{IJ}\left(\Omega\rho^J{}_K\right)_{\gamma\delta}\right.\\
&\left.-\frac{1}{4}t_{Jab}S_{KL}\left(\Omega\rho^{IJKL}\right)_{\gamma\delta}-\frac{1}{2}t^J{}_{ab}S^I{}_J\Omega_{\gamma\delta}+\frac{3}{4}t^K{}_{ab}S_{JK}\left(\Omega\rho^{IJ}\right)_{\gamma\delta}\right]\gamma^{ab}\epsilon^\delta=0.
\end{aligned}
\tag{92}
$$

Here, for convenience, we will write once again the covariant derivatives and the field strength

$$
\begin{aligned}
\mathcal{D}_\mu t^I{}_{ab}&=\partial_\mu t^I{}_{ab}-b_\mu t^I{}_{ab}+\left(\Omega_\mu\right)_a{}^c t^I{}_{cb}+\left(\Omega_\mu\right)_b{}^c t^I{}_{ac}-A_\mu{}^I{}_J t^J{}_{ab},\\
\mathcal{D}_\mu S_{IJ}&=\partial_\mu S_{IJ}-b_\mu S_{IJ}-A_{\mu I}{}^K S_{KJ}-A_{\mu J}{}^K S_{IK},\\
F_{\mu\nu}{}^I{}_J&=\partial_\mu A_\nu{}^I{}_J-\partial_\nu A_\mu{}^I{}_J-A_\mu{}^I{}_K A_\nu{}^K{}_J+A_\nu{}^I{}_K A_\mu{}^K{}_J.
\end{aligned}
$$

Appendix F. The SUSY transform of the mass term of the scalars

When the interested readers attempt to ensure the supersymmetry invariance of the actions L and S in Sections 3 and 4, respectively, it may be convenient to show how the mass term $\mathcal{M}_{BIJ}\phi^I\phi^J$ in the actions transforms under a supersymmetry transformation.[25]

$$
\begin{aligned}
\delta\left(\operatorname{tr}\left[\frac{1}{2}\mathcal{M}_{BIJ}\phi^I\phi^J\right]\right)=-\frac{i}{4}\operatorname{tr}\Bigg[\Bigg(&S^I{}_K S^K{}_J\phi^J+\frac{1}{20\alpha^2}G^{ab}G_{ab}\phi^I-4t^I{}_{ab}t_J{}^{ab}\phi^J\\
&+\frac{4}{15}M^I{}_J\phi^J+\frac{1}{5}R(\Omega)\phi^I\Bigg)(\bar{\chi}\cdot\rho_I\epsilon)\Bigg].
\end{aligned}
\tag{93}
$$

The two last terms on the right hand side of (93) depend on $M^I{}_J$ and $R(\Omega)$. If they are given in terms of the backgrounds S_{IJ}, G_{ab}, $t^I{}_{ab}$, they may cancel the supersymmetry variation of the other terms in the actions. In fact, this is the case, if one uses the supersymmetry condition (92) and the Killing spinor equation (7), as will seen below.

[25] For the abelian case, we need to regard the matrices ϕ^I and χ^α as 1×1 matrices and to forget the trace tr in the formulas here.

Using the supersymmetry condition (92), the term

$$-\frac{i}{4}\,\mathrm{tr}\left[\phi_I\,\bar{\chi}^\alpha\cdot\frac{4}{15}M^I{}_J(\Omega\rho^J)_{\alpha\beta}\epsilon^\beta\right]$$

on the right hand side of (93) can be straightforwardly replaced by terms depending on the backgrounds S_{IJ}, G_{ab}, $t^I{}_{ab}$.

The commutation relation of the covariant derivatives gives

$$\frac{1}{2}\gamma^{ab}[\mathcal{D}_a,\mathcal{D}_b]\epsilon^\alpha=-\frac{1}{4}R(\Omega)\epsilon^\alpha-\frac{1}{8}F_{ab}{}^{IJ}(\rho_{IJ})^\alpha{}_\beta\gamma^{ab}\epsilon^\beta,$$

and on the other hand, using the Killing spinor equation (7), one obtains

$$\begin{aligned}
\frac{1}{2}\gamma^{ab}[\mathcal{D}_a,\mathcal{D}_b]\epsilon^\alpha&=\gamma^{ab}\mathcal{D}_a\mathcal{D}_b\epsilon^\alpha\\
&=\mathcal{D}_aS^{IJ}(\rho_{IJ})^\alpha{}_\beta\gamma^a\epsilon^\beta-\frac{1}{2\alpha}\mathcal{D}_aG_{bc}\gamma^{ab}\gamma^c\epsilon^\alpha-\frac{1}{8\alpha}\mathcal{D}_aG_{bc}\gamma^{ad}\gamma_d{}^{bc}\epsilon^\alpha\\
&\quad+\frac{1}{2}\mathcal{D}_at^I{}_{bc}(\rho_I)^\alpha{}_\beta\gamma^{ad}\gamma_d{}^{bc}\epsilon^\beta+S^{IJ}(\rho_{IJ})^\alpha{}_\beta\gamma^a\mathcal{D}_a\epsilon^\beta-\frac{1}{2\alpha}G_{bc}\gamma^{ab}\gamma^c\mathcal{D}_a\epsilon^\alpha\\
&\quad-\frac{1}{8\alpha}G_{bc}\gamma^{ad}\gamma_d{}^{bc}\mathcal{D}_a\epsilon^\alpha+\frac{1}{2}t^I{}_{bc}(\rho_I)^\alpha{}_\beta\gamma^{ad}\gamma_d{}^{bc}\mathcal{D}_a\epsilon^\beta.
\end{aligned}$$

Comparing them, one obtains the formula

$$\begin{aligned}
&-\frac{2}{5}i\cdot(\Omega\rho_I)_{\alpha\beta}\,\mathrm{tr}\left[\phi^I\cdot\bar{\chi}^\alpha\left(-\frac{1}{4}R(\Omega)\epsilon^\beta\right)\right]\\
&=-\frac{i}{20}\,\mathrm{tr}\Big[\Big(F_{ab}{}^{KL}(\bar{\chi}\cdot\rho_I\rho_{KL}\gamma^{ab}\epsilon)+8\,\mathcal{D}_aS_{KL}(\bar{\chi}\cdot\rho_I\rho^{KL}\gamma^a\epsilon)\\
&\quad-4\,\mathcal{D}_aG_{bc}(\bar{\chi}\cdot\rho_I\gamma^{ab}\gamma^c\epsilon)-\mathcal{D}_aG_{bc}(\bar{\chi}\cdot\rho_I\gamma^{ad}\gamma_d{}^{bc}\epsilon)\\
&\quad+4\,\mathcal{D}_at^J{}_{bc}(\bar{\chi}\cdot\rho_I\rho_J\gamma^{ad}\gamma_d{}^{bc}\epsilon)+8\,S_{KL}(\bar{\chi}\cdot\rho_I\rho^{KL}\gamma^a\mathcal{D}_a\epsilon)\\
&\quad-4\,G_{bc}(\bar{\chi}\cdot\rho_I\gamma^{ab}\gamma^c\mathcal{D}_a\epsilon)-G_{bc}(\bar{\chi}\cdot\rho_I\gamma^{ad}\gamma_d{}^{bc}\mathcal{D}_a\epsilon)\\
&\quad+4\,t^J{}_{bc}(\bar{\chi}\cdot\rho_I\rho_J\gamma^{ad}\gamma_d{}^{bc}\mathcal{D}_a\epsilon)\Big)\phi^I\Big].
\end{aligned}$$

With the help of (7) once more, the last four terms on the right hand of the above equation yield

$$\begin{aligned}
&-\frac{2}{5}iS_{KL}(\bar{\chi}\cdot\rho_I\rho^{KL}\gamma^a\mathcal{D}_a\epsilon)\phi^I+\frac{i}{5\alpha}G_{bc}(\bar{\chi}\cdot\rho_I\gamma^{ab}\gamma^c\mathcal{D}_a\epsilon)\phi^I\\
&\quad+\frac{i}{20\alpha}G_{bc}(\bar{\chi}\cdot\rho_I\gamma^{ad}\gamma_d{}^{bc}\mathcal{D}_a\epsilon)\phi^I-\frac{i}{5}t^I{}_{bc}\phi^J(\bar{\chi}\cdot\rho_J\rho_I\gamma^{ad}\gamma_d{}^{bc}\mathcal{D}_a\epsilon)\\
&=-\frac{i}{2}S_{IJ}S_{MN}\phi^K(\bar{\chi}\cdot\rho_K\rho^{IJ}\rho^{MN}\epsilon)+\frac{9}{20\alpha}iS_{IJ}G_{ab}\phi^K(\bar{\chi}\cdot\rho_K\rho^{IJ}\gamma^{ab}\epsilon)\\
&\quad-\frac{3}{5}iS_{IJ}t^L{}_{ab}\phi^K(\bar{\chi}\cdot\rho_K\rho^{IJ}\rho_L\gamma^{ab}\epsilon)-\frac{i}{40\alpha^2}G_{ab}G_{cd}\phi^I\big(\bar{\chi}\cdot\rho_I(5\gamma^{abcd}+\delta^{ab}\delta^{cd})\epsilon\big)\\
&\quad-\frac{i}{10\alpha}G_{ab}t^I{}_{cd}\phi^J\big(\bar{\chi}\cdot\rho_J\rho_I(8\delta^{ac}\gamma^{bd}+6\delta^{ab}\delta^{bd})\epsilon\big)\\
&\quad+\frac{2}{5}it^I{}_{ab}t^J{}_{cd}\phi^K\big(\bar{\chi}\cdot\rho_K\rho_I\rho_J(\gamma^{abcd}+\delta^{ac}\gamma^{bd}+3\delta^{ac}\delta^{bd})\epsilon\big).
\end{aligned}$$

Using these formulas, it may be more accessible to verify the supersymmetry invariance of both the actions L and S.

Appendix G. Round, squashed, and ellipsoid 3-spheres

A 3-sphere S^3 is given by the set of solutions of $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$ to

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{94}$$

If we describe it in terms of complex variables $(z, w) \in \mathbf{C}^2$ as $z = x_4 + ix_3$, $w = x_2 + ix_1$, since the defining equation becomes $|z|^2 + |w|^2 = 1$, the two by two matrix

$$\begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}$$

yields an element of a Lie group of $SU(2)$. Conversely, any element of the $SU(2)$ group may take the form of the two by two matrix in the fundamental representation. More explicitly, if we introduce polar coordinates (ψ, θ, ϕ) and identify

$$z = e^{\frac{i}{2}(\psi+\phi)} \cos\left(\frac{\theta}{2}\right), \qquad w = e^{\frac{i}{2}(\phi-\psi)} \sin\left(\frac{\theta}{2}\right),$$

where $0 \le \psi \le 4\pi$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, the equivalence of the 3-sphere S^3 to the Lie group $SU(2)$ is understood by the mapping

$$U(\psi, \theta, \phi) = e^{\frac{i}{2}\phi\tau_3} e^{\frac{i}{2}\theta\tau_2} e^{\frac{i}{2}\psi\tau_3} = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}, \tag{95}$$

with the Pauli matrices τ_a $(a = 1, 2, 3)$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is a convenient parametrization for a round and a squashed S^3, as will be seen soon. On the other hand, for an ellipsoid S^3, we use another set of polar coordinates (ϕ, χ, θ) as

$$z = e^{i\varphi} \cos\theta, \qquad w = e^{i\chi} \sin\theta,$$

where $0 \le \varphi \le 2\pi$, $0 \le \chi \le 2\pi$, $0 \le \theta \le \pi/2$, and therefore, the former coordinates are related to the latter as

$$\theta\,|_{\text{former}} = 2\theta\,|_{\text{latter}}\,, \quad \phi\,|_{\text{former}} = (\varphi + \chi)\,|_{\text{latter}}\,, \quad \psi\,|_{\text{former}} = (\varphi - \chi)\,|_{\text{latter}}\,.$$

The mapping U gives the vielbeins $\mu_m^{(0)}$ $(m = 1, 2, 3)$ on a unit round sphere,

$$\mu^{(0)} = \sum_{a=1}^{3} \mu_a^{(0)} \tau^a = \left(\frac{1}{i}\right) U^{-1} dU = \tau_1 \left(\frac{\sin\theta\cos\psi d\phi - \sin\psi d\theta}{2}\right)$$
$$+ \tau_2 \left(\frac{\sin\theta\sin\psi d\phi + \cos\psi d\theta}{2}\right) + \tau_3 \left(\frac{\cos\theta d\phi + d\psi}{2}\right).$$

In terms of the vielbeins, the metric of a round sphere of radius r is given by

$$ds^2 = -r^2 \operatorname{tr}\left[\left(\mu^{(0)}\right)^2\right] = r^2\left[\left(\mu_1^{(0)}\right)^2 + \left(\mu_2^{(0)}\right)^2 + \left(\mu_3^{(0)}\right)^2\right] = (\mu_1)^2 + (\mu_2)^2 + (\mu_3)^2$$

with $\mu_m = r\mu_m^{(0)}$ $(m=1,2,3)$ and the spin connection,

$$\omega_{mn}^{(0)} = \varepsilon_{mnk}\mu_k^{(0)} = \frac{1}{r}\varepsilon_{mnk}\mu_k,$$

for $m,n,k=1,2,3$. The coframe ν^m $(m=1,2,3)$ is the inverse of the vielbein μ_m.

The isometry group of the round sphere is $SO(4) \simeq \left[SU(2)_L \times SU(2)_R\right]/\mathbf{Z}_2$, and it acts on the matrix $U(\psi,\theta,\phi)$ as

$$U(\psi,\theta,\phi) \quad \to \quad g_L \cdot U(\psi,\theta,\phi) \cdot g_R^{-1}$$

for $g_L \in SU(2)_L$ and $g_R \in SU(2)_R$, and one can see that the vielbeins are transformed as

$$\mu^{(0)} \quad \to \quad g_R \cdot \mu^{(0)} \cdot g_R^{-1}.$$

G.1. Killing spinors on a round 3-sphere

The Killing spinor equation on a round 3-sphere is given by

$$\left(d + \frac{1}{4}\omega_{mn}^{(0)}\tau^{mn}\right)\epsilon = \frac{i}{2r}\mu_m\tau^m\epsilon. \tag{96}$$

When ϵ satisfies the Killing equation, the spinor $C_3^{-1}\epsilon^*$ gives another solution to the equation. One solution to the equation is a constant spinor ϵ_0; $d\epsilon_0 = 0$.

Another Killing equation

$$\left(d + \frac{1}{4}\omega_{mn}^{(0)}\tau^{mn}\right)\epsilon = -\frac{i}{2r}\mu_m\tau^m\epsilon,$$

is rewritten into

$$d\epsilon = -i\mu^{(0)}\epsilon = -U^{-1}dU\cdot\epsilon,$$

which is solved by $\epsilon = U^{-1}\epsilon_0$, since one has

$$d(U^{-1}\epsilon_0) = -U^{-1}dU\cdot\left(U^{-1}\epsilon_0\right).$$

In the text, we make frequent use of the constant Killing spinor $\epsilon=\epsilon_0$ and its charge conjugate $\epsilon^c = C_3^{-1}\epsilon_0{}^*$. We normalize them such that $\epsilon^\dagger\epsilon = \epsilon^{c\dagger}\epsilon^c = 1$, and then the Fierz transformation gives

$$\epsilon\,\epsilon^\dagger + \epsilon^c\,\epsilon^{c\dagger} = \mathbf{1}_2. \tag{97}$$

We can make three Killing vectors $\epsilon^\dagger\tau^m\epsilon$, $\epsilon^{c\dagger}\tau^m\epsilon$, and $\epsilon^\dagger\tau^m\epsilon^c$ out of ϵ and ϵ^c; $\left(\epsilon^{c\dagger}\tau_m\epsilon\right)^* = \left(\epsilon^\dagger\tau_m\epsilon^c\right)$, and they obeys

$$\mathcal{D}_m\left(\epsilon^\dagger\tau_n\epsilon\right) = \frac{1}{r}\epsilon_{mnk}\left(\epsilon^\dagger\tau_k\epsilon\right), \qquad \mathcal{D}_m\left(\epsilon^{c\dagger}\tau_n\epsilon\right) = \frac{1}{r}\epsilon_{mnk}\left(\epsilon^{c\dagger}\tau_k\epsilon\right),$$
$$\mathcal{D}_m\left(\epsilon^\dagger\tau_n\epsilon^c\right) = \frac{1}{r}\epsilon_{mnk}\left(\epsilon^\dagger\tau_k\epsilon^c\right),$$

where ϵ_{mnk} is the constant antisymmetric tensor with $\epsilon_{123}=1$. From the norm of ϵ and its Fierz transformation, we can deduce that

$$\begin{aligned}
&\left(\epsilon^\dagger\tau^m\epsilon\right)\left(\epsilon^\dagger\tau^m\epsilon\right)=1, \qquad \left(\epsilon^\dagger\tau^m\epsilon\right)\left(\epsilon^{c\dagger}\tau^m\epsilon\right)=0, \qquad \left(\epsilon^\dagger\tau^m\epsilon\right)\left(\epsilon^\dagger\tau^m\epsilon^c\right)=0,\\
&\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\left(\epsilon^\dagger\tau^m\epsilon^c\right)=2, \qquad \left(\epsilon^{c\dagger}\tau^m\epsilon\right)\left(\epsilon^{c\dagger}\tau^m\epsilon\right)=0, \qquad \left(\epsilon^\dagger\tau^m\epsilon^c\right)\left(\epsilon^\dagger\tau^m\epsilon^c\right)=0,
\end{aligned} \tag{98}$$

and thus they span the three-dimensional space:

$$\left(\epsilon^\dagger\tau^m\epsilon\right)\left(\epsilon^\dagger\tau^n\epsilon\right)+\frac{1}{2}\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\left(\epsilon^\dagger\tau^n\epsilon^c\right)+\frac{1}{2}\left(\epsilon^\dagger\tau^m\epsilon^c\right)\left(\epsilon^{c\dagger}\tau^n\epsilon\right)=\delta^{mn},$$

so that we can expand a vector A_m in terms of the Killing vectors,

$$\begin{aligned}
A_m&=\left(\epsilon^\dagger\tau_m\epsilon\right)\left(\epsilon^\dagger\tau^n\epsilon\right)A_n+\frac{1}{2}\left(\epsilon^{c\dagger}\tau_m\epsilon\right)\left(\epsilon^\dagger\tau^n\epsilon^c\right)A_n+\frac{1}{2}\left(\epsilon^\dagger\tau_m\epsilon^c\right)\left(\epsilon^{c\dagger}\tau^n\epsilon\right)A_n\\
&=\left(\epsilon^\dagger\tau_m\epsilon\right)V_0+\left(\epsilon^{c\dagger}\tau_m\epsilon\right)V_-+\left(\epsilon^\dagger\tau_m\epsilon^c\right)V_+.
\end{aligned}$$

Similarly, the differential operator $\partial_m=\sum_{\mu=1}^{3}v_m{}^\mu\partial_\mu$ is expanded as

$$\partial_m=\left(\epsilon^\dagger\tau_m\epsilon\right)\left(\epsilon^\dagger\tau^n\epsilon\right)\partial_n+\frac{1}{2}\left(\epsilon^{c\dagger}\tau_m\epsilon\right)\left(\epsilon^\dagger\tau^n\epsilon^c\right)\partial_n+\frac{1}{2}\left(\epsilon^\dagger\tau_m\epsilon^c\right)\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\partial_n.$$

Since it satisfies the commutation relation

$$[\partial_m,\,\partial_n]=-\frac{2}{r}\epsilon_{mnk}\partial_k,$$

the covariant derivatives on a scalar field Φ commute with each other,

$$[\mathcal{D}_m,\,\mathcal{D}_n]\,\Phi=[\partial_m,\,\partial_n]\,\Phi+\left[(\omega_m)_n{}^k-(\omega_n)_m{}^k\right]\partial_k\Phi=0.$$

Using the properties

$$\begin{aligned}
&\epsilon^{mnk}\left(\epsilon^\dagger\tau_n\epsilon\right)\left(\epsilon^{c\dagger}\tau_k\epsilon\right)=-i\left(\epsilon^{c\dagger}\tau^m\epsilon\right), \qquad \epsilon^{mnk}\left(\epsilon^\dagger\tau_n\epsilon\right)\left(\epsilon^\dagger\tau_k\epsilon^c\right)=i\left(\epsilon^\dagger\tau^m\epsilon^c\right),\\
&\epsilon^{mnk}\left(\epsilon^{c\dagger}\tau_n\epsilon\right)\left(\epsilon^\dagger\tau_k\epsilon^c\right)=-2i\left(\epsilon^\dagger\tau^m\epsilon\right),
\end{aligned}$$

we can deduce the commutation relations among the differential operators $\left(\epsilon^\dagger\tau^n\epsilon\right)\mathcal{D}_n$, $\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}_n$, and $\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}_n$ on a scalar field,

$$\begin{aligned}
&\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}_m,\,\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}_n\right]\\
&\quad=\left[\left(\epsilon^\dagger\tau^n\epsilon\right)\mathcal{D}_n\left(\epsilon^{c\dagger}\tau^m\epsilon\right)-\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}_n\left(\epsilon^\dagger\tau^m\epsilon\right)\right]\partial_m\\
&\qquad+\left(\epsilon^\dagger\tau^m\epsilon\right)\left(\epsilon^{c\dagger}\tau^n\epsilon\right)[\mathcal{D}_m,\,\mathcal{D}_n]\\
&\quad=-\frac{2}{r}\epsilon_{mnk}\left(\epsilon^\dagger\tau^n\epsilon\right)\left(\epsilon^{c\dagger}\tau^k\epsilon\right)\mathcal{D}_m=\frac{2i}{r}\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m,\\
&\left[\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}_m,\,\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}_n\right]=-\frac{2}{r}\epsilon_{mnk}\left(\epsilon^\dagger\tau^n\epsilon\right)\left(\epsilon^\dagger\tau^k\epsilon^c\right)\mathcal{D}_m=-\frac{2i}{r}\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}_m,\\
&\left[\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}_m,\,\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}_n\right]=-\frac{2}{r}\epsilon_{mnk}\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\left(\epsilon^\dagger\tau^k\epsilon^c\right)\mathcal{D}_m=\frac{4i}{r}\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}_m.
\end{aligned} \tag{99}$$

Therefore, when we regard them as

$$L_3 = -i\frac{r}{2}\left(\epsilon^\dagger \tau^m \epsilon\right)\mathcal{D}_m, \qquad L_+ = -i\frac{r}{2}\left(\epsilon^{c\dagger} \tau^m \epsilon\right)\mathcal{D}_m, \qquad L_- = -i\frac{r}{2}\left(\epsilon^\dagger \tau^m \epsilon^c\right)\mathcal{D}_m,$$

they form the $SU(2)$ algebra,

$$\left[L_3,\, L_\pm\right] = \pm L_\pm, \qquad \left[L_+,\, L_-\right] = 2\,L_3.$$

G.2. *Killing spinors on a squashed 3-sphere*

Let us turn to a squashed 3-sphere. In terms of the Hopf fibration of a 3-sphere, the circle fiber in a squashed S^3 has the different radius $\tilde{r}$ from the radius r of the 2-sphere base, while $\tilde{r} = r$ for a round 3-sphere. The metric of the squashed S^3 is thus given by

$$ds^2 = r^2\left[\left(\mu_1^{(0)}\right)^2 + \left(\mu_2^{(0)}\right)^2\right] + \tilde{r}^2\left(\mu_3^{(0)}\right)^2 = (e_1)^2 + (e_2)^2 + (e_3)^2,$$

where $e_1 = r\mu_1^{(0)}$, $e_2 = r\mu_2^{(0)}$, and $e_3 = \tilde{r}\mu_3^{(0)}$.

Since the vielbeins $\mu^{(0)}$ are still invariant under the $SU(2)_L$ transformations, it shows that the isometry group $SO(4)$ is broken to $[SU(2)_L \times U(1)_R]/\mathbf{Z}_2$.

Here, we assume that $\tilde{r} \geq r$. Solving the equation $de^m + \omega^m{}_n \wedge e^n = 0$, one obtains the Levi-Civita spin connection ω^{ab},

$$(\omega_1)_{23} = (\omega_2)_{31} = \frac{\tilde{r}}{r^2}, \qquad (\omega_3)_{12} = \frac{\tilde{r}}{r^2} + \frac{2}{\tilde{r}}\left(1 - \frac{\tilde{r}^2}{r^2}\right).$$

The curvature tensor of the spin connection ω^{mn} on the squashed 3-sphere S^3

$$R^m{}_n = d\omega^m{}_n + \omega^m{}_k \wedge \omega^k{}_n,$$

is computed to give

$$R^1{}_2 = \frac{\tilde{r}^2}{r^4}\,e^1 \wedge e^2 + \frac{4}{r^2}\left(1 - \frac{\tilde{r}^2}{r^2}\right)e^1 \wedge e^2, \quad R^2{}_3 = \frac{\tilde{r}^2}{r^4}\,e^2 \wedge e^3, \quad R^3{}_1 = \frac{\tilde{r}^2}{r^4}\,e^3 \wedge e^1,$$

and the scalar curvature

$$R = R_{mn}{}^{mn} = \frac{6\tilde{r}^2}{r^4} + \frac{8}{r^2}\left(1 - \frac{\tilde{r}^2}{r^2}\right).$$

G.3. *Constant Killing spinors on a squashed 3-sphere*

The constant spinors $\epsilon = \epsilon_0$ and $\epsilon^c = C_3^{-1}\epsilon^*$ obey

$$\left(d + \frac{1}{4}\omega_{mn}\tau^{mn}\right)\epsilon = +\frac{i}{2}\frac{\tilde{r}}{r^2}e_m\tau^m\epsilon + \frac{i}{\tilde{r}}\left(1 - \frac{\tilde{r}^2}{r^2}\right)e_3\tau_3\,\epsilon,$$
$$\left(d + \frac{1}{4}\omega_{mn}\tau^{mn}\right)\epsilon^c = +\frac{i}{2}\frac{\tilde{r}}{r^2}e_m\tau^m\epsilon^c + \frac{i}{\tilde{r}}\left(1 - \frac{\tilde{r}^2}{r^2}\right)e_3\tau_3\epsilon^c,$$

respectively, with the spin connection ω^{mn} on the squashed 3-sphere. Therefore, if we impose the condition

$$\tau_3\epsilon = \epsilon, \qquad \tau_3\epsilon^c = -\epsilon^c, \tag{100}$$

we may regard the second terms on the right hand sides of the above two Killing spinor equations as the background gauge field:

$$\left[d+\frac{1}{4}\omega_{mn}\tau^{mn}-\frac{i}{\tilde{r}}\left(1-\frac{\tilde{r}^2}{r^2}\right)e_3\right]\epsilon=+\frac{i}{2}\frac{\tilde{r}}{r^2}e_m\tau^m\epsilon,$$
$$\left[d+\frac{1}{4}\omega_{mn}\tau^{mn}+\frac{i}{\tilde{r}}\left(1-\frac{\tilde{r}^2}{r^2}\right)e_3\right]\epsilon^c=+\frac{i}{2}\frac{\tilde{r}}{r^2}e_m\tau^m\epsilon^c,$$

and we may further regard the gauge field as the R-symmetry gauge field, when the spinors are embedded into a single five-dimensional spinor.

On the other hand, regarding the gauge field as a $U(1)$ gauge field,

$$V=-\frac{1}{\tilde{r}}\left(1-\frac{\tilde{r}^2}{r^2}\right)e_3$$

we can see that the Killing spinors ϵ and ϵ^c carry charge 1 and -1, respectively, and so obey

$$\mathcal{D}\epsilon=\left[d+\frac{1}{4}\omega_{mn}\tau^{mn}+iV\right]\epsilon=+\frac{i}{2}\frac{\tilde{r}}{r^2}e_m\tau^m\epsilon,$$
$$\mathcal{D}\epsilon^c=\left[d+\frac{1}{4}\omega_{mn}\tau^{mn}-iV\right]\epsilon^c=+\frac{i}{2}\frac{\tilde{r}}{r^2}e_m\tau^m\epsilon^c.$$

It follows from the Killing spinor equations that

$$[\mathcal{D}_m,\mathcal{D}_n]\epsilon=\left[\frac{1}{4}R_{mn}{}^{kl}\tau_{kl}+iV_{mn}\right]\epsilon=\frac{i}{2}\left(\frac{\tilde{r}}{r^2}\right)^2\epsilon_{mnk}\tau_k\epsilon=\frac{\tilde{r}}{r^2}\epsilon_{mnk}\mathcal{D}_k\epsilon,$$
$$[\mathcal{D}_m,\mathcal{D}_n]\epsilon^c=\left[\frac{1}{4}R_{mn}{}^{kl}\tau_{kl}-iV_{mn}\right]\epsilon^c=\frac{i}{2}\left(\frac{\tilde{r}}{r^2}\right)^2\epsilon_{mnk}\tau_k\epsilon^c=\frac{\tilde{r}}{r^2}\epsilon_{mnk}\mathcal{D}_k\epsilon^c,$$

where V_{mn} are the components of the field strength of the gauge field V,

$$dV=\frac{1}{2}V_{mn}e^m\wedge e^n=-\frac{2}{r^2}\left(1-\frac{\tilde{r}^2}{r^2}\right)e^1\wedge e^2.$$

Multiplying τ^{mn} from the left on both the left and right sides, we obtain

$$\left[\frac{1}{2}R(S^3)-iV_{mn}\tau^{mn}\right]\epsilon=3\left(\frac{\tilde{r}}{r^2}\right)^2\epsilon,\qquad\left[\frac{1}{2}R(S^3)+iV_{mn}\tau^{mn}\right]\epsilon^c=3\left(\frac{\tilde{r}}{r^2}\right)^2\epsilon^c.$$

From the conditions (100), we take

$$\epsilon=\begin{pmatrix}1\\0\end{pmatrix},\qquad\epsilon^c=\begin{pmatrix}0\\1\end{pmatrix},$$

and the Fierz transformation of ϵ and ϵ^c remains the same as in (97).

Three vectors $\epsilon^\dagger\tau^m\epsilon$, $\epsilon^{c\dagger}\tau^m\epsilon$, and $\epsilon^\dagger\tau^m\epsilon^c$ are of charge 0, 2, and -2, respectively under the gauge field V, and obey

$$\mathcal{D}_m\left(\epsilon^\dagger\tau_n\epsilon\right)=\frac{\tilde{r}}{r^2}\epsilon_{mnk}\left(\epsilon^\dagger\tau_k\epsilon\right),\qquad\mathcal{D}_m\left(\epsilon^{c\dagger}\tau_n\epsilon\right)=\frac{\tilde{r}}{r^2}\epsilon_{mnk}\left(\epsilon^{c\dagger}\tau_k\epsilon\right),$$
$$\mathcal{D}_m\left(\epsilon^\dagger\tau_n\epsilon^c\right)=\frac{\tilde{r}}{r^2}\epsilon_{mnk}\left(\epsilon^\dagger\tau_k\epsilon^c\right),$$

where the covariant derivatives contain the gauge field V as a connection according to their charges. They satisfy (98) and form an orthonormal basis

$$\left(\epsilon^\dagger\tau^m\epsilon\right)\left(\epsilon^\dagger\tau^n\epsilon\right)+\frac{1}{2}\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\left(\epsilon^\dagger\tau^n\epsilon^c\right)+\frac{1}{2}\left(\epsilon^\dagger\tau^m\epsilon^c\right)\left(\epsilon^{c\dagger}\tau^n\epsilon\right)=\delta^{mn},$$

and obey the relations of the cross product,

$$\epsilon^{mnk}\left(\epsilon^\dagger\tau_n\epsilon\right)\left(\epsilon^{c\dagger}\tau_k\epsilon\right)=-i\left(\epsilon^{c\dagger}\tau^m\epsilon\right),\qquad \epsilon^{mnk}\left(\epsilon^\dagger\tau_n\epsilon\right)\left(\epsilon^\dagger\tau_k\epsilon^c\right)=i\left(\epsilon^\dagger\tau^m\epsilon^c\right),$$
$$\epsilon^{mnk}\left(\epsilon^{c\dagger}\tau_n\epsilon\right)\left(\epsilon^\dagger\tau_k\epsilon^c\right)=-2i\left(\epsilon^\dagger\tau^m\epsilon\right). \tag{101}$$

We will denote the covariant derivative $\mathcal{D}$ on a scalar field Φ of charge q under the gauge field V as

$$\mathcal{D}^{(q)}{}_m\Phi=\partial_m\Phi+iqV_m\Phi.$$

The commutation relations of the differential operators $\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(q)}{}_m$, $\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(q)}{}_m$, and $\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(q)}{}_m$ essentially remains the same as in (99), if $1/r$ is replaced by $\tilde{r}/r^2$:

$$\begin{aligned}
&\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(q)}{}_n\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(q)}{}_m\\
&\quad=\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(q-2)}{}_m\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(q)}{}_n+2i\frac{\tilde{r}}{r^2}\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(q)}{}_n,\\
&\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}^{(q)}{}_n\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(q)}{}_m\\
&\quad=\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(q+2)}{}_m\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}^{(q)}{}_n-2i\frac{\tilde{r}}{r^2}\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}^{(q)}{}_n,\\
&\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}^{(q-2)}{}_n\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(q)}{}_m\\
&\quad=\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(q+2)}{}_m\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}^{(q)}{}_n+4i\frac{\tilde{r}}{r^2}\left(\epsilon^\dagger\tau^n\epsilon\right)\mathcal{D}^{(q)}{}_n+q\epsilon_{mnk}\left(\epsilon^\dagger\tau^m\epsilon\right)V_{nk},
\end{aligned} \tag{102}$$

except for the last term on the right hand side in the last equation.

G.4. Non-constant Killing spinors on a squashed 3-sphere

Similarly to the constant spinors $\epsilon=\epsilon_0$ and $C_3^{-1}\epsilon_0^*$, the spinor $\epsilon=U^{-1}\epsilon_0$ on the squashed 3-sphere gives the solution to the differential equation

$$\left(d+\frac{1}{4}\omega_{mn}\tau^{mn}\right)\epsilon=-\frac{i}{2r}\left(2-\frac{\tilde{r}}{r}\right)e_m\tau^m\epsilon+\frac{i}{r}\left(1-\frac{\tilde{r}}{r}\right)e_3\tau_3\,\epsilon.$$

However, the computation of the partition function with this Killing spinor would give the same result as with the constant spinor ϵ_0, and this Killing spinor isn't used in this paper.

As discussed in [22], there is another Killing spinor which is given by a Lorentz transform of $U^{-1}\epsilon_0$. As explained in Subsection 5.4, the slant periodic boundary condition is rotated to the time direction upon the dimensional reduction from the six-dimensional theory. The rotation induces the Lorentz transformation on the coframe e^m. This is quite parallel to the three-dimensional stray in [22]. Therefore, one obtains the Lorentz transformed Killing spinors on the squashed 3-sphere with the supersymmetry background in Subsection 5.4, which is five-dimensional analogues of a three-dimensional Killing spinor given in the paper [22].

It has been discussed in [22] that the spinor $\epsilon = e^{(1/2)\xi\tau_3}U^{-1}\epsilon_0$ with $\cosh\xi = \tilde{r}/r$ and $\sinh\xi = \sqrt{(\tilde{r}/r)^2 - 1}$ is the solution to

$$\left(d + \frac{1}{4}\omega_{mn}\tau^{mn}\right)\epsilon = -\frac{i}{2}\frac{\tilde{r}}{r^2}e_m\tau^m\epsilon + \frac{1}{r}\sqrt{\frac{\tilde{r}^2}{r^2} - 1}\,\varepsilon_{3mn}e^m\tau^n\epsilon,$$

and its charge conjugate $C_3^{-1}\epsilon^*$ satisfies

$$\left(d + \frac{1}{4}\omega_{mn}\tau^{mn}\right)(C_3^{-1}\epsilon^*) = -\frac{i}{2}\frac{\tilde{r}}{r^2}e_m\tau^m(C_3^{-1}\epsilon^*) - \frac{1}{r}\sqrt{\frac{\tilde{r}^2}{r^2} - 1}\,\varepsilon_{3mn}e^m\tau^n(C_3^{-1}\epsilon^*).$$

We have explained in Subsection 5.4 that the slant boundary condition with the parameter u on the round 3-sphere is transformed via the change of coordinates to the periodic boundary condition on the squashed 3-sphere with the fiber radius $\tilde{r}$ and the base radius r, as have been discussed in [22]. We have introduced the intermediate parameter ξ in Subsection 5.4, which explicitly appears in the Killing spinors. It could be convenient to summarize the relations of the parameters u, ξ and the ratio $\tilde{r}/r$ as

$$\cosh\xi = \frac{\tilde{r}}{r} = \frac{1}{\sqrt{1-u^2}}, \quad \sinh\xi = \frac{u}{\sqrt{1-u^2}},$$
$$e^{\pm\xi/2} = \sqrt{\frac{\tilde{r}+r}{2r}} \pm \sqrt{\frac{\tilde{r}-r}{2r}} = \left(\frac{1\pm u}{1\mp u}\right)^{\frac{1}{4}}.$$

G.5. *Killing spinors on an ellipsoid 3-sphere*

In order to obtain an ellipsoid S^3, following [21], we will deform the defining equation (94) to give

$$\frac{x_1^2 + x_2^2}{\tilde{r}^2} + \frac{x_3^2 + x_4^2}{r^2} = 1,$$

with the flat metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

Substituting the polar coordinates

$$z = x_4 + ix_3 = re^{i\varphi}\cos\theta, \qquad w = x_2 + ix_1 = \tilde{r}e^{i\chi}\sin\theta,$$

where $0 \le \varphi \le 2\pi$, $0 \le \chi \le 2\pi$, $0 \le \theta \le \pi/2$, into the metric, one obtains

$$ds^2 = \left(e^1\right)^2 + \left(e^2\right)^2 + \left(e^3\right)^2,$$

where the dreibeins $e^{1,2,3}$ are

$$e^1 = r\cos\theta\, d\varphi, \quad e^2 = \tilde{r}\sin\theta\, d\chi, \quad e^3 = f(\theta)d\theta,$$

with $f(\theta)=\sqrt{\tilde{r}^2\cos^2\theta+r^2\sin^2\theta}$. The coframes, which are the dual basis to the dreibeins, are given by

$$\theta_1=\frac{1}{r\cos\theta}\frac{\partial}{\partial\varphi},\qquad \theta_2=\frac{1}{\tilde{r}\sin\theta}\frac{\partial}{\partial\chi},\qquad \theta_3=\frac{1}{f(\theta)}\frac{\partial}{\partial\theta}.$$

The compatible spin connection is given by

$$\omega^{12}=0,\quad \omega^{23}=\frac{1}{f(\theta)\tan\theta}e^2,\quad \omega^{31}=\frac{\tan\theta}{f(\theta)}e^1,$$

and the curvature tensor $R^m{}_n=d\omega^m{}_n+\omega^m{}_k\wedge\omega^k{}_n$ by

$$R^{12}=\frac{1}{f(\theta)^2}e^1\wedge e^2,\quad R^{23}=\frac{r^2}{f(\theta)^4}e^2\wedge e^3,\quad R^{31}=\frac{\tilde{r}^2}{f(\theta)^4}e^3\wedge e^1,$$

and therefore one obtains the scalar curvature

$$R(S^3_{ell})=\frac{2}{f(\theta)^4}\left(f(\theta)^2+r^2+\tilde{r}^2\right).$$

When $r=\tilde{r}=f$, the ellipsoid S^3 becomes a round one, and then spinors

$$\epsilon=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{\frac{i}{2}(\chi-\varphi+\theta)}\\ -e^{\frac{i}{2}(\chi-\varphi-\theta)}\end{pmatrix},\qquad \epsilon^c=C_3^{-1}\epsilon^*=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-\frac{i}{2}(\chi-\varphi-\theta)}\\ e^{-\frac{i}{2}(\chi-\varphi+\theta)}\end{pmatrix},$$

obey the Killing spinor equations on the round S^3

$$\left(d+\frac{1}{4}\omega^{mn}\tau_{mn}\right)\epsilon=\frac{i}{2r}\mu^m\tau_m\epsilon,\qquad \left(d+\frac{1}{4}\omega^{mn}\tau_{mn}\right)\epsilon^c=\frac{i}{2r}\mu^m\tau_m\epsilon^c.$$

In order to keep these Killing spinors on the ellipsoid S^3 with $\tilde{r}\neq r$, turning on a background $U(1)$ gauge field

$$V=\frac{1}{2}\left[-\frac{1}{r\cos\theta}\left(\frac{r}{f}-1\right)e^1+\frac{1}{\tilde{r}\sin\theta}\left(\frac{\tilde{r}}{f}-1\right)e^2\right],\tag{103}$$

as in [21], one can see that they satisfy

$$D\epsilon\equiv\left(d+\frac{1}{4}\omega^{mn}\tau_{mn}+iV\right)\epsilon=\frac{i}{2f}e^m\tau_m\epsilon,$$
$$D\epsilon^c\equiv\left(d+\frac{1}{4}\omega^{mn}\tau_{mn}-iV\right)\epsilon^c=\frac{i}{2f}e^m\tau_m\epsilon^c.$$

The use of the Killing spinor equations twice for $DD\epsilon$ and $DD\epsilon^c$ leads to

$$\begin{aligned}\left(\frac{1}{2}R-i\tau^{mn}V_{mn}\right)\epsilon&=\left(\frac{3}{f^2}+\frac{2i}{f^2}\tau^m\partial_m f\right)\epsilon,\\ \left(\frac{1}{2}R+i\tau^{mn}V_{mn}\right)\epsilon^c&=\left(\frac{3}{f^2}+\frac{2i}{f^2}\tau^m\partial_m f\right)\epsilon^c,\end{aligned}\tag{104}$$

where V_{mn} is the field strength of V,

$$dV=\frac{1}{2}V_{mn}e^m\wedge e^n,$$

and the partial differential $\partial_m f$ is defined by $df=e^m\partial_m f$.

As on the squashed S^3 in Subsection G.3, three vectors $\epsilon^\dagger\tau^m\epsilon$, $\epsilon^{c\dagger}\tau^m\epsilon$, and $\epsilon^\dagger\tau^m\epsilon^c$ are of charge 0, 2, and -2, respectively under the gauge field V. They obey

$$\mathcal{D}_m\left(\epsilon^\dagger\tau_n\epsilon\right)=\frac{1}{f}\epsilon_{mnk}\left(\epsilon^\dagger\tau_k\epsilon\right),\qquad \mathcal{D}_m\left(\epsilon^{c\dagger}\tau_n\epsilon\right)=\frac{1}{f}\epsilon_{mnk}\left(\epsilon^{c\dagger}\tau_k\epsilon\right),$$
$$\mathcal{D}_m\left(\epsilon^\dagger\tau_n\epsilon^c\right)=\frac{1}{f}\epsilon_{mnk}\left(\epsilon^\dagger\tau_k\epsilon^c\right),$$

where the covariant derivatives contain the gauge field V as a connection according to their charges. They satisfy (98) and give an orthonormal basis

$$\left(\epsilon^\dagger\tau^m\epsilon\right)\left(\epsilon^\dagger\tau^n\epsilon\right)+\frac{1}{2}\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\left(\epsilon^\dagger\tau^n\epsilon^c\right)+\frac{1}{2}\left(\epsilon^\dagger\tau^m\epsilon^c\right)\left(\epsilon^{c\dagger}\tau^n\epsilon\right)=\delta^{mn},$$

and obey the same relations of the cross product,

$$\epsilon^{mnk}\left(\epsilon^\dagger\tau_n\epsilon\right)\left(\epsilon^{c\dagger}\tau_k\epsilon\right)=-i\left(\epsilon^{c\dagger}\tau^m\epsilon\right),\qquad \epsilon^{mnk}\left(\epsilon^\dagger\tau_n\epsilon\right)\left(\epsilon^\dagger\tau_k\epsilon^c\right)=i\left(\epsilon^\dagger\tau^m\epsilon^c\right),$$
$$\epsilon^{mnk}\left(\epsilon^{c\dagger}\tau_n\epsilon\right)\left(\epsilon^\dagger\tau_k\epsilon^c\right)=-2i\left(\epsilon^\dagger\tau^m\epsilon\right), \tag{105}$$

as in (101) for the squashed S^3.

More concretely, we have

$$\left(\left(\epsilon^\dagger\tau^1\epsilon\right),\left(\epsilon^\dagger\tau^2\epsilon\right),\left(\epsilon^\dagger\tau^3\epsilon\right)\right)=(-\cos\theta,\,\sin\theta,\,0),$$
$$\left(\left(\epsilon^{c\dagger}\tau^1\epsilon\right),\left(\epsilon^{c\dagger}\tau^2\epsilon\right),\left(\epsilon^{c\dagger}\tau^3\epsilon\right)\right)=ie^{i(\chi-\varphi)}(\sin\theta,\,\cos\theta,\,-i),$$
$$\left(\left(\epsilon^\dagger\tau^1\epsilon^c\right),\left(\epsilon^\dagger\tau^2\epsilon^c\right),\left(\epsilon^\dagger\tau^3\epsilon^c\right)\right)=-ie^{-i(\chi-\varphi)}(\sin\theta,\,\cos\theta,\,i),$$

and it implies that

$$\left(\epsilon^\dagger\tau^m\epsilon\right)\partial_m f=0, \tag{106}$$

where ∂_m is defined such that $d=e^m\partial_m$.

We will denote the covariant derivative $\mathcal{D}$ on a scalar field Φ of charge q under the gauge field V as

$$\mathcal{D}^{(q)}{}_m\Phi=\partial_m\Phi+iqV_m\Phi.$$

The commutation relations of the differential operators $\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(q)}{}_m$, $\left(\epsilon^{c\dagger}\tau^m\epsilon\right)\mathcal{D}^{(q)}{}_m$, and $\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(q)}{}_m$ slightly differ from (102) by the terms with the field strength V_{mn}, even if $\tilde{r}/r^2$ is replaced by $1/f$, and using (104), we will replace them by the terms including f to give

$$\begin{aligned}&\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(q)}{}_n\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(q)}{}_m\\&\quad=\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(q-2)}{}_m\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(q)}{}_n+\frac{2i}{f}\left(\epsilon^\dagger\tau^n\epsilon^c\right)\mathcal{D}^{(q)}{}_n+iq\left(\epsilon^\dagger\tau^n\epsilon^c\right)\partial_n\frac{1}{f},\\&\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}^{(q)}{}_n\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(q)}{}_m\\&\quad=\left(\epsilon^\dagger\tau^m\epsilon\right)\mathcal{D}^{(q+2)}{}_m\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}^{(q)}{}_n-\frac{2i}{f}\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}^{(q)}{}_n,+iq\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\partial_n\frac{1}{f},\end{aligned}$$

$$\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}^{(q-2)}{}_n\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(q)}{}_m$$
$$=\left(\epsilon^\dagger\tau^m\epsilon^c\right)\mathcal{D}^{(q+2)}{}_m\left(\epsilon^{c\dagger}\tau^n\epsilon\right)\mathcal{D}^{(q)}{}_n+\frac{4i}{f}\left(\epsilon^\dagger\tau^n\epsilon\right)\mathcal{D}^{(q)}{}_n+q\epsilon_{mnk}\left(\epsilon^\dagger\tau^m\epsilon\right)V_{nk}, \tag{107}$$

except for the last term on the right hand side in the last equation.

Let us consider the scalar spherical harmonics in the coordinates (χ,θ,φ) in the round limit $\tilde{r}\to r$ and set the radius to be unity; $r=1$. The mapping from (χ,θ,φ) to the 3-sphere is

$$U=e^{\frac{i}{2}(\chi+\varphi)\tau_3}\,e^{i\theta\tau_2}\,e^{\frac{i}{2}(\varphi-\chi)\tau_3},$$

and the $SU(2)_{\mathrm{L}}\times SU(2)_{\mathrm{R}}$ isometry of the round 3-sphere is given by $g_{\mathrm{L}}\in SU(2)_{\mathrm{L}}$ and $g_{\mathrm{R}}\in SU(2)_{\mathrm{R}}$ as

$$U\quad\to\quad g_{\mathrm{L}}\cdot U\cdot g_{\mathrm{R}}^{-1}.$$

The left- and the right-invariant one-forms are given by

$$\frac{1}{i}U^{-1}dU=(\tau_1+i\tau_2)\,e^{-}{}_{(\mathrm{L})}+(\tau_1-i\tau_2)\,e^{+}{}_{(\mathrm{L})}+\tau_3e^{3}{}_{(\mathrm{L})},$$
$$\frac{1}{i}dU\cdot U^{-1}=(\tau_1+i\tau_2)\,e^{-}{}_{(\mathrm{R})}+(\tau_1-i\tau_2)\,e^{+}{}_{(\mathrm{R})}+\tau_3e^{3}{}_{(\mathrm{R})},$$

where $e^a_{(\mathrm{L})}$, $e^a_{(\mathrm{R})}$ $(a=+,-,3)$ play the role of the dreibeins, respectively. Their coframes which are dual to $e^a_{(\mathrm{L,R})}$, respectively, are

$$\theta^{(\mathrm{L})}_{\pm}=e^{\mp i(\varphi-\chi)}\left(\theta_1\sin\theta+\theta_2\cos\theta\mp i\theta_3\right),\qquad\theta^{(\mathrm{L})}_3=\left(\theta_1\cos\theta-\theta_2\sin\theta\right),$$
$$\theta^{(\mathrm{R})}_{\pm}=e^{\pm i(\varphi+\chi)}\left(-\theta_1\sin\theta+\theta_2\cos\theta\mp i\theta_3\right),\qquad\theta^{(\mathrm{R})}_3=\left(\theta_1\cos\theta+\theta_2\sin\theta\right).$$

Then, identifying these coframes as

$$L_\pm=-\frac{1}{2i}\theta^{\mathrm{L}}_\pm,\quad L_3=-\frac{1}{2i}\theta^{\mathrm{L}}_3=\frac{1}{2i}\left(\frac{\partial}{\partial\chi}-\frac{\partial}{\partial\varphi}\right);$$
$$\tilde{L}_\pm=\frac{1}{2i}\theta^{\mathrm{R}}_\pm,\quad \tilde{L}_3=\frac{1}{2i}\theta^{\mathrm{R}}_3=\frac{1}{2i}\left(\frac{\partial}{\partial\chi}+\frac{\partial}{\partial\varphi}\right),$$

we can see that they form the Lie algebras of $SU(2)$:

$$\left[L_3,\,L_\pm\right]=\pm L_\pm,\quad\left[L_+,\,L_-\right]=2L_3;$$
$$\left[\tilde{L}_3,\,\tilde{L}_\pm\right]=\pm\tilde{L}_\pm,\quad\left[\tilde{L}_+,\,\tilde{L}_-\right]=2\tilde{L}_3.$$

Furthermore, a simple calculation shows that

$$L^2=\frac{1}{2}\left(L_+L_-+L_-L_+\right)+L_3L_3=\frac{1}{2}\left(\tilde{L}_+\tilde{L}_-+\tilde{L}_-\tilde{L}_+\right)+\tilde{L}_3\tilde{L}_3=\tilde{L}^2.$$

A spherical harmonics f_n corresponding to one of the highest weight states defined by

$$L_+f_n=\tilde{L}_+f_n=0,$$

is obtained by

$$f_n=c_ne^{in\chi}\sin^n\theta,\quad\text{with}\quad c_n=\frac{n+1}{2\pi^2},$$

where n is an integer required by the periodicity under $\chi \to \chi + 2\pi$ and is non-negative required by the normalizability of f_n; $n \in \mathbf{Z}_{\geq 0}$. It has the eigenvalues ($l = n/2, m = n/2, \tilde{m} = n/2$) for $L^2 = \tilde{L}^2$, L_3 and $\tilde{L}_3$,

$$L^2 f_n = \tilde{L}^2 f_n = \left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right) f_n, \qquad L_3 f_n = \frac{n}{2} f_n, \qquad \tilde{L}_3 f_n = \frac{n}{2} f_n,$$

which corresponds to the spherical harmonics $\varphi_{l,m,\tilde{m}} = \varphi_{n/2,n/2,n/2}$.

Therefore, a state corresponding to $\varphi_{l,l,\tilde{m}}$ is given by

$$\tilde{L}_-^n f_{n+m} = \frac{(m+n)!}{m!} c_{m+n} e^{im\chi - in\varphi} \cos^n\theta \cdot \sin^m\theta,$$

with[26] $m = l + \tilde{m}$ and $n = l - \tilde{m}$. The integers m and n run from 0 to ∞. A further calculation yields

$$L_-^k\left(\tilde{L}_-^n f_{n+m}\right) = (-)^k \frac{(m+n)!}{m!} c_{m+n} e^{i(m-k)\chi - i(n-k)\varphi} \frac{1}{\cos^{n-k}\theta \cdot \sin^{m-k}\theta} \times \left(\frac{1}{\sin 2\theta}\frac{d}{d\theta}\right)^k \cos^{2n}\theta \cdot \sin^{2m}\theta,$$

where the integer k runs from 0 to $n+m$. Multiplying it by an appropriate normalization constant, we refer to it as $h_{n,m,k}$ ($n, m = 0, 1, 2, 3, \cdots; k = 0, 1, 2, \cdots, n+m$).

G.6. From Killing spinors on 3-spheres to 5-dimensional Killing spinors

Introducing the basis of two-dimensional spinors

$$\zeta_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = (\zeta_\mp)^*, \qquad \tau_2 \zeta_\pm = \pm\zeta_\pm,$$

and combining them with the Killing spinors ϵ, $C^{-1}\epsilon^*$ on the 3-sphere to give the five-dimensional spinors

$$\epsilon \otimes \zeta_\pm, \qquad C_3^{-1}\epsilon^* \otimes \zeta_\pm.$$

The action of $i\gamma^{45} = \mathbf{1}_2 \otimes \tau_2$ on these spinors can be seen as

$$\gamma^{45}(\epsilon \otimes \zeta_\pm) = \mp i(\epsilon \otimes \zeta_\pm), \qquad \gamma^{45}(C_3^{-1}\epsilon^* \otimes \zeta_\pm) = \mp i(C_3^{-1}\epsilon^* \otimes \zeta_\pm),$$

and for the gamma matrices γ_a ($a = 1, 2, 3$),

$$\begin{aligned} &\gamma_a(\epsilon \otimes \zeta_\pm) = \pm\tau_a\epsilon \otimes \zeta_\pm, \\ &\gamma_a\left(C_3^{-1}\epsilon^* \otimes \zeta_\pm\right) = \pm\tau_a C_3^{-1}\epsilon^* \otimes \zeta_\pm = \mp C_3^{-1}(\tau_a)^*\epsilon^* \otimes \zeta_\pm \quad (a = 1, 2, 3) \end{aligned}$$

In the text, we assume that the Killing spinors $\varepsilon^{\dot{\alpha}}$ obey

$$i\gamma_{45}\varepsilon^{\dot{\alpha}} = (\tau_3)^{\dot{\alpha}}{}_{\dot{\beta}}\varepsilon^{\dot{\beta}},$$

or more explicitly, we make the ansatz

$$\varepsilon^{\dot{1}} = \epsilon \otimes \zeta_+, \qquad \varepsilon^{\dot{2}} = C_3^{-1}\epsilon^* \otimes \zeta_-.$$

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[26] Note that the integer m here is not the eigenvalue of L_3.

G.7. Auxiliary $SU(2)$ flavor spinors

In order to construct the off-shell supersymmetry of the five-dimensional theory, besides the supersymmetry parameter $\varepsilon^{\dot{\alpha}}$, we need to introduce an additional supersymmetry parameter $\varepsilon^{\check{\alpha}}$ ($\check{\alpha} = 1, 2$), with the index $\check{\alpha}$ of an additional $SU(2)$ flavor group which are not an subgroup of the $\mathrm{Spin}(5)_R$ R-symmetry group.

In this paper, we take the supersymmetry parameter $\varepsilon^{\dot{\alpha}}$ in the form

$$\varepsilon^{\dot{1}} = \epsilon \otimes \zeta_+, \qquad \varepsilon^{\dot{2}} = C_3^{-1}\epsilon^* \otimes \zeta_-,$$

where ϵ is an two-dimensional spinor. By a simple examination, one can see that ϵ and $C_3^{-1}\epsilon^*$ are linearly independent. For this parameter $\varepsilon^{\dot{\alpha}}$, we take $\varepsilon^{\check{\alpha}}$ to be

$$\varepsilon^{\check{1}} = \epsilon \otimes \zeta_-, \qquad \varepsilon^{\check{2}} = C_3^{-1}\epsilon^* \otimes \zeta_+; \qquad \varepsilon^{\check{\alpha}} = \gamma^5 \varepsilon^{\dot{\alpha}}$$

so that they obey

$$i\gamma^{45}\varepsilon^{\check{\alpha}} = -\,(\tau_3)^{\check{\alpha}}{}_{\check{\beta}}\varepsilon^{\check{\beta}}.$$

Since the two-dimensional spinors $\zeta_\pm$ are linearly independent, these supersymmetry parameters $\varepsilon^{\dot{\alpha}}$, $\varepsilon^{\check{\alpha}}$ form the basis of the four-dimensional linear space of five-dimensional spinors.

The covariant derivative $\mathcal{D}_\mu\varepsilon^{\check{\alpha}}$ is defined by

$$\mathcal{D}_\mu\varepsilon^{\check{\alpha}} = \partial_\mu\varepsilon^{\check{\alpha}} + \frac{1}{4}\omega_\mu{}^{bc}\gamma_{bc}\,\varepsilon^{\check{\alpha}} - \frac{1}{4}\check{A}_\mu{}^{ij}\left(\bar{\sigma}_{ij}\right)^{\check{\alpha}}{}_{\check{\beta}}\varepsilon^{\check{\beta}},$$

where we assume that the gauge field $\check{A}_\mu{}^{ij}$ takes the same as $A_\mu{}^{ij}$: $\check{A}_\mu{}^{ij} = A_\mu{}^{ij}$.

Appendix H. Dictionary among notations

In this paper, we follow the same procedure as in [24] to obtain five-dimensional $\mathcal{N} = 2$ supersymmetric Yang–Mills theory in the supergravity background, but our notations are different from the ones in [24]. The procedure is based on the dimensional reduction of the off-shell supergravity in [25], where different notations from [24] and ours however, are used. Therefore, the list of the different notations among [25,24] and ours may be convenient for the readers.

We use the common Lorentz metric with the signature $(-, +, \cdots, +)$. However, the indices of Lorentz vectors are different; for a Lorentz vector $\underline{V}^{\underline{a}}$ in our notations, it is V^a in [25] and $V^{\underline{a}}$ in [24]. Note here that the underline $\underline{}$ indicates that it is a six-dimensional one in our paper. That's why the above V carries the underline $\underline{V}$ in ours, but not in [25,24]. However, the underline $\underline{}$ carried by the Lorentz indices or the coordinate frame indices means that they are six-dimensional in [24] and ours, but not in [25].

We use the common algebra of the gamma matrices of the Lorentz group, but the different notations of the gamma matrices are used, as listed in Table 6. The properties (90) of our charge conjugation matrix $\underline{C}$ are also enjoyed by C in [25] and $\underline{C}$ in [24]. However, since our definition of the conjugate $\underline{\bar{\psi}}_\alpha$ of spinors is different from the ones in [25,24] by $i = \sqrt{-1}$; $\bar{\psi}^i \to i\underline{\bar{\psi}}^\alpha$, as is seen in Table 6, our charge conjugation matrix $\underline{C}$ multiplied by i is equal to the ones in [25,24].

Table 6
Dictionary of notations in [25,24] and ours.

Bergshoeff et al. [25]	Cordova–Jafferis [24]	Ours
The Lorentz vector indices		
a, b	$\underline{a}, \underline{b}$	$\underline{a}, \underline{b}$
The $\mathrm{Spin}(5)_R \simeq \mathrm{USp}(4) \simeq \mathrm{sp}(4)_R$ spinor indices		
i, j	m, n	α, β
The $\mathrm{Spin}(5)_R \simeq \mathrm{USp}(4) \simeq \mathrm{sp}(4)_R$ vector indices		
		I, J
The $\mathrm{Spin}(5)_R$ charge conjugation matrix $\Leftrightarrow$ the $\mathrm{USp}(4) \simeq \mathrm{sp}(4)_R$ invariant metric		
Ω_{ij}	Ω_{mn}	$\Omega_{\alpha\beta}$
The gamma matrices of the Lorentz group		
γ^a	$\Gamma^{\underline{a}}$	$\underline{\Gamma}^{\underline{a}}$
The charge conjugation matrix of the Lorentz group		
C	$\underline{C}$	$i\underline{C}$
The spinor indices of $\mathrm{Spin}(5)_R$		
$\psi^i = \Omega^{ij}\psi_j$, $\psi_i = \psi^j\Omega_{ji}$	$\psi^m = \Omega^{mn}\psi_n$, $\psi_m = \psi^n\Omega_{nm}$	$\psi^\alpha = \Omega^{\alpha\beta}\psi_\beta$, $\psi_\alpha = \psi^\beta\Omega_{\beta\alpha}$
The conjugate of a spinor $\underline{\psi}^\alpha$		
$\bar{\psi}^i = i(\psi_i)^\dagger\gamma^0 = \left(\psi^i\right)^T C$	$\bar{\psi}^m = i(\psi_m)^\dagger\Gamma^0 = (\psi^m)^T \underline{C}$	$\underline{\bar{\psi}}_\alpha = \left(\underline{\psi}^\alpha\right)^\dagger \underline{\Gamma}^0 = \left(\underline{\psi}^\beta\right)^T \underline{C}\Omega_{\beta\alpha}$
The local Lorentz (dependent) gauge field $\underline{\Omega_\mu}^{\underline{ab}}$		
${\omega_\mu}^{ab}$	${\omega_{\underline{\mu}}}^{\underline{ab}}$	$\underline{\Omega_\mu}^{\underline{ab}}$
The $\mathrm{Spin}(5)_R$ gauge field $\underline{V_\mu}^{IJ}$		
${V_\mu}^{ij} = {V_\mu}^{ji}$	${V_{\underline{\mu}}}^{mn} = {V_{\underline{\mu}}}^{nm}$	${\underline{V_\mu}^\alpha}_\beta = (1/2)\underline{V_\mu}^{IJ}{(\rho_{IJ})^\alpha}_\beta$, $\underline{V_\mu}^{IJ} = -\underline{V_\mu}^{JI}$
The auxiliary field ${\underline{M}^{\alpha\beta}}_{\gamma\delta}$		
${D^{ij}}_{kl}$	${D^{mn}}_{rs}$	${\underline{M}^{\alpha\beta}}_{\gamma\delta}$
The scalar field $\underline{\phi}^{\alpha\beta}$ of the tensor multiplet		
ϕ^{ij}	Φ^{mn}	$\underline{\phi}^{\alpha\beta}$
The scalar spinor $\underline{\chi}^\alpha$ of the tensor multiplet		
ψ^i	ϱ^m	$\underline{\chi}^\alpha$

The spinor indices α, β of the $\mathrm{Spin}(5)_R$ symmetry in our paper are i, j in [25] and m, n in [24], and they run from 1 to 4. Only in our paper, the vector indices I, J of the $\mathrm{Spin}(5)_R$ are introduced. Since there is the equivalence $\mathrm{Spin}(5) \simeq \mathrm{USp}(4)$, the $\mathrm{Spin}(5)_R$ group is referred to as USp(4) in [25], and its Lie algebra as $\mathrm{sp}(4)_R$ in [24]. The invariant metric Ω_{ij} of USp(4) in [25] is Ω_{mn} in [24] and is the charge conjugation matrix $\Omega_{\alpha\beta}$ of $\mathrm{Spin}(5)_R$ in ours.

As shown in Table 6, the gauge field $\underline{\Omega_\mu}^{\underline{ab}}$ of the local Lorentz symmetry is referred to as ${\omega_\mu}^{ab}$ in [25] and ${\omega_{\underline{\mu}}}^{\underline{ab}}$ in [24]. The curvature tensor of the gauge field ${\omega_\mu}^{ab}$ is defined in [25] by

$$R_{\mu\nu}{}^{ab}(M) = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_{\nu c}{}^{b} - \omega_\nu{}^{ac}\omega_{\mu c}{}^{b} \\ \underline{-2\left(f_\mu{}^a e_\nu{}^b - f_\mu{}^b e_\nu{}^a - f_\nu{}^a e_\mu{}^b + f_\nu{}^b e_\mu{}^a\right)} + \cdots ,$$

with the ellipses $\cdots$ denoting the contribution from the fermionic fields, following the usual construction of the curvature tensors in the conformal tensor calculus [26]. Upon the deformation of the superconformal algebra, the gauge field $f_\mu{}^a$ becomes dependent [25]:

$$f_\mu{}^a = -\frac{1}{8} R'_\mu{}^a(M) + \frac{1}{80} e_\mu{}^a R'(M) + \frac{1}{32} T^{ij}{}_{\mu cd} T_{ij}{}^{acd},$$

in their notations, where the curvature $R'_{\mu\nu}{}^{ab}(M)$ is $R_{\mu\nu}{}^{ab}(M)$ with the underlined terms omitted in the above. The Ricci curvature $R'_\mu{}^a(M) = e^\nu{}_b R'_{\mu\nu}{}^{ba}(M)$ and the scalar curvature $R'(M) = e^\mu{}_a R'_\mu{}^a(M)$ are also defined in [25].

In the notation of [24], $\underline{R}$ is the scalar curvature of $\omega_{\underline{\mu}}{}^{\underline{ab}}$ and is equal to $R'(M)$ with the fermionic contributions omitted.

The curvature tensor $\underline{R}_{\underline{\mu\nu}}{}^{\underline{ab}}(\underline{\Omega})$ of $\underline{\Omega}_{\underline{\mu}}{}^{\underline{ab}}$ in our conventions is the same as the bosonic contribution of $R'_{\mu\nu}{}^{ab}(M)$ in [25]; $\underline{R}_{\underline{\mu\nu}}{}^{\underline{ab}}(\underline{\Omega}) = R'_{\mu\nu}{}^{ab}(M)\big|$, with $\big|$ denoting the omission of the terms including fermionic fields. Our convention of the Ricci curvature is $\underline{R}_{\underline{\mu}}{}^{\underline{a}}(\underline{\Omega}) = \underline{\Theta}^{\underline{\nu}}{}_{\underline{b}} \underline{R}_{\underline{\nu\mu}}{}^{\underline{ba}}(\underline{\Omega}) = -R'_\mu{}^a(M)\big|$, and $\underline{R}(\underline{\Omega}) = -R'(M)\big|$ in [25] and $\underline{R}(\underline{\Omega}) = -\underline{R}$ in [24].

H.1. Difference in notations between the previous paper and this paper

Before estimating the partition function, we don't need to use the complex conjugate of the scalar fields ϕ^I and the hermitian conjugate of the spinor χ^α to obtain the supersymmetry transformations, the supersymmetry algebra, and the supersymmetric action. However, the scalar fields ϕ^I have the negative signature in the kinetic terms, and therefore, it is necessary to perform the "Wick's rotation" of the scalar fields ϕ^I to define the partition function. We will regard them as pure imaginary.

However, the four scalars ϕ^i $(i = 1, \cdots, 4)$ in the previous paper have the positive signature, and we can regard them as real scalars. See Table 7 for the notations in the previous paper [1]. Thus, they form the complex scalars $H^{\dot\alpha}$, $\bar H_{\dot\alpha}$, and the scalars $H^{\dot\alpha}$, $\bar H_{\dot\alpha}$ in the previous paper can be identified as

$$H_{\dot\alpha} = \pm\frac{i}{\sqrt{2}} (\sigma_i)^1{}_{\dot\alpha} \phi^i, \qquad \bar H^{\dot\alpha} = (H_{\dot\alpha})^* = \pm\frac{i}{\sqrt{2}} (\bar\sigma_i)^{\dot\alpha}{}_1 \phi^i,$$

in terms of ϕ^i $(i = 1, \cdots, 4)$ in this paper.

$$\mp\frac{i}{\sqrt{2}} (\sigma_i)^2{}_{\dot\alpha} \phi^i = \varepsilon_{\dot\alpha\dot\beta} \bar H^{\dot\beta}, \qquad \mp\frac{i}{\sqrt{2}} (\bar\sigma_i)^{\dot\alpha}{}_2 \phi^i = \varepsilon^{\dot\alpha\dot\beta} H_{\dot\beta},$$

$$\phi^i = \mp\frac{i}{\sqrt{2}} \left[\left(\bar\sigma^i\right)^{\dot\alpha}{}_1 H_{\dot\alpha} + \left(\sigma^i\right)^1{}_{\dot\alpha} \bar H^{\dot\alpha}\right], \qquad \phi^1 = \pm\frac{1}{\sqrt{2}}\left(H_2 - \bar H^2\right),$$

$$\phi^2 = \pm\frac{i}{\sqrt{2}}\left(H_2 + \bar H^2\right), \qquad \phi^3 = \pm\frac{1}{\sqrt{2}}\left(H_1 - \bar H^1\right), \qquad \phi^4 = \mp\frac{i}{\sqrt{2}}\left(H_1 + \bar H^1\right).$$

In order for the supersymmetry transformation and the action in the previous paper to be consistent with the ones in this paper, we need to identify

$$\Xi = \mp\frac{i}{2}\psi^{\tilde\alpha=1}, \qquad \Xi^\dagger = \pm\frac{i}{2}\left(\psi^{\tilde\alpha=2}\right)^T C.$$

Table 7
Difference of notations between [1] and this paper.

The previous paper [1]	This paper
The gamma matrices of the Lorentz group	
$\Gamma^M, C_5\ (M=1,\cdots,5)$	$\gamma^\mu, C, (\mu=1,\cdots,5)$
The supersymmetry parameter $\varepsilon^{\dot\alpha}$	
$\Sigma^{\dot\alpha}$	$\varepsilon^{\dot\alpha}/\sqrt{2}$
The non-abelian gauge field A_μ	
v_M	A_μ
The gaugino field $\lambda^{\dot\alpha}$	
$\Psi^{\dot\alpha}$	$\lambda^{\dot\alpha}/(2\sqrt{2})$
The scalar field σ	
σ	$-\sigma$
The scalar fields $\phi^i \quad (i=1,\cdots,4)$	
$H_{\dot\alpha}$	$\pm i(\sigma_i)^1{}_{\dot\alpha}\,\phi^i/\sqrt{2}$
$\bar H^{\dot\alpha}=(H_{\dot\alpha})^*$	$\pm i(\bar\sigma_i)^{\dot\alpha}{}_1\,\phi^i/\sqrt{2}$
The spinor field $\psi^{\tilde\alpha} \quad (i=1,\cdots,4)$	
Ξ	$\mp(i/2)\,\psi^{\tilde\alpha=1}$
$\Xi^\dagger$	$\pm(i/2)\left(\psi^{\tilde\alpha=2}\right)^T C$
The auxiliary $SU(2)$ indices ($\check\alpha=1,2$)	
α,β	$\check\alpha, \check\beta$
$F_{H\alpha}$	$F_{\check\alpha}$

But, the symplectic Majorana condition of $\psi^{\tilde\alpha}$ is inconsistent with the hermitian conjugation of Ξ. Since we don't use the symplectic Majorana condition of $\psi^{\tilde\alpha}$ to ensure the supersymmetry transformation, the supersymmetry algebra, and the supersymmetric action, this identification never causes any troubles.

Through the dictionary, the on-shell supersymmetry transformation (19) may be rewritten in terms of the notations in the previous paper as

$$\begin{aligned}
&\delta v_M=-i\,\bar\Sigma_{\dot\alpha}\Gamma_M\Psi^{\dot\alpha}, \qquad \delta\sigma=i\,\bar\Sigma_{\dot\alpha}\Psi^{\dot\alpha}, \qquad \delta H_{\dot\alpha}=-i\,\bar\Sigma_{\dot\alpha}\,\Xi \qquad (\delta\bar H^{\dot\alpha}=i\,\bar\Xi\Sigma^{\dot\alpha}),\\
&\delta\Psi^{\dot\alpha}=-\frac{1}{2}\left[\frac{1}{2}F_{MN}\Gamma^{MN}+\Gamma^M\mathcal{D}_M\sigma-\frac{1}{2\alpha}G_{MN}\Gamma^{MN}\sigma\right]\Sigma^{\dot\alpha}\\
&\qquad -\frac{1}{2}S_{ij}\sigma\left(\bar\sigma^{ij}\right)^{\dot\alpha}{}_{\dot\beta}\Sigma^{\dot\beta}-ig\left(\left[\bar H^{\dot\alpha},\,H_{\dot\beta}\right]-\frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\left[\bar H^{\dot\gamma},\,H_{\dot\gamma}\right]\right)\Sigma^{\dot\beta},\\
&\delta\Xi=\Bigg[\Gamma^M\mathcal{D}_M H_{\dot\alpha}+ig\,[\sigma,\,H_{\dot\alpha}]-\frac{1}{2\alpha}G_{MN}\Gamma^{MN}H_{\dot\alpha}-t_{MN}\Gamma^{MN}H_{\dot\alpha}\\
&\qquad -\frac{1}{2}\left(S^{ij}+\varepsilon^{ijkl}S_{kl}\right)\left[\left(\bar\sigma_j\right)^{\dot\beta}{}_1 H_{\dot\beta}+\left(\sigma_j\right)^1{}_{\dot\beta}\bar H^{\dot\beta}\right](\sigma_i)^1{}_{\dot\alpha}\Bigg]\Sigma^{\dot\alpha},
\end{aligned}$$

and the equations of motion (20) of $\lambda^{\dot\alpha}$, $\psi^{\tilde\alpha}$ as

$$\begin{aligned}
&\Gamma^M\mathcal{D}_M\Psi^{\dot\alpha}+ig\left[\sigma,\,\Psi^{\dot\alpha}\right]+ig\left[\bar H^{\dot\alpha},\,\Xi\right]-ig\varepsilon^{\dot\alpha\dot\beta}\left[H_{\dot\beta},\,C^{-1}\Xi^*\right]\\
&\quad=\frac{1}{8\alpha}G_{MN}\Gamma^{MN}\Psi^{\dot\alpha}+\frac{1}{4}S_{ij}\left(\bar\sigma^{ij}\right)^{\dot\alpha}{}_{\dot\beta}\Psi^{\dot\beta}+\frac{1}{2}t_{MN}\Gamma^{MN}\Psi^{\dot\alpha},
\end{aligned}$$

$$\Gamma^M \mathcal{D}_M \Xi - ig [\sigma, \Xi] - 2ig \left[H_{\dot{\alpha}}, \Psi^{\dot{\alpha}} \right]$$
$$= \frac{1}{8\alpha} G_{MN} \Gamma^{MN} \Xi + \frac{1}{4} S_{ij} \left(\left(\sigma^{ij}\right)^1{}_1 \Xi + \left(\sigma^{ij}\right)^2{}_2 C^{-1} \Xi^* \right) - \frac{1}{2} t_{MN} \Gamma^{MN} \Xi.$$

The Killing spinor equation (7) in the old notations becomes

$$\mathcal{D}_M \Sigma^{\dot{\alpha}} = \frac{1}{4} S_{ij} \left(\bar{\sigma}^{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \Gamma_M \Sigma^{\dot{\beta}} - \frac{1}{2\alpha} G_{MN} \Gamma^N \Sigma^{\dot{\alpha}}$$
$$- \frac{1}{8\alpha} G_{KL} \Gamma_M{}^{KL} \Sigma^{\dot{\alpha}} - \frac{1}{2} t_{KL} \Gamma_M{}^{KL} \Sigma^{\dot{\alpha}},$$

with the covariant derivative

$$\mathcal{D}_M \Sigma^{\dot{\alpha}} \equiv \partial_M \Sigma^{\dot{\alpha}} + \frac{1}{2} b_M \Sigma^{\dot{\alpha}} + \frac{1}{4} \Omega_M{}^{KL} \Gamma_{KL} \Sigma^{\dot{\alpha}} - \frac{1}{4} A_M{}^{ij} \left(\bar{\sigma}_{ij}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \Gamma_M \Sigma^{\dot{\beta}}.$$

Under the background

$$\alpha = 1 \quad (b_M = 0), \qquad \frac{1}{\alpha} G_{45} = -1, \qquad S_{12} = S_{34} = \frac{1}{2},$$
$$t_{45} = \frac{1}{4}, \qquad A^{12} = A^{34} = -\frac{1}{2} \omega^{45},$$

the Killing spinor equation is reduced to the one in the previous paper

$$\nabla_M \Sigma^{\dot{\alpha}} = -\frac{1}{2} \Gamma_M{}^{45} \Sigma^{\dot{\alpha}}.$$

The Killing spinor in the previous paper has the additional property

$$\Gamma^{45} \Sigma^{\dot{\alpha}} = -i \left(\tau_3\right)^{\dot{\alpha}}{}_{\dot{\beta}} \Sigma^{\dot{\beta}}$$
$$(\Gamma^4 \Sigma^{\dot{\alpha}} = i \left(\tau_3\right)^{\dot{\alpha}}{}_{\dot{\beta}} \Gamma^5 \Sigma^{\dot{\beta}}, \qquad \Gamma^5 \Sigma^{\dot{\alpha}} = -i \left(\tau_3\right)^{\dot{\alpha}}{}_{\dot{\beta}} \Gamma^4 \Sigma^{\dot{\beta}}).$$

The equations of motion are also reduced to the ones in the previous paper.

In order to lift the on-shell supersymmetry transformation to the off-shell one, introducing the auxiliary field $D^{\dot{\alpha}}{}_{\dot{\beta}}$ to replace

$$D^{\dot{\alpha}}{}_{\dot{\beta}} = i\sigma \left(\tau_3\right)^{\dot{\alpha}}{}_{\dot{\beta}} + 2ig \left(\left[\bar{H}^{\dot{\alpha}}, H_{\dot{\beta}} \right] - \frac{1}{2} \delta^{\dot{\alpha}}{}_{\dot{\beta}} \left[\bar{H}^{\dot{\gamma}}, H_{\dot{\gamma}} \right] \right),$$

and using the equations of motion of the spinors with the background, the supersymmetry transformation of $D^{\dot{\alpha}}{}_{\dot{\beta}}$ reproduces the previous one

$$\delta D^{\dot{\alpha}\dot{\beta}} = i \bar{\Sigma}^{(\dot{\alpha}} \left(\Gamma^M \mathcal{D}_M \Psi^{\dot{\beta})} + ig \left[\sigma, \Psi^{\dot{\beta})} \right] \right).$$

It yields the off-shell supersymmetry of the vector multiplet $(v_M, \lambda^{\dot{\alpha}}, \sigma, D^{\dot{\alpha}}{}_{\dot{\beta}})$ in the previous paper.

For the self-dual S_{ij}; $\varepsilon_{ij}{}^{kl} S_{kl} = 2S_{ij}$, the term in $\delta\Xi$,

$$-\frac{1}{2} \left(S^{ij} + \varepsilon^{ijkl} S_{kl} \right) \left[\left(\bar{\sigma}_j\right)^{\dot{\beta}}{}_1 H_{\dot{\beta}} + \left(\sigma_j\right)^1{}_{\dot{\beta}} \bar{H}^{\dot{\beta}} \right] \left(\sigma_i\right)^1{}_{\dot{\alpha}} \Sigma^{\dot{\alpha}}$$
$$= \mp \frac{i}{\sqrt{2}} \left(S^{ij} + \varepsilon^{ijkl} S_{kl} \right) \phi_j \left(\sigma_i\right)^1{}_{\dot{\beta}} \Sigma^{\dot{\beta}},$$

can be rewritten by using the formulas

$$\sigma^k \bar{\sigma}_{ij} = \delta^k{}_{[i}\sigma_{j]} + \varepsilon_{ij}{}^{kl}\sigma_l, \qquad \sigma_{ij}\sigma^k = \delta^k{}_{[i}\sigma_{j]} - \varepsilon_{ij}{}^{kl}\sigma_l,$$

as

$$\pm\frac{i}{\sqrt{2}}\cdot\frac{3}{4}S_{ij}\left(\sigma^k\right)^1{}_{\dot\alpha}\phi_k\left(\bar{\sigma}^{ij}\right)^{\dot\alpha}{}_{\dot\beta}\Sigma^{\dot\beta} = \frac{3}{4}S_{ij}\left(\bar{\sigma}^{ij}\right)^{\dot\alpha}{}_{\dot\beta}H_{\dot\alpha}\Sigma^{\dot\beta}.$$

It facilitates the computation to verify that the off-shell supersymmetry transformation of the hypermultiplet $(H_{\dot\alpha}, \Xi, F_{H\alpha})$ with the auxiliary field $F_{H\alpha}$ gives rise to the one in the previous paper. In fact, by modifying the on-shell supersymmetry transformation as

$$\delta\Xi \quad\to\quad \delta\Xi + F_{H\alpha}\check{\Sigma}^\alpha, \qquad \delta F_{H\alpha} = i\bar{\check{\Sigma}}_\alpha(\text{the e.o.m. of }\Xi),$$

one can lift it to the off-shell supersymmetry transformation.

In the previous paper, we have given the action[27]

$$\begin{aligned}\mathcal{L} = \int \sqrt{g}\,d^5x\,\mathrm{tr}\Big[&-\frac{1}{4}v_{MN}v^{MN} + \frac{1}{2}D_M\sigma D^M\sigma - \frac{1}{4}D^{\dot\alpha}{}_{\dot\beta}D^{\dot\beta}{}_{\dot\alpha} - D^M\bar{H}^{\dot\alpha}D_M H_{\dot\alpha} \\ &+ \bar{F}_H{}^\alpha F_{H\alpha} + \frac{1}{2}\omega_{\text{c.s.}} + \sigma^2 - \left(1+\frac{1}{4}R(\Sigma)\right)\bar{H}^{\dot\alpha}H_{\dot\alpha} + \left(\frac{i}{2}(\tau_3)^{\dot\alpha}{}_{\dot\beta}D^{\dot\beta}{}_{\dot\alpha} - v_{45}\right)\sigma \\ &+ i\bar{\Psi}_{\dot\alpha}\Gamma^M D_M\Psi^{\dot\alpha} - i\bar{\Xi}\Gamma^M D_M\Xi + \frac{1}{2}(\tau_3)^{\dot\alpha}{}_{\dot\beta}\bar{\Psi}_{\dot\alpha}\Psi^{\dot\beta} - \frac{i}{2}\bar{\Xi}\Gamma_{45}\Xi \\ &+ g^2\left[\sigma,\,\bar{H}^{\dot\alpha}\right][\sigma,\,H_{\dot\alpha}] + igD^{\dot\alpha}{}_{\dot\beta}\left[\bar{H}^{\dot\beta},\,H_{\dot\alpha}\right] \\ &- g\bar{\Psi}_{\dot\alpha}\left[\sigma,\,\Psi^{\dot\alpha}\right] - g\bar{\Xi}[\sigma,\,\Xi] - 2g\bar{\Xi}\left[H_{\dot\alpha},\,\Psi^{\dot\alpha}\right] + 2g\left[\bar{H}^{\dot\alpha},\,\bar{\Psi}_{\dot\alpha}\right]\Xi\Big],\end{aligned}$$

where the term $\omega_{\text{c.s.}}$ denotes

$$\begin{aligned}\int\sqrt{g}\,d^5x\,\mathrm{tr}\left[\varepsilon^{mnk}v_m\left(\partial_n v_k + \frac{i}{3}g\,[v_n,\,v_k]\right)\right] &= \int v\wedge\left(dv + \frac{2}{3}(ig)v\wedge v\right)\wedge\mathrm{vol}(\Sigma) \\ &= \int\sqrt{g}\,d^5x\;\;\mathrm{tr}[\omega_{\text{c.s.}}],\end{aligned}$$

with $\mathrm{vol}(\Sigma)$ the volume form of the Riemann surface Σ.

Integrating out the auxiliary fields $D^{\dot\alpha}{}_{\dot\beta}$, $F_{H\alpha}$ in the action, one obtains

$$\begin{aligned}\mathcal{L}\big| = \int\sqrt{g}\,d^5x\,\mathrm{tr}\Big[&-\frac{1}{4}v_{MN}v^{MN} - v_{45}\sigma + \frac{1}{2}D_M\sigma D^M\sigma - D^M\bar{H}^{\dot\alpha}D_M H_{\dot\alpha} \\ &+ \frac{1}{2}\omega_{\text{c.s.}} + \frac{1}{2}\sigma^2 - \left(1+\frac{1}{4}R(\Sigma)\right)\bar{H}^{\dot\alpha}H_{\dot\alpha} - g(\tau_3)^{\dot\alpha}{}_{\dot\beta}\sigma\left[\bar{H}^{\dot\beta},\,H_{\dot\alpha}\right] \\ &+ g^2\left[\sigma,\,\bar{H}^{\dot\alpha}\right][\sigma,\,H_{\dot\alpha}] - g^2\left[\bar{H}^{\dot\alpha},\,H_{\dot\beta}\right]\left[\bar{H}^{\dot\beta},\,H_{\dot\alpha}\right] + \frac{1}{2}g^2\left[\bar{H}^{\dot\alpha},\,H_{\dot\alpha}\right]\left[\bar{H}^{\dot\beta},\,H_{\dot\beta}\right] \\ &+ i\bar{\Psi}_{\dot\alpha}\Gamma^M D_M\Psi^{\dot\alpha} - i\bar{\Xi}\Gamma^M D_M\Xi + \frac{1}{2}(\tau_3)^{\dot\alpha}{}_{\dot\beta}\bar{\Psi}_{\dot\alpha}\Psi^{\dot\beta} - \frac{i}{2}\bar{\Xi}\Gamma_{45}\Xi \\ &- g\bar{\Psi}_{\dot\alpha}\left[\sigma,\,\Psi^{\dot\alpha}\right] - g\bar{\Xi}[\sigma,\,\Xi] - 2g\bar{\Xi}\left[H_{\dot\alpha},\,\Psi^{\dot\alpha}\right] + 2g\left[\bar{H}^{\dot\alpha},\,\bar{\Psi}_{\dot\alpha}\right]\Xi\Big].\end{aligned}$$

[27] The mass term of $H_{\dot\alpha}$ in the action $\mathcal{L}$ didn't include the curvature term $R(\Sigma)\bar{H}^{\dot\alpha}H_{\dot\alpha}$ in the previous paper [1].

It enables us to compare it easily with the action S in (22)–(25), and one can see that the substitution of the background on the round 3-sphere in Subsection 5.1 into the action S exactly reproduces the above action.

References

[1] Y. Fukuda, T. Kawano, N. Matsumiya, 5D SYM and 2D q-deformed YM, Nucl. Phys. B 869 (2013) 493, arXiv:1210.2855.

[2] T. Kawano, N. Matsumiya, 5D SYM on 3D sphere and 2D YM, Phys. Lett. B 716 (2012) 450, arXiv:1206.5966.

[3] A. Gadde, L. Rastelli, S.S. Razamat, W. Yan, The 4d superconformal index from q-deformed 2d Yang–Mills, Phys. Rev. Lett. 106 (2011) 241602, arXiv:1104.3850.

[4] J. Kinney, J.M. Maldacena, S. Minwalla, S. Raju, An index for 4 dimensional super conformal theories, Commun. Math. Phys. 275 (2007) 209, arXiv:hep-th/0510251.

[5] D. Gaiotto, $N = 2$ dualities, J. High Energy Phys. 1208 (2012) 034, arXiv:0904.2715.

[6] D. Gaiotto, G.W. Moore, A. Neitzke, Wall-crossing, Hitchin systems, and the WKB approximation, arXiv:0907.3987.

[7] M. Aganagic, H. Ooguri, N. Saulina, C. Vafa, Black holes, q-deformed 2d Yang–Mills, and non-perturbative topological strings, Nucl. Phys. B 715 (2005) 304, arXiv:hep-th/0411280.

[8] Y. Tachikawa, 4d partition function on $S^1 \times S^3$ and 2d Yang–Mills with nonzero area, PTEP, Proces. Teh. Energ. Poljopr. 2013 (2013) 013B01, arXiv:1207.3497.

[9] A. Gadde, E. Pomoni, L. Rastelli, S.S. Razamat, S-duality and 2d topological QFT, J. High Energy Phys. 1003 (2010) 032, arXiv:0910.2225.

[10] D. Gaiotto, G.W. Moore, Y. Tachikawa, On 6d $N = (2, 0)$ theory compactified on a Riemann surface with finite area, Prog. Theor. Exp. Phys. 2013 (2013) 013B03, arXiv:1110.2657.

[11] K. Yonekura, Supersymmetric gauge theory, $(2, 0)$ theory and twisted 5d super-Yang–Mills, J. High Energy Phys. 1401 (2014) 142, arXiv:1310.7943.

[12] C. Cordova, D.L. Jafferis, Complex Chern–Simons from M5-branes on the squashed three-sphere, arXiv:1305.2891.

[13] S. Lee, M. Yamazaki, 3d Chern–Simons theory from M5-branes, J. High Energy Phys. 1312 (2013) 035, arXiv:1305.2429.

[14] J. Yagi, 3d TQFT from 6d SCFT, J. High Energy Phys. 1308 (2013) 017, arXiv:1305.0291.

[15] L. Rastelli, S.S. Razamat, The superconformal index of theories of class $\mathcal{S}$, arXiv:1412.7131.

[16] F. Benini, Y. Tachikawa, B. Wecht, Sicilian gauge theories and $\mathcal{N} = 1$ dualities, J. High Energy Phys. 1001 (2010) 088, arXiv:0909.1327.

[17] I. Bah, B. Wecht, New $\mathcal{N} = 1$ superconformal field theories in four dimensions, J. High Energy Phys. 1307 (2013) 107, arXiv:1111.3402.

[18] I. Bah, C. Beem, N. Bobev, B. Wecht, AdS/CFT dual pairs from M5-branes on Riemann surfaces, Phys. Rev. D 85 (2012) 121901, arXiv:1112.5487.

[19] I. Bah, C. Beem, N. Bobev, B. Wecht, Four-dimensional SCFTs from M5-branes, J. High Energy Phys. 1206 (2012) 005, arXiv:1203.0303.

[20] C. Beem, A. Gadde, The $N = 1$ superconformal index for class S fixed points, J. High Energy Phys. 1404 (2014) 036, arXiv:1212.1467.

[21] N. Hama, K. Hosomichi, S. Lee, SUSY gauge theories on squashed three-spheres, J. High Energy Phys. 1105 (2011) 014, arXiv:1102.4716.

[22] Y. Imamura, D. Yokoyama, $\mathcal{N} = 2$ supersymmetric theories on squashed three-sphere, Phys. Rev. D 85 (2012) 025015, arXiv:1109.4734.

[23] G. Festuccia, N. Seiberg, Rigid supersymmetric theories in curved superspace, J. High Energy Phys. 1106 (2011) 114, arXiv:1105.0689.

[24] C. Cordova, D.L. Jafferis, Five-dimensional maximally supersymmetric Yang–Mills in supergravity backgrounds, arXiv:1305.2886.

[25] E. Bergshoeff, E. Sezgin, A. Van Proeyen, $(2, 0)$ tensor multiplets and conformal supergravity in $D = 6$, Class. Quantum Gravity 16 (1999) 3193, arXiv:hep-th/9904085.

[26] D.Z. Freedman, A. Van Proeyen, Supergravity, Part V, Cambridge University Press, Cambridge, 2012.

[27] T. Kugo, K. Ohashi, Supergravity tensor calculus in 5D from 6D, Prog. Theor. Phys. 104 (2000) 835, arXiv:hep-ph/0006231.

[28] Y. Imamura, H. Matsuno, Supersymmetric backgrounds from 5d $\mathcal{N} = 1$ supergravity, arXiv:1404.0210.

[29] K. Hosomichi, R.K. Seong, S. Terashima, Supersymmetric gauge theories on the five-sphere, Nucl. Phys. B 865 (2012) 376, arXiv:1203.0371.
[30] T. Kugo, The Quantum Theory of Gauge Fields, vol. 1, Appendix D, Baifukan, Tokyo, 1989 (in Japanese).
[31] M. Blau, G. Thompson, Derivation of the Verlinde formula from Chern–Simons theory and the G/G model, Nucl. Phys. B 408 (1993) 345, arXiv:hep-th/9305010;
M. Blau, G. Thompson, Lectures on 2d gauge theories: topological aspects and path integral techniques, arXiv:hep-th/9310144.
[32] N. Hama, K. Hosomichi, S. Lee, Notes on SUSY gauge theories on three-sphere, J. High Energy Phys. 1103 (2011) 127, arXiv:1012.3512.
[33] S. Kharchev, D. Lebedev, M. Semenov-Tian-Shansky, Unitary representations of $U(q)(sl(2,R))$, the modular double, and the multiparticle q deformed Toda chains, Commun. Math. Phys. 225 (2002) 573, arXiv:hep-th/0102180.
[34] A.G. Bytsko, J. Teschner, Quantization of models with non-compact quantum group symmetry: modular XXZ magnet and lattice sinh-Gordon model, J. Phys. A 39 (2006) 12927, arXiv:hep-th/0602093.
[35] Y. Tachikawa, A review of the T_N theory and its cousins, arXiv:1504.01481.