Remarks on equations of Benjamin–Bona–Mahony type

J. Límaco a, H.R. Clark a,*, L.A. Medeiros b

a Universidade Federal Fluminense, IM, 24.020-140 Praça do Valonguinho, Niterói, RJ, Brazil
b Universidade Federal do Rio de Janeiro, IM, RJ, Brazil

Received 20 July 2005
Available online 7 July 2006
Submitted by William F. Ames

Abstract

We investigate an initial-boundary value problem for equations of Benjamin–Bona–Mahony (BBM) type in two different physical situations. In the first, the mixed problem is considered on a cylinder domain $Q$ of $\mathbb{R}^n \times \mathbb{R}_t$. In the second one, the mixed problem is studied inside of an increasing noncylindrical domain $\tilde{Q}$ of $\mathbb{R}^n \times \mathbb{R}_t$. In both situations we show the existence of a unique nonlocal solution. In cylindrical case it is proved the existence of weak and strong solutions, regularity of strong solutions, and in noncylindrical case weak solutions. One of the goals of this paper is to show that the noncylindrical problem is well-posed by using the penalty method idealized by Lions [J.L. Lions, Une remarque sur les problèmes d’évolution non linéaires dans des domaines non cylindriques, Rev. Roumaine Math. Pures Appl. 9 (1964) 11–18].

© 2006 Elsevier Inc. All rights reserved.

Keywords: BBM equations; Cylindrical and noncylindrical domains; Existence uniqueness and regularity of solutions

1. Introduction

Benjamin, Bona and Mahony [2] considered the equation

$$u_t(x,t) + u_x(x,t) + u(x,t)u_x(x,t) - u_{xxt}(x,t) = 0,$$

today known as Benjamin–Bona–Mahony (BBM) equation, which is an alternative smoothness model for the KdV equation, see [10]. Equation (1.1) is included in the general class of equations of Sobolev type—see, for example, Brill [7], Showalter [18] and references therein. In [2] it was investigated existence and uniqueness of solutions for the Cauchy problem associated with
(1.1) in $\mathbb{R}_x \times \mathbb{R}_t$. Supposing the spacial variable $x$ in $[0, \infty[$ the BBM equation with boundary condition on $x = 0$ was investigated by Bona and Bryant [4] and the same precedent properties were established. With $x$ in $[0, 1]$ Medeiros and Milla Miranda [15] studied existence, uniqueness and regularity of solutions for BBM equation for general nonlinear term. Also with $x$ which belongs to $[0, 1]$ Bona and Dougalis [5] considered a mixed problem for (1.1) with inhomogeneous boundary condition, and also, proved existence and uniqueness of solutions. Goldstein and Wichenosi [8] generalized the one-dimensional BBM equation (1.1) for the case of open bounded sets $\Omega \in \mathbb{R}^n$, for $n \geq 1$, and more general nonlinear terms. That is, the initial boundary value problem for the generalized BBM equation

$$u'(x, t) - \Delta u'(x, t) + \text{div} \phi(u(x, t)) = 0 \quad \text{in } \Omega \times [0, T] \text{ for } T > 0.$$  

(1.2)

Boling [3] also studied an initial boundary value problem for Eq. (1.2) considering an external inhomogeneous force $f(u)$ acting into this equation.

Our main goal in this paper is to establish existence, uniqueness and regularity of solutions for an initial-boundary value problem for equations of BBM type given by

$$u'(x, t) + Au'(x, t) + \text{div} \phi(u(x, t)) = 0 \quad \text{in } \Omega \times [0, T] \text{ for } T > 0,$$  

(1.3)

where the objects of (1.3) are defined as follows: let $\Omega$ be an open-bounded subset of $\mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index where $\alpha_i$ is a nonnegative integer number with $i = 1, \ldots, n$. For any vector $x = (x_1, \ldots, x_n)$ which belongs to $\Omega$ we define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$, with $0! = 1$. Denoting $\frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}$ by $D^{\alpha_i}_{x_i}$ we define the derivative operator $D^\alpha = D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} \cdots D^{\alpha_n}_{x_n}$, and if $\alpha_i = 0$, then $D^{\alpha_i}_{x_i}$ is the identity operator $D^{\alpha_i}_{x_i} v = v$ for all $v$. The elliptic operator $A$ of Eq. (1.3) has order $2m$ and is defined by

$$A = (-1)^m \sum_{|\alpha| = m} D^{2\alpha}.$$  

(1.4)

On the vector valued function $\phi : \mathbb{R} \to \mathbb{R}^n$ there are assumed the following hypotheses:

$$\phi \in C^1(\mathbb{R}, \mathbb{R}^n) \quad \text{with} \quad \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s)), \quad \phi(0) = 0,$$

$$|\phi_i(s)|_{\mathbb{R}} \leq C_i(|s|_{\mathbb{R}} + |s|_{\mathbb{R}}^{\sigma_i+1}) \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and} \quad \phi_i : \mathbb{R} \to \mathbb{R},$$

$$|\phi_i'(s)|_{\mathbb{R}} \leq A_i (1 + |s|_{\mathbb{R}}^{\sigma_i}) \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and} \quad \phi_i' : \mathbb{R} \to \mathbb{R},$$  

(1.5)

where, for each $i, C_i, A_i$ are positive real constants, and $\sigma_i$ also is real constant satisfying

$$(i) \quad 0 \leq \sigma_i \leq \frac{4m - 2}{n - 2m} \quad \text{if } n > 2m,$$

$$(ii) \quad 0 \leq \sigma_i \leq m \quad \text{if } n \leq 2m.$$  

(1.6)

**Particular cases of (1.3).** If $\alpha_1 = 1, \alpha_2 = \cdots = \alpha_n = 0, n = 1$ and $m = 1$, then the elliptic operator defined in (1.4) becomes: $Au' = -D^2u' = -u''_{xx}$. Thus, considering $\phi(u) = u + \frac{1}{2}u^2$ we obtain the generic BBM equation (1.1). Now, supposing $n > 1$ and $m = 1$, then $|\alpha| = 1$ and $Au' = -\sum_{i=1}^n D^{\alpha_i}_{x_i} u' = -\Delta u'$, where $\Delta$ is the usual Laplace operator. In these conditions, since $\phi(u) = u + \frac{1}{2}u^2$ we have Eq. (1.2).

Another mathematical aspect related with (1.3) was investigated by Límaco et al. [12], where were established existence, uniqueness and regularity of solutions by means of the technique transforming the noncylindrical domain $\hat{Q}$ into a cylindrical one via a suitable diffeomorphism.
Some functional spaces. To formulate the mixed problem associated with (1.3) on $Q$ and $\hat{Q}$ we need some notations about Sobolev spaces. In fact, $H^m(\Omega)$ represents the Sobolev space of order $m$ on the open bounded set $\Omega$ of $\mathbb{R}^n$, with $m \in \mathbb{N}$. The space $L^2(\Omega)$ is the Lebesgue space of classes of functions $v: \Omega \rightarrow \mathbb{R}$ with square integrable on $\Omega$. The inner product and norm in $H^m(\Omega)$ and $L^2(\Omega)$ are represented, respectively, by $(\cdot, \cdot)_m$, $\| \cdot \|_m$ and $(\cdot, \cdot)$, $|\cdot|$. $D(\Omega)$ denotes the $C^\infty$ functions on $\Omega$ with compact support in $\Omega$. Thus, $H^m_0(\Omega)$ is the closure of $D(\Omega)$ in $H^m(\Omega)$. In these conditions, if $v \in H^m_0(\Omega)$ and the boundary $\Gamma$ of $\Omega$ is regular, then $D^\beta v = 0$ on $\Gamma$ for $|\beta| \leq m - 1$ see, for example, Lions [14]. For the spaces $L^p(0, T; H^m_0(\Omega))$ for $1 \leq p \leq \infty$, see also Lions [14]. Finally, in the case of noncylindrical domains $\hat{Q}$ the spaces $H^m(\Omega_t)$, $H^m_0(\Omega_t)$, and $L^p(0, T; H^m_0(\Omega_t))$ are defined, as, for instance, in [14].

Organization of the paper. This paper is organized in five sections regarding the mixed problem associated with Eq. (1.3), namely, Section 2 is devoted to investigate the existence and uniqueness of nonlocal weak solutions in the cylinder domain $Q$. In Section 3 the existence and uniqueness of weak solutions also are proved in the noncylindrical domain $\hat{Q}$. Section 4 is concerned with existence of strong solutions, and finally, in Section 5 the regularity of the strong solutions is investigated.

2. Weak solutions in the cylinder $Q$

The results in this section are established with the objective to be applied in the following one. The cylinder $Q$ is defined by a subset $\Omega \times [0, T]$ of $\mathbb{R}^n \times \mathbb{R}$, for $T > 0$, with lateral boundary $\Sigma = \Gamma \times [0, T]$ where $\Gamma$ is a smooth boundary of class $C^2$ of open-bounded set $\Omega$. To simplify the notation we denote by

$$ a(u, v) = \sum_{|\alpha|=m} \int_\Omega D^\alpha u D^\alpha v \, dx, \quad (2.1) $$

the bilinear form in $H^m_0(\Omega) \times H^m_0(\Omega)$ generated by operator $A$ defined in (1.4).

We propose the initial-boundary value problem

$$ u' + Au' + \text{div} \phi(u) = 0 \quad \text{in} \ Q, $$
$$ D^\beta u = 0 \quad \text{on} \ \Sigma \ \text{for all} \ |\beta| \leq m - 1, $$
$$ u(x, 0) = u_0(x) \quad \text{in} \ \Omega, \quad (2.2) $$

where $u = u(x, t)$ is defined for all $(x, t)$ in $Q$, $u' = \frac{\partial u}{\partial t}$ and all derivatives are in the sense of distributions.

Definition 2.1. A weak solution for (2.2) is a function $u: Q \rightarrow \mathbb{R}$ such that

$$ u, u' \ \text{belong to} \ L^\infty(0, T; H^m_0(\Omega)) \ \text{for} \ T > 0, $$
$$ \int_0^T [(u', \varphi) + a(u', \varphi) - (\phi(u), \nabla \varphi)] \, dt = 0 \quad \text{for all} \ \varphi \in L^2(0, T; H^m_0(\Omega)). $$

Moreover, $u$ verifies the initial condition $u(x, 0) = u_0(x)$ in $\Omega$. 
Theorem 2.1. If \( u_0 \in H_0^m(\Omega) \) and the hypotheses (1.5), (1.6) hold, then the mixed problem (2.2) has only one weak solution in the sense of Definition 2.1.

Proof of existence of solutions. We employ Faedo–Galerkin approximate method with a Hilbertian basis \((w_j)_{j \in \mathbb{N}}\) of \( H_0^m(\Omega) \), see, for example, Brezis [6]. Let \( V_N \) be the subspace of \( H_0^m(\Omega) \) spanned by the \( N \) first vectors of \((w_j)_{j \in \mathbb{N}}\). By using the notation fixed in (2.1) we will look for a function \( u_N(x,t) = \sum_{j=1}^N g_j(t) w_j(x) \) in \( V_N \) solution of the following system of ordinary differential equations

\[
\begin{align*}
(u_N'(t), w) + a(u_N'(t), w) - (\phi(u_N(t)), \nabla w) &= 0, \\
u_N(0) &= u_{0N} \rightarrow u_0 \quad \text{in } H_0^m(\Omega),
\end{align*}
\]

for all \( w \in V_N \). System (2.3) has local solution \( u_N \) in \( 0 \leq t < t_N \). The extension of the solutions \( u_N \) to the whole interval \([0, T]\), for all \( T > 0 \), and subsequences that converge in convenient spaces to the solution of (2.2) in the sense of Definition 2.1 are consequence of estimates established afterwards.

Estimate 2.1. Setting \( w = u_N(t) \in V_N \) in (2.3)1, yields

\[
\frac{1}{2} \frac{d}{dt} \left[ |u_N(t)|^2 + a(u_N(t), u_N(t)) \right] - (\phi(u_N(t)), \nabla u_N(t)) = 0.
\]

The nonlinear term of (2.4) is null. In fact, first note that

\[-(\phi(u_N(t)), \nabla u_N(t)) = -\sum_{i=1}^n \int_\Omega \phi_i(u_N(t)) \frac{\partial u_N(t)}{\partial x_i} dx.\]

Thus, setting

\[F_i(\xi) = \int_0^\xi \phi_i(s) ds\]

it implies

\[\frac{\partial F_i(u_N)}{\partial x_i} = \phi_i(u_N) \frac{\partial u_N}{\partial x_i}\]

and

\[\int_\Omega \frac{\partial F_i(u_N)}{\partial x_i} dx = 0,\]

by Gauss’ lemma, definition of \( F_i \) and for \( u_N \in H_0^m(\Omega) \). Therefore, in (2.4) there is no contribution of the nonlinear term. Thus, integrating (2.4) from 0 to \( t \leq t_N \) and observing that in \( H_0^m(\Omega) \) the norm \( \|u_N(t)\|_m \) and the seminorm \( a(u_N(t), u_N(t)) \) are equivalent, we get

\[|u_N(t)|^2 + \|u_N(t)w\|_m^2 \leq |u_0|^2 + \|u_0\|_m^2 \quad \text{for all } t \geq 0.\]

Estimate 2.2. Setting \( w = u'_N(t) \) into (2.3)1 yields

\[|u'_N(t)|^2 + \|u'_N(t)\|_m^2 = (\phi(u_N(t)), \nabla u'_N(t)).\]

The right-hand side of (2.6) will be analyzed by using the remark:
Remark 2.1. The hypothesis (1.6) ensures for \( \frac{1}{q} + \frac{1}{q'} = 1 \), with \( q \) and \( q' \) dependent on \( \sigma_i \), that the Sobolev embeddings

(I) \( H^m_0(\Omega) \hookrightarrow L^{q'(\sigma_i+1)}(\Omega) \),

(II) \( H^{-1}_0(\Omega) \hookrightarrow L^{q}(\Omega) \),

hold. In fact, since

(1) \( 1 \leq q'(\sigma_i + 1) \) for \( n \leq 2m \),
(2) \( 1 \leq q'(\sigma_i + 1) \leq \frac{2n}{n - 2m} \) for \( n > 2m \),
(3) \( 1 \leq q \) for \( n \leq 2(m - 1) \),
(4) \( 1 \leq q \leq \frac{2n}{n - 2(m - 1)} \) for \( n > 2(m - 1) \),

then (I) and (II) hold provided that (1), (2) and (3), (4) are assumed, respectively. Thus, when the order \( m \in \mathbb{N} \) of the operator \( A \) and the dimension \( n \in \mathbb{N} \) are fixed, then they are related for the three possibilities:

\( n \leq 2(m - 1) \), \( 2(m - 1) < n \leq 2m \) and \( n > 2m \).

Whence we obtain the following conditions on \( \sigma_i \):

(a) If \( n \leq 2(m - 1) \) we have the cases (1) and (3). Thus, considering \( q' \) big enough and being \( \frac{1}{q} + \frac{1}{q'} = 1 \) yield \( \sigma_i \geq 1 \).
(b) If \( 2(m - 1) < n \leq 2m \) we are in the cases (1) and (4). Again, for \( q' \) very large we have \( \sigma_i \geq 1 \).
(c) If \( n > 2m \), then the cases (2) and (4) hold. From (2), we can write \( \frac{1}{q'} \leq \sigma_i + 1 \leq \frac{2n}{n - 2m} \).

From this and considering \( q' \) big enough it yields \( \sigma_i + 1 \geq 0 \). Moreover, the maximum of \( 1/q' \) is the minimum of \( 1/q \). That is, \( q = \frac{2n}{n - 2(m - 1)} \) is maximum. From this and since \( \frac{1}{q} = 1 - \frac{1}{q'} \) we obtain \( \frac{1}{q'} = \frac{n + 2(m - 1)}{2n} \). Thus,

\( \sigma_i + 1 \leq \frac{n + 2(m - 1)}{2n} \frac{2n}{n - 2m} = \frac{n + 2(m - 1)}{n - 2m} \),

which gives \( \sigma_i \leq \frac{4m - 2}{n - 2m} \). So, if \( n \leq 2m \), \( \sigma_i \geq 1 \) and if \( n > 2m \), \( -1 \leq \sigma_i \leq \frac{4m - 2}{n - 2m} \). These conditions are according to hypothesis (1.6)

By hypothesis (1.5)2 we obtain

\[
\left( \phi(u_N(t)), \nabla u'_N(t) \right) = \sum_{i=1}^{n} \int_{\Omega} \phi_i(u_N(t)) \frac{\partial u'_N(t)}{\partial x_i} \, dx \\
\leq \sum_{i=1}^{n} C_i \int_{\Omega} \left( |u_N(t)|_{R} + |u_N(t)|_{\sigma_i+1} \right) \left| \frac{\partial u'_N(t)}{\partial x_i} \right|_{R} \, dx \\
\leq \sum_{i=1}^{n} C_i \left( |u_N(t)|_{\left\| u'_N(t) \right\|_1 + \left\| u_N(t) \right\|_{\sigma_i+1} \left\| \frac{\partial u'_N(t)}{\partial x_i} \right\|_{L^q}} \right).\
\]
where $\frac{1}{q} + \frac{1}{q'} = 1$. From this, using Remark 2.1 embeddings (I), (II) and estimate (2.5) we can write

$$
(\phi(u_N(t)), \nabla u'_N(t)) \leq \|u'_N(t)\|_m \sum_{i=1}^n \tilde{C}C_i(|u_N(t)| + \|u_N(t)\|_{m}^{\sigma_i+1})
$$

$$
\leq \frac{1}{2} \|u'_N(t)\|_m^2 + \bar{C}.
$$

Substituting this into (2.6) yields

$$
|u'_N(t)|^2 + \frac{1}{2} \|u'_N(t)\|_m^2 \leq \bar{C} \quad \text{for all } t \in [0, T].
$$

Note that $\bar{C}$ depends just on initial data $u_0$.

**The limit in the approximate problem (2.3).** Estimates 2.1 and 2.2 give a subsequences $(u_v)_{v \in \mathbb{N}}$ of $(u_N)_{N \in \mathbb{N}}$ and a function $u : Q \to \mathbb{R}$ satisfying

$$
\begin{align*}
    u_v &\to u \text{ weak star in } L^\infty(0, T; H^m_0(\Omega)), \\
    u'_v &\to u' \text{ weak star in } L^\infty(0, T; H^m_0(\Omega)).
\end{align*}
$$

(2.8)

From convergence (2.8) we are able to take the limits in the linear terms of (2.3). The nonlinear term needs a carefully analysis as follows: by (2.8) and Aubin compactness result, see Aubin [1] or Lions [13], we can extract a subsequence $(u_v)$ of $(u_N)$ such that

$$
    u_v \to u \text{ strongly } L^2(0, T; L^2(\Omega)) \quad \text{and} \quad u_v \to u \text{ a.e. in } Q.
$$

(2.9)

From this and continuity of $\phi_i$, for $i = 1, 2, \ldots, n$, we have

$$
    \phi_i(u_v(x, t)) \to \phi_i(u(x, t)) \quad \text{a.e. in } Q.
$$

Moreover, $\phi_i$ is bounded $L^{q'}(Q)$. In fact, by (1.5)$_2$, Remark 2.1(I) and (2.5) we have

$$
\int_Q |\phi_i(u_v(x, t))|^{q'} dx dt \leq \tilde{C} \int_0^T (|u_v(x, t)|^{q'} + |u_v(x, t)|^{(\sigma_i+1)}) dx dt \leq C,
$$

where $C$ depends just on $T$. Whence Lemma 1.3 of Lions [14]

$$
\phi_i(u_v) \rightharpoonup \phi_i(u) \text{ weakly in } L^{q'}(Q). \quad (2.10)
$$

Finally, by Remark 2.1(II), $\frac{\partial w}{\partial x_i}$ belongs to $L^q(\Omega)$ because $w \in V_N$. Thus, we are able to take the limit in the approximate system (2.3) for the sequence $(u_v)$, which proves the existence of solutions of (2.2) in the sense of Definition 2.1.

**Uniqueness.** Suppose $u$ and $\bar{u}$ are two weak solutions guaranteed by Theorem 2.1. Thus, the function $w = u - \bar{u}$ is a solution of (2.2) in the following sense

$$
\begin{align*}
    w, w' &\text{ belong to } L^\infty(0, T; H^m_0(\Omega)) \text{ for } T > 0, \\
    \int_0^T [(w', \varphi) + a(w', \varphi)] dt &= \int_0^T (\phi(u) - \phi(\bar{u}), \nabla \varphi) dt \quad \text{for all } \varphi \in L^2(0, T; H^m_0(\Omega)), \\
    w(x, 0) &= 0 \quad \text{in } \Omega_0.
\end{align*}
$$

(2.11)
Setting \( \varphi = w \) in (2.11) we have
\[
\frac{1}{2} \left[ |w(t)|^2 + a(w(t), w(t)) \right] = \int_0^t \left( \phi(u(s)) - \phi(\bar{u}(s)), \nabla w(s) \right) ds. \tag{2.12}
\]

The nonlinear term is upper bounded by using hypothesis (1.5) and Hölder’s inequality for \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \) as follows:
\[
\left| \int_0^t \left( \phi(u(s)) - \phi(\bar{u}(s)), \nabla w(s) \right) ds \right| 
\leq \int_0^T \int_\Omega \left| \phi'(u + \theta(\bar{u} - u)) \right| |\nabla w| \, dx \, dt
\leq \sum_{i=1}^n \tilde{\Lambda}_i \int_0^T \left( |\Omega|^{1/p} + \|u\|_{L^{p_q}(\Omega)}^{p_q} + \|\bar{u}\|_{L^{p_q}(\Omega)}^{p_q} \right) \|w\|_{L^q(\Omega)} \left\| \frac{\partial w}{\partial x_i} \right\|_{L^r(\Omega)} \, dt.
\]

By using the immersions \( H^m_0(\Omega) \hookrightarrow L^{p_q}(\Omega), \ H^m_0(\Omega) \hookrightarrow L^q(\Omega) \) and \( H^{m-1}_0(\Omega) \hookrightarrow L^r(\Omega) \), which hold in view of (1.6), we obtain
\[
\left| \int_0^t \left( \phi(u(s)) - \phi(\bar{u}(s)), \nabla w(s) \right) ds \right| \leq C \int_0^T \|w(t)\|_m^2 \, dt,
\]
also thanks to (2.11). Substituting these results into Eq. (2.12) we get
\[
|w(t)|^2 + \|w(t)\|_m^2 \leq C \int_0^T \|w(t)\|_m^2 \, dt.
\]
From this and Gronwall inequality we have the uniqueness. Thus, we have proved Theorem 2.1. \( \square \)

3. Weak solutions in the noncylinder \( \hat{Q} \)

Let us denote by \( \mathcal{O} \) an open, bounded and nonempty subset of \( \mathbb{R}_+^n \times \mathbb{R}_t \). Suppose also \( \Omega_s = \mathcal{O} \cap \{ t = s \}; \ s \in \mathbb{R} \) are open, bounded and nonempty sets with boundaries \( \Gamma_s \). We fix the interval \( [0, T] \) of \( \mathbb{R}_t \) and consider \( \hat{Q} = \bigcup_{0 \leq s < T} \Omega_s \times \{ s \} \) the noncylindrical domain contained in \( \mathbb{R}_+^n \times \mathbb{R}_t \) with lateral boundary defined by \( \tilde{\Sigma} = \bigcup_{0 \leq s < T} \Gamma_s \times \{ s \} \) and its boundary by \( \partial \hat{Q} = \Omega_0 \cup \tilde{\Sigma} \cup \Omega_T \).

In these conditions we are concerned with the existence and uniqueness of solutions for the following initial and moving-boundaries value problem
\[
\begin{align*}
& u' + Au' + \text{div} \phi(u) = 0 \quad \text{in} \ \hat{Q}, \\
& D^\beta u = 0 \quad \text{on} \ \tilde{\Sigma} \ \text{for all} \ |\beta| \leq m - 1, \\
& u(x, 0) = u_0(x) \quad \text{in} \ \Omega_0. \tag{3.1}
\end{align*}
\]
The methodology we will employ to solve the problem (3.1) consists in transforming it into a cylindrical problem by means of a perturbation of Eq. (3.1) adding a singular term depending on a parameter $\epsilon > 0$ which is destined to tend to zero. This method was idealized by Lions [13] and is called by him a penalty method. To apply the Lions’ method, some restrictive hypotheses on $\widehat{Q}$ are necessary. In fact, we suppose $\widehat{Q} \subset Q$ with $\Omega_0 \subset \Omega$. Note that $Q$ was defined in Section 2 by $\Omega \times [0, T]$. Moreover, we consider

**On the geometry of $\widehat{Q}$:**

If $t_1 \leq t_2$, then $\text{proj}_{t=0} \Omega_{t_1} \subseteq \text{proj}_{t=0} \Omega_{t_2}$. It means, the family $\{\Omega_t\}_{0 \leq t \leq T}$ is increasing.

**On regularity of $\widehat{Q}$:**

If $v \in H^m_0(\Omega)$ and $D^\beta v = 0$ on $\Omega - \Omega_t$ for almost all $t \in [0, T]$ and $|\beta| \leq m - 1$, then $v \in H^m_0(\Omega_t)$.

Regularity of this type is exemplified in Límaco et al. [11].

**Definition 3.1.** A weak solution for the initial moving boundary value problem (3.1) is a real valued function $u = u(x, t)$ defined for all $(x, t)$ in $\widehat{Q}$ such that

\[
\begin{align*}
  &u, u' \text{ belong to } L^\infty(0, T; H^m_0(\Omega_t)) \text{ for } T > 0, \\
  &\int_{\widehat{Q}} u' \varphi \, dx \, dt + \sum_{|\alpha| = m} \int_{\widehat{Q}} D^\alpha u' D^\alpha \varphi \, dx \, dt - \int_{\widehat{Q}} \phi(u) \nabla \varphi \, dx \, dt = 0
\end{align*}
\]

for all $\varphi \in L^2(0, T; H^m_0(\Omega_t))$.

Moreover, $u$ verifies the initial condition $u(x, 0) = u_0(x)$ in $\Omega_0$.

**Theorem 3.1.** Suppose $u_0 \in H^m_0(\Omega_0)$ and that hypotheses (1.5), (1.6), (3.2) and (3.3) hold. Then there exists a unique real function $u$ defined in $\widehat{Q}$ solution of problem (3.1) in the sense of Definition 3.1.

**Proof of existence of solutions.** Let $\chi: Q \to \mathbb{R}$ be a function defined by

\[
\chi(x, t) = \begin{cases} 1 & \text{in } Q \setminus (\widehat{Q} \cup (\Omega_0 \times \{0\})), \\ 0 & \text{in } \widehat{Q} \cup (\Omega_0 \times \{0\}). \end{cases}
\]

(3.4)

Note that $\chi \in L^\infty(\Omega \times [0, T])$. The mapping $\zeta(u) = \frac{1}{\epsilon} \chi u$, for $\epsilon > 0$, is called penalty operator. By means of $\zeta$ problem (3.1) is transformed from the noncylindrical domain $\widehat{Q}$ into another penalized one into a cylindrical domain $Q$ as follows: First note that $\zeta(u(x, t)) = 0$ for $(x, t) \in \widehat{Q} \cup \{\Omega_0 \times \{0\}\}$, which is the key point of the method. Second, denoting by $\tilde{u}_0$ the extension of $u_0$ to $\Omega$ defined zero in $\Omega \setminus \Omega_0$, it implies $\tilde{u}_0 \in H^m_0(\Omega)$.

**Penalized problem.** It consists in given $\epsilon > 0$ to look for a function $u_\epsilon: Q \to \mathbb{R}$ such that

\[
\begin{align*}
  &u_\epsilon, u'_\epsilon \text{ belong to } L^\infty(0, T; H^m_0(\Omega)) \text{ for } T > 0, \\
  &\int_0^T \left[ (u'_\epsilon(t), \varphi) + a(u'_\epsilon(t), \varphi) \right] \, dt + \frac{1}{\epsilon} \sum_{|\beta| \leq m - 1} \int_0^T (\chi(t) D^\beta u'_\epsilon(t), D^\beta \varphi) \, dt
\end{align*}
\]
\[ -\int_0^T \left( \phi(u_\epsilon(t)), \nabla \varphi \right) dt = 0, \]  
(3.6)

for all \( \varphi \in L^2(0, T; H^m_0(\Omega)) \). Moreover, \( u_\epsilon \) verifies the initial condition \( u_\epsilon(x, 0) = \tilde{u}_0(x) \) in \( \Omega \).

**Approximate problem.** The penalized problem is cylindrical. Therefore, we are in the case of Section 2. Thus, if \( (w_j)_{j \in \mathbb{N}} \) is a Hilbertian basis of \( H^m_0(\Omega) \) with \( V_N \) spanned by the \( N \) first vectors \( w_1, w_2, \ldots \), with \( w_1 = \tilde{u}_0 \), then the approximate problem associated with the penalized system (3.5), (3.6), for \( \epsilon > 0 \) fixed, is given by

\[
(u'_N(t), w) + a(u'_N(t), w) + \frac{1}{\epsilon} \sum_{|\beta| \leq m-1} \left( \chi(t)D^\beta u'_N(t), D^\beta w \right) - \left( \phi(u_N(t)), \nabla w \right) = 0,
\]

\[
u_\epsilon N(x, 0) = u_\epsilon N \rightarrow \tilde{u}_0 \quad \text{in} \ H^m_0(\Omega),
\]

(3.7)

for all \( w \in V_N \). The initial value problem (3.7) has a local solution \( u_\epsilon N \) defined on the interval \([0, t_N] \) for each \( \epsilon > 0 \) fixed. The extension to the whole interval \([0, T] \) for \( T > 0 \) depends on the estimates that we will find below. These estimates also are sufficient to pass to the limit as \( N \rightarrow \infty \) and \( \epsilon \rightarrow 0 \).

To obtain these a priori estimates we first must prove the following lemma.

**Lemma 3.1.** Let \( v \) and \( v' \) be functions which belong to \( L^{\infty}(0, T; L^2(\Omega)) \) with \( \Omega_0 \subset \Omega_t \subset \Omega \), for all \( 0 < t < T \). Suppose \( t = t(x) \) a parametric representation of the lateral boundary \( \hat{\Sigma} \) of \( \hat{Q} \).

Then

\[ \int_0^t \left( \chi(s)v(s), v'(s) \right) ds \geq \left| \chi(t)v(t) \right|^2 - \left| \chi(0)v(0) \right|^2. \]

**Proof.** First note that

\[ \int_0^t \left( \chi(s)v(s), v'(s) \right) ds = \frac{1}{2} \int_0^t \int_\Omega \left( \chi(x), s \right) \frac{d}{ds} \left[ v(x, s) \right]^2 \, dx \, ds. \]

As \( \Omega = (\Omega \setminus \Omega_t) \cup (\Omega_t \setminus \Omega_0) \cup \Omega_0 \) for all \( 0 < t < T \), and the integral on \( \Omega_0 \) is zero, because \( \Omega_0 \times \{ t \} \) is contained in \( \hat{Q} \cup (\Omega_0 \times \{ 0 \}) \) and there \( \chi(x, t) = 0 \), see (3.4). Then, the integral on the right-hand side above is modified as follows

\[ \frac{1}{2} \int_0^t \int_\Omega \chi(x, s) \frac{d}{ds} \left[ v(x, s) \right]^2 \, dx \, ds = \frac{1}{2} \int_0^t \int_{\Omega \setminus \Omega_t} \chi(x, s) \frac{d}{ds} \left[ v(x, s) \right]^2 \, dx \, ds \]

\[ + \frac{1}{2} \int_0^t \int_{\Omega_t \setminus \Omega_0} \chi(x, s) \frac{d}{ds} \left[ v(x, s) \right]^2 \, dx \, ds. \]

By hypothesis (3.2) the family \( \{ \Omega_t \}_{0 \leq t \leq T} \) is increasing, thus the integration domain is contained in \( \hat{Q} \setminus \hat{Q} \), it means \( x \in \Omega \setminus \Omega_t \) and \( 0 < s < t \), where \( t \) depends on \( x \), i.e., \( t = t(x) \) on the lateral boundary \( \hat{\Sigma} \) of \( \hat{Q} \). In this domain \( \chi(x, s) = 1 \) by definition (3.4). From this and Fubinni’s theorem the first integral on the right-hand side above can be modified as follows.
\[
\frac{1}{2} \int_{\Omega \setminus \Omega_0} \int_0^t \chi(x,s) \frac{d}{ds} [v(x,s)]^2 \, dx \, ds \\
= \frac{1}{2} \int_{\Omega \setminus \Omega_0} \int_0^t \frac{d}{ds} [v(x,s)]^2 \, dx \, ds \\
= \frac{1}{2} \int_{\Omega \setminus \Omega_0} [v(x,t)]^2 \, dx - \frac{1}{2} \int_{\Omega \setminus \Omega_0} [v(x,0)]^2 \, dx \\
= \frac{1}{2} \int_{\Omega \setminus \Omega_0} \chi(x,t)[v(x,t)]^2 \, dx - \frac{1}{2} \int_{\Omega \setminus \Omega_0} \chi(x,0)[v(x,0)]^2 \, dx.
\]

The second integral is analyzed as follows: first note that

\[
\chi(x,s) = 1 \quad \text{for } x \in \Omega \setminus \Omega_0 \text{ and } 0 < s < t(x), \\
\chi(x,s) = 0 \quad \text{for } x \in \Omega \setminus \Omega_0 \text{ and } t(x) < s < t.
\]

Whence we have

\[
\frac{1}{2} \int_{\Omega \setminus \Omega_0} \int_0^t \chi(x,s) \frac{d}{ds} [v(x,s)]^2 \, dx \, ds \\
= \frac{1}{2} \int_{\Omega \setminus \Omega_0} \int_0^{t(x)} \frac{d}{ds} [v(x,s)]^2 \, ds \, dx \\
= \frac{1}{2} \int_{\Omega \setminus \Omega_0} [v(x,t(x))]^2 \, dx - \frac{1}{2} \int_{\Omega \setminus \Omega_0} [v(x,0)]^2 \, dx \\
= \frac{1}{2} \int_{\Omega \setminus \Omega_0} \chi(x,t(x))[v(x,t(x))]^2 \, dx - \frac{1}{2} \int_{\Omega \setminus \Omega_0} \chi(x,0)[v(x,0)]^2 \, dx.
\]

Therefore, we can write

\[
\int_0^t (\chi(s)v(s), v'(s)) \, ds = \frac{1}{2} \left[ \int_{\Omega \setminus \Omega_0} \chi(x,t)[v(x,t)]^2 \, dx \right] + \int_{\Omega \setminus \Omega_0} \chi(x,0)[v(x,0)]^2 \, dx \\
- \frac{1}{2} \left[ \int_{\Omega \setminus \Omega_0} \chi(x,0)[v(x,0)]^2 \, dx \right] + \int_{\Omega \setminus \Omega_0} \chi(x,0)[v(x,0)]^2 \, dx.
\]

Note that

\[
\int_{\Omega \setminus \Omega_0} \chi(x,t)[v(x,t(x))]^2 \, dx \geq 0,
\]
\[ \frac{1}{2} \int_{\Omega \setminus \Omega_t} \chi(x, t)[v(x, t)]^2 \, dx = \frac{1}{2} \int_{\Omega \setminus \Omega_t} \chi(x, t)[v(x, t)]^2 \, dx + \frac{1}{2} \int_{\Omega_t} \chi(x, t)[v(x, t)]^2 \, dx = 0 \]

and

\[ \frac{1}{2} \int_{\Omega} \chi(x, 0)[v(x, 0)]^2 \, dx = \frac{1}{2} \int_{\Omega \setminus \Omega_t} \chi(x, 0)[v(x, 0)]^2 \, dx + \int_{\Omega_t \setminus \Omega_0} \chi(x, 0)[v(x, 0)]^2 \, dx = 0 \]

These all yield

\[ \int_0^t (\chi(s)v(s), v'(s)) \, ds \geq \frac{1}{2} \int_{\Omega} \chi(x, t)[v(x, t)]^2 \, dx - \frac{1}{2} \int_{\Omega} \chi(x, 0)[v(x, 0)]^2 \, dx = \frac{1}{2} |\chi(t)v(t)|^2 - \frac{1}{2} |\chi(0)v(0)|^2. \]

The proof of the lemma is finished. \( \square \)

Notice that the proof of Lemma 3.1, developed here, it is different from the one given by Nakao and Narazaki [16] and Rabelo [17]. Note also that is crucial in proofs above the fact that \( \hat{Q} \) is increasing.

**Estimate 3.1.** Setting \( w = u_{\epsilon N} \) in (3.7) and proceeding as in Estimate 2.1 we get

\[ \frac{1}{2} \frac{d}{dt} \left[ |u_{\epsilon N}(t)|^2 + a(u_{\epsilon N}(t), u_{\epsilon N}(t)) \right] + \frac{1}{\epsilon} \sum_{|\beta| \leq m-1} \left( \chi(t)D^\beta u_{\epsilon N}'(t), D^\beta u_{\epsilon N}(t) \right) = 0. \]

Applying Lemma 3.1 in each term of the sum \( \sum_{|\beta| \leq m-1} \left( \chi(t)D^\beta u_{\epsilon N}'(t), D^\beta u_{\epsilon N}(t) \right) \) we conclude that it is positive. Thus, integrating the resulting expression from 0 to \( t \) we obtain the same inequality (2.5) of Section 2 for \( u_{\epsilon N} \), namely

\[ |u_{\epsilon N}(t)|^2 + \|u_{\epsilon N}(t)\|^2_m \leq |\bar{u}_0|^2 + \|\bar{u}_0\|^2_m \quad \text{for all } t \in [0, T] \text{ with } T > 0. \]

**Estimate 3.2.** Setting \( w = u_{\epsilon N}'(t) \) in (3.7) we obtain

\[ |u_{\epsilon N}'(t)|^2 + a(u_{\epsilon N}'(t), u_{\epsilon N}'(t)) + \frac{1}{\epsilon} \sum_{|\beta| \leq m-1} \left| \chi(t)D^\beta u_{\epsilon N}'(t) \right|^2 = \left( \phi(u_{\epsilon N}(t)), \nabla u_{\epsilon N}'(t) \right). \]
The right-hand side of (3.10) is the same as that of (2.6) with $u'_{\epsilon N}(t)$. Thus, we have
\[
(\phi(u_{\epsilon N}(t)), \nabla u'_{\epsilon N}(t)) \leq \frac{1}{2} \|u'_{\epsilon N}(t)\|_m^2 + \tilde{C}.
\]
Substituting this inequality into (3.10) we have for all $t \in [0, T]$ that
\[
|u'_{\epsilon N}(t)|^2 + \frac{1}{2} \|u'_{\epsilon N}(t)\|_m^2 + \frac{1}{\epsilon} \sum_{|\beta| \leq m-1} |\chi(t) D^\beta u'_{\epsilon N}(t)|^2 \leq C.
\]
(3.11)

The limit in the approximate system (3.7). From (3.9) and (3.11) we can extract a subsequence $(u_{\epsilon \nu})_{\nu \in \mathbb{N}}$ of $(u_{\epsilon N})_{N \in \mathbb{N}}$, for $\epsilon$ fixed, and proceeding as in the convergences (2.9)–(2.11) of Section 2, yields as $\nu \to \infty$
\[
u
\]
\[
\begin{align*}
 u_{\epsilon \nu} & \rightharpoonup u \quad \text{weak star in } L^\infty(0, T; H^m_0(\Omega)), \\
 u'_{\epsilon \nu} & \rightharpoonup u' \quad \text{weak star in } L^\infty(0, T; H^m_0(\Omega)), \\
 u_{\epsilon \nu} & \to u \quad \text{strongly in } L^2(0, T; H^m_0(\Omega)) \text{ and a.e. in } Q, \\
 \phi_i(u_{\epsilon \nu}) & \to \phi_i(u) \quad \text{weak in } L^{q'}(Q).
\end{align*}
\]
(3.12)
Also from (3.11) we obtain
\[
\chi D^\beta u'_{\epsilon \nu} \rightharpoonup \chi D^\beta u' \quad \text{weak star in } L^\infty(0, T; L^2(\Omega)) \text{ as } \nu \to \infty.
\]
(3.15)
Setting $w \in H^m_0(\Omega)$ in (3.7), multiplying by $\theta \in \mathcal{D}(0, T)$, taking the limit as $\nu \to \infty$ and observing the convergence from (3.12) to (3.15) we obtain that $u_{\epsilon}$ is a weak solution of the penalized problem (3.5), (3.6).

The next step is an analysis of the penalized problem (3.5), (3.6) as $\epsilon \to 0$. In fact, from convergence (3.12), (3.14) and Banach–Steinhauss’ theorem we obtain a net $(u_{\epsilon})_{0<\epsilon<1}$, and a function $\omega: \Omega \to \mathbb{R}$ satisfying
\[
\begin{align*}
 u_{\epsilon} & \rightharpoonup \omega \quad \text{weak star in } L^\infty(0, T; H^m_0(\Omega)) \text{ as } \epsilon \to 0, \\
 u'_{\epsilon} & \rightharpoonup \omega' \quad \text{weak star in } L^\infty(0, T; H^m_0(\Omega)) \text{ as } \epsilon \to 0, \\
 \phi_i(u_{\epsilon}) & \to \phi_i(\omega) \quad \text{weak in } L^{q'}(Q) \text{ as } \epsilon \to 0, \\
 \chi D^\beta u'_{\epsilon} & \rightharpoonup \chi D^\beta \omega' \quad \text{weak star in } L^\infty(0, T; L^2(\Omega)) \text{ as } \epsilon \to 0.
\end{align*}
\]
(3.16)
From (3.11) we have
\[
\int_{\Omega \times [0,T]} |\chi(x,t) D^\beta u'_{\epsilon \nu}(x,t)|^2 \, dx \, dt \leq \epsilon C.
\]
From this, (3.15) and Banach–Steinhauss’ theorem we have
\[
\int_{\Omega \times [0,T]} |\chi(x,t) D^\beta u'_{\epsilon}(x,t)|^2 \, dx \, dt < \epsilon C.
\]
Whence we affirm that $\chi D^\beta u'_{\epsilon}$ converges weakly in $L^2(0, T; L^2(\Omega))$ to zero as $\epsilon \to 0$. Therefore, from (3.16)4 we get
\[
\int_0^T \int_{\Omega} |\chi(x,t) D^\beta \omega'(x,t)|^2 \, dx \, dt = 0.
\]
Consequently,
\[
\chi(x,t)D^\beta \omega'(x,t) = 0 \quad \text{a.e. in } \Omega \times [0,T[ = Q,
\]
and then
\[
D^\beta \omega'(x,t) = 0 \quad \text{in } Q \setminus \{ \hat{Q} \cup (\Omega_0 \times \{0\}) \} \quad \text{for } |\beta| \leq m - 1. \quad (3.17)
\]
From (3.17) it implies \( D^\beta \omega(x,t) = C \) in \( Q \setminus \{ \hat{Q} \cup (\Omega_0 \times \{0\}) \} \), with \( C \) constant. We also have \( D^\beta \omega(x,0) = D^\beta \tilde{u}_0(x) = 0 \). Then, \( C = 0 \) and so \( D^\beta \omega(x,t) = 0 \) in \( Q \setminus \{ \hat{Q} \cup (\Omega_0 \times \{0\}) \} \).
Since \( \hat{Q} \) is increasing, it implies \( D^\beta \omega(x,t) = 0 \) in \( \Omega \setminus \Omega_t \) for \( 0 < t < T \).
As \( w \in H^m_0(\Omega) \) then by regularity hypothesis on \( \hat{Q} \), \( w \in H^m_0(\Omega_t) \). Indeed \( w \in L^\infty(0,T;H^m_0(\Omega_t)) \) thanks to (3.16)_1.

From the above argument if \( u \) is the restriction of \( \omega \) to \( \hat{Q} \) then
\[
u \quad \text{belongs to } L^\infty(0,T;H^m_0(\Omega_t)). \quad (3.18)
\]
In a similar way, if \( u' \) is the restriction of \( \omega' \) to \( \hat{Q} \), then we get from (3.16)_2
\[
u' \quad \text{belongs to } L^\infty(0,T;H^m_0(\Omega_t)). \quad (3.19)
\]
Thus, by restriction to \( \hat{Q} \) of penalized Eq. (3.6) we obtain a function \( \hat{u}_\epsilon : \hat{Q} \rightarrow \mathbb{R} \), for \( \epsilon > 0 \) fixed, satisfying
\[
\int_{\hat{Q}} \hat{u}_\epsilon' \varphi \, dx \, dt + \sum_{|\alpha|=m} \int_{\hat{Q}} D^\alpha \hat{u}_\epsilon' D^\alpha \varphi \, dx \, dt - \int_{\hat{Q}} \phi(\hat{u}_\epsilon) \nabla \varphi \, dx \, dt = 0, \quad (3.20)
\]
for all \( \varphi \in L^2(0,T;H^m_0(\Omega_t)) \). Finally, from the convergence (3.16)_1–(3.16)_3 and taking the limit in (3.20) as \( \epsilon \rightarrow 0 \) we obtain a solution \( u \) of (3.1) in the sense of Definition 3.1.

**Proof of uniqueness of solutions.** The uniqueness of solutions for hyperbolic problems—as wave equation—via penalty method is, up to now, an open problem. For problem (3.1) this question is answered afterwards. In fact, suppose \( u \) and \( \bar{u} \) two weak solutions given by Theorem 3.1. Thus, the function \( w = u - \bar{u} \) is a solution of (3.1) in the following sense: For all \( \varphi \in L^2(0,T;H^m_0(\Omega_t)) \)
\[
w(x,0) = 0 \quad \text{in } \Omega_0. \quad (3.21)
\]
To show that the solution of (3.21) is null, first we prove this fact in the particular domain
\[
\hat{Q}_\rho = \bigcup_{0 \leq t < \rho} \Omega_t \times \{t\} \quad \text{of } \hat{Q} = \bigcup_{0 \leq t < T} \Omega_t \times \{t\} \quad \text{for } 0 \leq \rho < T.
\]
Second, we extend this result for whole \( \hat{Q} \). Therefore, considering (3.21) on \( \hat{Q}_\rho \) and replacing \( \varphi \) by \( w' \) we obtain
\[
\int_{\hat{Q}_\rho} |w'|^2 \, dx \, dt + \sum_{|\alpha|=m} \int_{\hat{Q}_\rho} |D^\alpha w'|^2 \, dx \, dt = \int_{\hat{Q}_\rho} \left[ \phi(u) - \phi(\bar{u}) \right] \nabla w' \, dx \, dt. \quad (3.22)
\]
To continue with the analysis of the term on the right-hand side of (3.22) we need two lemmas and a remark, namely

**Lemma 3.2.** If \( w \) is a weak solution of (3.21) and \( t < \rho < T \), then
\[
\sum_{|\alpha|=m} \int_{\tilde{Q}_\rho} |D^\alpha w|^2 dx dt \leq \rho^2 \sum_{|\alpha|=m} \int_{\tilde{Q}_\rho} |D^\alpha w'|^2 dx dt.
\]
(3.23)

**Lemma 3.3.** For all \( v \in H^m_0(\Omega_t) \), then
\[
\|v\|_{L^p(\Omega_t)} \leq C(m,n,|\Omega|)\|v\|_{H^m_0(\Omega_t)} \quad \text{for } 0 \leq t < T,
\]
(3.24)
provided that
\[
p \geq 1 \quad \text{if } n \leq 2m,
\]
\[
1 < p \leq \frac{2n}{n-2m} \quad \text{if } n > 2m,
\]
where \( |\Omega| \) is the measure of \( \Omega \) and \( C \) depends only on \( m, n \) and \( |\Omega| \).

The proofs of two lemmas will be given later.

**Remark 3.1.** The analysis of the nonlinear term is done by application of the following Sobolev embedding:

(I) \( H^m_0(\Omega_t) \hookrightarrow L^{p(\sigma_i)}(\Omega_t) \),

(II) \( H^m_0(\Omega_t) \hookrightarrow L^q(\Omega_t) \),

(III) \( H^{m-1}_0(\Omega_t) \hookrightarrow L^r(\Omega_t) \),

where \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \). The restrictions on \( \sigma_i \) assumed in hypothesis (1.6) are such that the Sobolev embeddings (I)–(III) hold. In fact, since

(1) \( 1 \leq p\sigma_i \) for \( n \leq 2m \),

(2) \( 1 \leq p\sigma_i \leq \frac{2n}{n-2m} \) for \( n > 2m \),

the first immersion (I) holds. Also, if

(3) \( 1 \leq q \) for \( n \leq 2m \),

(4) \( 1 \leq q \leq \frac{2n}{n-2m} \) for \( n > 2m \),

then it implies the second immersion (II), and finally, if

(5) \( 1 \leq r \) for \( n \leq 2(m-1) \),

(6) \( 1 \leq r \leq \frac{2n}{n-2(m-1)} \) for \( n > 2(m-1) \),

the third Sobolev embedding holds too. Thus, when the order \( m \in \mathbb{N} \) of the operator and the dimension \( n \in \mathbb{N} \) are fixed, then they are related to the three possibilities:

\( n \leq 2(m-1), \quad 2(m-1) < n \leq 2m \) and \( n > 2m \).

Whence we obtain the following conditions on \( \sigma_i \):
(a) If \( n \leq 2(m - 1) \) we have the cases (1), (3) and (5). Thus, considering \( q \) and \( r \) such that 
\[
\frac{1}{q} + \frac{1}{r} \to 1
\]
for instance \( q \) and \( r \) near to 2—then \( p \) is large enough such that by (1) it gives 
\( \sigma_i > 0 \).

(b) If \( 2(m - 1) < n \leq 2m \) we are in the cases (1), (3) and (6). Again, for \( q \) and \( r \) such that 
\[
\frac{1}{q} + \frac{1}{r} \to 1,
\]
then \( p \) is very large, and for (1) we have \( \sigma_i > 0 \).

(c) If \( n > 2m \), then the cases (2), (4) and (6) hold. From (2), we have 
\[
\frac{1}{p} \leq \sigma_i \leq \frac{2n}{n - 2m}.
\]
From this and considering \( \frac{1}{q} + \frac{1}{r} \to 1 \) it implies \( \sigma_i > 0 \).

On the other hand, the maximum of \( \sigma_i \) corresponds to the maximum of \( 1/p \). Since \( \frac{1}{p} = 1 - \frac{1}{q} - \frac{1}{r} \) then \( \frac{1}{q} + \frac{1}{r} \) is minimum, which implies \( q \) and \( r \) are maximum. Thus, by (4) and (6) we get 
\[
\frac{1}{p} = 1 - \frac{n - 2m}{2n} - \frac{n - 2(m - 1)}{2n} = \frac{4m - 2}{2n}.
\]

From this and (2) the maximum of \( \sigma_i \) is 
\[
\frac{1}{p} \frac{2n}{n - 2m} = \frac{4m - 2}{n - 2m},
\]
that is, \( 0 \leq \sigma_i \leq \frac{4m - 2}{n - 2m} \). Therefore, we have the hypothesis (1.6).

Returning to the analysis of the nonlinear term of (3.22) we have by hypotheses (1.5)\(_1\), (1.5)\(_3\)
and \( 0 < \theta < 1 \) that
\[
\left| \int_{\tilde{Q}_\rho} \left[ \phi(u) - \phi(\tilde{u}) \right] \nabla w' \, dx \, dt \right|_{\mathbb{R}} = \left| \sum_{i=1}^{n} \frac{\int_{\tilde{Q}_\rho} \phi_i'(\tilde{u} + \theta[u - \tilde{u}])w \frac{\partial w'}{\partial x_i} \, dx \, dt}{\partial x_i} \right|_{\mathbb{R}}
\]
\[
\leq \sum_{i=1}^{n} \tilde{A}_i \int_{\tilde{Q}_\rho} \left[ 1 + |u|_{\mathbb{R}}^{\sigma_i} + |\tilde{u}|_{\mathbb{R}}^{\sigma_i} \right] \left| w \right|_{\mathbb{R}} \left| \frac{\partial w'}{\partial x_i} \right|_{\mathbb{R}} \, dx \, dt
\]
\[
= \sum_{i=1}^{n} \tilde{A}_i \int_{\tilde{Q}_\rho} \left[ 1 + |u|_{\mathbb{R}}^{\sigma_i} + |\tilde{u}|_{\mathbb{R}}^{\sigma_i} \right] \left| w \right|_{\mathbb{R}} \left| \frac{\partial w'}{\partial x_i} \right|_{\mathbb{R}} \, dx \, dt.
\]

Applying Hölder’s inequality in the last integral for \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \) we get 
\[
\sum_{i=1}^{n} \tilde{A}_i \int_{0}^{\rho} \int_{\Omega_\rho} \left[ 1 + |u|_{\mathbb{R}}^{\sigma_i} + |\tilde{u}|_{\mathbb{R}}^{\sigma_i} \right] \left| w \right|_{\mathbb{R}} \left| \frac{\partial w'}{\partial x_i} \right|_{\mathbb{R}} \, dx \, dt
\]
\[
\leq \sum_{i=1}^{n} \tilde{A}_i \int_{0}^{\rho} \left[ |\Omega|^{1/p} + \|u\|_{L^p(\Omega)}^{\sigma_i} + \|\tilde{u}\|_{L^p(\Omega)}^{\sigma_i} \right] \left\| \frac{w}{\partial x_i} \right\|_{L^q(\Omega)} \, dt
\]
\[
\leq \sum_{i=1}^{n} \tilde{A}_i \int_{0}^{\rho} \left[ C_1(m, n, |\Omega|) + \left[ C_2(m, n, |\Omega|) \right]^{\sigma_i} \|u\|_{H^m_0(\Omega)} \right. 
\]
\[
\left. + \left[ C_3(m, n, |\Omega|) \right]^{\sigma_i} \|\tilde{u}\|_{H^m_0(\Omega)} \right] \left\| \frac{w}{\partial x_i} \right\|_{H^m_0(\Omega)} \, dt.
\]
where we have used in the last inequality Remark 3.1. From estimate (3.9) and denoting the constant
\[ C(u_0) = |u_0|^2 + \|u_0\|_{m}^2 \]
we obtain \( \|u\|_{H_0^m(\Omega_\rho)}^{\sigma_i} \leq C(u_0)^{\sigma_i} \) and \( \|\bar{u}\|_{H_0^m(\Omega_\rho)}^{\sigma_i} \leq C(u_0)^{\sigma_i} \).
From this and Hölder’s inequality we get
\[
\left|\int_{\hat{Q}_\rho} \left[ \phi(u) - \phi(\bar{u}) \right] \nabla w' \, dx \, dt \right| \leq K \left[ \int_0^\rho \int_{\Omega_\rho} \left| D^\alpha w \right|^2 dx \, dt \right]^{1/2} \left[ \int_0^\rho \int_{\Omega_\rho} \left| D^\alpha w' \right|^2 dx \, dt \right]^{1/2},
\]
where
\[
K = \sum_{i=1}^n \Lambda_i \left\{ C_1(m, n, |\Omega|) + \left[ C_2(m, n, |\Omega|) \right]^{\sigma_i} C(u_0)^{\sigma_i} \right\} + \left[ C_3(m, n, |\Omega|) \right]^{\sigma_i} C(u_0)^{\sigma_i},
\]
where \( K \) does not depend on \( T \). From this and Lemma 3.2 we get
\[
\left|\int_{\hat{Q}_\rho} \left[ \phi(u) - \phi(\bar{u}) \right] \nabla w' \, dx \, dt \right| \leq \rho K \sum_{|\alpha|=m} \int_{\hat{Q}_\rho} \left| D^\alpha w' \right|^2 dx \, dt.
\]
Finally, supposing \( \rho K = \frac{1}{2} \), that is, \( \rho \leq T_0 = \frac{1}{2K} \) we obtain from (3.22) that
\[
\int_{\hat{Q}_\rho} \left| w' \right|^2 dx \, dt + \frac{1}{2} \sum_{|\alpha|=m} \int_{\hat{Q}_\rho} \left| D^\alpha w' \right|^2 dx \, dt \leq 0 \quad \text{for } t \in [0, T_0].
\]
This and Lemma 3.2 yield
\[
\sum_{|\alpha|=m} \int_{\hat{Q}_\rho} \left| D^\alpha w \right|^2 dx \, dt \leq T_0^2 \sum_{|\alpha|=m} \int_{\hat{Q}_\rho} \left| D^\alpha w' \right|^2 dx \, dt \leq 0 \quad \text{for } t \in [0, T_0],
\]
which implies \( w(x, t) = 0 \) for all \( (x, t) \in \hat{Q}_\rho = \hat{Q}_{T_0} \). By Theorem 3.1, we get \( u \) and \( \bar{u} \) in \( C([0, T]; H_0^m(\Omega_\rho)) \). Consequently,
\[
w(x, T_0) = 0 \quad \text{for } T_0 < T. \quad (3.25)
\]
If the process above is repeated for Eq. (3.21) with initial data (3.25), now for \( 0 < T_0 \leq t < \rho \), we obtain
\[
\int_{T_0}^\rho \int_{\Omega_\rho} \left| w' \right|^2 dx \, dt + \sum_{|\alpha|=m} \int_{T_0}^\rho \int_{\Omega_\rho} \left| D^\alpha w' \right|^2 dx \, dt \leq K(\rho - T_0) \sum_{|\alpha|=m} \int_{T_0}^\rho \int_{\Omega_\rho} \left| D^\alpha w' \right|^2 dx \, dt.
\]
Therefore, for \( K(\rho - T_0) = \frac{1}{2} \), i.e., \( \rho = 2T_0 \), we get \( t \in [T_0, \rho] \) such that
\[
\sum_{|\alpha|=m} \int_{T_0}^\rho \int_{\Omega_\rho} \left| D^\alpha w \right|^2 dx \, dt \leq (2T_0)^2 \sum_{|\alpha|=m} \int_{T_0}^\rho \int_{\Omega_\rho} \left| D^\alpha w' \right|^2 dx \, dt \leq 0.
\]
which implies \( w(x, t) = 0 \) for all \( (x, t) \in \hat{Q}_\rho = \hat{Q}Q_{2T_0} \). And thus, \( w(x, t) = 0 \) for almost all \( t \) in \( [0, 2T_0] \). Therefore, if we repeat the process a finite number of times, that is, for \( n \) big enough such that \( nT_0 > T \) we will have \( w(x, t) = 0 \) for almost all \( t \) in \( [0, T] \).

This way the proof of Theorem 3.1 is ended. \( \Box \)

**Proof of Lemma 3.2.** From (3.21) we have \( w, w' \) belong to \( L^\infty(0, T; H^m_0(\Omega_\rho)) \) for \( T > 0 \). Since \( \Omega_\rho \subset \Omega \) and \( w = u - \bar{u} \), there exist \( \hat{u} \) and \( \hat{\bar{u}} \) in the class \( L^\infty(0, T; H^m_0(\Omega)) \) such that

\[
D^\alpha \hat{u} = D^\alpha \hat{\bar{u}} = 0 \quad Q \setminus \hat{Q} \quad \text{for} \ |\alpha| = m,
\]

\[
D^\alpha \hat{u}' = D^\alpha \hat{\bar{u}}' = 0 \quad Q \setminus \hat{Q} \quad \text{for} \ |\alpha| = m,
\]

where \( u = \hat{u}|_{\hat{Q}} \) and \( \bar{u} = \hat{\bar{u}}|_{\hat{Q}} \). Thus, if \( \hat{w} = \hat{u} - \hat{\bar{u}} \), then \( w = \hat{w}|_{\hat{Q}} \) and

\[
D^\alpha \hat{w}, D^\alpha \hat{w}' \quad \text{belong to} \quad L^\infty(0, T; L^2(\Omega_\rho)) \quad \text{for} \ T > 0 \quad \text{and} \ |\alpha| = m,
\]

\[
D^\alpha \hat{w} = D^\alpha \hat{w}' = 0 \quad Q \setminus \hat{Q} \quad \text{for} \ |\alpha| = m,
\]

with \( D^\alpha w = D^\alpha \hat{w}|_{\hat{Q}} \) and \( D^\alpha w' = D^\alpha \hat{w}'|_{\hat{Q}} \). In particular, we still have the same results in \( \hat{Q}Q_\rho \), that is

\[
D^\alpha w = D^\alpha \hat{w}|_{\hat{Q}_\rho}, \quad D^\alpha w' = D^\alpha \hat{w}'|_{\hat{Q}_\rho} \quad \text{for} \ |\alpha| = m. \tag{3.26}
\]

On the other hand, since \( D^\alpha \hat{w}(x, 0) = 0 \) we have, see, for instance, Temam [19, Lemma 1.1, p. 250], that

\[
D^\alpha \hat{w}(\cdot, \rho) = \int_0^\rho D^\alpha \hat{w}'(\cdot, t) \, dt \quad \text{almost all} \ t \leq \rho,
\]

and hence we obtain

\[
|D^\alpha \hat{w}(\cdot, \rho)|_{L^R} \leq \int_0^\rho |D^\alpha \hat{w}'(\cdot, t)|_{L^R} \, dt \leq \rho^{1/2} \left[ \int_0^\rho |D^\alpha \hat{w}'(\cdot, t)|^2_{L^R} \, dt \right]^{1/2}.
\]

Thus,

\[
|D^\alpha \hat{w}(\cdot, \rho)|^2_{L^R} \leq \rho \int_0^\rho |D^\alpha \hat{w}'(\cdot, t)|^2_{L^R} \, dt,
\]

and consequently

\[
\int_0^\rho \int_\Omega |D^\alpha \hat{w}|^2_{L^R} \, dx \, dt \leq \rho^2 \int_0^\rho \int_\Omega |D^\alpha \hat{w}'|^2_{L^R} \, dx \, dt.
\]

From this and (3.26) we get

\[
\int_{\hat{Q}_\rho} |D^\alpha w|^2_{L^R} \, dx \, dt \leq \rho^2 \int_{\hat{Q}_\rho} |D^\alpha w'|^2_{L^R} \, dx \, dt,
\]

which implies (3.23), and thus, the proof of the lemma is complete. \( \Box \)
Proof of Lemma 3.3. First, observe that if $v \in H^m_0(\Omega_t)$ for $0 \leq t < T$, then $\tilde{v} \in H^m(\mathbb{R}^n)$ and $\|v\|_{H^m_0(\Omega_t)} = \|\tilde{v}\|_{H^m(\mathbb{R}^n)}$. Moreover, we will also use the following Sobolev embedding immersions:

(i) $H^m(\mathbb{R}^n) \hookrightarrow C^0_b(\mathbb{R}^n)$ if $n < 2m$,
(ii) $H^m(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ if $n > 2m$ and $p = \frac{2n}{n-2m}$,
(iii) $H^m(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ if $n = 2m$ and $2 \leq p < \infty$.

In fact, by (i) we have $\|\tilde{v}\|_{C^0_b(\mathbb{R}^n)} \leq C_1(m,n)\|\tilde{v}\|_{H^m(\mathbb{R}^n)}$ for all $\tilde{v} \in H^m(\mathbb{R}^n)$. In particular, $\|v\|_{L^\infty(\Omega_t)} \leq C_1(m,n)\|v\|_{H^m_0(\Omega_t)}$ for all $v \in H^m_0(\Omega_t)$, which implies

$$\|v\|_{L^p(\Omega_t)} \leq C_1(m,n)|\Omega_t|^{1/p}\|v\|_{H^m_0(\Omega_t)} \quad \text{for all } p \geq 1,$$

$$\leq C_1(m,n)(1 + |\Omega_t|)\|v\|_{H^m_0(\Omega_t)} \quad \text{since } \Omega_t \subset \Omega \text{ for all } 0 \leq t < T.$$

Therefore, for $n < 2m$ and $\Omega_t \subset \Omega$ for all $0 \leq t < T$ we get

$$\|v\|_{L^p(\Omega_t)} \leq C_1(m,n,|\Omega_t|)\|v\|_{H^m_0(\Omega_t)} \quad \text{for all } v \in H^m_0(\Omega_t). \quad (3.27)$$

Now, (iii) implies $\|\tilde{v}\|_{L^p(\mathbb{R}^n)} \leq C_2(m,n)\|\tilde{v}\|_{H^m(\mathbb{R}^n)}$ for all $\tilde{v} \in H^m(\mathbb{R}^n)$ and $p \geq 2$. And hence, $\|v\|_{L^p(\Omega_t)} \leq C_2(m,n)\|v\|_{H^m_0(\Omega_t)}$ for all $v \in H^m_0(\Omega_t)$ and $p \geq 2$. Besides,

$$\|v\|_{L^p(\Omega_t)} \leq C_2(m,n)|\Omega_t|\|v\|_{H^m_0(\Omega_t)}^{(2-p)/2p} \quad \text{for all } 1 \leq p \leq 2,$$

$$\leq C_2(m,n)(1 + |\Omega_t|)\|v\|_{H^m_0(\Omega_t)} \quad \text{since } 2 - p < 2p.$$

Therefore, for $n = 2m$ and $\Omega_t \subset \Omega$ for all $0 \leq t < T$ we get

$$\|v\|_{L^p(\Omega_t)} \leq C_2(m,n,|\Omega_t|)\|v\|_{H^m_0(\Omega_t)} \quad \text{for all } v \in H^m_0(\Omega_t). \quad (3.28)$$

Finally, (ii) implies $\|\tilde{v}\|_{L^p(\mathbb{R}^n)} \leq C_3(m,n)\|\tilde{v}\|_{H^m(\mathbb{R}^n)}$ for all $\tilde{v} \in H^m(\mathbb{R}^n)$ and $p = \frac{2n}{n-2m}$. In particular, $\|v\|_{L^p(\Omega_t)} \leq C_3(m,n)\|v\|_{H^m_0(\Omega_t)}$ for all $v \in H^m_0(\Omega_t)$ and $p = \frac{2n}{n-2m}$. Therefore, if $p < \frac{2n}{n-2m} = q$, then

$$\|v\|_{L^p(\Omega_t)} \leq C_3(m,n)|\Omega_t|\|v\|_{H^m_0(\Omega_t)}^{(q-p)/pq} \quad \text{since } q - p < pq.$$

Thus, for $n > 2m$, $\Omega_t \subset \Omega$ for all $0 \leq t < T$ and $p = \frac{2n}{n-2m}$ we get

$$\|v\|_{L^p(\Omega_t)} \leq C_3(m,n,|\Omega_t|)\|v\|_{H^m_0(\Omega_t)} \quad \text{for all } v \in H^m_0(\Omega_t). \quad (3.29)$$

According to (3.27)–(3.29) we obtain (3.24), and thus the proof of the lemma is finished. \(\square\)

4. Strong solutions in the cylinder \(Q\)

Our objective now is to show the existence of strong solutions for problem (2.2). The hypothesis (1.6) is replaced by

(i) $0 \leq \sigma_i$ if $n \leq 2m$,
(ii) $0 \leq \sigma_i \leq \frac{n}{n-2m}$ if $2m < n \leq 4m - 2$, \(\quad (4.1)\)
(iii) $0 \leq \sigma_i \leq \frac{4m - 2}{n-2m}$ if $n > 4m - 2$. 


The concept of strong solution for system (2.2) is given by

**Definition 4.1.** A strong solution for (2.2) is a function \( u : Q \rightarrow \mathbb{R} \) such that
\[
u, u' \text{ belong to } L^\infty(0, T; H_0^m(\Omega) \cap H^{2m}(\Omega)) \text{ for } T > 0,\]
and Eq. (2.2)1 is verified a.e. in \( Q \).

**Theorem 4.1.** If \( u_0 \in H_0^m(\Omega) \cap H^{2m}(\Omega) \) and the hypotheses (1.5) and (4.1) hold, then the mixed problem (2.2) has only one solution in the sense of Definition 4.1.

**Proof.** We again employ Faedo–Galerkin method with a special hilbertian basis \((w_j)_{j \in \mathbb{N}}\) of \( H_0^m(\Omega) \), where \( w_j \) is a solution of the eigenvalue problem \( a(w_j, \psi) = \lambda_j (w_j, \psi) \) for all \( \psi \in H_0^m(\Omega) \). In these conditions, we have by hypothesis
\[
u_N(0) = u_{0N} \rightarrow u_0 \quad \text{in } H_0^m(\Omega) \cap H^{2m}(\Omega). \tag{4.2} \]

To reach our objective, two more estimates are necessary besides Estimates 2.1, Section 2. Thus, our task is to obtain an estimate for \( u_N \) and \( u'_N \) in \( L^\infty(0, T; H_0^m(\Omega) \cap H^{2m}(\Omega)) \). In fact, setting \( w = Au_N(t) \in V_N \) in (2.3)1, yields
\[
\left( u'_N(t), Au_N(t) \right) + \sum_{|\alpha|=m} \left( D^\alpha u'_N(t), D^\alpha \left[ Au_N(t) \right] \right) - \left( \phi(u_N(t)), \nabla \left[ Au_N(t) \right] \right) = 0. \tag{4.3} \]

The first term in (4.3) can be modified by
\[
\left( u'_N(t), Au_N(t) \right) = \frac{1}{2} \frac{d}{dt} a\left( u_N(t), u_N(t) \right). \]

The second one by
\[
\sum_{|\alpha|=m} \left( D^\alpha u'_N(t), D^\alpha \left[ Au_N(t) \right] \right) = \left( Au'_N(t), Au_N(t) \right) = \frac{1}{2} \frac{d}{dt} \left| Au_N(t) \right|^2. \]

The nonlinear term is upper bounded by using (1.5)3 as follows
\[
\left| \left( \phi(u_N(t)), \nabla \left[ Au_N(t) \right] \right) \right| = \left| -\left( \phi'(u_N(t)) \frac{\partial u_N(t)}{\partial x_i}, Au_N(t) \right) \right| \leq A_1 \left( |1 + |u_N(t)||_{L^{r'}} \right) \left| \frac{\partial u_N(t)}{\partial x_i} \right|_{L^{r'}} \left| Au_N(t) \right|_{L^{r'}} \leq A_1 \left| u_N(t) \right|_{m} \left| u_N(t) \right|_{2m} + A_1 \left( \left| u_N(t) \right|_{L^{r'}} \right) \left| \frac{\partial u_N(t)}{\partial x_i} \right|_{L^{r'}} \left| u_N(t) \right|_{2m} \leq A_1 \left[ \left| u_N(t) \right|_{m} \left| u_N(t) \right|_{2m} + \left| u_N(t) \right|_{L^{r'}} \right] \left| \frac{\partial u_N(t)}{\partial x_i} \right|_{L^{r'}} \left| u_N(t) \right|_{2m}, \]

where we have used above the Hölder’s inequality for \( \frac{1}{r'} + \frac{1}{r} + \frac{1}{2} = 1 \). Hence by using the Sobolev embedding \( H_0^m(\Omega) \hookrightarrow L^{r'}(\Omega) \) and \( H^{2m-1}(\Omega) \hookrightarrow L^{r}(\Omega) \) the nonlinear term is upper bounded by
\[
\left| \left( \phi(u_N(t)), \nabla \left[ Au_N(t) \right] \right) \right|_{L^{r'}} \leq C \left[ \left| u_N(t) \right|_{m} \left| u_N(t) \right|_{2m} + \left| u_N(t) \right|_{L^{r'}} \right] \left| u_N(t) \right|_{2m}^2. \]
If the results above are substituted into (4.3) it yields
\[
\frac{1}{2} \frac{d}{dt} \left\{ a(u_N(t), u_N(t)) + |A u_N(t)|^2 \right\} \leq C \left[ \| u_N(t) \|_m \| u_N(t) \|_{2m} + \| u_N(t) \|_{\sigma_i} \| u_N(t) \|_{2m}^2 \right].
\]
Integrating from 0 to \( t \), by using the estimate (2.5) in the term with exponent \( \sigma_i \) and observing that in \( H^m_0(\Omega) \cap H^{2m}(\Omega) \) the seminorm \( |Au_N(t)| \) and the norm \( \| u_N(t) \|_{2m} \) are equivalent, we get
\[
\| u_N(t) \|_m^2 + \| u_N(t) \|_{2m}^2 \leq \| u_{0N} \|_m^2 + \| u_{0N} \|_{2m}^2 + C \int_0^t \left( \| u_N(s) \|_m^2 + \| u_N(s) \|_{2m}^2 \right) ds.
\]
Hence, by Gronwall’s inequality and (4.2) we get
\[
\| u_N(t) \|_m^2 + \| u_N(t) \|_{2m}^2 < C \quad \text{for all } t \in [0, T]\] with \( T > 0 \).
\[(4.4)\]
The last estimate is obtained setting \( w = Au'_N(t) \) into (2.3)1, which yields
\[
\| u'_N(t) \|_m^2 + \| u'_N(t) \|_{2m}^2 - \left( \phi(u_N(t)), \nabla [Au'_N(t)] \right) = 0.
\]
\[(4.5)\]
The nonlinear term of (4.5) is upper bounded by
\[
\left| \left( \phi(u_N(t)), \nabla [Au'_N(t)] \right) \right|_\mathbb{R} \leq \Lambda_i \left[ \| u_N(t) \|_m \| u'_N(t) \|_{2m} + \| u_N(t) \|_{\sigma_i} \| u_N(t) \|_{2m} \right].
\]
From this and Sobolev embeddings \( H^m_0(\Omega) \hookrightarrow L^{r^{\prime}, \sigma_i}(\Omega) \) and \( H^{2m-1}(\Omega) \hookrightarrow L^{r}(\Omega) \), which hold by hypothesis (4.1), we can write
\[
\left| \left( \phi(u_N(t)), \nabla [Au'_N(t)] \right) \right|_\mathbb{R} \leq C \left[ \| u_N(t) \|_m \| u'_N(t) \|_{2m} + \| u_N(t) \|_{\sigma_i} \| u_N(t) \|_{2m} \right] \leq C + \frac{1}{2} \| u'_N(t) \|_{2m}^2,
\]
where we have used in the last inequality the estimate (4.4). Substituting the above results into (4.5) we get
\[
\| u'_N(t) \|_m^2 + \frac{1}{2} \| u'_N(t) \|_{2m}^2 < C \quad \text{for all } t \in [0, T]\] with \( T > 0 \).
\[(4.6)\]
From estimate (4.6) we are able to take the limit on the linear terms of (2.3). The limit on the nonlinear term is gotten by the same arguments employed in Section 2 thanks to estimate (4.4). The uniqueness is shown by using the energy method and similar argument employed in Section 2. Thus, the proof of Theorem 4.1 is complete. \( \square \)

5. Regularity of strong solutions

Our goal here is to prove a result of regularity for strong solutions established in Section 4. We will achieve this goal supposing the hypothesis (4.1) in the case
\[
\sigma_i \geq 1 \quad \text{for } n \leq 4m.
\]
Besides it is assumed that
\[
\left| \phi'_i(\tau_1) - \phi'_i(\tau_2) \right| \leq \tilde{C}_i \left( |\tau_1|^{\sigma_i-1} + |\tau_2|^{\sigma_i-1} \right) |\tau_1 - \tau_2|.
\]
In these conditions we have the following regularity result.
Theorem 5.1. Let \( u = u(x, t) \) be a strong solution of problem (2.2) guaranteed by Theorem 4.1, then \( u \in C^1([0, T]; H^m_0(\Omega) \cap H^{2m}(\Omega)) \) provided (5.1), (5.2) hold.

Proof. We will show that \( u \) and \( u' \) are limits of Cauchy sequences. In fact, suppose \( \mu, \nu \in \mathbb{N} \) are fixed with \( \mu > \nu \) and \( u_\mu, u_\nu \) are two solutions of (2.3). Thus, \( v_\nu = u_\mu - u_\nu \) satisfies

\[
(v_\nu'(t), w) + (Av_\nu'(t), w) = (P_\mu(\text{div}(\phi(u_\mu(t)))) - P_\nu(\text{div}(\phi(u_\nu(t)))), w)
\]

for all \( w \in V_\mu \subset L^2(\Omega) \), where \( V_\mu \) is spanned by the \( \mu \) first eigenvector of operator \( A \) in \( H^m_0(\Omega) \) and \( P_\mu, P_\nu \) are projection operators defined in \( L^2(\Omega) \) with values on \( V_\mu, V_\nu \) respectively, see, for example, Halmos [9].

It is opportune to observe that if on the right side of identity (5.3) it is not considered the projections operators this would not make sense, because the solutions \( u_\nu \) of approximate problem in \( \mu \) dimension does not satisfy the one in \( \nu \) dimension for \( \mu > \nu \). This is a fact in view of the nonlinear term \( \text{div}(\phi(u_\nu)) \).

Setting \( w = Av_\nu \) in (5.3) we get

\[
\frac{1}{2} \frac{d}{dt} \left\{ ||v_\nu(t)||^2_m + ||v_\nu(t)||^2_{2m} \right\} \leq |P_\mu(\text{div}(\phi(u_\mu(t)))) - P_\nu(\text{div}(\phi(u_\nu(t))))||v_\nu(t)||^2_{2m}.
\]

Note that

\[
|P_\mu(\text{div}(\phi(u_\mu(t)))) - P_\nu(\text{div}(\phi(u_\nu(t))))| \\
\leq |P_\mu(\text{div}(\phi(u_\mu(t)))) - \text{div}(\phi(u_\nu(t))))| + |(P_\mu - P_\nu)(\text{div}(\phi(u_\nu(t)))) - \text{div}(\phi(u(t)))| \\
+ |(P_\mu - P_\nu)(\text{div}(\phi(u_\nu(t))))|.
\]

Since \( P_\mu \) is a projection operator we have

\[
|P_\mu(\text{div}(\phi(u_\mu(t)))) - \text{div}(\phi(u_\nu(t))))|^2 \\
\leq |\text{div}(\phi(u_\mu(t)))) - \text{div}(\phi(u_\nu(t))))|^2 \\
= \int_\Omega \left| \phi_i'(u_\mu(x, t)) \frac{\partial u_\mu}{\partial x_i}(x, t) - \phi_i'(u_\nu(x, t)) \frac{\partial u_\nu}{\partial x_i}(x, t) \right|^2 dx \\
\leq 2 \int_\Omega \left| \phi_i'(u_\mu(x, t)) - \phi_i'(u_\nu(x, t)) \right|^2 \left| \frac{\partial u_\mu}{\partial x_i}(x, t) \right|^2 dx \\
+ 2 \int_\Omega \left| \phi_i'(u_\nu(x, t)) \right|^2 \left| \frac{\partial u_\mu}{\partial x_i}(x, t) - \frac{\partial u_\nu}{\partial x_i}(x, t) \right|^2 dx.
\]

Now we analyze the two last integrals of (5.6):

- The first can be upper bounded as follows: from (5.1), (5.2) we have

\[
2 \int_\Omega \left| \phi_i'(u_\mu(x, t)) - \phi_i'(u_\nu(x, t)) \right|^2 \left| \frac{\partial u_\mu}{\partial x_i}(x, t) \right|^2 dx.
\]
\[
\lesssim \tilde{C}_i \int_{\Omega} \left( |u_\mu(x,t)|^{\sigma_i-1} + |u_\nu(x,t)|^{\sigma_i-1} \right)^2 |u_\mu(x,t) - u_\nu(x,t)|^2 \left( \frac{\partial u_\mu}{\partial x_i}(x,t) \right)^2 dx \\
\lesssim \tilde{C}_i \left( \|u_\mu(t)\|_{L^{2\sigma_i-1}(\Omega)}^{2\sigma_i-1} + \|u_\nu(t)\|_{L^{2\sigma_i-1}(\Omega)}^{2\sigma_i-1} \right)^2 \left( \|v_\nu(t)\|_{L^{\sigma_i}(\Omega)}^2 \right)^2 \\
\leq C \left( \|u_\mu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} + \|u_\nu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} \right)^2 \left( \|v_\nu(t)\|_{L^{\sigma_i}(\Omega)}^2 \right)^2 \\
\leq C \left( \|u_\mu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} + \|u_\nu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} \right)^2 \left( \|v_\nu(t)\|_{L^{\sigma_i}(\Omega)}^2 \right)^2 \\
\leq C \left( \|u_\mu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} + \|u_\nu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} \right)^2 \left( \|v_\nu(t)\|_{L^{\sigma_i}(\Omega)}^2 \right)^2.
\]

where we have used the hypothesis on \(\sigma_i\), which allows us to obtain \(q, r, s\) such that the immersions \(H^{2m-1}(\Omega) \hookrightarrow L^q(\Omega), H^{2m}(\Omega) \hookrightarrow L^r(\sigma_i-1)(\Omega), H^{2m}(\Omega) \hookrightarrow L^s(\Omega)\), hold for \(\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = \frac{1}{2}\).

- In the second integral initially we use (1.5) and Hölder inequality for \(\frac{1}{q}\) and \(\frac{1}{q'}\) chosen in function of \(\sigma_i\) such that \(H^{2m}(\Omega) \hookrightarrow L^{q\sigma_i}(\Omega), H^{2m-1}(\Omega) \hookrightarrow L^{q'}(\sigma_i)(\Omega)\) for \(\frac{1}{q'} + \frac{1}{q} = \frac{1}{2}\). Thus we have

\[
2 \int_{\Omega} \left( \|u_\mu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} + \|u_\nu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} \right)^2 \left( \|v_\nu(t)\|_{L^{\sigma_i}(\Omega)}^2 \right)^2 dx \\
\leq 2^2 \int_{\Omega} \left( \|u_\mu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} + \|u_\nu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} \right)^2 \left( \|v_\nu(t)\|_{L^{\sigma_i}(\Omega)}^2 \right)^2 dx \\
\leq C \left( \|u_\mu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} + \|u_\nu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} \right)^2 \left( \|v_\nu(t)\|_{L^{\sigma_i}(\Omega)}^2 \right)^2 \\
\leq C \left( \|u_\mu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} + \|u_\nu(t)\|_{L^{2\sigma_i}(\Omega)}^{2\sigma_i} \right)^2 \left( \|v_\nu(t)\|_{L^{\sigma_i}(\Omega)}^2 \right)^2.
\]

Substituting these two results into (5.6) and by using the estimate (4.4) yields

\[
\|v_\nu(t)\|_{2m}^2 + \|v_\nu(t)\|_{2m-1}^2 \lesssim \|v_0\|_{2m}^2 + \|v_0\|_{2m}^2 + C \left[ \int_0^T \|u_\nu(s)\|_{2m}^2 ds + \int_0^T \|u_\nu(t) - u(t)\|_{2m-1}^2 dt \\
+ \int_0^T \left( (P_\mu - P_\nu)(\text{div}(\phi(u(t)))) \right)^2 dt \right].
\]

From this and Gronwall inequality we obtain

\[
\|v_\nu(t)\|_{2m}^2 + \|v_\nu(t)\|_{2m-1}^2 \lesssim \tilde{C} \left[ \|v_0\|_{2m}^2 + \|v_0\|_{2m}^2 + \int_0^T \|u_\nu(t) - u(t)\|_{2m-1}^2 dt \right]
\]
\[ + \int_{0}^{T} \left| (P_{\mu} - P_{\nu}) \left( \text{div}(\phi(u(t))) \right) \right|^{2} dt \]. \quad (5.9)

Note that from (4.4), (4.6) and Aubin–Lions theorem we get
\[ \int_{0}^{T} \| u_{\nu}(t) - u(t) \|^{2m-1}_{2m} \, dt \to 0 \quad \text{as} \quad \mu, \nu \to \infty. \]

Besides, we also have
\[ \int_{0}^{T} \left| (P_{\mu} - P_{\nu}) \left( \text{div}(\phi(u(t))) \right) \right|^{2} \to 0 \quad \text{as} \quad \mu, \nu \to \infty. \quad (5.10) \]

In fact, \( \text{div}(\phi(u)) \in L^{2}(0, T; L^{2}(\Omega)) \), and thus \( \text{div}(\phi(u(t))) \in L^{2}(\Omega) \) a.e. in \([0, T]\). Therefore, \( (P_{\mu} - P_{\nu}) \left( \text{div}(\phi(u(t))) \right) \to 0 \) in \( L^{2}(\Omega) \) a.e. in \([0, T]\) as \( \mu, \nu \to \infty \). As \( |\text{div}(\phi(u(t)))| \in L^{2}(0, T) \), then (5.10) is true thanks to Lebesgue’s dominated convergence theorem. From this and (5.9) \( (u_{\nu})_{\nu} \in \mathbb{N} \) is a Cauchy sequence in \( C^{0}(0, T; H_{0}^{m}(\Omega) \cap H^{2m}(\Omega)) \).

Now, setting \( w = Av_{\nu}' \) in (5.3) we get
\[ \| v_{\nu}'(t) \|^{2}_{m} + \| v_{\nu}'(t) \|^{2}_{2m} \leq \left| P_{\mu} \left( \text{div}(\phi(u_{\mu}(t))) \right) - P_{\nu} \left( \text{div}(\phi(u_{\nu}(t))) \right) \right| \| v_{\nu}'(t) \|^{2}_{2m}. \]

By using similar arguments that implicated (5.9) we get
\[ \| v_{\nu}'(t) \|^{2}_{m} + \| v_{\nu}'(t) \|^{2}_{2m} \leq C \left[ \| v_{\nu}(t) \|^{2}_{2m} + \| u_{\nu}(t) - u(t)w \|^{2}_{2m-1} + \left| (P_{\mu} - P_{\nu}) \left( \text{div}(\phi(u(t))) \right) \right|^{2} \right]. \]

Therefore, \( (u_{\nu}')_{\nu} \in \mathbb{N} \) is a Cauchy sequence in \( C^{0}(0, T; H_{0}^{m}(\Omega) \cap H^{2m}(\Omega)) \). Thus, we have the desired regularity. And so, the proof of Theorem 5.1 is completed. \( \square \)

Acknowledgment

We acknowledge C. Mathias from Universidade Federal Rural do Rio de Janeiro for the constructive conversation about the development of this paper.

References