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Quasifibrations and Bott periodicity

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Abstract

We give a proof of the Bott periodicity theorem, along the lines proposed by McDuff, based on the construction of a quasifibration over U with contractible total space and $\mathbb{Z} \times \mathrm{BU}$ as fiber. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The periodicity theorem of Raoul Bott is one of the most important results in algebraic topology. This theorem is used to define K-theory, which is a generalized cohomology theory that has enormous impact in topology, geometry and analysis. Bott's original proof [5] used Morse theory (see also [11]). The proofs of Toda [21], Cartan and Moore [8] and Dyer and Lashof [10] were based on homological calculations with spectral sequences. Atiyah and Bott [3] (see also [14,22]) obtained the result from a study of bundles over the product space $X \times \mathbb{S}^2$, in terms of bundles over X. In [2] Atiyah gave a proof using the index of a family of linear elliptic differential operators (cf. also [4]). More recently, other proofs have appeared. Kono and Tokunaga [15] use cohomology and Chern classes; Latour [16] works with the space of Lagrangians; Giffen [12] and Harris [13] use classifying spaces of categories defined via simplicial spaces; and Bryan and Sanders [7] and Tian [20] use moduli spaces of instantons. In this paper, we give a proof which uses only quasifibrations and linear algebra (and some basic differential topology). It is based on a very beautiful idea of McDuff [17], whose program we develop. This proof is both simpler and more elementary than previous proofs.

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We shall construct a quasifibration $p:\mathrm{E}\to\mathrm{U}$ over the infinite-dimensional unitary group, such that its total space E is contractible and has $\mathbb{Z}\times\mathrm{BU}$ as fiber. This way, we obtain a long exact sequence

$$\begin{aligned}\cdots\to\pi_i(\mathbb{Z}\times\mathrm{BU})\to\pi_i(\mathrm{E})\to\pi_i(\mathrm{U})\\ \to\pi_{i-1}(\mathbb{Z}\times\mathrm{BU})\to\pi_{i-1}(\mathrm{E})\to\cdots,\end{aligned}\tag{1.1}$$

where $\pi_i(\mathrm{E})=0=\pi_{i-1}(\mathrm{E})$, so that, for $i>1$,

$$\pi_i(\mathrm{U})\cong\pi_{i-1}(\mathbb{Z}\times\mathrm{BU})\cong\pi_{i-1}(\mathrm{BU}),\tag{1.2}$$

and, for $i=1$,

$$\pi_1(\mathrm{U})\cong\mathbb{Z}.\tag{1.3}$$

For the time being, one has (locally trivial) fibrations $\mathrm{E}_k(\mathbb{C}^\infty)\to\mathrm{BU}_k$ with fiber U_k, where the base spaces are the classifying spaces of the unitary groups given by the colimits of Grassmann manifolds, and the total spaces are the corresponding colimits of Stiefel manifolds, such that, by passing again to the colimit, they determine a (locally trivial) fibration $\mathrm{EU}\to\mathrm{BU}$ with EU a contractible space and U as fiber (see [19]).

On the other hand, consider the *path space*

$$\mathrm{PBU}=\{\omega:I\to\mathrm{BU}\mid\omega(0)=x_0\}$$

of BU, where $x_0\in\mathrm{BU}$ is the base point. One knows that PBU is contractible and the map $q:\mathrm{PBU}\to\mathrm{BU}$, such that $q(\omega)=\omega(1)$, is a Hurewicz fibration with fiber $\Omega\mathrm{BU}$.

Clearly, if $p:E\to B$ is a quasifibration with fiber F, and $p':E'\to B$ is a Hurewicz fibration with fiber F', such that their total spaces are contractible, then there is a weak homotopy equivalence $F\to F'$. Moreover, their homotopy groups satisfy $\pi_{i-1}(F)\cong\pi_i(B)\cong\pi_{i-1}(F')$, $i\geqslant 1$. Therefore, by [18] and the Whitehead theorem, one obtains homotopy equivalences $\Omega\mathrm{BU}\simeq\mathrm{U}$ and $\mathbb{Z}\times\mathrm{BU}\simeq\Omega\mathrm{U}$. So we have isomorphisms

$$\pi_{i-2}(\mathrm{U})\cong\pi_{i-2}(\Omega\mathrm{BU})\cong\pi_{i-1}(\mathrm{BU}),\quad i\geqslant 2.$$

Whence, we obtain the desired theorem, as stated below.

Theorem 1.4 (Bott periodicity). *There is a homotopy equivalence* $\mathbb{Z}\times\mathrm{BU}\simeq\Omega\mathrm{U}$; *hence, for* $i\geqslant 2$ *there is an isomorphism*

$$\pi_i(\mathrm{U})\cong\pi_{i-2}(\mathrm{U}),$$

or, equivalently,

$$\pi_i(\mathrm{BU})\cong\pi_{i-2}(\mathrm{BU}).$$

Or, in other terms, again by (1.2) one has that

$$\pi_i(\mathbb{Z}\times\mathrm{BU})\cong\pi_{i+1}(\mathrm{U})\cong\pi_{i+1}(\Omega\mathrm{BU})\cong\pi_i(\Omega^2\mathrm{BU});$$

i.e., we obtain an isomorphism

$$\pi_i(\mathbb{Z}\times\mathrm{BU})\cong\pi_i(\Omega^2\mathrm{BU}),$$

which is another usual version of Bott periodicity. In particular, from (1.3) and Theorem 1.4, we obtain that

$$\pi_i(\mathrm{BU}) = \begin{cases} \mathbb{Z} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

2. Preliminaries

Let $-\infty \leqslant p \leqslant q \leqslant \infty$ (not three of them equal) and define

$$\mathbb{C}_p^q = \{z : \mathbb{Z} \to \mathbb{C} \mid z_i = 0 \text{ for almost all } i \text{ and if } i \leqslant p \text{ or } i > q\},$$

with the usual topology in the finite-dimensional case and the topology of the union in the infinite-dimensional case. Therefore, $\mathbb{C}_0^q = \mathbb{C}^q$, $\mathbb{C}_0^\infty = \mathbb{C}^\infty$, $\mathbb{C}_0^1 = \mathbb{C}$, $\mathbb{C}_0^0 = \{0\}$, etc. All spaces $\mathbb{C}_p^q$ are then subspaces of $\mathbb{C}_{-\infty}^\infty$. With these definitions, we have that if $-\infty < p \leqslant q < \infty$, then $\dim \mathbb{C}_p^q = q - p$; moreover, if $p \leqslant q \leqslant r$, then $\mathbb{C}_p^q \oplus \mathbb{C}_q^r = \mathbb{C}_p^r$.

We have the Grassmann manifold $\mathrm{G}_n(\mathbb{C}_0^q) = \{W \mid W \text{ is a subspace of } \mathbb{C}_0^q \text{ of dimension } n\}$ and $\mathrm{BU}_n = \mathrm{G}_n(\mathbb{C}_0^\infty) = \operatorname{colim}_q \mathrm{G}_n(\mathbb{C}_0^q)$, where the colimit is taken with respect to the maps

$$\mathrm{G}_n(\mathbb{C}_0^q) \to \mathrm{G}_n(\mathbb{C}_0^{q+1})$$

given by sending $W \subset \mathbb{C}_0^q$ to $W = W \oplus 0 \subset \mathbb{C}_0^q \oplus \mathbb{C}_q^{q+1} = \mathbb{C}_0^{q+1}$. Hence, BU_n can be seen as the set $\{W \mid W \text{ is a subspace of } \mathbb{C}_0^\infty \text{ of dimension } n\}$.

If L is any linear operator, then we shall denote by $\mathcal{E}_1(L) = \ker(L - \mathrm{I})$ the space of eigenvectors of L with eigenvalue 1.

Definition 2.1. Given $k \in \mathbb{Z}$ we define the *shift operator* by k coordinates

$$t_k : \mathbb{C}_{-\infty}^\infty \to \mathbb{C}_{-\infty}^\infty$$

by $t_k(z)_i = z_{i-k}$. These shift operators are continuous isomorphisms such that $t_0 = \mathrm{I}$ and $t_k \circ t_l = t_l \circ t_k = t_{k+l}$.

Definition 2.2. Given n, there is a map $j_n^{n+1} : \mathrm{BU}_n \to \mathrm{BU}_{n+1}$ sending $W \subset \mathbb{C}^\infty$ to $\mathbb{C} \oplus t_1(W) \subset \mathbb{C}^\infty$. So, we define BU as

$$\mathrm{BU} = \operatorname*{colim}_k \mathrm{BU}_k.$$

In order to compare this definition with a different way of stabilizing, we have to prove a lemma. But before that, we give a definition.

Definition 2.3. Take $W \subset \mathbb{C}_k^l$ and let m be such that $\mathbb{C}_k^m \subset W$. Then $W/\mathbb{C}_k^m$ will denote the orthogonal complement of $\mathbb{C}_k^m$ in W, i.e., if $\{e_{k+1}, \ldots, e_m\}$ is the canonical basis for $\mathbb{C}_k^m$, we complete it to an orthonormal basis $\{e_{k+1}, \ldots, e_m, w_1, \ldots, w_q\}$ of W; then $W/\mathbb{C}_k^m$ is spanned by $\{w_1, \ldots, w_q\}$ and we have that $\mathbb{C}_k^m \oplus (W/\mathbb{C}_k^m) = W$.

Lemma 2.4. *There is a homeomorphism*

$$\Phi : \mathrm{BU} \to \overline{\mathrm{BU}}^{\,0} = \bigl\{W \subset \mathbb{C}_0^\infty \mid \dim W < \infty \text{ and } \mathbb{C}_0^k \subset W \Leftrightarrow k = 0\bigr\}.$$

Proof. Take $W \in \mathrm{BU}_n$ and let k be the largest integer such that $\mathbb{C}_0^k \subset W$. Define $\Phi_n(W) = t_{-k}(W/\mathbb{C}_0^k) \in \overline{\mathrm{BU}}^{\,0}$. Obviously, the map $\Phi_n : \mathrm{BU}_n \to \overline{\mathrm{BU}}^{\,0}$ determines in the colimit the desired map Φ.

Φ is surjective, since, if $W \in \overline{\mathrm{BU}}^{\,0}$ and $\dim W = n$, then $W \in \mathrm{BU}_n$ and $\Phi_n(W) = W$, because in this case $k = 0$. (In fact, the map $\Psi : \overline{\mathrm{BU}}^{\,0} \to \mathrm{BU}$ such that $W \mapsto W$ is the inverse.)

It is also injective, since, if $V \in \mathrm{BU}_m$ and $W \in \mathrm{BU}_n$ are such that $\Phi_m(V) = \Phi_n(W)$, then, if p and q are the largest integers such that $\mathbb{C}_0^p \subset V$ and $\mathbb{C}_0^q \subset W$, one has

$$t_{-p}(V/\mathbb{C}_0^p) = t_{-q}(W/\mathbb{C}_0^q). \tag{2.5}$$

Therefore, the dimensions $m - p$ and $n - q$ coincide. Without losing generality, we may assume that $p \leqslant q$, so that, in particular, $q - p = n - m \geqslant 0$. Applying t_q and adding $\mathbb{C}_0^q$ on the left to both sides of (2.5), yields, on the left,

$$\begin{aligned}\mathbb{C}_0^q \oplus t_{q-p}(V/\mathbb{C}_0^p) &= \mathbb{C}_0^{q-p} \oplus \mathbb{C}_{q-p}^q \oplus t_{q-p}(V/\mathbb{C}_0^p)\\ &= \mathbb{C}_0^{q-p} \oplus t_{q-p}(\mathbb{C}_0^p \oplus V/\mathbb{C}_0^p) = \mathbb{C}_0^{q-p} \oplus t_{q-p}(V),\end{aligned}$$

which is the image of V in $\mathrm{BU}_{m+q-p} = \mathrm{BU}_n$; and, on the right,

$$\mathbb{C}_0^q \oplus t_0(W/\mathbb{C}_0^q) = W;$$

hence $j_m^n(V) = W$, where $j_m^n = j_{n-1}^n \circ \cdots \circ j_m^{m+1}$, and, thus, V and W represent the same element in BU. □

Definition 2.6. Define $\widetilde{\mathrm{BU}} = \{W \mid \mathbb{C}_{-\infty}^p \subset W \subset \mathbb{C}_{-\infty}^q, -\infty < p \leqslant q < \infty\}$, which is covered by the subspaces $\widetilde{\mathrm{BU}}^p = \{W \in \widetilde{\mathrm{BU}} \mid \mathbb{C}_{-\infty}^p \subset W$ and p is maximal$\}$, $p \in \mathbb{Z}$.

Clearly, the map $W \mapsto \mathbb{C}_{-\infty}^0 \oplus W$ determines a homeomorphism $\overline{\mathrm{BU}}^{\,0} \to \widetilde{\mathrm{BU}}^0$; similarly, $W \mapsto t_{-k}(W)$ determines a homeomorphism $\widetilde{\mathrm{BU}}^k \to \widetilde{\mathrm{BU}}^0$, so that one has a canonical homeomorphism

$$\mathbb{Z} \times \widetilde{\mathrm{BU}}^0 \to \widetilde{\mathrm{BU}}$$

given by the composite $(k, W) \mapsto t_k(W) \in \widetilde{\mathrm{BU}}^k \hookrightarrow \widetilde{\mathrm{BU}}$. By Lemma 2.4, we have proved the following.

Theorem 2.7. *There is a homeomorphism*

$$\mathbb{Z} \times \mathrm{BU} \to \widetilde{\mathrm{BU}}.$$

We define an operator U in $\mathbb{C}_{-\infty}^\infty$ as *unitary* if $\langle Uz, Uz'\rangle = \langle z, z'\rangle$ or, equivalently, if $UU^* = \mathrm{I}$, the identical operator in $\mathbb{C}_{-\infty}^\infty$, where U^* denotes the transposed conjugate operator of U. Let $\mathrm{U} = \{U \mid U$ is unitary and of finite type$\}$, where we understand by a unitary operator of *finite type* one for which there exist $r < s$ such that $Ue^i = e^i$ if $i \leqslant r$ or $i > s$, where $\{e^i\}$ denotes the canonical basis in $\mathbb{C}^{\infty}_{-\infty}$. In other words, a unitary operator of finite type can be represented as a direct sum of the form

$$\mathrm{I}^r_{-\infty} \oplus \widetilde{U} \oplus \mathrm{I}^{\infty}_s,$$

where I^n_m represents the identical operator or identity matrix on $\mathbb{C}^n_m$ and $\widetilde{U}$ is a unitary operator on $\mathbb{C}^s_r$ (or an $(s-r) \times (s-r)$ unitary matrix).

For convenience, we denote by U_n the *unitary group* of $n \times n$ matrices acting on $\mathbb{C}^{n/2}_{-n/2}$ if n is even, and on $\mathbb{C}^{(n-1)/2}_{-(n+1)/2}$ if n is odd, and consider the inclusions

$$i^{n+1}_n : \mathrm{U}_n \to \mathrm{U}_{n+1}$$

such that

$$U \mapsto \mathrm{I}^{-n/2}_{-(n+2)/2} \oplus U$$

if n is even, and

$$U \mapsto U \oplus \mathrm{I}^{(n+1)/2}_{(n-1)/2}$$

if n is odd.

The inclusions $i_n : \mathrm{U}_n \to \mathrm{U}$ such that $U \mapsto \mathrm{I}^{-n/2}_{-\infty} \oplus U \oplus \mathrm{I}^{\infty}_{n/2}$ if n is even, and $U \mapsto \mathrm{I}^{-(n+1)/2}_{-\infty} \oplus U \oplus \mathrm{I}^{\infty}_{(n-1)/2}$ if n is odd, determine an isomorphism

$$\operatorname*{colim}_n \mathrm{U}_n \cong \mathrm{U}.$$

Remark 2.8. Let $S : \mathbb{C}^{\infty}_{-\infty} \to \mathbb{C}^{\infty}_0$ be such that

$$Se^i = \begin{cases} e^{2i} & \text{if } i > 0, \\ e^{2|i|+1} & \text{if } i \leqslant 0; \end{cases}$$

then the usual inclusions $k^{n+1}_n : \mathrm{U}_n \to \mathrm{U}_{n+1}$ of the usual unitary groups acting on $\mathbb{C}^n_0$ and $\mathbb{C}^{n+1}_0$, respectively, such that $U \mapsto U \oplus I^{n+1}_n$, and the induced inclusion in the colimit, $k_n : \mathrm{U}_n \to \mathrm{U}$, up to a shuffling of the intermediate coordinates, induce i^{n+1}_n and i_n; that is, essentially, $i^{n+1}_n(U) = Sk^{n+1}_n(U)S^{-1}$, respectively $i_n(U) = Sk_n(U)S^{-1}$. Therefore, algebraically as well as topologically, the given definition of U coincides with the classical one.

Before passing to the proof of the main result of this paper, let us state a criterion to determine when a given map is a quasifibration, which is an easy consequence of theorem [9, 2.15] (see also [1, A.1.19]).

Lemma 2.9. *Let $B = \bigcup_n B_n$, where $B_{n-1} \subset B_n$ is a closed subspace, with the topology of the union and take a (surjective) map $p : E \to B$. Assume that there are trivializations $p^{-1}(B_n - B_{n-1}) \approx (B_n - B_{n-1}) \times F$. Furthermore, assume that for every n there is a neighborhood V_n of B_{n-1} in B_n and a strong deformation retraction $r_n : V_n \to B_{n-1}$ in B_n such that it has a lifting $\tilde{r}_n : p^{-1}(V_n) \to p^{-1}(B_{n-1})$, inducing a homotopy equivalence on the fibers. Then p is a quasifibration.*

3. Proof of the main theorem

Before constructing the desired quasifibration mentioned in the introduction, we shall study, more as a motivation, the finite-dimensional case; afterwards we shall show the stabilization.

Recall that an $n \times n$ matrix C with complex entries is *Hermitian* if $C = C^*$, where C^* denotes, as before, the transposed conjugate matrix of C. If $\langle -, - \rangle$ denotes the canonical Hermitian product in $\mathbb{C}_0^n$, then for any $v, w \in \mathbb{C}_0^n$, C satisfies the identity $\langle Cv, w \rangle = \langle v, Cw \rangle$. This implies, in particular, that the eigenvalues of the matrix C are real.

The set $\mathrm{H}_n(\mathbb{C})$ of all Hermitian $n \times n$ matrices has the structure of a real vector space. Let E_n be the topological subspace of $\mathrm{H}_n(\mathbb{C})$ consisting of matrices whose eigenvalues lie in the interval I. The space E_n is contractible through the homotopy $h : \mathrm{E}_n \times I \to \mathrm{E}_n$ such that $h(C, \tau) = (1 - \tau)C$, which starts with the identity and ends with the constant map with value the matrix 0.

Let $M_{n\times n}(\mathbb{C})$ be the complex vector space of complex $n \times n$ matrices and let $\mathrm{GL}_n(\mathbb{C})$ be the subgroup of the invertible ones (general linear group). One has a (differentiable) map

$$\exp : M_{n\times n}(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$$

given by

$$\exp(B) \equiv \mathrm{e}^B = \sum_{i=0}^{\infty} \frac{B^i}{i!} = \mathrm{I}_n + B + \frac{B^2}{2!} + \cdots,$$

which fulfills the exponential laws, whenever the matrices taken as exponents commute among themselves. One can easily check the following properties:

$$\mathrm{e}^{TBT^{-1}} = T\mathrm{e}^B T^{-1}$$

for any invertible operator T, and

$$\mathrm{e}^D = \begin{pmatrix} \mathrm{e}^{\lambda_1} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_n} \end{pmatrix} \quad \text{if } D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Let $M^a_{n\times n}(\mathbb{C}) \subset M_{n\times n}(\mathbb{C})$ be the real subspace of *skew-Hermitian* matrices, that is, of matrices A such that $A^* = -A$. If A is skew-Hermitian, then $(\mathrm{e}^A)^* = \mathrm{e}^{A^*} = \mathrm{e}^{-A}$, so that

$$(\mathrm{e}^A)^*\mathrm{e}^A = \mathrm{e}^{-A}\mathrm{e}^A = \mathrm{e}^0 = \mathrm{I}_n.$$

Therefore, the map exp defined above restricts to

$$\exp : M^a_{n\times n}(\mathbb{C}) \to \mathrm{U}_n.$$

One has an isomorphism $\mathrm{H}_n(\mathbb{C}) \to M^a_{n\times n}(\mathbb{C})$, given by $C \mapsto 2\pi \mathrm{i} C$. We define a map $p_n : \mathrm{E}_n \to \mathrm{U}_n$, by $p_n(C) = \exp(2\pi \mathrm{i} C)$; then the diagram

$$\begin{array}{ccc} M^a_{n\times n}(\mathbb{C}) & \xrightarrow{\exp} & \mathrm{U}_n \\ \uparrow \cong & & \\ \mathrm{H}_n(\mathbb{C}) & & \nearrow p_n \\ \uparrow & & \\ \mathrm{E}_n & & \end{array}$$

commutes.

Proposition 3.1. *The map p_n is surjective.*

Proof. Take $U \in \mathrm{U}_n$; we diagonalize this matrix taking another matrix $T \in \mathrm{U}_n$ and the product $T^{-1}UT$. Since the eigenvalues of a unitary matrix have norm 1, we have that

$$T^{-1}UT = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\lambda_1} & & & 0 \\ & \mathrm{e}^{2\pi\mathrm{i}\lambda_2} & & \\ & & \ddots & \\ 0 & & & \mathrm{e}^{2\pi\mathrm{i}\lambda_n} \end{pmatrix},$$

where $\lambda_i \in I$, $i = 1, 2, \ldots, n$. Take

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

and consider the matrix TDT^{-1}. Since $T \in \mathrm{U}_n$, then $T^{-1} = T^*$ and, therefore,

$$(TDT^{-1})^* = (TDT^*)^* = TD^*T^* = TDT^{-1},$$

that is, TDT^{-1} is Hermitian; thus, $TDT^{-1} \in \mathrm{E}_n$. Whence, we have that

$$\begin{aligned} p_n(TDT^{-1}) &= \mathrm{e}^{2\pi\mathrm{i}(TDT^{-1})} = \mathrm{e}^{T(2\pi\mathrm{i}D)T^{-1}} = T\mathrm{e}^{2\pi\mathrm{i}D}T^{-1} \\ &= T \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\lambda_1} & & & 0 \\ & \mathrm{e}^{2\pi\mathrm{i}\lambda_2} & & \\ & & \ddots & \\ 0 & & & \mathrm{e}^{2\pi\mathrm{i}\lambda_n} \end{pmatrix} T^{-1} = U. \quad \square \end{aligned} \tag{3.2}$$

Let us now analyze the fibers of p_n. For this, given a matrix $C \in \mathrm{E}_n$, consider the subspaces $\ker(C - \mathrm{I})$ and $\ker(p_n(C) - \mathrm{I})$.

If $v \in \ker(C - \mathrm{I})$, then $Cv = v$ and one has that

$$\begin{aligned} p_n(C)v = (\mathrm{e}^{2\pi\mathrm{i}C})v &= \left(\mathrm{I} + 2\pi\mathrm{i}C + \frac{(2\pi\mathrm{i})^2}{2!}C^2 + \cdots\right)v \\ &= \mathrm{I}v + 2\pi\mathrm{i}Cv + \frac{(2\pi\mathrm{i})^2}{2!}C^2v + \cdots \end{aligned}$$

$$= v + 2\pi\mathrm{i}v + \frac{(2\pi\mathrm{i})^2}{2!}v + \cdots$$
$$= \left(1 + 2\pi\mathrm{i} + \frac{(2\pi\mathrm{i})^2}{2!} + \cdots\right)v = \mathrm{e}^{2\pi\mathrm{i}}v = v.$$

Therefore, $\ker(C - \mathrm{I}) \subset \ker(p_n(C) - \mathrm{I})$. Hence, if $U \in \mathrm{U}_n$, we may define a map $g : p_n^{-1}(U) \to \mathrm{G}(\ker(U - \mathrm{I}))$, the *Grassmann space* of all finite-dimensional vector subspaces of $\ker(U - \mathrm{I})$, which maps $C \in p_n^{-1}(U)$ to the subspace $\ker(C - \mathrm{I}) = \mathcal{E}_1(v)$.

Lemma 3.3. *The map* $g : p_n^{-1}(U) \to \mathrm{G}(\ker(U - \mathrm{I}))$ *is bijective.*

Proof. g is surjective; indeed, let $V \subset \ker(U - \mathrm{I})$ be a subspace. We wish to construct a matrix $C_V \in p_n^{-1}(U) \subset \mathrm{E}_n$ such that $\ker(C_V - \mathrm{I}) = V$. For this, we construct a matrix T diagonalizing U adequately.

Let $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_{r+s}\}$ be an orthonormal basis of $\mathcal{E}_1(U)$ such that $\{v_1, \ldots, v_r\}$ is a basis of V. Since U is unitary, the orthogonal complement of $\mathcal{E}_1(U)$ in $\mathbb{C}^n$, $\mathcal{E}_1(U)^\perp$, is an invariant subspace under U; indeed, if $w \in \mathcal{E}_1(U)^\perp$ and $v \in \mathcal{E}_1(U)$, then $\langle Uw, v\rangle = \langle w, U^*v\rangle = \langle w, U^{-1}v\rangle = \langle w, v\rangle = 0$. That is, $U(\mathcal{E}_1(U)^\perp) \subset \mathcal{E}_1(U)^\perp$ and, therefore, we may find an orthonormal basis $\{v_{r+s+1}, \ldots, v_n\}$ of $\mathcal{E}_1(U)^\perp$ built up by eigenvectors of U, with eigenvalues different from 1.

Let $T \in \mathrm{U}_n$ be such that $T\mathrm{e}^i = v_i$, $i = 1, \ldots, n$, where e^i denotes the vectors of the canonical basis of $\mathbb{C}^n$. Then $T^{-1}UT = \widehat{D}$ is a diagonal matrix,

$$\widehat{D} = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \mathrm{e}^{2\pi\mathrm{i}\lambda_{r+s+1}} & & \\ & & & & \ddots & \\ 0 & & & & & \mathrm{e}^{2\pi\mathrm{i}\lambda_n} \end{pmatrix},$$

with $r + s$ ones in the diagonal and all other eigenvalues different from 1.

Take now

$$D = \begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & & \lambda_{r+s+1} & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & \lambda_n \end{pmatrix},$$

with r ones in the diagonal and zeros in the places $r + 1$ through $r + s$.

We shall see that $C_V = TDT^{-1}$ is the desired matrix. Clearly $\mathrm{e}^{2\pi\mathrm{i}D} = \widehat{D}$, so that

$$p_n(C_V) = \mathrm{e}^{2\pi\mathrm{i}C_V} = \mathrm{e}^{T(2\pi\mathrm{i}D)T^{-1}} = T\mathrm{e}^{2\pi\mathrm{i}D}T^{-1} = T\widehat{D}T^{-1} = U.$$

On the other hand, it is immediate to verify that $\mathcal{E}_1(C_V) = T(\mathcal{E}_1(D))$. But $\mathcal{E}_1(D) = \{z \in \mathbb{C}^n \mid z_j = 0 \text{ for } r < j \leqslant n\}$; since $Te_i = v_i$ for $i = 1, \ldots, n$ and V is the subspace generated by $v_1, \ldots, v_r$, then $T(\mathcal{E}_1(D)) = V$.

To check that the map $C \mapsto \mathcal{E}_1(C)$ is bijective, one only has to show that $C = C_{\mathcal{E}_1(C)}$. For that, observe that if C is Hermitian, then λ is an eigenvalue of C if and only if $\mathrm{e}^{2\pi\mathrm{i}\lambda}$ is an eigenvalue of $\mathrm{e}^{2\pi\mathrm{i}C}$. To see this, let $R \in \mathrm{U}_n$ be such that $D = R^{-1}CR$ is a diagonal matrix, then

$$R^{-1}\mathrm{e}^{2\pi\mathrm{i}C}R = \mathrm{e}^{R^{-1}2\pi\mathrm{i}CR} = \mathrm{e}^{2\pi\mathrm{i}R^{-1}CR} = \mathrm{e}^{2\pi\mathrm{i}D},$$

which is a diagonal matrix with an entry $\mathrm{e}^{2\pi\mathrm{i}\lambda}$ for each entry λ of D. Let now $C_1, C_2 \in \mathrm{E}_n$ be such that $\mathrm{e}^{2\pi\mathrm{i}C_1} = \mathrm{e}^{2\pi\mathrm{i}C_2}$ and that $\mathcal{E}_1(C_1) = \mathcal{E}_1(C_2)$, then $C_1 = C_2$. To see this, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of C_1 and $\mu_1, \ldots, \mu_n$ those of C_2. Since $\mathcal{E}_1(C_1) = \mathcal{E}_1(C_2)$, we may assume that $\lambda_k = \mu_k = 1$ for $1 \leqslant k \leqslant r = \dim \mathcal{E}_1(C_1)$ and, then, $\lambda_k \neq 1 \neq \mu_k$ if $r < k \leqslant n$. As we proved before, $\mathrm{e}^{2\pi\mathrm{i}\lambda_k}$ and $\mathrm{e}^{2\pi\mathrm{i}\mu_k}$, $1 \leqslant k \leqslant n$, are the eigenvalues of $\mathrm{e}^{2\pi\mathrm{i}C_1}$ and $\mathrm{e}^{2\pi\mathrm{i}C_2}$, respectively. Consequently, $\mathrm{e}^{2\pi\mathrm{i}\lambda_k} = \mathrm{e}^{2\pi\mathrm{i}\mu_k}$ for all k and, taking $r < k \leqslant n$, this implies that $\lambda_k = \mu_k$. We have proved, therefore, that $\lambda_k = \mu_k$ for all k, so that $C_1 = C_2$.

If, in particular, we apply what we did to $C_1 = C$ and $C_2 = C_{\mathcal{E}_1(C)}$, then we have that $C = C_{\mathcal{E}_1(C)}$. □

We may summarize all in the following theorem.

Theorem 3.4. *Let* E_n *be the space of Hermitian* $n \times n$ *matrices whose eigenvalues lie in the unit interval and let* $p_n : \mathrm{E}_n \to \mathrm{U}_n$ *be such that* $p_n(C) = \mathrm{e}^{2\pi\mathrm{i}C}$. *Then* E_n *is contractible,* p_n *is surjective and the fiber over each matrix* $U \in \mathrm{U}_n$ *is homeomorphic to the Grassmann space of* $\mathrm{G}(\mathcal{E}_1(U))$.

Let us now study two ways of stabilizing this result. The usual one is to take the canonical embeddings $\rho^n_{n+1} : \mathrm{E}_n \to \mathrm{E}_{n+1}$ and $\tau^n_{n+1} : \mathrm{U}_n \to \mathrm{U}_{n+1}$, given by

$$\rho_n(C) = \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathrm{E}_{n+1},$$

and by

$$\tau_n(U) = \left(\begin{array}{c|c} U & 0 \\ \hline 0 & \mathrm{I} \end{array}\right) \in \mathrm{U}_{n+1},$$

or as it was described above. It is immediate to check that the diagram

$$\begin{array}{ccc} \mathrm{E}_n & \xrightarrow{\rho^n_{n+1}} & \mathrm{E}_{n+1} \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p_{n+1}} \\ \mathrm{U}_n & \xrightarrow[\tau^n_{n+1}]{} & \mathrm{U}_{n+1} \end{array}$$

commutes. This way, one obtains a map $p' : \mathrm{colim}_n \mathrm{E}_n \to \mathrm{colim}_n \mathrm{U}_n$ such that $p' \circ \rho_n = p'|\mathrm{E}_n = p_n$.

Let us now analyze the fibers of p'. It is clear that if $U \in \mathrm{U}_n$, then $\mathcal{E}_1(\tau(U)) = \mathcal{E}_1(U) \oplus \mathbb{C}$, and that $\mathcal{E}_1(\rho(C)) = \mathcal{E}_1(C) \oplus 0$; therefore, we have the following commutative diagram:

$$\begin{array}{ccc} p_n^{-1}(U) & \xrightarrow{\rho_{n+1}^n} & p_{n+1}^{-1}(\tau(U)) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{G}(\mathcal{E}_1(U)) & \xrightarrow[\delta]{} & \mathrm{G}(\mathcal{E}_1(U) \oplus \mathbb{C}) \end{array}$$

where $\delta(V) = V \oplus 0$. For instance, if we take $U = \mathrm{I}$, we have $\delta : \mathrm{G}(\mathbb{C}_0^n) \to \mathrm{G}(\mathbb{C}_0^{n+1})$. This way, the fibers of p' are homeomorphic to $\coprod_{r \geqslant 0} \mathrm{G}_r(\mathbb{C}_0^\infty) = \coprod_{r \geqslant 0} \mathrm{BU}_r$.

Now we shall construct a new map $p : \mathrm{E} \to \mathrm{U}$ whose fiber will be $\mathbb{Z} \times \mathrm{BU}$, which is, in a certain way, a completion of p'. We define an operator C in $\mathbb{C}_{-\infty}^\infty$ as *Hermitian* if $\langle Cz, z' \rangle = \langle z, Cz' \rangle$ or, equivalently, if $C = C^*$. Let $\mathrm{E} = \{C \mid C$ is Hermitian, of finite type and with eigenvalues in $I\}$, where we understand for a Hermitian operator of *finite type* one for which there exist $r < s$ such that $Ce^i = 0$ if $i \leqslant r$ or $i > s$. In other words, a Hermitian operator of finite type is a sum

$$0_{-\infty}^r \oplus \widetilde{C} \oplus 0_s^\infty,$$

where 0_k^l denotes the 0-operator on $\mathbb{C}_k^l$ (the zero $(l-k) \times (l-k)$ matrix) and $\widetilde{C}$ is a Hermitian $(s-r) \times (s-r)$ matrix acting on $\mathbb{C}_r^s$. Again, as E_n, E is contractible.

We can define a map $p : \mathrm{E} \to \mathrm{U}$ by $p(C) = \exp(2\pi \mathrm{i} C)$, that is

$$p(C) = \mathrm{I}_{-\infty}^r \oplus \mathrm{e}^{2\pi \mathrm{i} \widetilde{C}} \oplus \mathrm{I}_s^\infty.$$

Take $U \in \mathrm{U}$; the space of eigenvectors of U with eigenvalue 1, $\ker(U - \mathrm{I})$, is obviously $\mathbb{C}_{-\infty}^r \oplus \ker(\widetilde{U} - \mathrm{I}_r^s) \oplus \mathbb{C}_s^\infty$ and, thus, isomorphic to $\mathbb{C}_{-\infty}^\infty$. We define the *Grassmannian*

$$\mathrm{G}_\infty\big(\ker(U - \mathrm{I})\big) = \big\{W \subset \ker(U - \mathrm{I}) \mid \mathbb{C}_{-\infty}^{r'} \subset W \text{ and } \dim(W/\mathbb{C}_{-\infty}^{r'}) < \infty\big\}.$$

We clearly have the following result.

Lemma 3.5. *For each $U \in \mathrm{U}$, there is a homeomorphism*

$$\varphi_U : \mathrm{G}_\infty(\ker(U - \mathrm{I})) \approx \widetilde{\mathrm{BU}}.$$

Analogously to Lemma 3.3, we have the following result.

Proposition 3.6. *Let $U \in \mathrm{U}$. Then $p^{-1}(U) \approx \widetilde{\mathrm{BU}} = \mathbb{Z} \times \mathrm{BU}$.*

Proof. It is enough to show that there is a homeomorphism

$$g_U : p^{-1}(U) \to \mathrm{G}_\infty\big(\ker(U - \mathrm{I})\big).$$

Observe first that if $C \in \mathrm{E}$, then $V_C = \ker(C - \mathrm{I}) = \ker(\widetilde{C} - \mathrm{I}_r^s) \subset \mathbb{C}_r^s$. Take $C \in p^{-1}(U)$, then define $g_U(C) = \mathbb{C}_{-\infty}^r \oplus V_C \in \mathrm{G}_\infty(\ker(U - \mathrm{I}))$.

We shall now show that g_U is surjective. Take $W \in \mathrm{G}_\infty(\ker(U - \mathrm{I}))$; hence, $W = \mathbb{C}^{r'}_{-\infty} \oplus \widetilde{W}$, with $\dim \widetilde{W} < \infty$.

Without loss of generality, we may assume $r' = r$. Since $W \subset \ker(U - \mathrm{I}) = \mathbb{C}^r_{-\infty} \oplus \ker(\widetilde{U} - \mathrm{I}^s_r) \oplus \mathbb{C}^\infty_s$, then, taking s sufficiently large, one has that $\widetilde{W} \subset \ker(\widetilde{U} - \mathrm{I}^s_r) = \mathcal{E}_1(\widetilde{U}) \subset \mathbb{C}^s_r$.

As in Lemma 3.3, let $\{v_1, \ldots, v_m\}$ be an orthonormal basis of $\widetilde{W}$, $\{v_{m+1}, \ldots, v_{m+n}\}$ an orthonormal basis of the orthogonal complement of $\widetilde{W}$ in $\mathcal{E}_1(\widetilde{U})$ and $\{v_{m+n+1}, \ldots, v_{s-r}\}$ an orthonormal basis of the orthogonal complement of $\mathcal{E}_1(\widetilde{U})$ in $\mathbb{C}^s_r$ (which is invariant under $\widetilde{U}$), this last built up by eigenvectors with eigenvalues different from 1.

If we define $T \in \mathrm{U}$ such that

$$Te^i = \begin{cases} e^i & \text{if } i \leqslant r, \\ v_{i-r} & \text{if } r < i \leqslant s, \\ e^i & \text{if } i > s \end{cases}$$

we have that $T^{-1}UT = \widehat{D}$ is diagonal of the form

$$\mathrm{I}^{r+m+n}_{-\infty} \oplus \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\lambda_{m+n+1}} & & \\ & \ddots & \\ & & \mathrm{e}^{2\pi\mathrm{i}\lambda_{s-r}} \end{pmatrix} \oplus \mathrm{I}^\infty_s.$$

If we take now

$$D = 0^r_{-\infty} \oplus \mathrm{I}^{r+m}_r \oplus 0^{r+m+n}_{r+m} \oplus \begin{pmatrix} \lambda_{m+n+1} & & \\ & \ddots & \\ & & \lambda_{s-r} \end{pmatrix} \oplus 0^\infty_{s-r},$$

we define $C_W = TDT^{-1}$, which is such that $p(C_W) = \mathrm{e}^{2\pi\mathrm{i}C_W} = T\mathrm{e}^{2\pi\mathrm{i}D}T^{-1} = T\widehat{D}T^{-1} = U$. Moreover, $g_U(C_W) = \mathbb{C}^r_{-\infty} \oplus \ker(C_W - \mathrm{I})$. The same argument as in Lemma 3.3 shows that $\ker(C_W - \mathrm{I}) = \widetilde{W}$, whence $g_U(C_W) = W$.

The map g_U is injective, since if C_1 and C_2 are such that $\exp(2\pi\mathrm{i}C_1) = \exp(2\pi\mathrm{i}C_2) = U$ and $\mathcal{E}_1(C_1) = \ker(C_1 - \mathrm{I}) = \ker(C_2 - \mathrm{I}) = \mathcal{E}_1(C_2)$; then we can prove in the same way as in the corresponding part of Lemma 3.3 that $C_1 = C_2$. □

In order to show that $p : \mathrm{E} \to \mathrm{U}$ is a quasifibration applying Lemma 2.9, we need two facts. First, we shall see that $p|_{\mathrm{U}_n - \mathrm{U}_{n-1}}$ is trivial, that is, we will define a homeomorphism

$$h : p^{-1}(\mathrm{U}_n - \mathrm{U}_{n-1}) \to (\mathrm{U}_n - \mathrm{U}_{n-1}) \times \widetilde{\mathrm{BU}},$$

such that $\mathrm{proj}_1 \circ h = p$.

For this, assume that n is even; the odd case is analogous. Let C be a matrix in $p^{-1}(\mathrm{U}_n - \mathrm{U}_{n-1})$; therefore, $U = p(C)$ is of the form $\mathrm{I}^{-n/2}_{-\infty} \oplus \widetilde{U}$, where $-n/2$ is maximal, i.e., $\widetilde{U}$ is not of the form $\mathrm{I}^{-n/2+1}_{-n/2} \oplus \widetilde{U}'$.

Take the homeomorphism $g_U : p^{-1}(U) \to \mathrm{G}_\infty(\ker(U - \mathrm{I}))$, given in the proof of Proposition 3.6. Since $\ker(U - \mathrm{I}) = \mathbb{C}^{-n/2}_{-\infty} \oplus \ker(\widetilde{U} - \mathrm{I}^\infty_{-n/2})$, it depends continuously on U, as does the homeomorphism g_U. Analogously, one can choose the homeomorphism $\varphi_U : \mathrm{G}_\infty(\ker(U - \mathrm{I})) \to \widetilde{\mathrm{BU}}$ of Lemma 3.5, depending continuously on U.

Consequently, the map h defined by $h(C) = (p(C), \varphi(C))$, where $\varphi(C) = \varphi_U(g_U(C))$, is a homeomorphism, since, fiberwise, it is one.

Second, we have to show that there is a neighborhood V_n of U_{n-1} in U_n and a strong deformation retraction of V_n onto U_{n-1}, which lifts to a strong deformation retraction of $p^{-1}(V_n)$ onto $p^{-1}(\mathrm{U}_{n-1})$ in $p^{-1}(\mathrm{U}_n)$. To see this, since U_{n-1} is a submanifold of U_n, we shall construct a tubular neighborhood V_n of the first in the second as follows.

Recall $\mathrm{H}_n(\mathbb{C})$, the space of Hermitian $n \times n$ matrices, and define $f : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{H}_n(\mathbb{C})$ by $f(A) = A^*A$. One can easily verify that f is smooth and has I as a regular value; therefore, $\mathrm{U}_n = f^{-1}(\mathrm{I})$ is a smooth manifold and if $W \in \mathrm{U}_n$, then the tangent space of U_n at W, $\mathrm{T}_W(\mathrm{U}_n)$, is the kernel of the differential of f at W, that is,

$$\mathrm{T}_W(\mathrm{U}_n) = \{A \in M_{n\times n}(\mathbb{C}) \mid A^*W = -W^*A\}.$$

Now recall that there is a Hermitian product in $M_{n\times n}(\mathbb{C})$, given by $\langle A, B\rangle = \mathrm{trace}(AB^*)$; thus, taking the real part of this product, we get an inner product $M_{n\times n}(\mathbb{C}) \times M_{n\times n}(\mathbb{C}) \to \mathbb{R}$. The restriction of this inner product to each tangent space $\mathrm{T}_W(\mathrm{U}_n) \subset M_{n\times n}(\mathbb{C})$ defines a Riemannian metric on U_n. Let $i : \mathrm{U}_{n-1} \hookrightarrow \mathrm{U}_n$ be the inclusion, such that $i(U) = U \oplus I$; then the differential $\mathrm{d}i : \mathrm{T}_U(\mathrm{U}_{n-1}) \to \mathrm{T}_{i(U)}(\mathrm{U}_n)$ is an inclusion mapping a matrix R to $R \oplus 0$, that is

$$\mathrm{d}i(R) = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}.$$

One can easily check that the orthogonal complement of $\mathrm{T}_U(\mathrm{U}_{n-1})$ in $\mathrm{T}_{i(U)}(\mathrm{U}_n)$ is given by

$$\mathrm{T}_U(\mathrm{U}_{n-1})^\perp = \left\{ \begin{pmatrix} 0 & b \\ -b^*U & \mathrm{i}t \end{pmatrix} \in M_{n\times n}(\mathbb{C}) \,\middle|\, b = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} \in \mathbb{C}^{n-1} \text{ and } t \in \mathbb{R} \right\},$$

which is a real $(2n-1)$-dimensional vector space. We denote by $N = \bigcup_{U\in \mathrm{U}_{n-1}} \mathrm{T}_U(\mathrm{U}_{n-1})^\perp$, the normal bundle of U_{n-1} in U_n.

Any vector space basis of $\mathrm{T}_\mathrm{I}(\mathrm{U}_n)$ provides a parallelization of U_n which defines a connection on it. This connection does not depend on the chosen basis and determines a spray on U_n. By [6], there exists $\varepsilon > 0$, such that $N_\varepsilon = \{v \in N \mid \|v\| < \varepsilon\}$ is an open neighborhood of the 0-section, and the exponential map associated to the spray, $\mathrm{Exp} : N_\varepsilon \to \mathrm{U}_n$, is an embedding onto a neighborhood of U_{n-1} in U_n. Now, since the geodesics of this spray are the integral curves of the left-invariant vector fields, then $\mathrm{Exp}(A) = L_U \exp((\mathrm{d}L_U)^{-1}(A))$, where $A \in \mathrm{T}_U(\mathrm{U}_{n-1})^\perp$, $L_U : \mathrm{U}_n \to \mathrm{U}_n$ is given by $L_U(W) = UW$, and exp is the usual exponential map defined above. Evaluating the differential of L_U, we obtain $\mathrm{Exp}(A) = U\exp(U^*A)$.

Therefore, we have the following description of a tubular neighborhood $V_n = \mathrm{Exp}(N_\varepsilon)$ of U_{n-1} in U_n as

$$\left\{ U\exp\left(U^*\begin{pmatrix} 0 & b \\ -b^*U & \mathrm{i}t \end{pmatrix}\right) \,\middle|\, U \in \mathrm{U}_{n-1},\ (b,t) \in \mathbb{C}^{n-1} \times \mathbb{R} \text{ and } \|(b,t)\| < \varepsilon \right\}.$$

In order to compute $U\exp(U^*\begin{pmatrix} 0 & b \\ -b^*U & \mathrm{i}t \end{pmatrix})$, first note that

$$U^*\begin{pmatrix} 0 & b \\ -b^*U & \mathrm{i}t \end{pmatrix} = \begin{pmatrix} 0 & U^*b \\ -b^*U & \mathrm{i}t \end{pmatrix}.$$

Set

$$A(b,t) = \begin{pmatrix} 0 & b \\ -b^* & \mathrm{i}t \end{pmatrix}.$$

Assume $b \neq 0$. To diagonalize this matrix, one takes an orthonormal basis of eigenvectors and uses it to form a matrix. The $n \times n$ matrix $A(b,t)$ has $n-2$ eigenvalues equal to 0 and two eigenvalues λ_1, λ_2, such that

$$\lambda_\nu = \frac{t + (-1)^\nu \sqrt{4|b|^2 + t^2}}{2}\mathrm{i},$$

so that the matrix

$$W(b,t) = \begin{pmatrix} v_1 & \cdots & v_{n-2} & \mu_1 b & \mu_2 b \\ 0 & \cdots & 0 & \mu_1\lambda_1 & \mu_2\lambda_2 \end{pmatrix},$$

where $\{v_1, \ldots, v_{n-2}\} \subset \mathbb{C}^{n-1}$ is an orthonormal basis of the space $b^\perp = \{v \mid v \perp b\} \subset \mathbb{C}^{n-1}$ and $\mu_\nu = (|b|^2 + |\lambda_\nu|^2)^{-1/2}$, is a unitary $n \times n$ matrix which satisfies

$$D(b,t) = W(b,t)^* A(b,t) W(b,t) = \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & & \lambda_1 & \\ 0 & & & & \lambda_2 \end{pmatrix}.$$

Since we can write $D(b,t) = W(b,t)^*UU^*A(b,t)UU^*W(b,t)$, and $A(U^*b,t) = U^*A(b,t)U$, then $A(U^*b,t) = U^*W(b,t)D(b,t)(U^*W(b,t))^*$. Therefore, the points in the tubular neighborhood are of the form $U\exp(A(U^*b,t)) = UU^*W(b,t)\exp(D(b,t)) W(b,t)^*U = \exp(A(b,t))U$.

Hence, every element in V_n coming from the fiber over U in N_ε is right translation by U of an element coming from the fiber over I. It is thus enough to study the situation over the identity matrix.

Since we may linearly deform the neighborhood N_ε to the zero section, simply by $v \mapsto (1-\tau)v$, $0 \leqslant \tau \leqslant 1$, we obtain a strong deformation retraction $r_n^\tau : V_n \to V_n$, such that

$$r_n^\tau\big(\exp(A(b,t))U\big) = \exp\big(A((1-\tau)b, (1-\tau)t)\big)U.$$

Observe that for $\tau = 1$, $r_n^1(\exp(A(b,t))U) = \exp(0)U = \mathrm{I}U = U$, so that it is a retraction of V_n onto U_{n-1}.

In what follows, we define the lifting $\tilde{r}_n^\tau : p^{-1}(V_n) \to p^{-1}(V_n)$. Since fiberwise, $p^{-1}(V_n)$ consists of spaces homeomorphic to the Grassmannians $G_\infty(\mathcal{E}_1(U'))$, $U' \in V_n$, we will show how $\tilde{r}_n^\tau$ acts on these spaces. It is clearly enough to study the case $\tau = 0$.

Take $U' = \exp(A(b,t))U \in V_n$ and let $G_{U,b,t} = G_\infty(\mathcal{E}_1(U'))$; we also have to show that the restriction of the lifting $\tilde{r}_n^1$, $\tilde{r}_n^1| : G_{U,b,t} \to G_{U,0,0} = G_\infty(\mathcal{E}(U))$ is a homotopy equivalence.

Since $\mathcal{E}_1(\exp(A(b,t))U) = U\mathcal{E}_1(\exp(A(U^*b,t))$ and for $b \neq 0$, $t \neq 0$, $\mathcal{E}_1(\exp(A(U^*b, t))) = \mathbb{C}_{-\infty}^{(n-4)/2} \oplus \mathbb{C}_{n/2}^{\infty}$, because $\mathrm{e}^{\lambda_1} \neq 1 \neq \mathrm{e}^{\lambda_2}$, we have that the Grassmannians $G_{U,b,t}$ and $G_{\mathrm{I},b,t}$ differ only by left multiplication by U. It is thus enough to study the case $U = \mathrm{I}$, namely the map

$$\tilde{r} : G_\infty(\mathbb{C}_{-\infty}^{(n-4)/2} \oplus \mathbb{C}_{n/2}^{\infty}) \to G_\infty(\mathbb{C}_{-\infty}^{\infty}).$$

If $V \subset \mathbb{C}_{-\infty}^{(n-4)/2} \oplus \mathbb{C}_{n/2}^{\infty}$ is a subspace, then we define $\tilde{r}(V) = V \subset \mathbb{C}_{-\infty}^{\infty}$, i.e., the map induced by the inclusion $\mathbb{C}_{-\infty}^{(n-4)/2} \oplus \mathbb{C}_{n/2}^{\infty} \hookrightarrow \mathbb{C}_{-\infty}^{\infty}$. The result now follows from the next proposition.

Proposition 3.7. *The inclusion $\mathbb{C}_{-\infty}^{r} \oplus \mathbb{C}_{s}^{\infty} \hookrightarrow \mathbb{C}_{-\infty}^{\infty}$, $r \leqslant s \in \mathbb{Z}$, induces a homotopy equivalence between the Grassmannians*

$$\alpha : G_\infty(\mathbb{C}_{-\infty}^{r} \oplus \mathbb{C}_{s}^{\infty}) \to G_\infty(\mathbb{C}_{-\infty}^{\infty}).$$

Proof. Take $V \in G_\infty(\mathbb{C}_{-\infty}^{\infty})$ and decompose it as $V = V_1 \oplus V_2$, where $V_1 \subset \mathbb{C}_{-\infty}^{r}$ and $V_2 \subset \mathbb{C}_{r}^{\infty}$, and define $\beta : G_\infty(\mathbb{C}_{-\infty}^{\infty}) \to G_\infty(\mathbb{C}_{-\infty}^{r} \oplus \mathbb{C}_{s}^{\infty})$ such that $\beta(V) = V_1 \oplus t_{s-r}V_2$, where t_{s-r} is the shift by $s-r$ coordinates (see Definition 2.1). Then $\alpha\beta(V) = V_1 \oplus t_{s-r}V_2 \subset \mathbb{C}_{-\infty}^{\infty}$ and $\beta\alpha(W) = W_1 \oplus t_{s-r}W_2$, if $W = W_1 \oplus W_2 \subset \mathbb{C}_{-\infty}^{r} \oplus \mathbb{C}_{s}^{\infty}$. The proposition now follows immediately from the next lemma. □

Lemma 3.8. *The map $\gamma : G_\infty(\mathbb{C}_{-\infty}^{r} \oplus \mathbb{C}_{s}^{\infty}) \to G_\infty(\mathbb{C}_{-\infty}^{r} \oplus \mathbb{C}_{s}^{\infty})$, $r \leqslant s \in \mathbb{Z}$, given by $\gamma(V) = V_1 \oplus t_k(V_2)$, $k \geqslant 0$, where $V = V_1 \oplus V_2$, $V_1 \subset \mathbb{C}_{-\infty}^{r}$ and $V_2 \subset \mathbb{C}_{s}^{\infty}$, is homotopic to the identity.*

Proof. The homotopy $h_\tau^1 = \sin(\frac{1}{2}\pi\tau)\mathrm{I} + \cos(\frac{1}{2}\pi\tau)t_1 : \mathbb{C}_{s}^{\infty} \to \mathbb{C}_{s}^{\infty}$, $0 \leqslant \tau \leqslant 1$, starts with t_1 and ends with the identity through monomorphisms, and $h_\tau^k = h_\tau^1 \circ \cdots \circ h_\tau^1$ (k times) is such that $h_0^k = t_k$ and $h_1^k = \mathrm{I}$. Then $\hat{h}_\tau(V) = V_1 \oplus h_\tau^k(V_2)$ is a homotopy as desired. □

We have thus shown that $\tilde{r}_n^1 : p^{-1}(V_n) \to p^{-1}(\mathrm{U}_{n-1})$ is fiberwise a homotopy equivalence. It should be remarked that for $s < 1$, the deformation $\tilde{r}_n^s : p^{-1}(V_n) \to p^{-1}(V_n)$ is fiberwise a homeomorphism, since after identifying the fibers with the associated Grassmannians, it is the identity. This behavior is congruent with the first fact needed for the verification of Lemma 2.9.

Thus we have our main theorem, which, as already seen in Section 1, implies Bott periodicity in the complex case.

Theorem 3.9. *Let* E *be the space of Hermitian operators on $\mathbb{C}_{-\infty}^{\infty}$ of finite type with eigenvalues in the unit interval, and let $p : \mathrm{E} \to \mathrm{U}$ be such that $p(C) = \exp(2\pi\mathrm{i}C)$. Then p is a quasifibration such that* E *is contractible and for each $U \in \mathrm{U}$, $p^{-1}(U) \approx \mathbb{Z} \times \mathrm{BU}$.*

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