DECOMPOSITIONS OF HYPERGRAPHS INTO HYPERSTARS

Zbigniew LONC
Institute of Mathematics, Technical University of Warsaw, Warsaw, Poland

Received 20 June 1984
Revised 19 November 1986

In the paper we investigate decompositions of hypergraphs into hyperstars.

A hyperstar with center $F$ and size $c$ is every hypergraph $(X, \mathcal{E})$ such that $F \subseteq \bigcap \mathcal{E}$ and $|\mathcal{E}| = c$. A decomposition of a hypergraph $(X, \mathcal{E})$ into hypergraphs from a certain class $\mathcal{K}$ is a family of hypergraphs $\{H_i = (X, \mathcal{E}_i) : i \in I\}$ such that $\{\mathcal{E}_i : i \in I\}$ is a partition of $\mathcal{E}$ and each $H_i$ is isomorphic to a hypergraph in $\mathcal{K}$.

In the paper we find necessary and sufficient conditions for existence of a decomposition of a hypergraph into hyperstars with given centers and sizes. This result is then applied to obtain sufficient conditions for existence of a hyperstar decomposition of the hypergraphs $P_m = (X, \mathcal{P}(X) \setminus \{\emptyset\})$ and $K_m^n = (X, \mathcal{P}_k(X))$, $|X| = m$. As a corollary, these results give a partial solution of a problem of Yamamoto and Tazawa [7] related to hyperstar decompositions of $K_m^n$.

1. Introduction

All undefined notions concerning hypergraphs can be found in the text book by Berge [1]. By $\mathcal{P}(X)$ (respectively $\mathcal{P}_k(X)$) we mean the family of all subsets (respectively $k$-element subsets) of a set $X$. For any real $x$ and positive integer $s$ we define

$$\binom{x}{s} = \frac{x(x-1) \cdots (x-s+1)}{s!}.$$

A hyperstar with center $F$ and size $c$ is every hypergraph $H = (X, \mathcal{E})$ such that $F \subseteq \bigcap \mathcal{E}$ and $|\mathcal{E}| = c$. Note that every hypergraph is a hyperstar with the empty center.

Let $\mathcal{K}$ be a family of hypergraphs. By a decomposition of a hypergraph $H = (X, \mathcal{E})$ into hypergraphs from $\mathcal{K}$ we mean a family of hypergraphs $\{H_i = (X, \mathcal{E}_i) : i \in I\}$ such that $\{\mathcal{E}_i : i \in I\}$ is a partition of $\mathcal{E}$ and each $H_i$ is isomorphic to a hypergraph in $\mathcal{K}$. Such decompositions were considered mainly in the case of graphs, an exhaustive list of references can be found in [3]. The general case of decompositions of hypergraphs was investigated by Bermond [2] and Yamamoto and Tazawa [7]. The latter authors deal with complete $n$-uniform hypergraphs $K_m^n = (X, \mathcal{P}_n(X))$, where $|X| = m$, and investigate the problem of existence of decompositions of $K_m^n$ into isomorphic hyperstars of size $c$ and $(n-1)$-element center. For $n = 2$, i.e., for complete graphs this problem was solved by Yamamoto.
et al. [6]. They proved that the complete graph of order $n$ has a decomposition into stars of size $c$ if and only if $c \mid \binom{m}{c}$ and $m \geq 2c$. In the case of $n \geq 3$ Yamamoto [8] gave the following necessary conditions of the existence of a decomposition of $K_n^m$ into hyperstars of size $c$ and $(n-1)$-element center:

\[
c \mid \binom{m}{c}
\]

and

\[
c \leq B(n, m) = \begin{cases} 
\frac{3}{8}(m-1), & \text{for } n = 3, \\
\frac{1}{8}(2m - 5 + \sqrt{7m^2 - 35m + 43}), & \text{for } n = 4, \\
m - n + 1, & \text{for } n \geq 5.
\end{cases}
\]

Yamamoto and Tazawa [7] raised a problem: what are the sufficient conditions for a decomposition of $K_n^m$ into hyperstars of size $c$ and $(n-1)$-element center to exist? In [7] and [8] the authors constructed suitable decompositions for some particular values of $m$, $n$ and $c$. These constructions allow to conjecture that the conditions (1) and (2) are the solution of the problem of Yamamoto and Tazawa. The case $n = 3$ was studied in more detail in [7]. The construction of the decomposition of $K_n^3$ into hyperstars of size $c = \frac{3}{8}(m-1)$ and 2-element center was given there.

The basic role in our considerations play theorems obtained in Section 2. They are simple generalizations for the case of hypergraphs of theorems on decompositions of graphs into stars proved by Tarsi [5]. In the remaining two sections we apply these results for some particular hypergraphs.

In Section 3 we consider decompositions of the hypergraph $P_m = (X, \mathcal{P}(X) \setminus \{\emptyset\})$, where $|X| = m$, into hyperstars with 1-element centers, i.e., partitions of the family of all nonempty subsets of an $m$-element set into subfamilies with nonempty intersection. In Section 4 we consider decompositions of $K_m^n$, $n < m$, into hyperstars with centres consisting of $k$ elements, $k < n$, i.e., partitions of the family of all $n$-element subsets of an $m$-element set into subfamilies whose intersection consists of at least $k$ elements. The theorems obtained are of two types. Theorems of the first type give sufficient conditions for the existence of a decomposition of a hypergraph into hyperstars under the assumption that each subset of the vertex set of the hypergraph is the center of at most one hyperstar. From these theorems follow the theorems of the second type, where we allow a subset of the vertex set to be a center of more than one hyperstar. Theorems of the same type are stated and proved in the similar way. As a corollary of the results obtained we have that (1) and the additional condition $c \leq (m - n + 1)/n(n - 1)$ ensures the existence of a decomposition of $K_n^m$ into hyperstars of size $c$ and $(n-1)$-element centers.
2. General theorems

Let $H = (X, \mathcal{E})$ be a hypergraph and let $\mathcal{F} \subseteq \bigcup_{E \in \mathcal{E}} \mathcal{P}(E)$, $\mathcal{P}(E) \cap \mathcal{F} \neq \emptyset$, for every $E \in \mathcal{E}$. For every family of sets $\mathcal{F} \subseteq \mathcal{E}$ we define

$$\mathcal{F} = \{F \in \mathcal{F} : F \subseteq E \text{ for some } E \in \mathcal{F}\},$$

and for every family of sets $\mathcal{G} \subseteq \mathcal{F}$ we define

$$\mathcal{G}^* = \{E \in \mathcal{E} : \mathcal{P}(E) \cap \mathcal{F} \subseteq \mathcal{G}\}.$$

It may readily be verified that $\mathcal{F} \subseteq (\mathcal{F})^*$ for $\mathcal{F} \subseteq \mathcal{E}$ and $(\mathcal{G}^*) \subseteq \mathcal{G}$ for $\mathcal{G} \subseteq \mathcal{F}$.

**Theorem 1.** Let $\delta$ be a function on $\mathcal{F}$ into non-negative integers. There is a decomposition $\{H_S : S \in \mathcal{F} \text{ and } \delta(S) > 0\}$ of $H$, where each $H_S$ is a hyperstar with center $S$ and size $\delta(S)$ if and only if

$$|\mathcal{G}^*| \leq \sum_{F \in \mathcal{G}} \delta(F), \quad \text{for every } \mathcal{G} \subseteq \mathcal{F},$$

and

$$|\mathcal{F}| = \sum_{F \in \mathcal{F}} \delta(F).$$

**Proof.** Let $G$ be a bipartite graph with vertex classes $\mathcal{F}$ and $Y = \bigcup_{F \in \mathcal{F}} \{x_F^1, \ldots, x_F^{\delta(F)}\}$ such that $E \in \mathcal{F}$ and $x_F^i \in Y$ are adjacent if and only if $F \in E$. Clearly there exists a required decomposition of $H$ into hyperstars if and only if $G$ has a perfect matching. This, by Hall's theorem, is equivalent to the conditions

$$|\mathcal{F}| \leq |\Gamma(\mathcal{F})|, \quad \text{for every } \mathcal{F} \subseteq \mathcal{E}$$

and

$$|\mathcal{G}| = |Y|,$$

where $\Gamma(\mathcal{F})$ denotes the set of vertices of $G$ adjacent to vertices in $\mathcal{F}$. We shall show that the conditions (5) and (6) are equivalent to (3) and (4). The equivalence of (4) and (6) is obvious. So, let us suppose that (5) holds. It is easily seen that $|\Gamma(\mathcal{F})| = \sum_{F \in \mathcal{F}} \delta(F)$. Hence for every $\mathcal{G} \subseteq \mathcal{F}$ we have

$$|\mathcal{G}^*| \leq \sum_{F \in (\mathcal{G}^*)} \delta(F) \leq \sum_{F \in \mathcal{G}} \delta(F),$$

the first inequality follows from (5) and the second one from the inclusion $(\mathcal{G}^*) \subseteq \mathcal{G}$. Conversely, let us assume (3). Then for $\mathcal{F} \subseteq \mathcal{E}$ we have

$$|\mathcal{F}| \leq |(\mathcal{F})^*| \leq \sum_{F \in \mathcal{F}} \delta(F) = |\Gamma(\mathcal{F})|. \quad \square$$

Let us define the following function

$$f(p, H) = \max\{|\mathcal{G}^*| : \mathcal{G} \subseteq \mathcal{F} \text{ and } |\mathcal{G}| = p\}, \quad \text{for } 1 \leq p \leq |\mathcal{F}|.$$
Let $t$ be the first number for which $f(t, H) \neq 0$; let us note that $t$ is well defined since $f(|\mathcal{S}|, H) = |\mathcal{E}|$. Now, we define

$$g(H) = \min \left\{ \left\lfloor \frac{|\mathcal{E}| - f(p, H) + 1}{|\mathcal{S}| - p} \right\rfloor - \left\lfloor \frac{f(p, H) - 1}{p} \right\rfloor : t \leq p \leq |\mathcal{S}| - 1 \right\}.$$  

(It may happen that $t = |\mathcal{S}|$. To cover this case we adopt the convention that $\min \emptyset = \infty$.) Let $(d_1, \ldots, d_{|\mathcal{S}|})$ be a sequence such that $d_1 = \cdots = d_r = 0 < d_{r+1} \leq \cdots \leq d_{|\mathcal{S}|}$. We say that a decomposition \{\mathcal{H}_1, \ldots, \mathcal{H}_{|\mathcal{S}|-r}\} of $H$ into hyperstars with centers in $\mathcal{S}$ corresponds to the sequence $(d_1, \ldots, d_{|\mathcal{S}|})$ if for every $1 \leq i \leq |\mathcal{S}| - r$ the size of $\mathcal{H}_i$ is equal to $d_{i+r}$.

**Theorem 2.** Let $0 \leq d_1 \leq \cdots \leq d_{|\mathcal{S}|}$. If

$$|\mathcal{E}| = \sum_{i=1}^{|\mathcal{S}|} d_i, \quad (7)$$

and either

$$f(p, H) \leq \sum_{i=1}^{p} d_i, \quad \text{for } p = 1, \ldots, |\mathcal{S}| - 1, \quad (8)$$

or

$$d_{|\mathcal{S}|} - d_1 \leq g(H) \quad (9)$$

holds, then there is a decomposition of $H$ into hyperstars with centers in $\mathcal{S}$ corresponding to the sequence $(d_1, \ldots, d_{|\mathcal{S}|})$.

**Proof.** We shall use Theorem 1. First we shall assume that (8) holds. Let us label the sets in $\mathcal{S}$ in an arbitrary way, say $\mathcal{S} = \{S_1, \ldots, S_{|\mathcal{S}|}\}$, and define $\delta(S_i) = d_i$, for $1 \leq i \leq |\mathcal{S}|$. Let $\mathcal{G} \subseteq \mathcal{S}$. Then we have

$$|\mathcal{G}^*| \leq f(|\mathcal{G}|, H) \leq \sum_{i=1}^{|\mathcal{S}|} d_i \leq \sum_{F \in \mathcal{G}} \delta(F),$$

the first inequality follows from the definition of $f$, the second one from (8) and the third one from the fact that $d_1 \leq \cdots \leq d_{|\mathcal{S}|}$ and from the definition of $\delta$. The condition (4) follows immediately from (7) so the assertion follows by Theorem 1. Now, let us assume (9). We shall show that (9) implies (8). To this end let us suppose that for some $p = 1, \ldots, |\mathcal{S}| - 1$ we have $f(p, H) > \sum_{i=1}^{p} d_i$. This implies that $p \geq t$ ($t$ occurs in the definition of $g$) and that

$$f(p, H) - 1 \geq \sum_{i=1}^{p} d_i \geq pd_1.$$

From (7) it follows that

$$|\mathcal{E}| - f(p, H) + 1 \leq \sum_{i=p+1}^{|\mathcal{S}|} d_i \leq (|\mathcal{S}| - p)d_{|\mathcal{S}|},$$
hence we have
\[ g(H) > d_{i_{\mathcal{F}}} - d_i = \left[ \frac{|\mathcal{F}| - f(p, H) + 1}{|\mathcal{F}| - p} \right] - \left[ \frac{f(p, H) - 1}{p} \right] \geq g(H). \]

This contradiction completes the proof. □

In the next two sections we shall apply the results obtained here to some particular hypergraphs.

3. Decomposition of \( P_m \)

Let us recall that \( P_m = (X, \mathcal{P}(X) - \emptyset) \), where \( |X| = m \).

**Theorem 3.** Let \( 0 \leq d_1 \leq \cdots \leq d_m \). If \( 2^m - 1 = \sum_{i=1}^{m} d_i \) and either (a) \( 2^p - 1 \leq \sum_{i=1}^{p} d_i \), for \( p = 1, \ldots, m - 1 \), or (b) \( d_m - d_1 < \frac{2^m}{m} \), then there is a decomposition of \( P_m \) into hyperstars with 1-element centers corresponding to the sequence \( (d_1, \ldots, d_m) \).

**Proof.** Let \( \mathcal{F} = \mathcal{P}_1(X) \). Sufficiency of (a) follows by Theorem 2 from the equality \( f(p, P_m) = 2^p - 1 \). Now, let us assume (b). By an easy but rather lengthy reasoning one can show that
\[ \left[ \frac{2^m - 2^p + 1}{m - p} \right] - \left[ \frac{2^p - 2}{p} \right] \geq \frac{2^m}{m}, \]
for \( 1 \leq p \leq m - 1 \). Hence
\[ g(P_m) = \min \left\{ \left[ \frac{2^m - 2^p + 1}{m - p} \right] - \left[ \frac{2^p - 2}{p} \right] : 1 \leq p \leq m - 1 \right\} \geq \frac{2^m}{m} > d_m - d_1, \]
and the assertion follows by Theorem 2 again. □

**Theorem 4.** Let \( (m_1, \ldots, m_r) \) be a sequence of positive integers such that \( \sum_{i=1}^{r} m_i = 2^m - 1 \) and \( m_i < \frac{2^m}{m} \) for \( i = 1, \ldots, r \). Then there is a decomposition of \( P_m \) into \( r \) hyperstars with 1-element centers and sizes \( m_1, \ldots, m_r \).

**Proof.** Let us divide the set \( M = \{m_1, \ldots, m_r\} \) into \( m \) disjoint parts \( B_1, \ldots, B_m \) (not necessarily nonempty). Denote by \( d_i \) the sum of elements in \( B_i \) and assume that \( d_1 \leq \cdots \leq d_m \). Suppose that the sets \( B_1, \ldots, B_m \) are chosen so that \( d_m - d_1 \) is least possible and the number of sets \( B_i \) having maximum sum of elements, is least possible. Now, suppose that \( d_m - d_1 \geq \frac{2^m}{m} \). Then after shifting an arbitrary \( m_i \) from \( B_m \) to \( B_1 \) we obtain either a partition with less difference between the greatest and the least \( d_i \) or a partition with the least possible difference but with less sets having maximum sum, in both cases this contradicts to the choice of \( B_1, \ldots, B_m \). Hence \( d_m - d_1 < \frac{2^m}{m} \) and the result follows by Theorem 3. □
4. Decompositions of complete \( n \)-uniform hypergraphs

In this section we shall consider decompositions of complete \( n \)-uniform hypergraphs \( K'_m = (X, \mathcal{P}_n(X)) \), where \( |X| = m \), into hyperstars with \( k \)-element centers. We shall assume that \( m > n > k > 0 \) to exclude trivial cases. Since for \( k = 1 \) we can obtain slightly stronger results than for arbitrary \( k \), the cases \( k = 1 \) and \( k \geq 2 \) will be dealt with separately.

**Theorem 5.** Let \( 0 \leq d_1 \leq \cdots \leq d_m \). If \( \binom{m}{n} = \sum_{i=1}^{m} d_i \) and either (a) \( \binom{m}{n} \leq \sum_{i=1}^{m} d_i \), for \( i = 1, \ldots, m - 1 \), or (b) \( d_m - d_1 \leq \lceil \binom{m}{n}/(m - n) \rceil \), then there is a decomposition of \( K'_m \) into hyperstars with 1-element centers corresponding to the sequence \( (d_1, \ldots, d_m) \).

**Proof.** Put \( \mathcal{S} = \mathcal{P}_1(X) \). It is easy to see that \( f(p, K'_m) = \binom{m}{n} \) and so sufficiency of (a) follows immediately from Theorem 2. Let us now suppose (b). First we shall prove the following auxiliary inequality:

\[
\binom{m}{n} \frac{p(p-n)}{m(m-n)} + 1 \geq \binom{p}{n},
\]

for \( m \geq p \geq n \geq 3 \) and \( m > n \). To this end let us consider the function

\[
\varphi(x) = \binom{x}{n} / x(x - n).
\]

Since

\[
\varphi'(x) \geq \frac{1}{n!} \frac{(x-1) \cdots (x-n+3) \cdot ((x-n)^2 - 2)}{(x-n)^3} \geq 0,
\]

for \( x \geq n + 2 \), \( \varphi \) is nondecreasing in the interval \( (n + 2, \infty) \). Moreover, it can easily be verified that if \( n \geq 3 \), then we have \( \varphi(n+1) \leq \varphi(n+2) \). Hence, for \( p = n + 1, \ldots, m \), \( \varphi(p) \leq \varphi(m) \) and (10) follows. For \( p = n \), (10) can be verified directly. Now we shall consider two cases.

**Case 1.** \( n \geq 3 \)

In this case, by (10), we obtain

\[
\binom{m}{n} \frac{p(p-n)}{m(m-n)} + 1 - \binom{p}{n} - 1 \geq \frac{\binom{m}{n} - \binom{p}{n}}{p} \frac{p(p-n)/m(m-n) + 1 + 1}{m-p}.
\]

\[
\binom{m}{n} \frac{p(p-n)/m(m-n) + 1 - 1}{p} = \binom{m}{n} \frac{m}{m-n},
\]

(11)
for \( n \leq p \leq m - 1 \). Since \( f(p, K_m^n) = 0 \) for \( p < n \), we have \( t = n \) and by (11)

\[
g(K_m^n) \geq \left\lceil \frac{\binom{m}{n}}{m-n} \right\rceil > d_m - d_1.
\]

So, the assertion follows from Theorem 2.

Case 2. \( n = 2 \)

For \( m \geq 4 \) we have

\[
\left\lceil \frac{\binom{m}{2} - \binom{p}{2} + 1}{m - p} \right\rceil - \left\lceil \frac{\binom{p}{2} - 1}{p} \right\rceil = \left\lceil \frac{m + p - 1}{2} + \frac{1}{m - p} \right\rceil - \left\lfloor \frac{p - 1}{2} - \frac{1}{p} \right\rfloor
\]

\[
\geq \frac{m + p}{2} - \frac{p - 2}{2} = \frac{m + 2}{2} \geq \frac{m}{2}/(m-2),
\]

and similarly as in 1 the assertion follows by Theorem 2. If \( m = 3 \), then \( d_3 - d_1 < 3 \) and consequently either \( d_1 = d_2 = d_3 = 1 \) or \( d_1 = 0, d_2 = 1, d_3 = 2 \). In both cases easy verification shows that the theorem holds. \( \square \)

**Theorem 6.** Let \((m_1, \ldots, m_r)\) be a sequence of positive integers such that

\[ \sum_{i=1}^{r} m_i = \binom{m}{n} \quad \text{and} \quad m_i < \left\lceil \frac{\binom{m}{n}}{(m-n)} \right\rceil, \quad \text{for} \quad i = 1, \ldots, r. \]

Then there is a decomposition of \( K_m^n \) into hyperstars with 1-element centers and sizes \( m_1, \ldots, m_r \).

This theorem is analogous to Theorem 4 and may be proved by exactly the same reasoning.

**Remarks.**

(1) Theorem 6 is best possible in the sense that with the condition \( m_i < \left\lceil \frac{\binom{m}{n}}{(m-n)} \right\rceil + 1 \) instead of \( m_i < \left\lceil \frac{\binom{m}{n}}{(m-n)} \right\rceil \) the assertion becomes false. To see this let \( m_1 = \cdots = m_{r-1} = \left\lceil \frac{\binom{m}{n}}{(m-n)} \right\rceil \) and \( m_r = \binom{m}{n} - (r-1) \left\lceil \frac{\binom{m}{n}}{(m-n)} \right\rceil \) and choose \( r \) so that \( 0 < m_r < \left\lceil \frac{\binom{m}{n}}{(m-n)} \right\rceil \). It is clear that \( r \leq m-n \). Hence if there existed a decomposition of \( K_m^n \) into hyperstars with 1-element centers then the set of centers of these hyperstars would consist of at most \( r \leq m-n \) elements and there would be an edge in \( K_m^n \) not containing any center. This edge would not belong to any hyperstar of the decomposition which is impossible.

(2) For \( n = 2 \) the condition of Theorem 6 reduces to

\[ m_i \leq \frac{1}{2}(m+1), \quad \text{for} \quad i = 1, \ldots, r, \]

which is slightly weaker than the condition

\[ m_i \leq \frac{1}{2}m, \quad \text{for} \quad i = 1, \ldots, r, \]

occurring in the theorem of Tarsi for graphs (see [5, Theorem 1, pp. 299–300]).
In the remaining part of this section we consider the problem of decomposing
the hypergraph $K^m_n$ into hyperstars with $k$-element centers, where $k$ is greater
than 1. This case is substantially more difficult than the case $k = 1$ and the results
are not so complete.

It is not difficult to prove (see e.g. [4, pp. 251–252]) that, for a fixed positive
integer $r$, every positive integer $a$ can uniquely be represented as the following
sum of binomial symbols:

$$a = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_1}{t},$$

for some $t, r \geq t \geq 1$, and $a_r > a_{r-1} > \cdots > a_t \geq t$.

For every $a$ represented as in (12) we define a fractional pseudopower
of $a$ as follows

$$a^{(r/r')} = \binom{a_r}{r'} + \binom{a_{r-1}}{r'-1} + \cdots + \binom{a_t}{r' - r + t}.$$

In the definition of fractional pseudopower we use the convention that $\binom{m}{n}$ = 0 for
$m < n$ or for $n < 0$. In particular we easily see that $a^{(r/r')} = 0$ if $r' > a_r$. The notion
of fractional pseudopower was introduced and its properties investigated by
Kruskal [4].

Now we are in a position to state and prove our results.

**Theorem 7.** Let $0 \leq d_1 \leq \cdots \leq d_{(r)}$. If $\binom{m}{n} = \sum_{i=1}^d d_i$ and $p^{(r/k)} \leq \sum_{i=1}^d d_i$, for
$p = 1, \ldots, (r/k)$, then there is a decomposition of $K^m_n$ into hypergraphs with
$k$-element centers corresponding to the sequence $(d_1, \ldots, d_{(r/)}$).

**Proof.** Put $\mathcal{G} = \mathcal{P}_k(X)$. Clearly, for any $\mathcal{G} \subseteq \mathcal{P}_k(X)$, $\mathcal{G}^* = \{E \subseteq \mathcal{P}_k(X): \mathcal{P}_k(E) \subseteq \mathcal{G}\}$. Then the well-known Kruskal–Katona theorem (see [4, p. 252]) asserts that

$$f(p, K^m_n) = \max \{|\mathcal{G}^*|: \mathcal{G} \subseteq \mathcal{P}_k(X) \text{ and } |\mathcal{G}| = p\} = p^{(r/k)}.$$

This in turn gives the assertion via Theorem 2 (condition (8)). \qed

Condition (9) of Theorem 2 enables us to obtain another sufficient condition
for the existence of a decomposition of $K^m_n$ into hyperstars with $k$-element
centers. To get it we will use a little bit weaker but easier to handle version of
Kruskal–Katona theorem due to Lovász [9].

Let $r$ and $r'$ be fixed positive integers. Notice that every positive integer can
uniquely be represented as $(\cdot)$, where $x$ is a real number and $x \geq r$. Denote by
$\varphi_{r,r'}$, a function such that for every positive integer $a = (\cdot)$, $\varphi_{r,r'}(a) = (\cdot)$.

Using the terminology defined above the theorem of Lovász can be formulated
as follows:

$$\max \{|\mathcal{G}^*|: \mathcal{G} \subseteq \mathcal{P}_k(X) \text{ and } |\mathcal{G}| = p\} \leq \varphi_{k,n}(p),$$

(13)
for \( p = 1, \ldots, \binom{m}{k} \), where \( \mathcal{G}^* \) is defined as in the proof of Theorem 7.\(^1\)

We shall need the following technical lemma.

**Lemma 8.** If \( 0 < k < n < m \) and \( 0 < a < \binom{m}{k} \), then

\[
\varphi_{k,n}(a) < \left( \frac{m}{n} \right) \left( \binom{x}{k} / \binom{m}{k} \right)^{n/k}.
\]

**Proof.** Let \( a = \binom{x}{k}, x > k \). It suffices to show that

\[
\left( \frac{x}{n} \right) < \left( \frac{m}{n} \right) \left( \frac{x}{k} / \binom{m}{k} \right)^{n/k}, \text{ for } k < x < n.
\]

To this end we prove that the function \( \psi(t) = \binom{t}{k}^k / (\binom{x}{k})^n \) is increasing for \( t \in (n-1, m) \). Notice that

\[
\psi(t) = \frac{(k!)^n t^k (t-1)^k \cdots (t-n+1)^k}{(n!)^k r^n (t-1)^n \cdots (t-k+1)^n}.
\]

Since \( k < n \), \( \psi \) is a product of a constant \( (k!)^n/(n!)^k \) and nondecreasing functions \( (t-\beta)/(t-\gamma) \), where \( \beta \geq \gamma \). Moreover the functions \( (t-\beta)/(t-\gamma) \) are positive for \( t \in (n-1, m) \) and at least one of them is increasing. Thus \( \psi \) is also increasing in the interval \( (n-1, m) \).

Now, inequality (14) follows from the inequality \( \psi(x) < \psi(m) \) for \( n-1 < x < m \). For \( k < x < n-1 \), \( \binom{x}{k} = 0 \) and \( (\binom{x}{k}) > 0 \) so (14) holds too. \( \square \)

**Theorem 9.** Let \( 0 \leq d_1 \leq \cdots \leq d(\mathcal{T}) \). If \( \binom{m}{n} = \sum_{i=1}^{\mathcal{T}} d_i \) and

\[
d(\mathcal{T}) - d_1 \leq \begin{cases} \left( \frac{m}{n} \right) / \binom{m}{k}, & \text{for } n \geq 2k, \\ \left( \frac{n}{k-1} \right) \cdot \left( \frac{m}{n} \right) / \binom{m}{k}, & \text{for } n < 2k, \end{cases}
\]

then there is a decomposition of \( K_m^n \) into hyperstars with \( k \)-element centers, corresponding to the sequence \( (d_1, \ldots, d(\mathcal{T})) \).

\(^1\) As a matter of fact, Lovász [9] proves a slightly different theorem, namely:

\[
\min \{|\tilde{\mathcal{F}}| : \tilde{\mathcal{F}} \subseteq \mathcal{P}_n(X) \text{ and } |\mathcal{F}| = p \} \geq \varphi_{k,n}(p).
\]

for \( p = 1, \ldots, \binom{m}{k} \), where \( \tilde{\mathcal{F}} = \{F \subseteq \mathcal{P}_n(X) : F \subseteq E \text{ for some } E \in \mathcal{F} \} \). Nevertheless, inequality (13) follows easily from (15). Using, in turn, the equality \( \varphi_{k,n}(\varphi_{n,k}(a)) = a \) (which is valid for every positive integer \( a \) and \( k \leq n \)), (15), the fact that \( \varphi_{k,n} \) is nondecreasing and the inclusion \( \mathcal{G}^* \subseteq \mathcal{G} \) we get

\[
|\mathcal{G}^*| = \varphi_{k,n}(\varphi_{n,k}(|\mathcal{G}^*|)) \leq \varphi_{k,n}(|\mathcal{G}^*|) \leq \varphi_{k,n}(|\mathcal{G}|),
\]

for every \( \mathcal{G} \subseteq \mathcal{P}_n(X) \).
Proof. Let \( \mathcal{F} = \mathcal{P}_k(X) \). By virtue of the theorem of Lovász we have

\[
f(p, K^n_m) \leq \varphi_{k,n}(p), \quad \text{for } p = 1, \ldots, \binom{m}{k} - 1.
\]

Using this inequality and Lemma 8 we obtain

\[
\frac{\binom{m}{n} - f(p, K^n_m)}{\binom{m}{k} - p} - \frac{f(p, K^n_m)}{p} \geq \frac{\binom{m}{n} - \varphi_{k,n}(p)}{\binom{m}{k} - p} - \frac{\varphi_{k,n}(p)}{p}
\]

\[
> \frac{(\binom{m}{n} - (\binom{m}{n} p / \binom{m}{k})^{n/k})}{\binom{m}{k} - p} - \frac{(\binom{m}{n} p / \binom{m}{k})^{n/k}}{p}
\]

\[
= \frac{\binom{m}{n}}{\binom{m}{k}} \frac{1 - (p / \binom{m}{k})^{(n/k) - 1}}{1 - p / \binom{m}{k}}.
\]

It is easy to see that

\[
\inf \left\{ \frac{1 - t^\alpha}{1 - t} : t \in (0, 1) \right\} = \begin{cases} \alpha, & \text{for } \alpha < 1, \\ 1, & \text{for } \alpha \geq 1, \end{cases}
\]

hence

\[
\frac{\binom{m}{n} - f(p, K^n_m)}{\binom{m}{k} - p} - \frac{f(p, K^n_m)}{p} \geq \begin{cases} \left( \frac{\binom{m}{n}}{\binom{m}{k}} \right), & \text{for } n \geq 2k, \\ \left( \frac{n - 1}{k} \right) \frac{\binom{m}{n}}{\binom{m}{k}}, & \text{for } n < 2k. \end{cases}
\]

This, together with Theorem 2 proves the assertion. \( \square \)

Theorem 10. Let \((m_1, \ldots, m_r)\) be a sequence of positive integers such that \( \sum_{i=1}^r m_i = \binom{m}{n} \) and

\[
m_i \leq \begin{cases} \left( \frac{\binom{m}{n}}{\binom{m}{k}} \right), & \text{for } n \geq 2k, \\ \left( \frac{n - 1}{k} \right) \frac{\binom{m}{n}}{\binom{m}{k}}, & \text{for } n < 2k. \end{cases}
\]

for \( i = 1, \ldots, r \). Then there is a decomposition of \( K^n_m \) into \( r \) hyperstars with \( k \)-element centers and sizes \( m_1, \ldots, m_r \).

We omit the proof which is almost the same as that of Theorem 4.

Let us observe that if we assume in Theorem 10, \( k = n - 1 \) and \( m_1 = \cdots = m_r = c \), then we obtain a condition for the existence of a decompositions of \( K^n_m \).
into hyperstars of \((n-1)\)-element centers and the same size \(c\), which were considered by Yamamoto and Tazawa [7] and Yamamoto [8].

**Corollary 11.** If \(c \mid \binom{m}{n}\) and \(c \leq \frac{(m-n+1)}{n(n-1)}\), then \(K^a_n\) has a decomposition into isomorphic hyperstars with \((n-1)\)-element center and size \(c\). \(\square\)

There still remains a question whether such decomposition exists for \(c\) satisfying \(\frac{(m-n+1)}{n(n-1)} < c \leq B(n, m)\). It is rather large gap since for \(n \geq 5\)

\[
B(n, m) \geq \frac{m-n+1}{n(n-1)} = n(n-1).
\]

Nevertheless, in case \(n = 3\), Corollary 11 proves that if \(c \mid \binom{m}{3}\), then a decomposition of \(K^3_m\) into stars of size \(c\) and \((n-1)\)-element center exists for \(c \leq \frac{1}{2}(m-2)\) which, roughly speaking, is one fourth of all possible values for \(c\).

## 5. Concluding remarks

Theorem 2 plays the crucial role in obtaining the results concerning the existence of decompositions of hypergraphs considered, i.e., \(P_n\) and \(K^a_n\). All results obtained are corollaries of this general fact. Unfortunately the sufficient conditions given in Theorem 2, and consequently in all subsequent results, are very strong. As a matter of fact, for every one-to-one correspondence \(\lambda : \mathcal{S} \rightarrow \{d_1, \ldots, d_{|\mathcal{S}|}\}\), they ensure the existence of a decomposition which has the additional property that every \(S \in \mathcal{S}\) is the center of a hyperstar with size \(\lambda(S)\). Hence they are far from being necessary. Here lies the reason for which the gap uncovered by Corollary 11 is so large. On the other hand Theorem 1 gives necessary and sufficient conditions for the existence of a decomposition of a hypergraph into hyperstars. Thus the general problem arises to infer from Theorem 1 sufficient conditions for the existence of a hyperstar decomposition which would be weaker than those provided in Theorem 2.

**Acknowledgment**

I am grateful to Miroslaw Truszczynski for help in the preparation of this paper.

**References**


