The Structure of Inner Multipliers on Spaces with Complete Nevanlinna Pick Kernels[1]

Devin C. V. Greene

Department of Mathematics, University of Nebraska, Lincoln, Nebraska 68588
E-mail: dgreene@math.unl.edu

and

Stefan Richter[2] and Carl Sundberg

Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1300
E-mail: richter@math.utk.edu, sundberg@math.utk.edu

Communicated by D. Sarason

Let k be the reproducing kernel for a Hilbert space $\mathscr{H}(k)$ of analytic functions on B_d, the open unit ball in $\mathbb{C}^d$, $d \geqslant 1$. k is called a complete NP kernel if $k_0 \equiv 1$ and if $1 - 1/k_\lambda(z)$ is positive definite on $B_d \times B_d$. Let $\mathscr{D}$ be a separable Hilbert space, and consider $\mathscr{H}(k) \otimes \mathscr{D} \cong \mathscr{H}(k, \mathscr{D})$, and think of it as a space of $\mathscr{D}$-valued $\mathscr{H}(k)$-functions. A theorem of McCullough and Trent (*J. Funct. Anal.* **178** (2000), 226–249) partially extends the Beurling–Lax–Halmos theorem for the invariant subspaces of the Hardy space $H^2(\mathscr{D})$. They show that if k is a complete NP kernel and if $\mathscr{D}$ is a separable Hilbert space, then for any scalar multiplier invariant subspace $\mathscr{M}$ of $\mathscr{H}(k, \mathscr{D})$ there exists an auxiliary Hilbert space $\mathscr{E}$ and a multiplication operator $\Phi: \mathscr{H}(k, \mathscr{E}) \to \mathscr{H}(k, \mathscr{D})$ such that Φ is a partial isometry and $\mathscr{M} = \Phi \mathscr{H}(k, \mathscr{E})$. Such multiplication operators are called inner multiplication operators and they satisfy $\Phi\Phi^* = $ the orthogonal projection onto $\mathscr{M}$. In this paper, we shall show that for many interesting complete NP kernels the analogy with the Beurling–Lax–Halmos theorem can be strengthened. We show that for almost every $z \in \partial B_d$ the nontangential limit $\phi(z)$ of the multiplier $\phi: B_d \to \mathscr{B}(\mathscr{E}, \mathscr{D})$ associated with an inner multiplication operator Φ is a partial isometry and that rank $\phi(z)$ is equal to a constant almost everywhere. The result applies to certain weighted Dirichlet spaces and to the space H_d^2, which is determined by the kernel $k_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle_d}$, $\lambda, z \in B_d$. In particular, our result implies that the curvature invariant of Arveson (*Proc. Natl. Acad. Sci. USA* **96** (1999), 11,096–11,099) of a pure contractive Hilbert module of finite rank is an integer. This answers a question of W. Arveson (*Proc. Natl Acad. Sci. USA* **96** (1999), 11096–11099). © 2002 Elsevier Science (USA)

1. INTRODUCTION

For a positive integer d we denote the unit ball in $\mathbb{C}^d$ by $B_d = \{\lambda \in \mathbb{C}^d : |\lambda| < 1\}$. If $\mathscr{H}$ is a Hilbert space of analytic functions on B_d such that for each $\lambda \in B_d$ the point evaluation $\lambda \mapsto f(\lambda)$ is a continuous linear functional on $\mathscr{H}$, then $\mathscr{H}$ has a reproducing kernel k; that is, for each $\lambda \in B_d$ there is a $k_\lambda \in \mathscr{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for each $f \in \mathscr{H}$. As a function of λ and z in B_d, $k_\lambda(z)$ is a positive definite function which is analytic in z and anti-analytic in λ. It is well known that k determines the space $\mathscr{H}$. Thus, we shall write $\mathscr{H}(k)$ for the space of analytic functions with reproducing kernel k.

An analytic function φ on B_d is a multiplier of $\mathscr{H}(k)$ if $\varphi f \in \mathscr{H}(k)$ for every $f \in \mathscr{H}(k)$. We shall write $M(k)$ for the collection of all multipliers. A standard argument with the closed graph theorem shows that each $\varphi \in M(k)$ defines a bounded linear operator $M_\varphi : f \mapsto \varphi f$ on $\mathscr{H}(k)$. Thus we define the multiplier norm by $\|\varphi\|_M = \|M_\varphi\|$. A subspace $\mathscr{M}$ of $\mathscr{H}(k)$ is called a multiplier invariant subspace if $\varphi \mathscr{M} \subseteq \mathscr{M}$, and we shall denote the collection of all multiplier invariant subspaces by Lat $M(k)$.

A reproducing kernel k on B_d is called a *complete Nevanlinna–Pick kernel* (*complete NP kernel*), if $k_0(z) = 1$ for all $z \in B_d$ and if there exists a sequence of analytic functions $\{b_n\}_{n \geqslant 1}$ on B_d such that

$$1 - \frac{1}{k_\lambda(z)} = \sum_{n \geqslant 1} b_n(z)\overline{b_n(\lambda)} \quad \text{for all } \lambda, z \in B_d. \tag{1.1}$$

We note that the sequence may be finite, and that this condition is actually equivalent to the assumption that $1 - 1/k$ is positive definite. Complete NP kernels have been investigated in connection with Nevanlinna–Pick and Caratheodory interpolation and commutant lifting properties (see [Agla, Aglb, BTV, McC92, Qui93]). Examples of spaces with complete NP kernels on the unit disk $\mathbb{D} = B_1$ are the Hardy space $H^2(\mathbb{D})$ of the unit disk $\mathbb{D}$, the Dirichlet space of all analytic functions on $\mathbb{D}$ whose derivative is square area integrable, or, more generally, the weighted Dirichlet spaces D_α, $\alpha \geqslant 0$ on the unit disk (see Example 4.4 for definitions and details). For $d \geqslant 1$ we mention the space H_d^2 on B_d, which is defined by the kernel $k_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle_d}$, $\langle z, \lambda \rangle_d = \sum_{i=1}^d z_i \overline{\lambda}_i$. The space H_d^2 was investigated in [Arv98, Arv99, Arv00] because of its connection to the dilation theory of certain commuting operator tuples, the so-called d-contractions, or row contractions. In Proposition 2.3 we shall see that for all complete NP kernels k one has $\mathscr{H}(k) \subseteq H^2(\partial B_d)$, the ordinary Hardy space of the unit ball. But we note that for $d > 1$, the reproducing kernel for $H^2(\partial B_d)$ is not a complete NP kernel.

We shall now fix a complete NP kernel k and a sequence $\{b_n\}$ as in (1.1). One shows that each $b_n \in M(k)$ and that $P_0 = \sum_{n \geqslant 1} M_{b_n} M_{b_n}^*$ (SOT), where P_0 is the projection onto the multiplier invariant subspace $\mathcal{H}_0 = \{f \in \mathcal{H}(k) : f(0) = 0\}$ (see Lemma 1.4 of [MT00]). We have used the term (SOT) to denote that the sum converges in the strong operator topology. It is remarkable that it follows that the projection onto every multiplier invariant subspace can be written in a similar manner. The general case of the following theorem is due to McCullough and Trent [MT00], and for the special case of $\mathcal{H}(k) = H_d^2$ it was found by Arveson in [Arv00].

THEOREM 1.1.   *Let k be a complete NP kernel and let $\mathcal{M}$ be a multiplier invariant subspace. Then there exists a sequence $\{\varphi_n\} \subseteq M(k) \cap \mathcal{M}$ such that*

$$P_{\mathcal{M}} = \sum_{n \geqslant 1} M_{\varphi_n} M_{\varphi_n}^* (\text{SOT}), \tag{1.2}$$

where $P_{\mathcal{M}}$ is the projection onto $\mathcal{M}$.

We make several remarks. First, McCullough and Trent prove this theorem in a somewhat more general setting; it is not even necessary that the kernel k is reproducing for a space of *analytic* functions. Secondly, by applying expression (1.2) to the reproducing kernel k_λ, $\lambda \in B_d$, one obtains

$$\sum_{n \geqslant 1} |\varphi_n(\lambda)|^2 = \frac{\|P_{\mathcal{M}} k_\lambda\|^2}{\|k_\lambda\|^2} \leqslant 1. \tag{1.3}$$

Thus, each function φ_n is in the unit ball of $H^\infty(B_d)$ and therefore for a.e. $z \in \partial B_d$ the nontangential limit $\varphi_n(z)$ of φ exists. Here, and in what follows a.e. stands for a.e. $[\sigma]$ where σ is the rotationally invariant probability measure on ∂B_d.

Of course, Beurling's theorem implies that for $H^2(\mathbb{D})$ the sequence $\{\varphi_n\}$ can be chosen to be a single inner function φ, which satisfies $|\varphi(z)| = 1$ for a.e. $z \in \partial \mathbb{D}$. Our first main result is the following.

THEOREM 1.2.   *Let k be a complete NP kernel on B_d and assume that there is a set $\mathscr{P} \subseteq \mathcal{H}(k) \cap C(\overline{B_d})$ which is dense in $\mathcal{H}(k)$ and such that for all $p \in \mathscr{P}$ and $z \in \partial B_d$, $\lim_{\lambda \to z} \frac{\|p k_\lambda\|}{\|k_\lambda\|} = |p(z)|$. Then any sequence $\{\varphi_n\}$ which is associated with a nonzero multiplier invariant subspace $\mathcal{M}$ as in Theorem (1.1) is an inner sequence, i.e.*

$$\sum_{n \geqslant 1} |\varphi_n(z)|^2 = 1 \quad \text{for } [\sigma] \text{ a.e. } z \in \partial B_d.$$

We shall show in Section 4 that the hypothesis of the theorem is satisfied for many complete NP kernels that are invariant under the unitary group $\mathcal{U}$. We shall call k unitary invariant, or $\mathcal{U}$-invariant, if $k_\lambda(z) = f(\langle z, \lambda \rangle_d)$ for some function f on the unit disk. One can show that the $\mathcal{U}$-invariant kernels are precisely the kernels that satisfy $k_{U\lambda}(Uz) = k_\lambda(z)$ for each unitary map $U : \mathbb{C}^d \to \mathbb{C}^d$. More precisely, we shall see that whenever a complete NP kernel k is of the form

$$k_\lambda(z) = \sum_{n=0}^{\infty} a_n(\langle z, \lambda \rangle_d)^n,$$

where $a_n > 0$, $\sum_{n=0}^{\infty} a_n = \infty$, and $\lim_{n \to \infty} a_n/a_{n+1} = 1$, then the hypothesis of Theorem 1.2 is satisfied. In particular, the theorem applies to the weighted Dirichlet spaces D_α, $0 \leqslant \alpha \leqslant 1$, and to the space H_d^2, and we shall see that if k is $\mathcal{U}$-invariant then the multiplier invariant subspaces are exactly the subspaces which are invariant under the multiplication by all the coordinate functions $z \mapsto z_i$, $i = 1, \ldots, d$.

However, we shall see that the conclusion of Theorem 1.2 does not hold for the weighted Dirichlet spaces D_α, $\alpha > 1$.

For the space H_d^2 this theorem was conjectured by Arveson [Arv99, Arv00]. He proved the theorem for invariant subspaces $\mathcal{M}$ of H_d^2 which contain a polynomial. In [MT00] the theorem is proved for certain $\mathcal{M}$ of finite codimension in spaces with complete NP kernels k such that $k_\lambda(\lambda) \to \infty$ as $\lambda \to \partial B_d$.

It turns out that vector-valued analogs of Theorems 1.1 and 1.2 are true. Before we can explain this, we need a few more definitions.

If $\mathscr{D}$ is a separable complex Hilbert space, then $\mathscr{H}(k, \mathscr{D})$ is the space of $\mathscr{D}$-valued $\mathscr{H}(k)$-functions. It is the set of all analytic functions $f : B_d \to \mathscr{D}$ such that for each $x \in \mathscr{D}$ the function $f_x(\lambda) = \langle f(\lambda), x \rangle_{\mathscr{D}}$ defines a function in $\mathscr{H}(k)$ and such that

$$\|f\|^2 = \sum_{n=1}^{\infty} \|f_{e_n}\|^2 < \infty$$

for some orthonormal basis $\{e_n\}_{n \geqslant 1}$ of $\mathscr{D}$. One shows that the above expression is independent of the choice of orthonormal basis. In particular, one has for $f \in \mathscr{H}(k)$, $x \in \mathscr{D}$ the function $fx : \lambda \to f(\lambda)x$ is in $\mathscr{H}(k, \mathscr{D})$ and $\|fx\| = \|f\| \|x\|_{\mathscr{D}}$. If $f \in \mathscr{H}(k, \mathscr{D})$, $x \in \mathscr{D}$, and $\lambda \in B_d$ we have $\langle f(\lambda), x \rangle_{\mathscr{D}} = \langle f, k_\lambda x \rangle$. There is an obvious identification of the tensor product $\mathscr{H}(k) \otimes \mathscr{D}$ with $\mathscr{H}(k, \mathscr{D})$, where one identifies the elementary tensors $f \otimes x$ with the functions fx. Considering the definition of the norm in $\mathscr{H}(k, \mathscr{D})$, one may also think of $\mathscr{H}(k, \mathscr{D})$ as a direct sum of $\dim \mathscr{D}$ copies of the scalar valued space $\mathscr{H}(k)$.

Each (scalar valued) multiplier $\varphi \in M(k)$ defines an operator on $\mathscr{H}(k, \mathscr{D})$ of the same norm, and we shall also denote this operator by M_φ. Again, we shall say that a subspace $\mathscr{M}$ of $\mathscr{H}(k, \mathscr{D})$ is multiplier invariant if $M_\varphi \mathscr{M} \subset \mathscr{M}$ for each $\varphi \in M(k)$.

Let $\mathscr{D}$ and $\mathscr{E}$ be two separable Hilbert spaces, and let $\phi : B_d \to \mathscr{B}(\mathscr{E}, \mathscr{D})$ be an operator valued analytic function. For $\lambda \in B_d$ and $f \in \mathscr{H}(k, \mathscr{E})$ we define $(\Phi f)(\lambda) = \phi(\lambda) f(\lambda)$, then Φf is a $\mathscr{D}$-valued analytic function. If $\Phi f \in \mathscr{H}(k, \mathscr{D})$ for every $f \in \mathscr{H}(k, \mathscr{E})$, then ϕ is called an operator-valued multiplier, and one shows that the associated multiplication operator $\Phi : \mathscr{H}(k, \mathscr{E}) \to \mathscr{H}(k, \mathscr{D})$ is bounded. It is clear that such multiplication operators $\Phi \in \mathscr{B}(\mathscr{H}(k, \mathscr{E}), \mathscr{H}(k, \mathscr{D}))$ intertwine the (scalar) multiplication operators M_φ, $\varphi \in M(k)$. It will follow from Lemma 2.2 that for spaces with complete NP kernels the multipliers $M(k)$ are dense in $\mathscr{H}(k)$. It then follows from standard arguments that a bounded linear operator $A : \mathscr{H}(k, \mathscr{E}) \to \mathscr{H}(k, \mathscr{D})$ intertwines every M_φ, $\varphi \in M(k)$ (i.e. $AM_\varphi = M_\varphi A$) if and only if $A = \Phi$ for some multiplication operator.

A short calculation shows that for any multiplication operator $\Phi \in \mathscr{B}(\mathscr{H}(k, \mathscr{E}), \mathscr{H}(k, \mathscr{D}))$ one has $\Phi^*(k_\lambda x) = k_\lambda \phi(\lambda)^* x$ for all $x \in \mathscr{D}$, $\lambda \in B_d$. Thus, we have $\|\phi(\lambda)\|_{\mathscr{D}} \leqslant \|\Phi\|$ for all $\lambda \in B_d$ and it follows from standard arguments that for a.e. $z \in \partial B_d$, $\phi(\lambda)$ converges in the strong operator topology to an operator $\phi(z)$ as λ approaches z nontangentially (for the scalar case see [Rud80], then see [RR97, pp. 81–84], on how to get the operator-valued version). Similarly, by applying this reasoning to $\phi(\bar{\lambda})^*$ one sees that also $\phi(\lambda)^* \to \phi(z)^*$ (SOT) for a.e. $z \in \partial B_d$ as λ approaches z nontangentially. Actually, the limits exist a.e. if the approach is from within certain nonisotropic approach regions which for $d > 1$ are larger than the standard nontangential approach regions (see Section 2 for definitions).

A multiplication operator Φ is called *inner*, if it is a partial isometry as an operator $\mathscr{H}(k, \mathscr{E}) \to \mathscr{H}(k, \mathscr{D})$. Since partial isometries have closed range it is clear that every inner multiplier defines a multiplier invariant subspace $\mathscr{M} = \Phi \mathscr{H}(k, \mathscr{E}) \subseteq \mathscr{H}(k, \mathscr{D})$. Again, it is a remarkable fact that the converse to this theorem is true, if k is a complete NP kernel.

THEOREM 1.3. *Let k be a complete NP kernel, let $\mathscr{D}$ be a separable Hilbert space, and let $\mathscr{M} \subseteq \mathscr{H}(k, \mathscr{D})$ be a multiplier invariant subspace.*

Then there is an auxiliary Hilbert space $\mathscr{E}$ and an inner multiplication operator $\Phi \in \mathscr{B}(\mathscr{H}(k, \mathscr{E}), \mathscr{H}(k, \mathscr{D}))$ such that $\mathscr{M} = \Phi \mathscr{H}(k, \mathscr{E})$ and $P_{\mathscr{M}} = \Phi \Phi^*$. Furthermore, if $\mathscr{F}$ is another Hilbert space and $\Psi \in \mathscr{B}(\mathscr{H}(k, \mathscr{F}), \mathscr{H}(k, \mathscr{D}))$ is another inner multiplication operator such that $\Phi \mathscr{H}(k, \mathscr{E}) = \Psi \mathscr{H}(k, \mathscr{F})$, then there is a partial isometry $V \in \mathscr{B}(\mathscr{E}, \mathscr{F})$ such that $\phi(\lambda) = \psi(\lambda) V$ for all $\lambda \in B_d$.

This theorem is from [MT00], see [Arv00] for the case of H_d^2. Theorem 1.3 implies Theorem 1.1. To see this we take $\mathscr{D} = \mathbb{C}$, fix an orthonormal

basis $\{e_n\}$ of $\mathcal{E}$ and set $\varphi_n(\lambda) = \phi(\lambda)e_n$ for $n \geqslant 1$ and $\lambda \in B_d$. With this notation it is easy to verify that each $\varphi_n \in M(k) \cap \mathcal{M}$ and $P_{\mathcal{M}} = \Phi\Phi^* = \sum_{n \geqslant 1} M_{\varphi_n} M_{\varphi_n}^* (\text{SOT})$.

We note that in the classical Beurling–Lax–Halmos theorem for $H^2(\mathbb{D})$ one may take $\mathcal{E} = \mathcal{D}$, but for general complete NP kernels other than the Szegö kernel that may not be possible if $\dim \mathcal{D} < \infty$. In fact, one may have to take $\mathcal{E}$ to be infinite dimensional even if $\dim \mathcal{D} < \infty$. This happens for example for the classical Dirichlet space.

Since the existence of these inner multiplication operators is important for our paper, we give a brief outline of the proof of Theorem 1.3. The proof will explain where the space $\mathcal{E}$ comes from.

We already mentioned that the functions b_n, $n \geqslant 1$ in (1.1) are multipliers and $P_0 = \sum_{n \geqslant 1} M_{b_n} M_{b_n}^* (\text{SOT})$ is the projection onto $\{f \in \mathcal{H}(k): f(0) = 0\}$. Similarly, if one thinks of the operators M_{b_n} as multipliers on $\mathcal{H}(k, \mathcal{D})$, then $E_0 = \sum_{n \geqslant 1} M_{b_n} M_{b_n}^* (\text{SOT})$ is the projection onto $\{f \in \mathcal{H}(k, \mathcal{D}): f(0) = 0\}$, and it is easy to see that $Q(A) = \sum_{n \geqslant 1} M_{b_n} A M_{b_n}^*$ defines a bounded operator $\mathcal{B}(\mathcal{H}(k, \mathcal{D})) \to \mathcal{B}(\mathcal{H}(k, \mathcal{D}))$. Now if $\mathcal{M} \subseteq \mathcal{H}(k, \mathcal{D})$ is a multiplier invariant subspace, then one computes

$$P_{\mathcal{M}} - Q(P_{\mathcal{M}}) = P_{\mathcal{M}}(I - E_0)P_{\mathcal{M}} + P_{\mathcal{M}}Q(I - P_{\mathcal{M}})P_{\mathcal{M}} \geqslant 0.$$

We set $S = (P_{\mathcal{M}} - Q(P_{\mathcal{M}}))^{1/2}$, $\mathcal{E} = (\ker S)^{\perp} \subseteq \mathcal{H}(k, \mathcal{D})$, and for $\lambda \in B_d$, $x \in \mathcal{D}$,

$$\phi(\lambda)^* x = S(k_\lambda x).$$

With these definitions one verifies that ϕ is an operator valued multiplier and that the associated multiplication operator Φ satisfies $\Phi\Phi^* = P_{\mathcal{M}}$.

The vector analogue of Theorem 1.2 is that under certain circumstances the analytic functions associated with inner multiplication operators deserve to be called inner functions. In fact, we shall prove the following theorem.

THEOREM 1.4. *Let k be a complete NP kernel on B_d and assume that there is a set $\mathcal{P} \subseteq \mathcal{H}(k) \cap C(\overline{B_d})$ which is dense in $\mathcal{H}(k)$ and such that for all $p \in \mathcal{P}$ and $z \in \partial B_d$, $\lim_{\lambda \to z} \frac{\|pk_\lambda\|}{\|k_\lambda\|} = |p(z)|$. Let $\mathcal{E}$ and $\mathcal{D}$ be separable Hilbert spaces, and let $\Phi \in \mathcal{B}(\mathcal{H}(k, \mathcal{E}), \mathcal{H}(k, \mathcal{D}))$ be an inner multiplication operator with associated operator-valued multiplier $\phi: B_d \to \mathcal{B}(\mathcal{E}, \mathcal{D})$. Then for a.e. $z \in B_d$, $\phi(z)$ is a partial isometry with*

$$m = \operatorname{rank} \phi(z) = \sup\{\operatorname{rank} \phi(\lambda): \lambda \in B_d\} = \sup\{\dim E_\lambda \mathcal{M}: \lambda \in B_d\},$$

where $\mathcal{M} = \Phi\mathcal{H}(k, \mathcal{E})$ and E_λ denotes the point evaluation map $E_\lambda: \mathcal{H}(k, \mathcal{D}) \to \mathcal{D}$, $E_\lambda f = f(\lambda)$, $f \in \mathcal{H}(k, \mathcal{D})$. In particular, $m \leqslant \dim \mathcal{D}$.

This will be an immediate consequence of Lemma 2.1 and Theorem 3.2. Of course, as was the case with Theorem 1.2, this theorem applies to the spaces D_α, $0 \leqslant \alpha \leqslant 1$, and H_d^2. In the classical Beurling–Lax–Halmos theorem for $H^2(\mathbb{D})$ it is also true that the initial space of $\phi(z)$ is a.e. equal to one fixed space $\mathscr{K} \subseteq \mathscr{E} = \mathscr{D}$. In the general situation of invariant subspaces $\mathscr{M}$ of $H_d^2(\mathscr{D})$, $d > 1$, the initial space of $\phi(z)$ may vary with $z \in \partial B_d$.

Section 5 contains our results on the curvature invariant of contractive Hilbert modules.

2. SOME PRELIMINARIES

Let k be an analytic reproducing kernel on B_d with associated Hilbert space $\mathscr{H}(k)$. We will also assume that $k_0(\lambda) = 1$ for all $\lambda \in B_d$, but we will not necessarily assume that k is a complete NP kernel.

Let $\mathscr{M}$ be multiplier invariant subspace of $\mathscr{H}(k)$. If $\mathscr{M}$ contains a function that does not vanish at 0, then the function $\varphi_{\mathscr{M}} = \frac{P_{\mathscr{M}}1}{\sqrt{P_{\mathscr{M}}1(0)}}$ has norm 1 and solves the extremal problem

$$\sup\{\operatorname{Re} f(0) : f \in \mathscr{M}, \ \|f\| = 1\}. \tag{2.1}$$

In fact, if $f \in \mathscr{M}$, then one calculates that $\langle f, \varphi_{\mathscr{M}} \rangle = \frac{f(0)}{\varphi_{\mathscr{M}}(0)}$, and so $\|f\| \geqslant |\frac{f(0)}{\varphi_{\mathscr{M}}(0)}|$, which implies the extremal property of $\varphi_{\mathscr{M}}$.

LEMMA 2.1. *Let k be a complete NP kernel, let $\mathscr{M} \subseteq \mathscr{H}(k)$ be a multiplier invariant subspace. Then $P_{\mathscr{M}}1 \in M(k)$ and $\|(P_{\mathscr{M}}1)f\|^2 \leqslant |(P_{\mathscr{M}}1)(0)| \|f\|^2$ for all $f \in \mathscr{H}(k)$.*

This is proved in [MT00], and it follows immediately from Theorem 1.1 or Theorem 1.3. In fact, let Φ be the inner multiplication operator associated with $\mathscr{M}$ as in Theorem 1.3, let ϕ be the associated operator-valued multiplier, take $\mathscr{D} = \mathbb{C}$, and set $\varphi_n(\lambda) = \phi(\lambda)e_n$, where $\{e_n\}$ is some orthonormal basis for the auxiliary space $\mathscr{E}$.

Then for all $\lambda \in B_d$ we have $(P_{\mathscr{M}}1)(\lambda) = \sum_{n \geqslant 1} \varphi_n(\lambda)\overline{\varphi_n(0)}$. It follows that for $f \in \mathscr{H}(k)$,

$$\|(P_{\mathscr{M}}1)f\|^2 = \left\| \sum_{n \geqslant 1} \varphi_n \overline{\varphi_n(0)} f \right\|^2 \leqslant \|\Phi\|^2 \sum_{n \geqslant 1} \|\overline{\varphi_n(0)}f\|^2$$

$$\leqslant \sum_{n \geqslant 1} |\varphi_n(0)|^2 \|f\|^2 = |P_{\mathscr{M}}1)(0)| \|f\|^2.$$

For $\lambda \in B_d$ we define $\mathcal{M}_\lambda = \{f \in \mathcal{H}(k) : f(\lambda) = 0\}$. Then each $\mathcal{M}_\lambda$ is a multiplier invariant subspace of $\mathcal{H}(k)$ with $\mathcal{M}_\lambda = \{k_\lambda\}^\perp$. Thus, for $\lambda \neq 0$, one obtains

$$\varphi_\lambda(z) = \varphi_{\mathcal{M}_\lambda}(z) = \frac{1 - k_\lambda(z)/k_\lambda(\lambda)}{\sqrt{1 - 1/k_\lambda(\lambda)}}. \tag{2.2}$$

We shall refer to φ_λ as the *one point extremal function*. Note that Lemma 2.1 implies that if k is a complete NP kernel, then all one point extremal functions φ_λ, $\lambda \neq 0$, are contractive multipliers.

LEMMA 2.2. *Let k be such that for each $\lambda \in B_d \backslash \{0\}$ the one point extremal function φ_λ is a contractive multiplier on $\mathcal{H}(k)$, let $\mathcal{D}$ be a separable Hilbert space, and let $f \in \mathcal{H}(k, \mathcal{D})$.*
   Then

   (a) *for each $\lambda \in B_d$, $k_\lambda \in M(k)$ and $\|k_\lambda\|_M \leqslant 2k_\lambda(\lambda)$,*

   (b) *for each $\lambda \in B_d$, $\|f(\lambda)\|_{\mathcal{D}}^2 \leqslant \dfrac{\|k_\lambda f\|^2}{\|k_\lambda\|^2} \leqslant 2\mathrm{Re}\langle f, k_\lambda f \rangle - \|f\|^2$,*

   (c) *the function $F : B_d \to \mathbb{C}$, $F(\lambda) = \langle f, k_\lambda f \rangle$ is analytic on B_d.*

Proof. For $\lambda = 0$, (a) and (b) are clear since $k_0 \equiv 1$. For $\lambda \neq 0$, (a) follows from (2.2) since $\varphi_\lambda \in M(k)$ and $\|\varphi_\lambda\|_M \leqslant 1$. Furthermore, the hypothesis also implies that $\|f\|^2 - \|\varphi_\lambda f\|^2 \geqslant 0$. After a short calculation this leads to the right inequality of (b). To see the left inequality in (b), note that $k_\lambda f(\lambda) : z \mapsto k_\lambda(z)f(\lambda)$ defines a function in $\mathcal{H}(k, \mathcal{D})$ which is orthogonal to $(k_\lambda f) - k_\lambda f(\lambda)$. Hence $\|k_\lambda f\|^2 = \|k_\lambda f - k_\lambda f(\lambda)\|^2 + \|k_\lambda f(\lambda)\|^2 \geqslant \|k_\lambda\|^2 \|f(\lambda)\|_{\mathcal{D}}^2$.

We now prove (c). If $f \in \mathcal{H}(k, \mathcal{D})$, $\varphi \in M(k)$, and $x \in \mathcal{D}$, then $\langle f, k_\lambda \varphi x \rangle = \langle (M_\varphi^* f)(\lambda), x \rangle_{\mathcal{D}}$ is an analytic function in $\lambda \in B_d$. Hence if $\mathcal{L} \subseteq \mathcal{H}(k, \mathcal{D})$ is the set of finite linear combinations of elements of the form φx, $\varphi \in M(k)$, $x \in \mathcal{D}$, then for each $f \in \mathcal{H}(k, \mathcal{D})$ and $g \in \mathcal{L}$ the function $\lambda \mapsto \langle f, k_\lambda g \rangle$ is analytic in B_d. Finite linear combinations of the functions k_λ are dense in $\mathcal{H}(k)$, hence it follows from (a) that $M(k)$ is dense in $\mathcal{H}(k)$, and so $\mathcal{L}$ is dense in $\mathcal{H}(k, \mathcal{D})$. The unit ball in $\mathcal{H}(k)$ is a normal family, thus the uniform boundedness principle implies that for each compact subset $K \subseteq B_d$ there is C_K such that $k_\lambda(\lambda) = \|k_\lambda\|^2 \leqslant C_K$. This implies that for each compact set $K \subseteq B_d$, $\lambda \in K$, $f \in \mathcal{H}(k, \mathcal{D})$, and $g \in \mathcal{L}$, we have by (a)

$$|\langle f, k_\lambda f \rangle - \langle f, k_\lambda g \rangle| \leqslant \|f\| \|k_\lambda\|_M \|f - g\| \leqslant 2C_K \|f\| \|f - g\|,$$

i.e. for each $f \in \mathcal{H}(k, \mathcal{D})$, $F(\lambda) = \langle f, k_\lambda f \rangle$ is analytic as it is a local uniform limit of analytic functions. This concludes the proof of the lemma. ∎

This lemma has a number of implications for the regularity of the functions in $\mathscr{H}(k)$.

PROPOSITION 2.3. *Let k be such that for each $\lambda \in B_d \backslash \{0\}$ the one point extremal function φ_λ is a contractive multiplier on $\mathscr{H}(k)$. Then $\mathscr{H}(k)$ is contractively contained in $H^2(\partial B_d)$. In fact, for every $z \in \partial B_d$, the slice function f_z, $f_z(\zeta) = f(\zeta z)$, $\zeta \in \mathbb{D}$, is in $H^2(\mathbb{D})$, and satisfies $\|f_z\|_{H^2} \leqslant \|f\|$.*

Proof. We use the scalar version of Lemma 2.2(b). Hence for each $f \in \mathscr{H}(k)$ and $\lambda \in B_d$ we have $|f(\lambda)|^2 \leqslant u(\lambda) = 2\mathrm{Re}\langle f, k_\lambda f \rangle - \|f\|^2$. As before, let σ denote the rotationally invariant probability measure on ∂B_d. We fix $0 < r < 1$ and integrate over ∂B_d and obtain

$$\int_{\partial B_d} |f(rz)|^2 \, d\sigma(z) \leqslant \int_{\partial B_d} u(rz) \, d\sigma(z) = u(0) = \|f\|^2,$$

since the integrand on the right is the real part of an analytic function. We now take the supremum over $0 < r < 1$ and obtain $\|f\|_{H^2(\partial B_d)} \leqslant \|f\|$ for all $f \in \mathscr{H}(k)$.

Furthermore, if $z \in \partial B_d$, then $u_z(\zeta) = u(\zeta z)$, $\zeta \in \mathbb{D}$, defines a positive harmonic function in the unit disk $\mathbb{D} \subseteq \mathbb{C}$. Thus $|f_z(\zeta)|^2 = |f(\zeta z)|^2 \leqslant u_z(\zeta)$, hence $\|f_z\|_{H^2}^2 \leqslant u_z(0) = \|f\|^2$. ∎

Functions in $H^2(\partial B_d)$ have a.e. limits from within certain approach regions that contain the standard nontangential approach regions (see [Rud80]). For $\alpha > 1$ and $z \in \partial B_d$, define $\Omega_\alpha(z)$ to be the set of all $\lambda \in B_d$ such that $|1 - \langle \lambda, z \rangle_d| < \frac{\alpha}{2}(1 - |\lambda|^2)$. We say that a function $f : B_d \to \mathbb{C}$ has a K-limit A at $z \in \partial B_d$, $(K - \lim f)(z) = A$, if for every $\alpha > 1$ and for every sequence $\{\lambda_n\} \subseteq \Omega_\alpha(z)$ that converges to z, we have $f(\lambda_n) \to A$ as $n \to \infty$.

Let k be a reproducing kernel as in Proposition 2.3. It is well known that every function in $H^2(\partial B_d)$ has finite K-limits at a.e. every point $z \in \partial B_d$ [Rud80], hence the same is true for every $f \in \mathscr{H}(k)$. Furthermore, if $\mathscr{D}$ is a separable Hilbert space, and $f \in \mathscr{H}(k, \mathscr{D})$, then the arguments given in [RR97, p. 84] show that for a.e. $z \in \partial B_d$ there is an $f(z) \in \mathscr{D}$ such that $f(z)$ is the K-norm-limit of $f(\lambda)$ at z.

PROPOSITION 2.4. *Let k be such that for each $\lambda \in B_d \backslash \{0\}$ the one point extremal function φ_λ is a contractive multiplier on $\mathscr{H}(k)$, and assume that there is a set $\mathscr{P} \subseteq \mathscr{H}(k) \cap C(\overline{B_d})$ which is dense in $\mathscr{H}(k)$ and such that for all $p \in \mathscr{P}$ and $z \in \partial B_d$, $\lim_{\lambda \to z} \frac{\|pk_\lambda\|}{\|k_\lambda\|} = |p(z)|$. Let $\mathscr{D}$ be a separable Hilbert*

space. Then for every $f \in \mathcal{H}(k, \mathcal{D})$ we have

$$K - \lim \frac{\|fk_\lambda\|}{\|k_\lambda\|} = \|f(z)\|_{\mathcal{D}} \quad for \ a.e. \ z \ \in \partial B_d.$$

Proof. Because of the hypothesis and Lemma 2.2, we can use standard techniques. We briefly outline the details of the proof.

If $f \in \mathcal{H}(k, \mathcal{D})$, $\alpha > 1$, we define the maximal function

$$M_\alpha f(z) = \sup \left\{ \frac{\|fk_\lambda\|}{\|k_\lambda\|} : \lambda \in \Omega_\alpha(z) \right\}.$$

The right-hand side in Lemma 2.2(b) is positive and the real part of an analytic function (i.e. it is pluriharmonic), hence it can be represented as the invariant Poisson integral of a positive measure μ on ∂B_d, $P\mu(\lambda) = 2\text{Re}\langle f, k_\lambda f \rangle - \|f\|^2$ (see [Rud80]). Furthermore, we note that $\|\mu\| = P\mu(0) = \|f\|^2$, and that for all $\alpha > 1$, the Ω_α-maximal function of $P\mu$ satisfies a weak-type estimate with constant C_α (see [Rud80]). Hence by Lemma 2.2(b) we obtain for all $\alpha > 1$, $\varepsilon > 0$, and $f \in \mathcal{H}(k, \mathcal{D})$ the weak-type estimate

$$\sigma(\{z \in \partial B_d : M_\alpha f(z) > \varepsilon\}) \leqslant C_\alpha \frac{\|f\|^2}{\varepsilon^2}.$$

Next, let $\mathcal{P}' \subseteq \mathcal{H}(k, \mathcal{D})$ be the set of all finite linear combinations of the form px, where $p \in \mathcal{P} \subseteq \mathcal{H}(k)$ and $x \in \mathcal{D}$. Then $\mathcal{P}'$ is dense in $\mathcal{H}(k, \mathcal{D})$. We shall first show that $\lim_{\lambda \to z} \frac{\|(p - p(\lambda))k_\lambda\|}{\|k_\lambda\|} = 0$ for all $p \in \mathcal{P}'$ and $z \in \partial B_d$.

Let $z \in \partial B_d$, and note that if $q \in \mathcal{P}$, then

$$\frac{\|(q - q(\lambda))k_\lambda\|^2}{\|k_\lambda\|^2} = \frac{\|qk_\lambda\|^2}{\|k_\lambda\|^2} - |q(\lambda)|^2 \to 0 \quad \text{as } \lambda \to z,$$

because $q \in \mathcal{P}$ and q is continuous at z. Now let $p = \sum_{i=1}^{n} p_i x_i$, where $p_i \in \mathcal{P}$ and $x_i \in \mathcal{D}$, then

$$\frac{\|(p - p(\lambda))k_\lambda\|}{\|k_\lambda\|} \leqslant \sum_{i=1}^{n} \|x_i\|_{\mathcal{D}} \frac{\|(p_i - p_i(\lambda))k_\lambda\|}{\|k_\lambda\|} \to 0 \quad \text{as } \lambda \to z.$$

Finally, let $f \in \mathcal{H}(k, \mathcal{D})$. Then f has K-limit $f(z)$ at a.e. $z \in \partial B_d$ and $\frac{\|(f - f(\lambda))k_\lambda\|^2}{\|k_\lambda\|^2} = \frac{\|fk_\lambda\|^2}{\|k_\lambda\|^2} - \|f(\lambda)\|^2$, thus we must show that the K-limsup of $\frac{\|(f - f(\lambda))k_\lambda\|^2}{\|k_\lambda\|^2}$ equals 0 at σ a.e. $z \in \partial B_d$. Using Lemma 2.2(b), we see that

for every $p \in \mathscr{P}'$ we have for all $\lambda \in B_d$,

$$\frac{\|(f - f(\lambda))k_\lambda\|}{\|k_\lambda\|} \leqslant \frac{\|(f - p)k_\lambda\|}{\|k_\lambda\|} + \frac{\|(p - p(\lambda))k_\lambda\|}{\|k_\lambda\|} + \|p(\lambda) - f(\lambda)\|_{\mathscr{D}}$$

$$\leqslant 2\frac{\|(f - p)k_\lambda\|}{\|k_\lambda\|} + \frac{\|(p - p(\lambda))k_\lambda\|}{\|k_\lambda\|}.$$

Hence for $z \in \partial B_d$, we obtain for every $\alpha > 1$,

$$\limsup_{\substack{\lambda \to z \\ \lambda \in \Omega_\alpha}} \frac{\|(f - f(\lambda))k_\lambda\|}{\|k_\lambda\|} \leqslant 2M_\alpha(f - p)(z),$$

and so the weak-type estimate implies that for every $\varepsilon > 0$, we have for every $p \in \mathscr{P}'$,

$$\sigma\left(\left\{z \in \partial B_d : \limsup_{\substack{\lambda \to z \\ \lambda \in \Omega_\alpha}} \frac{\|(f - f(\lambda))k_\lambda\|}{\|k_\lambda\|} > \varepsilon\right\}\right) \leqslant 4C_\alpha \frac{\|f - p\|^2}{\varepsilon^2}.$$

Since $\mathscr{P}'$ is dense in $\mathscr{H}(k, \mathscr{D})$ the result follows. ∎

3. INNER MULTIPLICATION OPERATORS AND INNER MULTIPLIERS

As in Section 2, in this section k will denote an analytic reproducing kernel on B_d with $k_0 \equiv 1$.

LEMMA 3.1. *Let $\mathscr{D}$, $\mathscr{E}$ be separable Hilbert spaces, and let $\Phi \in \mathscr{B}(\mathscr{H}(k, \mathscr{E}), \mathscr{H}(k, \mathscr{D}))$ be a multiplication operator with associated operator-valued multiplier ϕ, $\phi(\lambda) \in \mathscr{B}(\mathscr{E}, \mathscr{D})$, $\lambda \in B_d$. For $\lambda \in \overline{B_d}$ let* rank $\phi(\lambda) =$ dim rank $\phi(\lambda)$, *and set*

$$m = \sup\{\text{rank } \phi(\lambda) : \lambda \in B_d\}.$$

Then rank $\phi(\lambda) = m$ *on $B_d \backslash E$, where E is at most a countable union of zero varieties of nonzero bounded analytic functions in B_d and* rank $\phi(z) = m$ *for σ a.e. $z \in \partial B_d$.*

Proof. First note that if T_n, $T \in \mathscr{B}(\mathscr{E}, \mathscr{D})$ such that $T_n \to T$ (SOT), then rank $T \leqslant \liminf$ rank T_n. In fact, for the proof we may assume that rank $T_n = r < \infty$ for all n. Suppose rank $T > r$. Then let $\{Tf_j\}_{j=1}^{r+1}$ be an

orthonormal set in the range of T, $f_j \in \mathscr{E}$. Then $d_n = \det[(\langle T_n f_j,$ $T f_k \rangle_{\mathscr{D}})_{1 \leqslant j,k \leqslant r+1}] = 0$ for each n since rank $T_n = r < r + 1$. But this leads to a contradiction since $d_n \to \det(\langle T f_j, T f_k \rangle_{\mathscr{D}}) = 1$. Thus, at each point $z \in \partial B_d$ where the K-limit of $\phi(\lambda)$ exists in the strong operator topology, we have rank $\phi(z) \leqslant m$.

Now assume that $1 \leqslant m < \infty$. Then there is a $\lambda_0 \in B_d$ such that rank $\phi(\lambda_0) = m$. Let $\{e_n\}_{n \geqslant 1}^m$ be an orthonormal basis for ker $\phi(\lambda_0)^{\perp} \subseteq \mathscr{E}$, and let $\{d_k\}_{k \geqslant 1}^m$ be an orthonormal basis for rank $\phi(\lambda_0) \subseteq \mathscr{D}$.

We define $D(\lambda) = \det[(\langle \phi(\lambda) e_n, d_k \rangle_{\mathscr{D}})_{1 \leqslant n,k \leqslant m}]$. Then D is a bounded analytic function in B_d with $D(\lambda_0) \neq 0$. It is clear that $m \leqslant$ rank $\phi(\lambda)$ whenever $D(\lambda) \neq 0$, $\lambda \in B_d$. But since m was the supremum of rank $\phi(\lambda)$ for $\lambda \in B_d$ we actually get $m =$ rank $\phi(\lambda)$ whenever $D(\lambda) \neq 0$, $\lambda \in B_d$.

Furthermore, since the determinant is a polynomial in its entries it is clear that the K-limit of $D(\lambda)$ exists, is nonzero at a.e. $z \in \partial B_d$, and equals $D(z) = \det[(\langle \phi(z) e_n, d_k \rangle_{\mathscr{D}})_{1 \leqslant n,k \leqslant m}]$, (see [Rud80]). Hence rank $\phi(z) \geqslant m$ for σ a.e. $z \in \partial B_d$. We already explained the other inequality, thus this proves the lemma when $m < \infty$.

If $m = \infty$, then for any integer $s > 0$ we can find λ_s such that rank $\phi(\lambda_s) \geqslant s$. Thus, as above, we obtain a bounded analytic function $D_s(\lambda)$ with $D_s(\lambda_s) \neq 0$. The boundary value function of D_s is not identically zero, hence rank $\phi(z) \geqslant s$ for a.e. $z \in \partial B_d$. This implies that rank $\phi(z) = \infty$ for a.e. $z \in \partial B_d$. It also follows that rank $\phi(\lambda) = \infty$ for all $\lambda \in B_d$, $\lambda \notin \bigcap_{k=1}^{\infty} \bigcup_{s=k}^{\infty} Z(D_s)$. ∎

For $\lambda \in B_d$, let $E_\lambda : \mathscr{H}(k, \mathscr{D}) \to \mathscr{D}$, $E_\lambda f = f(\lambda)$, let $\Phi \in \mathscr{B}(\mathscr{H}(k, \mathscr{E}), \mathscr{H}(k, \mathscr{D}))$ be an inner multiplication operator with associated operator-valued multiplier ϕ, and let $\mathscr{M} = \Phi \mathscr{H}(k, \mathscr{E})$. Then for all $\lambda \in B_d$, rank $\phi(\lambda)$ $= \{\phi(\lambda) y : y \in \mathscr{E}\} = \{\phi(\lambda) f(\lambda) : f \in \mathscr{H}(k, \mathscr{E})\} = \{E_\lambda(\Phi f) : f \in \mathscr{H}(k, \mathscr{E})\}$ $= E_\lambda \mathscr{M}$.

THEOREM 3.2. *Let k be such that for each $\lambda \in B_d \backslash \{0\}$ the one point extremal function φ_λ is a contractive multiplier on $\mathscr{H}(k)$ and assume that there is a set $\mathscr{P} \subseteq \mathscr{H}(k) \cap C(\overline{B_d})$ which is dense in $\mathscr{H}(k)$ and such that for all $p \in \mathscr{P}$ and $z \in B_d$, $\lim_{\lambda \to z} \frac{\|p k_\lambda\|}{\|k_\lambda\|} = |p(z)|$. Let $\mathscr{E}$ and $\mathscr{D}$ be separable Hilbert spaces, and let $\Phi \in \mathscr{B}(\mathscr{H}(k, \mathscr{E}), \mathscr{H}(k, \mathscr{D}))$ be an inner multiplication operator with associated operator-valued multiplier ϕ, and let $m = \sup\{$rank $\phi(\lambda) : \lambda \in B_d\} = \sup\{\dim E_\lambda \mathscr{M} : \lambda \in B_d\}$. Then for σ a.e. $z \in \partial B_d$, $\phi(z)$ is a partial isometry with rank $\phi(z) = m$.*

Proof. The statement about the rank follows from Lemma 3.1. Let $z \in \partial B_d$ be such that the K-limits of $\phi(\lambda)$ and $\phi(\lambda)^*$ exist at z in the strong

operator topology. We have to show that for a.e. $z \in \partial B_d$ $\phi(z)^*$ is an isometry on ran $\phi(z)$. Since $\|\phi(z)^*\| \leqslant 1$ it suffices to show that $\|\phi(z)^*\phi(z)y\|_{\mathscr{E}} \geqslant \|\phi(z)y\|_{\mathscr{D}}$ for all $y \in \mathscr{E}$ with $\phi(z)y \neq 0$.

Let $\mathscr{M} = $ ran $\Phi \subseteq \mathscr{H}(k, \mathscr{D})$, then $P_{\mathscr{M}} = \Phi\Phi^*$ and $\mathscr{M}$ is a multiplier invariant subspace. Note that for $\lambda \in B_d$ and $x \in \mathscr{D}$ we have

$$\|P_{\mathscr{M}}k_\lambda x\| = \sup\{|\langle f(\lambda), x\rangle_{\mathscr{D}}| : f \in \mathscr{M}, \|f\| \leqslant 1\}, \tag{3.1}$$

because for $f \in \mathscr{M}$, $|\langle f(\lambda), x\rangle_{\mathscr{D}}| = |\langle f, k_\lambda x\rangle| = |\langle f, P_{\mathscr{M}}k_\lambda x\rangle| \leqslant \|f\|\|P_{\mathscr{M}}k_\lambda x\|$ with equality if $f = P_{\mathscr{M}}k_\lambda x$.

Hence if $f \in \mathscr{M}$ is nonzero, if $\lambda \in B_d$, then $k_\lambda f \in \mathscr{M}$ and for $x \in \mathscr{D}$

$$\begin{aligned}
\|\phi(\lambda)^* x\|_{\mathscr{E}} &= \frac{\|k_\lambda \phi(\lambda)^* x\|}{\|k_\lambda\|} = \frac{\|\Phi^*(k_\lambda x)\|}{\|k_\lambda\|} = \frac{\|P_{\mathscr{M}}(k_\lambda x)\|}{\|k_\lambda\|} \\
&\geqslant \frac{|\langle (k_\lambda f)(\lambda), x\rangle_{\mathscr{D}}|}{\|k_\lambda\|\|k_\lambda f\|} = \frac{|\langle f(\lambda), x\rangle_{\mathscr{D}}|}{\|k_\lambda f\|/\|k_\lambda\|}.
\end{aligned}$$

Now let $y \in \mathscr{E}$ with $\phi(z)y \neq 0$. Then $f = \Phi y$, $f(\lambda) = \phi(\lambda)y$, is a nonzero function in $\mathscr{M}$ with K-limit $f(z) = \phi(z)y$ as $\lambda \to z$. Hence Proposition 2.4 and the above imply $\|\phi(z)^* x\|_{\mathscr{D}} \geqslant \dfrac{|\langle \phi(z)y, x\rangle_{\mathscr{D}}|}{\|\phi(z)y\|}$ for a.e. such $z \in \partial B_d$. This concludes the proof since we may take $x = \phi(z)y$. ∎

4. $\mathscr{U}$-INVARIANT COMPLETE NP KERNELS

In this section, we shall verify that the hypothesis of Theorem 3.2 is satisfied for many complete NP kernels that are invariant under unitary maps. If k is an analytic reproducing kernel on B_d that is invariant under every unitary map $U : \mathbb{C}^d \to \mathbb{C}^d$, i.e. $k_{U\lambda}(Uz) = k_\lambda(z)$ for all $\lambda, z \in B_d$, then one can show that $k_\lambda(z) = f(\langle \lambda, z\rangle_d)$ for some function f of type $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n \geqslant 0$. Such a kernel will be a complete NP kernel if and only if there exists a sequence $\{b_n\}_n$, $b_n \geqslant 0$ such that $f(x) = \dfrac{1}{1 - \sum_{n=1}^{\infty} b_n x^n}$. Note that $a_1 = b_1$. Finite linear combinations of the kernels k_λ, $\lambda \in B_d$, are dense in $\mathscr{H}(k)$ and the evaluations of partial derivatives at 0 are continuous linear functionals, hence if $a_1 = 0$, it would follow that the coordinate functions $z_i(\lambda) = \lambda_i$, $i = 1, \ldots, d$ are not in $\mathscr{H}(k)$. On the other hand, if $a_1 = b_1 \neq 0$, then each $z_i \in M(k)$. This follows because we have $1 - \dfrac{1}{k_\lambda(w)} = b_1 \sum_{i=1}^{d} w_i \bar{\lambda}_i +$ higher order terms. Thus $\sum_{i=1}^{d} M_{z_i} M_{z_i}^*$ $\leqslant (1/b_1)I$ because, as we already mentioned, each of the functions b_n in the representation (1.1) is a multiplier with $\sum_{n \geqslant 1} M_{b_n} M_{b_n}^* = P_0 \leqslant I$. It follows

from this, or it is easy to see anyway, that the hypothesis $b_1 \neq 0$ implies that $a_n > 0$ for all n.

Thus in this section we will assume that

$$k_\lambda(z) = \sum_{n=0}^{\infty} a_n(\langle z, \lambda \rangle_d)^n = \frac{1}{1 - \sum_{n=1}^{\infty} b_n(\langle z, \lambda \rangle_d)^n}, \qquad (4.1)$$

where $a_n, b_n \geq 0$, $a_0 = 1$, and $a_1 = b_1 > 0$. In particular, k is a complete NP kernel, and the space $\mathcal{H}(k)$ contains the polynomials.

At this point, we should mention that if only the sequence $\{a_n\}$ is given, then it may be difficult to determine whether k is a complete NP kernel, i.e. for which $\{a_n\}$ it follows that a sequence $\{b_n\}$ can be found such that $b_n \geq 0$ for each n and such that (4.1) holds. However, it was pointed out in [SS62] that if a_{n+1}/a_n increases to 1, then the existence of nonnegative $\{b_n\}$ follows by a theorem that can be found in [Har92]. On the other hand, if the sequence $\{b_n\}$ is given, $b_n \geq 0$, one always obtains a complete NP kernel.

In order to compute the norm of polynomials we need to recall multiindex notation. Let $j = (j_1, j_2, \ldots, j_d)$ be a multiindex of nonnegative integers, then $|j| = j_1 + j_2 + \cdots + j_d$, $j! = j_1! j_2! \cdots j_d!$, and for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d$, $\lambda^j = \lambda_1^{j_1} \lambda_2^{j_2} \cdots \lambda_d^{j_d}$, and the multinomial formula implies that for $z, \lambda \in B_d$ and $n \geq 0$

$$\langle z, \lambda \rangle_d^n = \sum_{|j|=n} \frac{|j|!}{j!} z^j \bar{\lambda}^j.$$

Thus $k_\lambda(z) = \sum_j a_{|j|} \frac{|j|!}{j!} z^j \bar{\lambda}^j$, where the sum is taken over all multiindices j with entries in the nonnegative integers. Since $k_\lambda(z) = \langle k_\lambda, k_z \rangle$ it follows that monomials in $\mathcal{H}(k)$ are mutually orthogonal and

$$\|z^j\|^2 = \frac{j!}{a_{|j|}|j|!}.$$

If $f_n(z) = \sum_{|j|=n} c_j z^j$ is a homogeneous polynomial of degree n, then $\|f_n\|^2 = \frac{1}{a_n} \sum_{|j|=n} \frac{j!}{|j|!} |c_j|^2$, and it follows that an analytic function f on B_d with homogeneous expansion $f(z) = \sum_{n=0}^{\infty} f_n(z)$ is in $\mathcal{H}(k)$ if and only if $\|f\|^2 = \sum_{n=0}^{\infty} \|f_n\|^2 < \infty$, and that the polynomials are dense in $\mathcal{H}(k)$.

We need a few technical lemmas.

LEMMA 4.1. *Let $\mathcal{D}$ be a separable Hilbert space, and for $n \in \mathbb{N}$ let $F_n(e^{it}) = \frac{1}{n+1} \frac{\sin^2((n+1)t/2)}{\sin^2(t/2)}$ be the Fejér kernel.*

If $\varphi \in M(k)$, $n \in \mathbb{N}$, *and* $p_n(z) = \int_0^{2\pi} \varphi(e^{it}z)F_n(e^{it})\,dt$, *then* p_n *is a polynomial and* $M_{p_n} \to M_{\varphi}$ *(WOT) as* $n \to \infty$ *in* $\mathscr{H}(k)$ *or* $\mathscr{H}(k, \mathscr{D})$.

In particular, if a subspace $\mathscr{M}$ *is* M_{z_i} *invariant for each* $i = 1, \ldots, d$, *then it is multiplier invariant.*

Proof. It follows from the well-known properties of the Fejér kernel that each p_n is a polynomial and that $p_n(\lambda) \to \varphi(\lambda)$ for each $\lambda \in B_d$. Thus, it will suffice to show that for each $f \in \mathscr{H}(k, \mathscr{D})$ the sequence $\{\|p_n f\|\}_{n \in \mathbb{N}}$ is norm bounded.

For $t \in \mathbb{R}$, $f \in \mathscr{H}(k, \mathscr{D})$ let $f_t(z) = f(e^{it}z)$. One checks that $f_t \in \mathscr{H}(k, \mathscr{D})$, $\|f_t\| = \|f\|$, and $f_t \to f_{t_0}$ in norm as $t \to t_0$. Hence if $\varphi \in M(k)$, then $\varphi_t f = (\varphi f_{-t})_t$, hence $\varphi_t \in M(k)$ with $\|\varphi_t\|_M = \|\varphi\|_M$, and $\varphi_t f \to \varphi_{t_0} f$ in norm as $t \to t_0$. Hence for each $n \in \mathbb{N}$ the integral $p_n f = \int_0^{2\pi} \varphi_t f F_n(e^{it})\,dt$ converges in the norm of $\mathscr{H}(k, \mathscr{D})$, and we have $\|p_n f\| \leqslant \|\varphi\|_M \|f\|$. ∎

LEMMA 4.2. *Suppose k satisfies* (4.1). *Let p be a homogeneous polynomial of degree n. Then*

$$\sum_{i=1}^{d} \|z_i p\|^2 - \|p\|^2 = \left(\frac{a_n}{a_{n+1}} \frac{n+d}{n+1} - 1 \right) \|p\|^2. \tag{4.2}$$

Hence, if $a_n/a_{n+1} \to 1$ *as* $n \to \infty$, *then* $\sum_{i=1}^{d} M_{z_i}^* M_{z_i} - I$ *is a compact operator on* $\mathscr{H}(k)$.

Proof. For $1 \leqslant i \leqslant d$ and any multiindex $j = (j_1, \ldots, j_d)$, we obtain $\|z_i z^j\|^2 = \dfrac{a_{|j|}}{a_{|j|+1}} \dfrac{j_i + 1}{|j| + 1} \|z^j\|^2$. Thus, if p is a homogeneous polynomial of degree $n \geqslant 0$, then $\sum_{i=1}^{d} \|z_i p\|^2 = \dfrac{a_n}{a_{n+1}} \dfrac{n+d}{n+1} \|p\|^2$. ∎

THEOREM 4.3. *Let $\mathscr{E}$ and $\mathscr{D}$ be separable Hilbert spaces, and let k be a complete NP kernel that satisfies the hypothesis* (4.1), $k_\lambda(\lambda) \to \infty$ *as* $|\lambda| \to 1$, *and* $\lim_{n \to \infty} a_n/a_{n+1} = 1$.

If $\Phi \in \mathscr{B}(\mathscr{H}(k, \mathscr{E}), \mathscr{H}(k, \mathscr{D}))$ *is an inner multiplication operator with associated operator-valued multiplier* ϕ, *and* $m = \sup\{\operatorname{rank} \phi(\lambda) : \lambda \in B_d\}$, *then for* σ *a.e.* $z \in B_d$, $\phi(z)$ *is a partial isometry with* $\operatorname{rank} \phi(z) = m$.

Furthermore, Theorem 1.3 applies to every subspace $\mathscr{M} \subseteq \mathscr{H}(k, \mathscr{D})$ *that is invariant for every* M_{z_i}, $i = 1, \ldots, d$. *In this case,* $m = \sup\{\dim E_\lambda \mathscr{M} : \lambda \in B_d\}$.

Proof. The statement of the last paragraph follows from Theorem 1.3, Lemma 4.1, and the remarks made in the paragraph before Lemma 2.2.

We shall show that $\lim_{\lambda \to w} \frac{\|pk_\lambda\|}{\|k_\lambda\|} = |p(w)|$ for every $w \in \partial B_d$ and every polynomial p.

If p is a polynomial and $w \in \partial B_d$, then there are polynomials q_i, $i = 1, \ldots, d$ such that $p(z) - p(w) = \sum_{i=1}^d (z_i - w_i)q_i(z)$. Then for $\lambda \in B_d$,

$$\frac{\|(p - p(w))k_\lambda\|}{\|k_\lambda\|} \leqslant \sum_{i=1}^d \|q_i\|_M \frac{\|(z_i - w_i)k_\lambda\|}{\|k_\lambda\|}$$

$$\leqslant C \left[\sum_{i=1}^d \frac{\|(z_i - w_i)k_\lambda\|^2}{\|k_\lambda\|^2} \right]^{1/2}$$

$$= C \left[\sum_{i=1}^d \frac{\|z_i k_\lambda\|^2}{\|k_\lambda\|^2} - 2\operatorname{Re} \lambda_i \bar{w}_i + |w_i|^2 \right]^{1/2}$$

$$= C \left[\sum_{i=1}^d \|z_i \frac{k_\lambda}{\|k_\lambda\|}\|^2 - 2\operatorname{Re}\langle \lambda, w \rangle + 1 \right]^{1/2}.$$

Now the hypothesis $k_\lambda(\lambda) \to \infty$ and the density of the polynomials implies that $k_\lambda/\|k_\lambda\| \to 0$ weakly as $|\lambda| \to 1$, and by Lemma 4.2 we have that $\sum_{i=1}^d M_{z_i}^* M_{z_i} - I$ is compact, so $\sum_{i=1}^d \|z_i(k_\lambda/\|k_\lambda\|)\|^2 \to 1$ as $|\lambda| \to 1$. Hence $\frac{\|(p - p(w))k_\lambda\|}{\|k_\lambda\|} \to 0$ as $\lambda \to w$. ∎

EXAMPLE 4.4. Let $a_n = (n + 1)^{-\alpha}$, $\alpha \geqslant 0$. Then the corresponding kernel k is a complete NP kernel. This follows from the theorem of [Har92] that we already mentioned after (4.1). For $d = 1$ the spaces $\mathcal{H}(k) = D_\alpha$ are weighted Dirichlet spaces with $D = D_1$ being the classical Dirichlet space. We note that for $0 \leqslant \alpha \leqslant 1$ the coefficients a_n satisfy the hypothesis of the theorem. However, if $\alpha > 1$, then $k_\lambda(\lambda)$ stays bounded as $|\lambda| \to 1$.

We shall show now that the conclusion of Theorem 4.3 does not hold in these cases.

Assume that $k_\lambda(z)$ is a reproducing kernel of the type considered in (4.1) and $k_\lambda(\lambda)$ stays bounded as $|\lambda| \to 1$. The $\sum_{n=0}^\infty a_n < \infty$. Thus, the power series for $k_\lambda(z) = \sum_{n=0}^\infty a_n \langle z, \lambda \rangle^n$ converges absolutely and uniformly on $\bar{B}_d \times \bar{B}_d$, and for $\lambda \in \bar{B}_d$, $k_\lambda \in \mathcal{H}(k)$. It follows that all functions in $\mathcal{H}(k)$ extend to be continuous on $\bar{B}_d$, where $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}(k)$, $\lambda \in \bar{B}_d$. Let $z \in \partial B_d$ and $\mathcal{M} = \{f \in \mathcal{H}(k) : f(z) = 0\} = \{k_z\}^\perp$. Clearly, this is an invariant subspace.

Let $\{\varphi_n\}$ be the sequence that is associated with $\mathcal{M}$ according to Theorem 1.1. Then for $\lambda \in B_d$, $\sum_{n \geqslant 1} |\varphi_n(\lambda)|^2 = \frac{\|P_\mathcal{M} k_\lambda\|^2}{\|k_\lambda\|^2} = 1 - \frac{|k_\lambda(z)|^2}{k_\lambda(\lambda)k_z(z)}$. This is a

continuous function on $\overline{B_d}$, which is zero at $\lambda = z$. Thus its boundary values cannot be zero a.e. $[\sigma]$.

Note that for kernels considered in (4.1) the condition $k_\lambda(\lambda) \to \infty$ as $|\lambda| \to 1$ is equivalent to $\sum_{n=1}^{\infty} b_n = 1$. Sometimes the hypothesis $\lim_{n\to\infty} a_n/a_{n+1} = 1$ in Theorem 4.3 is implied by another condition. We prove the following proposition.

PROPOSITION 4.5.   *Suppose $\sum_{n=1}^{\infty} b_n = 1$, and either $\sum_{n=1}^{\infty} nb_n < \infty$ or $\{a_n\}$ is eventually nonincreasing.*
   Then $a_n/a_{n+1} \to 1$ as $n \to \infty$.

Proof.   Suppose first that $\sum_{n=1}^{\infty} b_n = 1$ and $c = \sum_{n=1}^{\infty} nb_n < \infty$. For $z \in \mathbb{D}$ let $g(z) = \dfrac{1 - \sum_{n=1}^{\infty} b_n z^n}{1 - z} = \sum_{n=1}^{\infty} b_n \dfrac{1 - z^n}{1 - z} = \sum_{n=1}^{\infty} b_n \sum_{j=0}^{n-1} z^j$. Then $|g(z)| \leqslant c$, and, in fact, g has a Taylor series that converges absolutely and uniformly in $\overline{\mathbb{D}}$. Note that $g(1) = c > 0$ and $\operatorname{Re}(1 - z)g(z) = 1 - \sum_{n=1}^{\infty} b_n \operatorname{Re} z^n \geqslant b_1(1 - \operatorname{Re} z)$, so $g(z) \neq 0$ in $\overline{\mathbb{D}}$. It follows from Wiener's Lemma (see [Rud91, Theorem 11.6]), that the function $h(z) = \dfrac{1}{g(z)} - \dfrac{1}{c}$ has a Taylor series $h(z) = \sum_{k=0}^{\infty} \hat{h}(k) z^k$ with $\sum_{k=0}^{\infty} |\hat{h}(k)| < \infty$, and $\sum_{k=0}^{\infty} \hat{h}(k) = h(1) = 0$.

We have $h(z) + 1/c = \dfrac{1}{g(z)} = (1 - z) \sum_{n=0}^{\infty} a_n z^n = 1 + \sum_{n=1}^{\infty} (a_n - a_{n-1}) z^n$. We compare coefficients and evaluate at $z = 1$ to obtain $1/c = 1 + \sum_{n=1}^{\infty} (a_n - a_{n-1}) = \lim_{n\to\infty} a_n$, hence $a_n/a_{n+1} \to 1$ as $n \to \infty$.

Now suppose that $\sum_{n=1}^{\infty} b_n = 1$ and that $a_n \geqslant a_{n+1}$ for all $n \geqslant N$. Then since $\sum_{n=0}^{\infty} a_n z^n = \dfrac{1}{1 - \sum_{n=1}^{\infty} b_n z^n}$ we may multiply through and compare coefficients. We obtain $a_{n+1} = \sum_{k=1}^{n+1} b_k a_{n+1-k}$. Thus for $n \geqslant N$ we have $1 \geqslant a_{n+1}/a_n = \sum_{k=1}^{n+1} b_k a_{n+1-k}/a_n \geqslant \sum_{k=1}^{n-N} b_k \to 1$ as $n \to \infty$. ∎

5. AN APPLICATION TO CONTRACTIVE HILBERT MODULES

In [Arv00] Arveson defines a contractive Hilbert module to be a Hilbert space $\mathscr{H}$ which is also a module over $\mathscr{A} = \mathbb{C}[z_1, \ldots, z_d]$, the algebra of complex polynomials in d variables, and has the property that

$$\|z_1\xi_1 + \cdots + z_d\xi_d\|^2 \leqslant \|\xi_1\|^2 + \cdots + \|\xi_d\|^2 \quad \text{for all } \xi_1, \ldots, \xi_d \in \mathscr{H}. \quad (5.1)$$

Hence, the actions of $z_1, \ldots, z_d$ define bounded linear operators on $\mathscr{H}$ which we denote by $T_1, \ldots, T_d$, respectively. From the definition of a

contractive Hilbert module and (5.1) the d-tuple of operators $(T_1, \ldots, T_d)$ satisfies the properties

$$\sum_{i=1}^{\infty} T_i T_i^* \leqslant 1_{\mathscr{H}} \quad \text{and} \quad T_i T_j = T_j T_i \text{ for all } 1 \leqslant i,j \leqslant d.$$

Such d-tuples have been called d-contractions or row contractions. Associated with any Hilbert module $\mathscr{H}$ there is a completely positive map $\Psi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ defined by $\Psi(X) = \sum_{i=1}^{d} T_i X T_i^*$, $X \in \mathscr{B}(\mathscr{H})$. A Hilbert module is said to be pure if $\lim_{n \to \infty} \Psi^n(1_{\mathscr{H}}) = 0$ (SOT). The rank of $\mathscr{H}$ is defined as the rank of the defect operator $\Delta = (1_{\mathscr{H}} - \Psi(1_{\mathscr{H}}))^{1/2}$.

Of course, the spaces of the form $\mathscr{H} = H_d^2(\mathscr{D})$ come with a natural Hilbert module structure: If $\xi = f \in H_d^2(\mathscr{D})$, then $z_i \xi = M_{z_i} f$, $i = 1, \ldots, d$. One verifies that $H_d^2(\mathscr{D})$ is contractive, pure, and rank $H_d^2(\mathscr{D}) = \dim \mathscr{D}$. These modules serve a universal role in the category of pure contractive Hilbert modules. The following theorem makes this precise.

THEOREM 5.1.   *Let $\mathscr{H}$ be a pure contractive Hilbert module and let $\mathscr{D}$ be a Hilbert space with $\dim \mathscr{D} = \operatorname{rank} \mathscr{H}$. Then there exists a coisometric module homomorphism $U : H_d^2(\mathscr{D}) \to \mathscr{H}$ that is minimal in the sense that $U^* \mathscr{H}$ generates $H_d^2(\mathscr{D})$ as a Hilbert module. Furthermore, if $U' : H_d^2(\mathscr{D}') \to \mathscr{H}$ is another such map, then there exists a unitary operator $V : \mathscr{D} \to \mathscr{D}'$ such that $U = U' \tilde{V}$, where $\tilde{V}(fx) = fVx$ for all $f \in H_d^2$, $x \in \mathscr{D}$.*

Theorem 5.1 is a well-known result in dilation theory. For example, it can easily be derived from the results in [Agl82], or, for a precise statement in the language of Hilbert modules, see [Arv98]. In fact, if k is a complete NP kernel, then one can define a category of Hilbert modules where the spaces of the type $\mathscr{H}(k, \mathscr{D})$ play the role of the universal object. In this case one uses the functions $\{b_n\}$ of (1.1) to define the completely positive map Ψ, and an analogue of the above theorem holds (see [Agl82]).

Thus, any pure contractive Hilbert module $\mathscr{H}$ can be associated with a submodule $\mathscr{M} = \ker U$ of $H_d^2(\mathscr{D})$. It follows from Lemma 4.1 that $\mathscr{M}$ is a multiplier invariant subspace of $H_d^2(\mathscr{D})$, so by Theorem 1.3 there exists an auxiliary Hilbert space $\mathscr{E}$ and an inner multiplication operator $\Phi \in \mathscr{B}(H_d^2(\mathscr{E}), H_d^2(\mathscr{D}))$ with associated operator valued multiplier $\phi \in \mathscr{B}(\mathscr{E}, \mathscr{D})$ such that $\mathscr{M} = \Phi H_d^2(\mathscr{E})$.

The *curvature invariant* of a finite rank Hilbert module $\mathscr{H}$ was introduced in [Arv00]. To review the definition we need to fix some more notation. If $T_1, \ldots, T_d$ are the operators associated with $\mathscr{H}$, then for $\lambda \in B_d$ we set $T(\lambda) = \bar{\lambda}_1 T_1 + \cdots + \bar{\lambda}_d T_d$. Since $\mathscr{H}$ has finite rank the space $\Delta \mathscr{H}$ is finite

dimensional. We define a $\mathcal{B}(\Delta\mathcal{H})$-valued function on B_d by

$$F(\lambda) = (1 - |\lambda|^2)\Delta(1_{\mathcal{H}} - T(\lambda)^*)^{-1}(1_{\mathcal{H}} - T(\lambda))^{-1}\Delta.$$

It can be shown that $F(\lambda)$ is unitarily equivalent to $1_{\mathcal{D}} - \phi(\lambda)\phi(\lambda)^*$, where ϕ is the operator valued multiplier as in the previous paragraph (see [Arv00]). Thus, the radial limit (or even K-limit) of F exists in the strong operator topology for a.e. $z \in \partial B_d$. The curvature invariant of $\mathcal{H}$ is defined as

$$K(\mathcal{H}) = \int_{\partial B_d} \text{trace } F(z) \, d\sigma(z).$$

It is clear that $0 \leqslant K(\mathcal{H}) \leqslant \text{rank } \mathcal{H}$, and it follows that $K(\mathcal{H}) = \int_{\partial B_d} \text{trace} \, (1_{\mathcal{D}} - \phi(z)\phi(z)^*) \, d\sigma(z)$. The following theorem resolves Problem 1 of [Arv99].

THEOREM 5.2. *If $\mathcal{H}$ is a contractive, pure Hilbert module of finite rank, then $K(\mathcal{H})$ is an integer. In particular, if $\mathcal{M}$ is the multiplier invariant subspace associated with $\mathcal{H}$ as above, then*

$$K(\mathcal{H}) = \text{rank } \mathcal{H} - \sup\{\dim E_\lambda \mathcal{M} : \lambda \in B_d\}$$
$$= \inf\{\dim \mathcal{D}_\lambda \cap \mathcal{M}^\perp : \lambda \in B_d\},$$

where $\mathcal{D}_\lambda = k_\lambda \mathcal{D}$.

Proof. This follows immediately from Theorem 4.3. Recall that for $\lambda \in B_d$, $E_\lambda : H_d^2(\mathcal{D}) \to \mathcal{D}$ denotes the point evaluation, $E_\lambda f = f(\lambda)$.

For a.e. $z \in \partial B_d$, $F(z) = 1_{\mathcal{D}} - \phi(z)\phi(z)^*$ is a projection equal to rank $\mathcal{H} - \sup\{\dim E_\lambda \mathcal{M} : \lambda \in B_d\} = \inf\{\dim(E_\lambda \mathcal{M})^\perp : \lambda \in B_d\} = \inf\{\dim \mathcal{D}_\lambda \cap \mathcal{M}^\perp : \lambda \in B_d\}$, since one easily sees that $k_\lambda(E_\lambda \mathcal{M})^\perp = \mathcal{D}_\lambda \cap \mathcal{M}^\perp$. Hence $K(\mathcal{H}) = \int_{\partial B_d} \text{trace } F(z) \, d\sigma(z) = \inf\{\dim \mathcal{D}_\lambda \cap \mathcal{M}^\perp : \lambda \in B_d\}$. ∎

EXAMPLE 5.3. Let $\mathcal{H}$ be a pure contractive Hilbert module of finite rank, let U be as in Theorem 5.1, and, as above, set $\mathcal{M} = \ker U$.

(a) If φ is a nonzero scalar multiplier of H_d^2 such that $\varphi\mathcal{H} = 0$, then $K(\mathcal{H}) = 0$.

(b) If $\mathcal{M}$ is generated by a family of functions $\{f_n\}_{n \geqslant 1}$ such that there is a nonempty open set $\Omega \subseteq B_d$ such that the dimension of the linear span of $\{f_n(\lambda)\}_{n \geqslant 1}$ in $\mathcal{D}$ equals m for each $\lambda \in \Omega$, then $K(\mathcal{H}) = \text{rank } \mathcal{H} - m$.

Proof. (a) Let $\lambda \in B_d$ such that $\varphi(\lambda) \neq 0$. According to Theorem 5.2 it suffices to show that $\mathcal{D}_\lambda \cap \mathcal{M}^\perp = (0)$. Thus, let $x \in \mathcal{D}$ be such that

$k_\lambda x \in \mathcal{M}^\perp$. Then for any $y \in \mathcal{D}$, φy is a function in $\mathcal{M}$ since $\varphi \mathcal{H} = 0$. Hence $0 = \langle \varphi y, k_\lambda x \rangle = \varphi(\lambda) \langle y, x \rangle_{\mathcal{D}}$ and it follows that $x = 0$.

(b) It follows from the hypothesis that $\dim E_\lambda \mathcal{M} = m$ for each $\lambda \in \Omega$. Thus, it follows from Lemma 3.1 that $\sup\{\dim E_\lambda \mathcal{M} : \lambda \in B_d\} = m$, and the result follows from Theorem 5.2. ∎

It is sometimes possible for $K(\mathcal{H})$ to be defined and finite when $\dim \mathcal{D} = \infty$. Thus, a more general resolution of Problem 1 in [Arv99] would follow from an answer to the following question.

Question 5.4. Is $\dim(\operatorname{ran} \phi(z))^\perp$ almost everywhere equal to a constant even if $\dim \mathcal{D} = \infty$?

Note that Theorem 4.3 implies that $\dim \operatorname{ran} \phi(z)$ is a.e. equal to a constant.

REFERENCES

[Ag182]   J. Agler, The Arveson extension theorem and coanalytic models, *Integral Equations Operator Theory* **5** (1982), 608–631. MR 84g:47011.

[Agla]   J. Agler, Interpolation, unpublished.

[Aglb]   J. Agler and J. McCarthy, Complete Nevanlinna–Pick kernels, *J. Funct. Anal.* **175** (2000), 111–124. MR 2001h:47019.

[Arv98]   W. Arveson, Subalgebras of C^*-algebras III. Multivariable operator theory, *Acta Math.* **181** (1998), 159–228.

[Arv99]   W. Arveson, The curvature of a Hilbert module over $\mathbb{C}[z_1, z_2, \ldots, z_d]$, *Proc. Natl. Acad. Sci. USA* **96** (1999), 11 096–11 099.

[Arv00]   W. Arveson, The curvature invariant of a Hilbert module over $\mathbb{C}[z_1, \ldots, z_d]$, *J. Reine Angew. Math.* **522** (2000), 173–236. MR 1 758 582.

[BTV]   J. Ball, T. Trent and V. Vinnikov, Interpolation and commutant lifting for multipliers on reproducing kernel hilbert spaces—"Operator Theory and Analysis," pp. 89–138, Amsterdam, 1997. *Oper. Theory Adv. Appl.* **122**, Birkhäuser, Basel, 2001.

[Har92]   G. H. Hardy, "Divergent Series," Éditions Jacques Gabay, Sceaux, 1992, With a preface by J. E. Littlewood and a note by L. S. Bosanquet, Reprint of the revised (1963) edition. MR 93g:01100.

[McC92]   S. McCullough, Carathéodory interpolation kernels, *Integral Equations Operator Theory* **15** (1992), 43–71. MR 92m:47030.

[MT00]   S. McCullough and T. T. Trent, Invariant subspaces and Nevanlinna–Pick kernels, *J. Funct. Anal.* **178** (2000), 226–249. MR 2002b:47006.

[Qui93]   P. Quiggin, For which reproducing kernel Hilbert spaces is Pick's theorem true? *Integral Equations Operator Theory* **16** (1993), 244–266. MR 94a:47026.

[RR97]   M. Rosenblum and J. Rovnyak, "Hardy Classes and Operator Theory," Dover Publications Inc., Mineola, NY, 1997, Corrected reprint of the 1985 original. MR 97j:47002.

[Rud80] W. Rudin, "Function Theory in the Unit Ball of $\mathbb{C}^n$," Springer-Verlag, NewYork/ Berlin, 1980.

[Rud91] W. Rudin, "Functional Analysis," 2nd ed., McGraw-Hill Inc., New York, 1991. MR 92k:46001.

[SS62] H. S. Shapiro and A. L. Shields, On the zeros of functions with finite Dirichlet integral and some related function spaces, *Math. Z.* **80** (1962), 217–229. MR 26 #2617.