Hermitian presentations of Chevalley groups II

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Received 26 May 2005
Available online 17 November 2005
Communicated by Eva Bayer-Fluckiger

Abstract

A Chevalley group is called Hermitian if its root system is 3-graded. In this case, the roots of degree 0 are called compact and the remaining ones (those of degree 1 or \(-1\)) are called noncompact. Here, we modify the classical Steinberg presentation of a Chevalley group in a way that its generators become the symbols indexed by noncompact roots (the noncompact symbols); the new presentation is referred to as Hermitian. Either a compact symbol (that is, a symbol indexed by a compact root) or the commutator of a compact symbol and a noncompact one can be expressed as a product of noncompact symbols by Chevalley’s commutator formula. Combining these expressions, one obtains a formula that displays a pair of concatenated commutators and involves noncompact symbols only. We get a presentation of the same Chevalley group when we replace Chevalley’s commutator formula by this new double commutator formula and restrict the other relations to noncompact symbols. The simply-laced case has already been treated in [M.P. De Oliveira, E.W. Ellers, Hermitian presentations of Chevalley groups I, J. Algebra 276 (2004) 371–382. MR2054401 (2005b:20084)]. Here we proceed with an intrinsic investigation of the general case. In the process, we give a detailed analysis of the structure constants as well as higher order constants of the Chevalley algebra associated with our Chevalley group. In particular, we see that there are fewer choices for the signs of the coefficients appearing in the double commutator formula than there are in Chevalley’s commutator formula; actually, we show that the former are in one-to-one correspondence with the choice of signs produced by the noncompact vectors in a Chevalley basis of the above Chevalley algebra when seen

✩ This research has been supported in part by NSERC Canada Grant A7251.
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as a basis for the Lie triple system they span. In the end we give examples of Hermitian presentations
for the types $B_n$ and $C_n$ and a review of basic properties of 3-graded root systems.

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Keywords: Chevalley group; Hermitian semisimple Lie algebra; Group presentation

1. Introduction

The generators in Steinberg’s presentation for a Chevalley group $G$ are symbols $x_{\alpha}(t)$, where $\alpha$ belongs to the root system of $G$ and $t$ is a field element. Our goal is to give a presentation for Hermitian Chevalley groups that uses only noncompact symbols and relations for noncompact symbols. A symbol $x_{\alpha}(t)$ is called noncompact if $\alpha$ is a noncompact root.

We shall describe the background and derive some basic information.

Presentations of Chevalley groups are useful in proving results on representations of these groups. In [9], R. Steinberg has given presentations that today bear his name. Employing these presentations, he has shown that, for important classes of Chevalley groups, any projective representation can be lifted to a linear one ([9,10]; for other interesting applications see [11]). In [5], we studied a variation of the Steinberg presentation under the assumption that the root system $\Phi$ is of single length and 3-graded. The central idea in [5] is to restrict the set of generators to formal exponentials corresponding to roots of nonzero degrees (the noncompact ones), adjusting the set of relations accordingly. Here we extend this study, supplying similar presentations for Chevalley groups of type $B_n$ and $C_n$ (see Appendix A). In Steinberg’s classical presentation of a general Chevalley group, the most striking feature is Chevalley’s commutator formula. It gives the commutator of two symbols in terms of a product of symbols corresponding to roots that are positive linear combinations of the roots in the commutator. Now, as in [5], the new relations are simply restrictions of those in Steinberg’s presentation to noncompact roots, except for Chevalley’s commutator formula. Here we use a double commutator formula with a correcting factor to suppress the compact symbols. This factor occurs in the formula whenever the (noncompact) roots involved have two different lengths. Finally, an expression for the conjugation of a generator associated with a noncompact root by the reflection relative to the same root (seen as a group element), is added here to the set of defining relations. This expression is classically obtained as a corollary of the full Steinberg presentation. We call the presentation developed here ‘Hermitian’ because of its connection with the theory of Hermitian symmetric spaces, which comprises these methods and also these Chevalley groups when the underlying field is $\mathbb{C}$.

We start by introducing some notation. Let $g$ be a simple Lie algebra over $\mathbb{C}$, $\Phi$ its root system with respect to a Cartan subalgebra $h$ and let $\Phi^+$ be the set of positive roots with respect to some fundamental system $\Pi = \{s_1, \ldots, s_n\}$. Given $\alpha \in \Phi$, then $t_\alpha$ denotes its dual with respect to the Killing form $(\ ,\ )$ and

$$h_\alpha = \frac{2 t_\alpha}{(\alpha, \alpha)}.$$
Recall that a Chevalley basis is a basis
\[ \{ e_\alpha \mid \alpha \in \Phi \} \cup \{ h_i = h_{\alpha_i} \mid 1 \leq i \leq n \} \]
of \( g \) with

(i) \[ [e_\alpha, e_{-\alpha}] = h_\alpha, \]
(ii) if \( \alpha, \beta, \alpha + \beta \in \Phi \), then \[ [e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha + \beta}, \]
where \( N_{\alpha,\beta} = \pm (p + 1) \), \( p \) being the largest nonnegative integer such that \( \beta - p\alpha \in \Phi \).

Let us put \( N_{\alpha,\beta} = 0 \) for roots \( \alpha, \beta \) if \( \alpha + \beta / \notin \Phi \) and \( \alpha + \beta \neq 0 \). The structure constants \( N_{\alpha,\beta} \) \( (\alpha, \beta \in \Phi, \alpha \neq -\beta) \) of the complex Lie algebra \( g \) relative to some Chevalley basis, satisfy the following properties [3]:

(a) \( N_{\alpha,\beta} = \pm (p + 1) \) if \( \alpha + \beta \in \Phi \); \( N_{\alpha,\beta} = 0 \) otherwise,
(b) \( N_{\alpha,\beta} = -N_{\beta,\alpha} \),
(c) \( (\alpha, \alpha)N_{\alpha,\beta} + (\alpha + \beta, \alpha + \beta)N_{\alpha + \beta, -\beta} \) if \( \alpha + \beta \in \Phi \),
(d) \( N_{-\alpha, -\beta} = -N_{\alpha, \beta} \),
(e) \( \frac{N_{\alpha,\beta}N_{\gamma,\delta}}{(\alpha + \beta, \alpha + \beta)} + \frac{N_{\beta,\gamma}N_{\alpha,\delta}}{(\beta + \gamma, \beta + \gamma)} + \frac{N_{\gamma,\alpha}N_{\beta,\delta}}{(\alpha + \gamma, \alpha + \gamma)} = 0 \)
if \( \alpha + \beta + \gamma + \delta = 0 \).

Further, any solution \( N_{\alpha,\beta} \) for Eqs. (a)–(e) above constitutes a choice of structure constants for some Chevalley basis of \( g \). Moreover, the structure constants are defined but not uniquely determined by the above relations. As a matter of fact, for each nonsimple root \( \phi \) of \( \Phi^+ \), fix positive roots \( \alpha_\phi, \beta_\phi \) such that
\[ \phi = \alpha_\phi + \beta_\phi. \]
Then the signs of the \( N_{\alpha_\phi,\beta_\phi} \) can be chosen arbitrarily; the other structure constants are uniquely determined by the \( N_{\alpha_\phi,\beta_\phi} \) and the relations (a)–(e) above (see Lemma 1.2 below).

Although not explicitly laid out in [3], the next two lemmas follow promptly from the developments there.

**Lemma 1.1.** Numbers \( N_{\alpha,\beta} \) \( (\alpha, \beta \text{ roots, } \alpha + \beta \text{ nonzero}) \) are structure constants for some Chevalley basis of \( g \) if and only if they satisfy Eqs. (a)–(e) for \( \Phi \).

**Proof.** If \( N_{\alpha,\beta} \) are structure constants arising from some Chevalley basis, then they satisfy (a)–(e). On the other hand, suppose some \( N_{\alpha,\beta} \) satisfy (a)–(e). Let \( M_{\alpha,\beta} \) be the structure constants arising from some Chevalley basis. As these also satisfy (a)–(e), they satisfy (a) in particular. Therefore the absolute values of \( M_{\alpha,\beta} \) and \( N_{\alpha,\beta} \) are equal. Proposition 4.2.2 of [3] guarantees that the signs of the \( M_{\alpha,\beta} \) may be arbitrarily changed for extraspecial pairs, so we may assume
\[ M_{\alpha,\beta} = N_{\alpha,\beta}. \]
for extraspecial pairs. Now the developments on p. 59 of [3] imply that a solution for (a)–(e) is determined by the values at extraspecial pairs, so

\[ M_{\alpha,\beta} = N_{\alpha,\beta} \]

in general and the converse of the lemma is proved. □

In other words, the set of all possible structure constants coincides with the set of all solutions for (a)–(e). The degree of freedom in the system (a)–(e) is described by the following result:

**Lemma 1.2.** For each root \( \varphi \in \Phi^+ \setminus \Pi \), fix positive roots \( \alpha_{\varphi}, \beta_{\varphi} \) such that

\[ \varphi = \alpha_{\varphi} + \beta_{\varphi}. \]

Then for each choice of the signs of the \( N_{\alpha_{\varphi},\beta_{\varphi}} \), there is a unique solution for (a)–(e) extending the \( N_{\alpha_{\varphi},\beta_{\varphi}} \).

**Proof.** Let \( N_{\alpha,\beta} \) be a solution for (a)–(e). Then the \( N_{\alpha,\beta} \) are structure constants arising from some Chevalley basis, by Lemma 1.1. Since Proposition 4.2.2 of [3] holds for any choice of the pairs \( (\alpha_{\varphi}, \beta_{\varphi}) \), not necessarily extraspecial, we can change the sign of \( N_{\alpha_{\varphi},\beta_{\varphi}} \) arbitrarily and still claim that they come from a choice of Chevalley structure constants. The second statement of that proposition assures that the other values of \( N_{\alpha,\beta} \) are uniquely determined. (That Proposition 4.2.2 of [3] holds for an arbitrary choice of pairs \( (\alpha_{\varphi}, \beta_{\varphi}) \) of positive roots as above, is easy to prove. It is the same as in [3], except that, in the proof here, one should sort the root vectors in increasing order according to some total ordering of the root space that is compatible with \( \Phi^+ \) before adjusting their signs.) □

Let \( K \) be an arbitrary field. By means of a Chevalley basis, one can define a Lie algebra over \( K \) and its automorphisms \( x_\alpha(t) = \exp(t \text{ad} e_\alpha) \) \((t \in K, \alpha \in \Phi)\) in a natural way. These automorphisms generate a group \( G_\alpha \), which has trivial center and is simple, except for the types \( A_1, B_2, G_2 \) if \( K = \mathbb{F}_2 \), or \( A_1 \) if \( K = \mathbb{F}_3 \) (cf. [3,4]).

In the general case of a nonzero complex finite-dimensional representation \( \pi : g \to V \), let \( v_+ \in V \) be a nonzero highest weight vector and \( V(\mathbb{Z}) \) the lattice spanned by the vectors of the form

\[ \frac{(\pi e_{-\alpha_1})^{i_1}}{i_1!} \cdots \frac{(\pi e_{-\alpha_k})^{i_k}}{i_k!} v_+, \quad \text{where} \ \alpha_j \in \Phi^+. \]

As the elements

\[ \frac{e_{-\alpha_1}^{i_1}}{i_1!} \cdots \frac{e_{-\alpha_m}^{i_m}}{i_m!} (h_1)_{j_1} \cdots (h_l)_{j_l} \frac{e_{\alpha_1}^{k_1}}{k_1!} \cdots \frac{e_{\alpha_n}^{k_n}}{k_n!}, \]
form a \(\mathbb{Z}\)-basis of a \(\mathbb{Z}\)-form of the universal enveloping algebra of \(\mathfrak{g}\), the lattice \(V(\mathbb{Z})\) is invariant under the action of

\[
(\pi e_\alpha)^k / k!, \quad \text{where } \alpha \in \Phi \text{ and } k \in \mathbb{Z}^+ 
\]

(see [1]). The exponentials

\[
x_\alpha(t) = x_\alpha^\pi(t) = \exp(t \pi e_\alpha) = 1 + t \pi e_\alpha + t^2 (\pi e_\alpha)^2 / 2! + \cdots, \quad \alpha \in \Phi,
\]

span a group of matrices with entries in \(\mathbb{Z}[t]\); specializing \(t\) to elements in an arbitrary field \(K\), we obtain a group \(G_\pi\) which is called a Chevalley group of type \(\pi\).

Let \(L_\pi\) denote the \(\mathbb{Z}\)-lattice in \(\mathfrak{h}^*\) spanned by the weights of \(\pi\). Let \(\pi\) and \(\rho\) be two finite-dimensional representations of \(\mathfrak{g}\). If \(L_\pi \supset L_\rho\), then there exists a homomorphism \(\varphi: G_\pi \to G_\rho\) such that \((\varphi, G_\pi)\) is a central extension of \(G_\rho\), satisfying \(\varphi(x_\alpha^\pi(t)) = x_\rho^\alpha(t)\) for any \(\alpha \in \Phi\) and \(t \in K\). If \(L_\pi = L_\rho\) then \(\varphi\) is an isomorphism from \(G_\pi\) onto \(G_\rho\) (see [10]).

There are two extremes for such a lattice: \(L_a\), the lattice generated by the roots, and \(L_u\), the lattice generated by the fundamental weights. Any \(L_\pi\) contains \(L_a\) and is contained in \(L_u\), and any lattice satisfying this condition arises from some nonzero finite-dimensional representation \(\pi\) of \(\mathfrak{g}\). The Chevalley groups \(G_a\) and \(G_u\) corresponding to the lattices \(L_a\) and \(L_u\) are called Chevalley group of adjoint type and universal Chevalley group, respectively. Consider the special case \(\rho = \text{ad}\) in the previous paragraph. Then \(L_\rho = L_a\), \(G_\rho = G_a\), and the homomorphism \(\varphi: G_\pi \to G_a\) has kernel \(Z(G_\pi)\) (the center of \(G_\pi\)) for any \(\pi\). In particular, \(G_a \cong G_a / Z(G_u)\).

The reader is referred to [4], the original reference to Chevalley groups of adjoint type, and also to Steinberg’s articles [7,9,11], and his notes [10] to obtain other relevant information. Here we list some relations satisfied by the exponential elements in any Chevalley group \(G_\pi\):

\[
(A) \quad x_\alpha(t)x_\alpha(u) = x_\alpha(t + u), \quad \text{where } \alpha \in \Phi; t, u \in K.
\]

\[
(B) \quad [x_\alpha(t), x_\beta(u)] = \prod_{i, j > 0} x_i\alpha + j\beta (-C_{ij}\alpha\beta t^i (-u)^j),
\]

where \(\alpha, \beta \in \Phi, \alpha + \beta \neq 0; t, u \in K, \text{ and } [x, y] = xyx^{-1}y^{-1}.
\]

In this product \(i\) and \(j\) range over all pairs of positive integers such that \(i\alpha + j\beta \in \Phi\). (If there are no such integers, then \([x_\alpha(t), x_\beta(u)] = 1\). The terms occur in any order with \(i + j\) nonincreasing (cf. [3], where the order of the terms is any such that \(i + j\) is nondecreasing). The scalars \(C_{ij}\alpha\beta\) are defined in Section 2 (see [3]). They are independent of \(t\) and \(u\), and up to the sign depend only on \(i, j\) and the geometry of \(\alpha\) and \(\beta\).

\[
(C) \quad h_\alpha(t)h_\alpha(u) = h_\alpha(tu) \quad (\alpha \in \Phi, tu \neq 0),
\]

\[
(W) \quad n_\alpha(t)x_\alpha(u)n_\alpha(t)^{-1} = x_{-\alpha}(-t^{-2}u) \quad (\alpha \in \Phi, t, u \in K, t \neq 0),
\]

where
\[ n_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t), \]
\[ h_\alpha(t) = n_\alpha(t)n_\alpha(1)^{-1}, \quad \alpha \in \Phi, \ t \neq 0. \]

The relations (A), (B), (C) suffice to characterize the universal Chevalley groups:

**Theorem 1.3.** (Steinberg [10]) If \( \Phi \) is not of type \( A_1 \), then the abstract group generated by the symbols \( x_\alpha(t)(\alpha \in \Phi, t \in K) \) subject to the relations (A), (B), and (C) is the universal Chevalley group \( G_u \) associated with \( g \).

If \( \Phi \) is of type \( A_1 \), then \( G_u \) is abstractly generated by these symbols subject to (A), (W), and (C). Other combinations of the relations are of interest as well: the group \( G_c \) generated by the \( x_\alpha(t) \) subject to the relations (A), (B) if \( \Phi \) is not of type \( A_1 \) (or (A), (W) if \( \Phi \) is of type \( A_1 \)) is the universal central extension of \( G_a \) under the natural projection, provided that the field \( K \) is not very small. If \( K \) is an algebraic extension of a finite field, then \( G_c = G_u \) (see [10]).

Suppose now that the complex simple Lie algebra \( g \) is Hermitian; this is equivalent to saying that the root system \( \Phi \) of \( g \) admits a 3-gradation

\[ \Phi = \Phi_{-1} \cup \Phi_0 \cup \Phi_1, \]

as defined in [5]. Since this gradation arises from symmetries of a Hermitian symmetric space, we can call the roots in

\[ \Phi_n = \Phi_{-1} \cup \Phi_1 \]

noncompact and the others (those in \( \Phi_0 \)) compact (see [8]). It is possible to choose a positive system for \( \Phi \) relative to which the roots of \( \Phi_1 \) are all positive. For such a positive system of roots, there is exactly one simple root in \( \Phi_1 \).

Let \( (A^*) \), \( (C^*) \), and \( (W^*) \) be the families of relations obtained by restricting the families (A), (C), and (W), respectively, to \( \Phi_n \), the noncompact roots. From (B) we derive the families of relations \( (B_1^*) \) and \( (B_2^*) \) given by

\[ (B_1^*) \left[ x_\beta(t), x_\gamma(u) \right] = 1 \text{ if } 0 \neq \beta + \gamma \notin \Phi, \]
\[ (B_2^*) \left[ x_\beta(t), x_\gamma(u) \right] \cdot \prod \limits_{\substack{i+j>2 \\\ i\beta+j\gamma \in \Phi}} x_{i\beta+j\gamma}
(C_{1i+j\gamma} t^i (-u)^j, x_\delta(s)) = \prod \limits_{\substack{i,j>0 \\\ i(\beta+\gamma)+j\delta \in \Phi}} x_{i(\beta+\gamma)+j\delta}
(-C_{ij(\beta+\gamma)} \delta (C_{11\beta\gamma} tu)^i (-s)^j) \quad \text{if } \beta + \gamma \in \Phi, \]

where \( \beta, \gamma, \delta \in \Phi_n \) and \( s, t, u \in K \).

The scalars \( C_{ij\alpha\beta} \) will be defined in Section 2 (see also [3]). Here a double bracket is employed in order to obtain relations involving noncompact roots only. Notice that the expression paired with \( x_\delta(s) \) in \( (B_2^*) \) corresponds to the compact symbol \( x_{\beta+\gamma}(C_{11\beta\gamma} tu) \).
in \((B)\). The product on the left-hand side of Eq. \((B^*_2)\) is either 1 or a single generator. Similarly, the right-hand side of \((B^*_2)\) is a product of none, one, or two commuting symbols, depending on how many roots are positive linear combinations of \(\beta + \gamma\) and \(\delta\). It is easy to see that only noncompact symbols appear in \((B^*_2)\).

We may restate \((B^*_2)\) as

\[
(B^*_2) = [x_{\beta}(t), x_{\gamma}(u)]x_{\beta+2\gamma}(C_{12}\beta\gamma tu^2)x_{2\beta+\gamma}(-C_{21}\beta\gamma t^2u), x_{\delta}(s)]
\]

for \(\beta, \gamma \in \Phi_n; \beta + \gamma \in \Phi; s, t, u \in K,\)

where any symbol whose index is not a root is interpreted as 1.

We state our main result:

**Theorem 1.4.** Let \(\Phi\) be a 3-graded root system. Let \(G\) be a group generated by elements \(x_{\beta}(t), \) where \(\beta \in \Phi_n\) and \(t \in K,\) satisfying the relations \((A^*), (B^*_1), (B^*_2),\) and \((W^*)\). Then there are elements \(x_\alpha(t) \in G,\) where \(\alpha \in \Phi_0\) and \(t \in K,\) which, together with the above generators, satisfy the relations \((A)\) and \((B).\) Moreover, if the relations \((C^*)\) also hold, so do all relations \((C)\).

The assertion of Theorem 1.4 has already been proved in the simply-laced case (see [5]). If \(\Phi\) is of type \(C_n,\) the relations \((W^*)\) follow from \((A^*), (B^*_1), (B^*_2),\) so they can be omitted in the statement of the theorem. This last remark, Theorem 1.4, and Steinberg’s notes [10, p. 45], imply the following result:

**Corollary 1.5.** If \(\Phi\) is of type \(B_n,\) then the group generated by the noncompact symbols \((the x_{\beta}(t) with \beta \in \Phi_n and t \in K)\) subject to the relations \((A^*), (B^*_1), (B^*_2), (C^*), (W^*)\) is isomorphic to \(\text{Spin}_{2n+1}(K).\) If \(\Phi\) is of type \(C_n,\) then the group generated by the noncompact symbols subject to the relations \((A^*), (B^*_1), (B^*_2), (C^*), (W^*)\) is isomorphic to \(\text{Sp}_{2n}(K).\)

The study of Hermitian Chevalley groups has started only recently. More information is provided in [7] where the authors examine the structure of these groups.

### 2. Structure constants

As the relations \((a)-(e)\) from Section 1 characterize the structure constants arising from a Chevalley basis, we can use them to prove any result about these constants. Here we restrict our attention to the Hermitian situation. In what follows, \(\Phi\) denotes an irreducible 3-graded root system unless otherwise specified.

#### 2.1. Let \(\alpha, \beta, \alpha + \beta\) be roots, let

\[-p\alpha + \beta, \ldots, q\alpha + \beta\]
be the $\alpha$-chain through $\beta$ and
\[
-p\beta + \alpha, \ldots, q\beta + \alpha
\]
the $\beta$-chain through $\alpha$, with nonnegative numbers $p, q, q'$ (Eqs. (a) and (b) imply that the same number $p$ appears at the bottom of the chains).

Equations (a) and (c) yield
\[
N_{\alpha,\beta}N_{\alpha+\beta,-\beta} = (1 + p)^2 \frac{(\alpha, \alpha)}{(\alpha + \beta, \alpha + \beta)},
\]
and
\[
N_{\alpha+\beta,-\beta}N_{\alpha,\beta} = q' \frac{(\alpha + \beta, \alpha + \beta)}{(\alpha, \alpha)}
\]
since $\alpha + \beta - (q' - 1)(-\beta) = \alpha + q'\beta$.

Therefore
\[
\frac{(\alpha, \alpha)}{(\alpha + \beta, \alpha + \beta)} = \frac{q'}{1 + p} \tag{2.1}
\]
and
(c') $N_{\alpha,\beta}N_{\alpha+\beta,-\beta} = q'(1 + p)$

which clearly hold for any finite root system.

2.2. As before, let $\alpha, \beta \in \Phi$, let
\[
-p\alpha + \beta, \ldots, q\alpha + \beta
\]
be the $\alpha$-chain through $\beta$ and
\[
-p\beta + \alpha, \ldots, q\beta + \alpha
\]
the $\beta$-chain through $\alpha$. Define (see [3])
\[
C_{i1\alpha\beta} = \frac{1}{i!} N_{\alpha,\beta} N_{\alpha,\alpha+\beta} \cdots N_{\alpha, (i-1)\alpha+\beta}, \quad 1 \leq i \leq q + 1. \tag{2.2}
\]
\[
C_{i'1\alpha\beta} = \frac{1}{i'!} N_{\alpha,\beta} N_{\alpha+\beta,\beta} \cdots N_{\alpha+(i-1)\beta,\beta}, \quad 1 \leq i' \leq q' + 1. \tag{2.3}
\]
Then
\[ C_{i1\alpha\beta} = (-1)^i C_{1i\beta\alpha}, \]  
(2.4)

\[ C_{i1\alpha\beta} = \pm \left( \frac{p+i}{i} \right), \]  
(2.5)
as long as they are defined. Given a subset \( S \) of \( \Phi \), let \( \Phi(S) \) be the subsystem of \( \Phi \) consisting of the roots that are integral linear combinations of elements of \( S \).

- If \( \alpha, \beta \in \Phi \) have the same length and are not orthogonal, with \( \alpha + \beta \in \Phi \), then \( \{\alpha, \beta\} \) is a system of fundamental roots for \( \Phi(\alpha, \beta) \) of type \( A_2 \). Hence \( p = 0 \), \( q = q' = 1 \).

\[ N_{\beta,\alpha} N_{\alpha,\beta} = q(1 + p) = 1, \]
\[ N_{\alpha,\beta} N_{\alpha,\beta} = q'(1 + p) = 1, \]
\[ C_{11\alpha\beta} = -C_{11\beta\alpha} = N_{\alpha,\beta} = \pm 1. \]  
(2.6)

- If \( \alpha, \beta \in \Phi \) are orthogonal and \( \alpha + \beta \in \Phi \), then they are both short, \( p = q = q' = 1 \), either \( \alpha \) or \( \beta \) is compact, and \( \Phi(\alpha, \beta) \) is of type \( B_2 \).

\[ N_{\beta,\alpha} N_{\alpha,\beta} = q(1 + p) = 2, \]
\[ N_{\alpha,\beta} N_{\alpha,\beta} = q'(1 + p) = 2, \]
\[ C_{11\alpha\beta} = -C_{11\beta\alpha} = N_{\alpha,\beta} = \pm 2. \]  
(2.7)

- If \( \alpha \in \Phi \) is short and \( \beta \in \Phi \) is long, with \( \alpha + \beta \in \Phi \), then \( \{\alpha, \beta\} \) is a system of fundamental roots for \( \Phi(\alpha, \beta) \) of type \( B_2 \) and consequently \( p = 0 \), \( q = 2 \), \( q' = 1 \).

\[ N_{\beta,\alpha} N_{\alpha,\beta} = q(1 + p) = 2, \]
\[ N_{\alpha,\beta} N_{\alpha,\beta} = q'(1 + p) = 1, \]
\[ C_{11\alpha\beta} = -C_{11\beta\alpha} = N_{\alpha,\beta} = \pm 1, \]
\[ C_{21\alpha\beta} = C_{12\beta\alpha} = \frac{1}{2} N_{\alpha,\beta} N_{\alpha,\alpha} = \pm 1. \]  
(2.8)

2.3. We may endow \( \Phi \) with a positive system such that all roots of \( \Phi_1 \) are positive. We represent the simple roots as in [2].

If \( \Phi \) is of type \( B_n \), then the only positive simple root in \( \Phi_1 = \Phi^+_n = \beta_1 = e_1 - e_2 \) and its 3-graded root system is given by

\[ \Phi_0 = \{ \pm e_i \mid 1 < i < n \} \cup \{ e_i \pm e_j \mid 1 < i, j < n, i \neq j \}, \]
\[ \Phi_1 = \Phi^+_n = \{ e_1 \} \cup \{ e_1 \pm e_j \mid 1 < j \leq n \}. \]

If \( \Phi \) is of type \( C_n \), then the only positive simple root in \( \Phi_1 = \Phi^+_n = \beta_n = 2e_n \) and its 3-graded root system is given by
\[ \Phi_0 = \{ e_i - e_j \mid 1 \leq i, j \leq n, \ i \neq j \}, \]
\[ \Phi_1 = \Phi_n^+ = \{ e_i + e_j \mid 1 \leq i < j \leq n \} \cup \{ 2e_i \mid 1 \leq i \leq n \}. \]

Suppose \( \beta, \gamma, \delta, \varepsilon \) are noncompact roots such that
\[ \beta + \gamma + \delta + \varepsilon = 0. \]

We recall that in the decomposition of a noncompact root into a sum of simple ones the (unique) noncompact simple root always appears with multiplicity \( \pm 1 \). Hence there must be exactly two positive noncompact roots in the sum above and we may assume, without loss of generality, that \( \beta, \delta \in \Phi_n^+ \) and \( \gamma, \varepsilon \in \Phi_n^- \). Clearly, \( \beta + \delta \) is not a root, thus \( N_{\beta, \delta} = 0 \).

If no two of them are opposite, we rewrite \((e)\) as
\[ \frac{N_{\beta, \gamma} N_{\delta, \varepsilon}}{(\beta + \gamma, \beta + \gamma)} + \frac{N_{\gamma, \delta} N_{\beta, \varepsilon}}{(\gamma + \delta, \gamma + \delta)} = 0. \]

Looking at \( \Phi_0, \Phi_n^+, \) and \( \Phi_n^- \) for \( B_n \) and \( C_n \) we see that
\[ (e') \ N_{\beta, \gamma} N_{\delta, \varepsilon} + N_{\gamma, \delta} N_{\beta, \varepsilon} = 0 \]
for \( \beta, \gamma, \delta, \varepsilon \) as above, i.e., \( \beta, \delta \in \Phi_n^+ \), \( \gamma, \varepsilon \in \Phi_n^- \), \( \beta + \gamma + \delta + \varepsilon = 0 \), where no two of them are opposite. Indeed, if \( \beta + \gamma \) and \( \gamma + \delta \) are roots, then the denominators are equal; if either \( \beta + \gamma \) or \( \gamma + \delta \) is not a root, then both terms of \((e')\) are zero and \((e')\) holds trivially.

If the roots \( \alpha \) and \( \alpha' \) are compact, then we can calculate the structure constant \( N_{\alpha, \alpha'} \) from those whose index pair contains at least one noncompact root.

**Lemma 2.1.** Assume \( \Phi \) is a 3-graded root system. Let \( \alpha \in \Phi \) and let \( \alpha' = \beta + \gamma \) be a compact root, where \( \beta, \gamma \in \Phi_n \).

(a) If \( \alpha + \beta, \alpha + \gamma \in \Phi \) and \( \alpha + \alpha' \neq 0 \), then
\[ N_{\beta, \gamma} (N_{\alpha, \beta} N_{\alpha + \beta, \gamma} + N_{\alpha, \gamma} N_{\beta, \alpha + \gamma}) = N_{\alpha, \alpha'}. \]

(b) If \( \alpha + \beta \in \Phi \) and \( \alpha + \gamma \notin \Phi \), then \( \alpha + \alpha' \neq 0 \) and
\[ N_{\beta, \gamma} N_{\alpha, \beta} N_{\alpha + \beta, \gamma} = N_{\alpha, \alpha'}. \]

(c) If \( 0 \neq \alpha + \beta \notin \Phi \) and \( 0 \neq \alpha + \gamma \notin \Phi \), then \( \alpha + \alpha' \neq 0 \) and \( N_{\alpha, \alpha'} = 0 \).

**Proof.** If \( \alpha + \alpha' \) is not a root, then all the statements are trivially true. Now we assume that \( \alpha + \alpha' \in \Phi \).

First we prove (a).

Put \( r = -(\alpha + \beta + \gamma) = -(\alpha + \alpha') \). Then by Section 1(e) we get
\[(i) \ \frac{N_{\alpha, \beta} N_{\gamma, r}}{(\alpha + \beta, \alpha + \beta)} + \frac{N_{\beta, \gamma} N_{\alpha, r}}{(\beta + \gamma, \beta + \gamma)} + \frac{N_{\gamma, \alpha} N_{\beta, r}}{(\alpha + \gamma, \alpha + \gamma)} = 0. \]
Also, Section 1(b), (c), and (d) yield

\[(ii) \quad \frac{N_{r,\gamma}}{(\alpha + \beta, \alpha + \beta)} = \frac{N_{\gamma,\alpha+\beta}}{(\alpha + \alpha', \alpha + \alpha')},\]

\[(iii) \quad \frac{N_{r,\alpha}}{(\beta + \gamma, \beta + \gamma)} = \frac{N_{\alpha,\beta+\gamma}}{(\alpha + \alpha', \alpha + \alpha')},\]

and

\[(iv) \quad \frac{N_{r,\beta}}{(\alpha + \gamma, \alpha + \gamma)} = \frac{N_{\beta,\alpha+\gamma}}{(\alpha + \alpha', \alpha + \alpha')}.\]

Thus

\[(v) \quad \frac{N_{\alpha,\beta}N_{\gamma,\alpha+\beta}}{(\alpha + \alpha', \alpha + \alpha')} + \frac{N_{\beta,\gamma}N_{\alpha,\beta+\gamma}}{(\alpha + \alpha', \alpha + \alpha')} + \frac{N_{\gamma,\alpha}N_{\beta,\alpha+\gamma}}{(\alpha + \alpha', \alpha + \alpha')} = 0\]

and so

\[(vi) \quad N_{\alpha,\beta}N_{\alpha+\beta,\gamma} + N_{\alpha,\gamma}N_{\beta,\alpha+\gamma} = N_{\beta,\gamma}N_{\alpha,\alpha'}.\]

Also, $N_{\beta,\gamma} = \pm 1$; indeed, $-\beta$ and $\gamma$ are both either noncompact positive or noncompact negative and this implies that $-\beta + \gamma$ is not a root.

Therefore,

$$N_{\beta,\gamma}(N_{\alpha,\beta}N_{\alpha+\beta,\gamma} + N_{\alpha,\gamma}N_{\beta,\alpha+\gamma}) = N_{\beta,\gamma}N_{\beta,\gamma}N_{\alpha,\alpha'} = N_{\alpha,\alpha'}.$$

In order to see that (b) and (c) hold, we observe that now only two and one nonzero terms are left in (i) and (v), respectively (in particular, the condition in (c) never happens if $\alpha + \alpha' \in \Phi$).

3. Products of structure constants

The constants $C_{21\alpha\beta}$ have properties reminiscent of the properties (a) to (e) in Section 1 for the structure constants $N_{\alpha,\beta}$.

**Lemma 3.1.** The constants $C_{21\alpha\beta}$ satisfy the following properties: In (a)–(d), suppose \{\alpha, \beta\} is a system of fundamental roots of type B_2, where the root $\alpha$ is short.

(a) $C_{21\alpha\beta} = \pm 1$,

(b) $C_{21\alpha\beta} = C_{12\beta\alpha}$,

(c) $C_{21\alpha\beta} = -C_{21(\alpha+\beta)(-\beta)} = C_{21(-\alpha)(2\alpha+\beta)}$,

(d) $C_{21(-\alpha)(-\beta)} = C_{21\alpha\beta}$. 

In (e), (f), suppose \{\alpha, \beta, \gamma\} is a system of fundamental roots of type $B_3$, where $\alpha$ is short and orthogonal to $\gamma$.

(e) $C_{21\alpha\beta}N_{\alpha,\gamma} = -C_{21(\alpha+\beta)}N_{\gamma,\gamma - \alpha}$ if $\alpha + \beta + \gamma + \delta = 0$

(f) $C_{21\alpha\alpha'} = C_{21\alpha\beta}N_{\beta,\gamma}N_{2\alpha+\beta,\gamma}$ if $\alpha' = \beta + \gamma$

In (g), suppose \{\beta, \gamma, \delta\} is a system of fundamental roots of type $C_3$, where $\beta$ is the long root and orthogonal to $\delta$, and $\alpha + \beta + \gamma + \delta = 0$.

(g) $C_{21\alpha\beta}N_{\beta+2\gamma,\delta} = C_{21\gamma\beta}N_{\beta+2\alpha,\delta}$.

**Proof.** (a) and (b) are immediate consequences of (2.8).

(c)

\[
C_{21\alpha\beta} = (1/2)N_{\alpha,\beta}N_{\alpha,\alpha + \beta} = -(1/2)N_{\alpha + \beta, -\beta}N_{\alpha + \beta, \alpha} = -C_{21(\alpha + \beta)(-\beta)}
\]
\[
C_{21\alpha\beta} = (1/2)N_{\alpha,\beta}N_{\alpha,\alpha + \beta} = (1/2)(N_{-\alpha,\alpha + \beta}/2)(2N_{-\alpha,2\alpha + \beta})
\]
\[
= (1/2)N_{-\alpha,2\alpha + \beta}N_{-\alpha,\alpha + \beta} = C_{21(-\alpha)(2\alpha + \beta)}.
\]

(d)

\[
C_{21(-\alpha)(-\beta)} = (1/2)N_{-\alpha, -\beta}N_{-\alpha, -\alpha - \beta} = (1/2)N_{\alpha, \beta}N_{\alpha, \alpha + \beta} = C_{21\alpha\beta}.
\]

(e) Since $\alpha + (\alpha + \beta) + \gamma + (\delta - \alpha) = 0$ and $\alpha + \beta + (\delta - \alpha - \beta) - \delta = 0$, property (e) for structure constants yields

\[
N_{\alpha,\alpha + \beta}N_{\gamma,\delta - \alpha} = -2N_{\alpha + \beta,\gamma}N_{\alpha,\delta - \alpha}
\]

and

\[
N_{\beta,\delta - \alpha - \beta}N_{\alpha, -\delta} = -2N_{\alpha,\beta}N_{\delta - \alpha - \beta, -\delta}.
\]

Thus

\[
N_{\alpha,\alpha + \beta}N_{\gamma,\delta - \alpha}N_{\beta,\delta - \alpha - \beta}N_{\alpha, -\delta} = 4N_{\alpha + \beta,\gamma}N_{\alpha,\delta - \alpha}N_{\alpha,\beta}N_{\delta - \alpha - \beta, -\delta}
\]

or

\[
N_{\alpha,\alpha + \beta}N_{\gamma,\delta - \alpha}N_{\beta,\delta - \alpha - \beta} = N_{\alpha + \beta,\gamma}N_{\alpha,\beta}N_{\alpha,\alpha + \beta, -\delta}
\]
because $N_{\alpha,-\delta} = -2N_{\alpha,\delta-\alpha}$ and $N_{\alpha+\beta,-\delta} = -2N_{\delta-\alpha-\beta,-\delta}$.

Hence

$$N_{\alpha,\beta}N_{\alpha,\alpha+\beta}N_{\gamma,\delta-\alpha}N_{\beta,\delta-\alpha-\beta} = N_{\alpha+\beta,\gamma}N_{\alpha+\beta,-\delta}, \text{ or}$$

$$C_{21\alpha\beta}N_{\beta,\alpha-\delta} = -C_{21(\alpha+\beta)\gamma}N_{\gamma,\delta-\alpha}.$$

**(f)** We can endow the root system of type $B_3$ with a 3-grading such that $\beta, \gamma$ are noncompact roots. Then $\alpha$ and $\alpha' = \beta + \gamma$ are compact roots. We use Lemma 2.1(b), along with $\alpha' = \beta + \gamma$ and $\alpha + \alpha' = (\alpha + \beta) + \gamma$ and obtain

$$N_{\alpha,\alpha'} = N_{\beta,\gamma}N_{\alpha,\beta}N_{\alpha+\beta,\gamma},$$

$$N_{\alpha,\alpha+\alpha'} = N_{\alpha+\beta,\gamma}N_{\alpha,\alpha+\beta}N_{2\alpha+\beta,\gamma}.$$

Hence

$$N_{\alpha,\alpha'}N_{\alpha,\alpha+\alpha'} = N_{\beta,\gamma}N_{\alpha,\beta}N_{\alpha+\beta,\gamma}N_{\alpha,\alpha+\beta}N_{2\alpha+\beta,\gamma} = N_{\beta,\gamma}N_{\alpha,\beta}N_{\alpha,\alpha+\beta}N_{2\alpha+\beta,\gamma},$$

that is, $C_{21\alpha\alpha'} = C_{21\alpha\beta}N_{\beta,\gamma}N_{2\alpha+\beta,\gamma}$.

**(g)** We have

$$\alpha + \beta + \gamma + \delta = 0,$$

$$-\alpha + (\beta + 2\alpha) + \gamma + \delta = 0,$$

$$\alpha + (\beta + 2\gamma) - \gamma + \delta = 0.$$

From property (e) of the structure constants, one gets

$$N_{\alpha,\beta}N_{\gamma,\delta} = -N_{\beta,\gamma}N_{\alpha,\delta},$$

$$N_{-\alpha,\beta+2\alpha}N_{\gamma,\delta} = -N_{\gamma,-\alpha}N_{\beta+2\alpha,\delta},$$

$$N_{-\gamma,\alpha}N_{\beta+2\gamma,\delta} = -N_{\beta+2\gamma,-\gamma}N_{\alpha,\delta}.$$  

Then

$$N_{\alpha,\beta}N_{\gamma,\delta}N_{-\alpha,\beta+2\alpha}N_{\gamma,\delta}N_{-\gamma,\alpha}N_{\beta+2\gamma,\delta} = -N_{\beta,\gamma}N_{\alpha,\delta}N_{\gamma,-\alpha}N_{\beta+2\alpha,\delta}N_{\beta+2\gamma,-\gamma}N_{\alpha,\delta}$$

or

$$N_{\alpha,\beta}N_{-\alpha,\beta+2\alpha}N_{\beta+2\gamma,\delta} = N_{\beta,\gamma}N_{\beta+2\gamma,-\gamma}N_{\beta+2\alpha,\delta}.$$
Now $2N_{\alpha,2\gamma} = N_{\alpha,\alpha+2\alpha}$. We hence have

$$C_{21\alpha\beta}N_{\beta+2\gamma,\gamma} = C_{21\gamma\beta}N_{\beta+2\alpha,\alpha}.$$

### 4. Obtaining the Steinberg relations from the Hermitian ones

We now give a proof of Theorem 1.4. This has already been done in the simply-laced case (see [5]). In this section, we make all the assumptions of Theorem 1.4.

Besides $(B^*_1)$ and $(B^*_2)$ as in Section 1, we assume the following relations for $\beta \in \Phi_n$ and $t, u, v \in K$:

$$(A^*) \ x_\beta(t)x_\beta(u) = x_\beta(t + u),$$

$$(W^*) \ n_\beta(t)x_\beta(u)n_\beta(t)^{-1} = x_\gamma(-t^{-2}u),$$

where

$$n_\beta(t) = x_\beta(t)x_\gamma(t) - t^{-1}x_\beta(t), \quad t \neq 0,$$

and, for the second assertion in Theorem 1.4, additionally

$$(C^*) \ h_\beta(t)h_\beta(u) = h_\beta(tu), \quad tu \neq 0,$$

where

$$h_\beta(t) = n_\beta(t)n_\beta(1)^{-1}, \quad t \neq 0.$$
(g) $[x_{-\beta}(v), [x_{\gamma}(u), x_\beta(t)]] = x_\gamma(-2tuv) \cdot [x_{-\beta}(v), x_{\gamma + 2\beta}(-C_{21}\beta\gamma t^2u)]$.

(h) $[x_{-\gamma}(v), [x_{\gamma}(u), x_\beta(t)]] = x_\beta(tuv) \cdot x_{2\beta + \gamma}(C_{21}\beta\gamma t^2u^2v)$.

**Proof.** The set $\{\beta, \gamma\}$ is a system of fundamental roots for $\Phi(\beta, \gamma)$ of type $B_2$ since $\beta \in \Phi_n$ is short and $\gamma \in \Phi_n$ is long, with $\beta + \gamma \in \Phi$. For $w \in K$, we get

$$[[x_\beta(t), x_\gamma(u)], x_\delta(v)] = [[x_\beta(t), x_\gamma(u)] x_\gamma \gamma + 2\beta(-w), x_\delta(v)]$$

$$= [[[x_\beta(t), x_\gamma(u)] x_\gamma \gamma + 2\beta(-w), x_\delta(v)]$$

$$\cdot [[[x_\beta(t), x_\gamma(u)] x_\gamma \gamma + 2\beta(-w), x_\delta(v)]$$.  \hspace{1cm} (4.1)

In (a) and (b), substitute $\beta$ and $\gamma$, respectively, for $\delta$; in (c) and (d) substitute $-\beta$ and $-\gamma$, respectively, for $\delta$. In all these cases, put $w = C_{12}\beta\gamma tu^2$. We obtain (e), (f), (g), and (h) from (a), (b), (c), and (d), respectively, by conjugation using the commutator identity (iv).

(a) $[[[x_\beta(t), x_\gamma(u)], x_\beta(v)]$

$$= [[x_\beta(t), x_\gamma(u)] x_\gamma \gamma + 2\beta(-C_{21}\beta\gamma t^2u), x_\beta(v)]$$

$$\cdot [[[x_\beta(t), x_\gamma(u)] x_\gamma \gamma + 2\beta(-C_{21}\beta\gamma t^2u), x_\beta(v)]$$

Now $(B_2^{**})$ implies

$$[[x_\beta(t), x_\gamma(u)] x_{\gamma + 2\beta}(-C_{21}\beta\gamma t^2u), x_\beta(v)] \cdot x_{\gamma + 2\beta}(C_{21}\beta\gamma t^2u), x_\beta(v)].$$

Since $C_{11}(\beta + \gamma)C_{11}\beta\gamma = N_{\beta, \gamma}N_{\beta + \gamma, \beta} = -2C_{12}\beta\gamma = -2C_{21}\beta\gamma$, one has

$$[[x_\beta(t), x_\gamma(u)], x_\beta(v)] = x_{\gamma + 2\beta}(-2C_{12}\beta\gamma tuv).$$

(b) $[[[x_\beta(t), x_\gamma(u)], x_\gamma(v)]$

$$= [[x_\gamma \gamma + 2\beta(C_{21}\beta\gamma t^2u), [[x_\beta(t), x_\gamma(u)] x_\gamma \gamma + 2\beta(-C_{21}\beta\gamma t^2u), x_\gamma(v)]$$

$$\cdot [[[x_\beta(t), x_\gamma(u)] x_\gamma \gamma + 2\beta(-C_{21}\beta\gamma t^2u), x_\gamma(v)]$$

By $(B_2^{**})$, $[[x_\beta(t), x_\gamma(u)], x_\gamma(v)] = 1$. 
Hence

\[ [[x_\beta(t), x_\gamma(u)], x_\gamma(v)] = 1. \]

(c)

\[
[[x_\beta(t), x_\gamma(u)], x_{-\beta}(v)]
= [x_{\gamma+2\beta}(C_{21\beta\gamma}t^2u), [[x_\beta(t), x_\gamma(u)]x_{\gamma+2\beta}(\beta_21\beta\gamma t^2u), x_{-\beta}(v)]] \\
\cdot [[x_\beta(t), x_\gamma(u)]x_{\gamma+2\beta}(\beta_21\beta\gamma t^2u), x_{-\beta}(v)] \\
\cdot [x_{\gamma+2\beta}(\beta_21\beta\gamma t^2u), x_{-\beta}(v)].
\]

Again, \((B_2^{**})\) implies

\[
[[x_\beta(t), x_\gamma(u)]x_{\gamma+2\beta}(\beta_21\beta\gamma t^2u), x_{-\beta}(v)] = x_\gamma(C_{11(\beta+\gamma)(-\beta)}C_{11\beta\gamma} tuv) = x_\gamma(-2tuv)
\]
since

\[
C_{11(\beta+\gamma)(-\beta)}C_{11\beta\gamma} = N_{\beta+\gamma,-\beta}N_{\beta,\gamma} = -2.
\]

Therefore

\[
[[x_\beta(t), x_\gamma(u)], x_{-\beta}(v)] = x_\gamma(-2tuv) \cdot [x_{\gamma+2\beta}(\beta_21\beta\gamma t^2u), x_{-\beta}(v)].
\]

(d)

\[
[[x_\beta(t), x_\gamma(u)], x_{-\gamma}(v)]
= [x_{\gamma+2\beta}(C_{21\beta\gamma}t^2u), [[x_\beta(t), x_\gamma(u)]x_{\gamma+2\beta}(\beta_21\beta\gamma t^2u), x_{-\gamma}(v)]] \\
\cdot [[x_\beta(t), x_\gamma(u)]x_{\gamma+2\beta}(\beta_21\beta\gamma t^2u), x_{-\gamma}(v)] \\
\cdot [x_{\gamma+2\beta}(\beta_21\beta\gamma t^2u), x_{-\gamma}(v)].
\]

Now

\[
[[x_\beta(t), x_\gamma(u)]x_{2\beta+\gamma}(\beta_21\beta\gamma t^2u), x_{-\gamma}(v)] \\
= x_\beta(C_{11(\beta+\gamma)(-\gamma)}C_{11\beta\gamma} tuv) \cdot x_{2\beta+\gamma}(C_{11(\beta+\gamma)(-\gamma)}t^2u^2v) \\
= x_\beta(tuv) \cdot x_{2\beta+\gamma}(\beta_21\beta\gamma t^2u^2v)
\]
since

\[
C_{11(\beta+\gamma)(-\gamma)}C_{11\beta\gamma} = N_{\beta,\gamma}N_{\beta+\gamma,-\gamma} = 1
\]

and
\[ C_{21(\beta + \gamma)(-\gamma)} = \frac{1}{2} N_{\beta + \gamma, -\gamma} N_{\beta + \gamma, \beta} = \frac{1}{2} N_{\beta, \gamma} N_{\beta, \gamma} N_{\beta + \gamma, \beta} \]

\[ = -\frac{1}{2} N_{\beta, \gamma} N_{\beta, \beta + \gamma} = -C_{21\beta}. \]

Hence

\[ [[x_\beta(t), x_\gamma(u)], x_{-\gamma}(v)] = x_\beta(tuv) \cdot x_{2\beta + \gamma}\left(-C_{21\beta} t^2 u^2 v\right). \]

\((e)\) follows directly from \((b)\) by conjugation.

\((f)\) gives

\[ [x_{-\beta}(v), [x_\gamma(u), x_\beta(t)]] = [x_\gamma(u), x_\beta(t)] [[x_\beta(t), x_\gamma(u)], x_{-\beta}(v)] [x_\beta(t), x_\gamma(u)] = x_{\gamma + 2\beta}(-2C_{12\gamma \beta} tuv). \]

\((g)\)

\[ [x_{-\beta}(v), [x_\gamma(u), x_\beta(t)]] \]

\[ = [x_\gamma(u), x_\beta(t)] \cdot [[x_\beta(t), x_\gamma(u)], x_{-\beta}(v)] \cdot [x_\beta(t), x_\gamma(u)] \]

\[ = [x_\gamma(u), x_\beta(t)] \cdot [x_\gamma(-2tuv) \cdot [x_{\gamma + 2\beta}(C_{21\beta\gamma} t^2 u), x_{-\beta}(v)] \cdot [x_\beta(t), x_\gamma(u)]] \]

\[ = x_\gamma(-2tuv) \cdot [x_{\gamma + 2\beta}(C_{21\beta\gamma} t^2 u), [[x_\gamma(u), x_\beta(t)], x_{-\beta}(v)] \cdot x_{-\beta}(v)] \]

\[ = x_\gamma(-2tuv) \cdot x_{\gamma + 2\beta}(C_{21\beta\gamma} t^2 u) \cdot [[x_\gamma(u), x_\beta(t)], x_{-\beta}(v)] \cdot x_{-\beta}(v) \]

\[ \cdot x_{\gamma + 2\beta}(-C_{21\beta\gamma} t^2 u) \cdot x_{-\beta}(-v) \cdot [x_{-\beta}(v), [x_\gamma(u), x_\beta(t)]] \]

Canceling \([x_{-\beta}(v), [x_\gamma(u), x_\beta(t)]]\) on both sides of the above equation gives

\[ 1 = x_\gamma(-2tuv) \cdot x_{\gamma + 2\beta}(C_{21\beta\gamma} t^2 u) \cdot [[x_\gamma(u), x_\beta(t)], x_{-\beta}(v)] \cdot x_{-\beta}(v) \]

\[ \cdot x_{\gamma + 2\beta}(-C_{21\beta\gamma} t^2 u) \cdot x_{-\beta}(-v), \]

i.e.,

\[ [[x_\gamma(u), x_\beta(t)], x_{-\beta}(v)] = x_\gamma(2tuv) \cdot [x_{\gamma + 2\beta}(-C_{21\beta\gamma} t^2 u), x_{-\beta}(v)]. \]

\((h)\)

\[ [x_{-\gamma}(v), [x_\gamma(u), x_\beta(t)]] \]

\[ = [x_\gamma(u), x_\beta(t)] \cdot [[x_\beta(t), x_\gamma(u)], x_{-\gamma}(v)] \cdot [x_\beta(t), x_\gamma(u)] \]

\[ = [x_\gamma(u), x_\beta(t)] \cdot x_\beta(tuv) \cdot [x_\beta(t), x_\gamma(u)] \cdot x_{2\beta + \gamma}(-C_{21\beta\gamma} t^2 u^2 v) \]
Proof. (a) Now suppose \( n_\gamma(u)x_\beta(t) = x_\beta(tuv) - x_\gamma(u)x_\beta(t)x_\gamma(u) \).

\[
= x_{\gamma + 2\beta}(2C_{12}\gamma t^2u^2v) \cdot x_\beta(tuv) \cdot x_{2\beta + \gamma}(-C_{12}\beta t^2u^2v)
= x_\beta(tuv) \cdot x_{2\beta + \gamma}(C_{12}\gamma t^2u^2v).
\]

\[
\text{Lemma 4.2. Let } \beta, \gamma \in \Phi_n \text{ where } \beta \text{ is short and } \gamma \text{ is long. Let } t, u \in K \text{ with } u \neq 0. \text{ Then}
\]

\[
n_\gamma(u)x_\beta(t)n_\gamma(u)^{-1} = \begin{cases} 
[x_\gamma(u), x_\beta(t)]x_{\gamma + 2\beta}(C_{12}\gamma t^2u) & \text{if } \beta + \gamma \in \Phi, \quad (a) \\
x_{\gamma + 2\beta}(-C_{12}(-\gamma) t^2u^{-1})x_{\gamma}(-u^{-1}), x_\beta(t) & \text{if } \beta - \gamma \in \Phi, \quad (b) \\
x_\beta(t) & \text{if } \beta \pm \gamma \notin \Phi. \quad (c)
\end{cases}
\]

Proof. Observe that either \( \beta + \gamma \) or \( \beta - \gamma \) can be a root, but not both.

(a) Suppose \( \beta + \gamma \in \Phi \). Then, using Lemma 4.1(h), (e), (f), and (2.8), we obtain

\[
n_\gamma(u)x_\beta(t)n_\gamma(u)^{-1} = x_\gamma(u)x_{\gamma + 2\beta}(-C_{21}\beta t^2u)[x_\gamma(u), x_\beta(t)]x_{\gamma + 2\beta}(C_{12}\gamma t^2u)\]

(b) Now suppose \( \beta - \gamma \in \Phi \). Then, using Lemma 4.1(h), (e), and (2.8), we obtain

\[
n_\gamma(u)x_\beta(t)n_\gamma(u)^{-1} = x_\gamma(u)x_{\gamma}(-u^{-1})[x_\gamma(u), x_\beta(t)]x_{\gamma + 2\beta}(C_{12}\gamma t^2u).
\]
\[= x_\beta(-t)[x_{-\gamma}(-u^{-1}), x_\beta(t)]x_\beta(t)x_{-\gamma+2\beta}(C_{21\beta(-\gamma)}t^2u^{-1})\]
\[= [x_\beta(-t), [x_{-\gamma}(-u^{-1}), x_\beta(t)]]x_{-\gamma+2\beta}(C_{21\beta(-\gamma)}t^2u^{-1})\]
\[= x_{-\gamma+2\beta}(-2C_{12(-\gamma)\beta}t^2u^{-1})[x_{-\gamma}(-u^{-1}), x_\beta(t)]x_{-\gamma+2\beta}(C_{21\beta(-\gamma)}t^2u^{-1})\]
\[= x_{-\gamma+2\beta}(-C_{12(-\gamma)\beta}t^2u^{-1})[x_{-\gamma}(-u^{-1}), x_\beta(t)].\]

(c) Finally, if \( \beta \pm \gamma \notin \Phi \), then
\[n_\gamma(u)x_\beta(t)n_\gamma(u)^{-1} = x_\gamma(u)x_{-\gamma}(-u^{-1})x_\gamma(u)x_\beta(t)x_\gamma(-u)x_{-\gamma}(u^{-1})x_\gamma(-u)\]
\[= x_\beta(t). \quad \square\]

Now we introduce symbols connected with two noncompact roots which will serve as substitutes for compact symbols.

Definition 4.3. For \( \beta, \gamma \in \Phi_n \) such that \( \beta + \gamma \in \Phi \) and \( t, u \in K \), let

\[x^{t,u}_{\beta,\gamma} = [x_\beta(C_{11\beta\gamma}t), x_\gamma(u)]x_{\gamma+2\beta}(C_{12\beta\gamma}C_{11\beta\gamma}tu^2)x_{2\gamma+\beta}(-C_{21\beta\gamma}t^2u).\]

Put \( n_\beta = n_\beta(1) \) for \( \beta \in \Phi_n \).

Lemma 4.4. Let \( \beta, \gamma \in \Phi_n \) such that \( \beta \) is short, \( \gamma \) is long and \( \beta + \gamma \in \Phi \). Let \( t, u \in K \). Then

(a) \( n_\gamma x^{t,u}_{\beta,\gamma} n_\gamma^{-1} = x_\beta(N_{\beta,\gamma}tu), \)

(b) \( n_{-\gamma} x^{t,u}_{\beta,\gamma} n_{-\gamma}^{-1} = x_\beta(N_{\gamma,\beta}tu). \)

Proof. We shall use the following formulas:

\[x^{t,u}_{\beta,\gamma} = [x_\beta(C_{11\beta\gamma}t), x_\gamma(u)]x_{\gamma+2\beta}(C_{12\beta\gamma}C_{11\beta\gamma}tu^2)x_{2\gamma+\beta}(-C_{21\beta\gamma}t^2u),\]
\[n_\gamma x_\gamma(u)n_\gamma^{-1} = n_{-\gamma} x_\gamma(u)n_{-\gamma}^{-1} = x_{-\gamma}(-u),\]
\[n_\gamma x_\beta(t)n_\gamma^{-1} = [x_\gamma(1), x_\beta(t)]x_{\gamma+2\beta}(C_{12\beta\gamma}t^2),\]
\[n_{-\gamma} x_\beta(t)n_{-\gamma}^{-1} = [x_\gamma(-1), x_\beta(t)]x_{\gamma+2\beta}(-C_{12\beta\gamma}t^2).\]

(a) \( n_\gamma x^{t,u}_{\beta,\gamma} n_\gamma^{-1} \]
\[= [[x_\gamma(1), x_\beta(C_{11\beta\gamma}t)]x_{\gamma+2\beta}(C_{12\beta\gamma}t^2), x_{-\gamma}(-u)] \cdot x_{\gamma+2\beta}(-C_{21\beta\gamma}t^2u)\]
\[= x_\beta(-C_{11(\gamma+\beta)(-\gamma)}C_{11\beta\gamma}C_{11\beta\gamma}tu) \]
\[\cdot x_{\gamma+2\beta}(-C_{21(\gamma+\beta)(-\gamma)}C_{11\beta\gamma}t^2u)x_{\gamma+2\beta}(-C_{21\beta\gamma}t^2u).\]
that \( \Phi \) is simply-laced has been dealt with in \([5]\), hence we can assume 
\( \Phi(\beta, \gamma) \) is of type \( B_n \) or \( C_n \).

Remark 4.5. If \( \beta, \gamma \in \Phi_n \) and \( \beta + \gamma \in \Phi \), then \( \{ \beta, \gamma \} \) is a system of fundamental roots for \( \Phi(\beta, \gamma) \) which is either of type \( A_2 \) or of type \( B_2 \). The previous results that were proved in the case where \( \beta \) is short and \( \gamma \) is long, also hold if, instead, \( \beta \) and \( \gamma \) are linearly independent roots of the same length and the symbols \( x_{\gamma + 2\beta}(t), t \in K \), are interpreted as \( 1 \). In this case, Lemmas 4.1, 4.2, 4.4 and their proofs are still valid. Alternatively, one can notice that Lemma 4.1 becomes a straightforward consequence of \((B_2^{**})\), and that Lemmas 4.2 and 4.4 were already proved in \([5]\). For instance, Lemma 4.2 becomes

\[
n_{-\gamma} x_{\beta, \gamma} n_{-\gamma}^{-1} = \begin{cases} 
[x_\gamma(u), x_\beta(t)] & \text{if } \beta + \gamma \in \Phi, \\
[x_{-\gamma}(-u^{-1}), x_\beta(t)] & \text{if } \beta - \gamma \in \Phi, \\
x_\beta(t) & \text{if } 0 \neq \beta \pm \gamma \notin \Phi,
\end{cases}
\]

where \( \beta, \gamma \in \Phi_n \) are linearly independent roots of the same length and \( t, u \in K \), \( u \neq 0 \).

Lemma 4.6. Suppose \( t, u \in K \) and \( \beta, \gamma \in \Phi_n \) such that \( \beta + \gamma \in \Phi \). The expression \( x_{\beta, \gamma}^{t,u} \) depends only on the sum \( \beta + \gamma \) and the product \( t \cdot u \).

Proof. The case where \( \Phi \) is simply-laced has been dealt with in \([5]\), hence we can assume that \( \Phi \) is of type \( B_n \) or \( C_n \).

1. Assume that \( \Phi \) is of type \( B_n \). Recall that its 3-graded root system is given by

\[
\Phi_0 = \{ \pm e_i \mid 2 \leq i \leq n \} \cup \{ e_i \pm e_j \mid 2 \leq i, j \leq n, i \neq j \},
\]

\[
\Phi_1 = \Phi_n^+ = \{ e_1 \} \cup \{ e_i \pm e_j \mid 2 \leq j \leq n \},
\]

\[
\Phi_{-1} = \Phi_n^- = \{ -e_1 \} \cup \{ -e_i \pm e_j \mid 2 \leq j \leq n \}.
\]

The compact root \( \alpha = \beta + \gamma \) is either short or long. If \( \alpha \) is long, so are all \( \beta_1, \beta_2 \in \Phi_n \) such that \( \beta_1 + \beta_2 = \alpha \), and those form a 3-graded subsystem of \( \Phi \) which is of type \( A_3 \).
This situation has already been treated in [5, Lemma 2.4]. If $\alpha$ is a short root, then one of the summands, say $\gamma$, is long and the other, $\beta$, is short. Notice that if

$$\alpha = \beta_1 + \beta_2, \quad \beta_i \in \Phi_n,$$

then there are only the following possibilities for $\beta_1, \beta_2$:

1. $\beta_1 = \gamma, \beta_2 = \beta$,
2. $\beta_1 = \beta, \beta_2 = \gamma$,
3. $\beta_1 = \gamma', \beta_2 = \beta'$,
4. $\beta_1 = \beta', \beta_2 = \gamma'$,

where $\gamma' = \gamma + 2\beta$ and $\beta' = -\beta$.

All those solutions lie in the 3-graded subsystem of $\Phi$ generated by $\beta$ and $\gamma$, which is of type $B_2$. Therefore we can restrict our analysis to that case. We can write

$$x^{t,u}_{\beta,\gamma} = \left[ x_{\beta(C_{11}\beta\gamma t)}, x_{\gamma(u)} \right] \cdot x_{\gamma+2\beta} (-C_{21}\beta\gamma t^2 u),$$

$$x^{t,u}_{\gamma,\beta} = \left[ x_{\gamma(C_{11}\gamma\beta t)}, x_{\beta(u)} \right] \cdot x_{\gamma+2\beta} (C_{12}\gamma\beta C_{11}\gamma\beta t u^2).$$

From Lemma 4.4(a), we conclude that

(a) $x^{t,u}_{\beta,\gamma}$ depends only on $t \cdot u$ for fixed $\beta, \gamma$,

(b) $(x^{t,u}_{\beta,\gamma})^{-1} = x^{-t,u}_{\beta,\gamma} = x^{t,-u}_{\beta,\gamma}$,

(c) $[x_{\beta(t)}, x_{\gamma(-u)}] = [x_{\gamma(u)}, x_{\beta(t)}] = [x_{\beta(t)}, x_{\gamma(u)}]^{-1}$.

Now

$$x^{t,u}_{\beta,\gamma} = x^{u,t}_{\beta,\gamma} = (x^{u,-t}_{\beta,\gamma})^{-1} = \left( [x_{\beta(C_{11}\beta\gamma tu)}, x_{\gamma(-t)} x_{\gamma+2\beta} (C_{21}\beta\gamma t^2 u)] \right)^{-1}$$

$$= \left[ x_{\gamma(-t)}, x_{\beta(C_{11}\beta\gamma tu)} x_{\gamma+2\beta} (-C_{12}\gamma\beta t u^2) \right].$$

But $x^{t,u}_{\beta,\gamma} = x^{C_{11}\beta\gamma t, C_{11}\beta\gamma u}_{\beta,\gamma}$ and hence

$$x^{t,u}_{\beta,\gamma} = \left[ x_{\gamma(C_{11}\gamma\beta t)}, x_{\beta(u)} \right] x_{\gamma+2\beta} (C_{12}\gamma\beta C_{11}\gamma\beta t u^2) = x^{t,u}_{\gamma,\beta}.$$

Now we fix $t, u \in K$ and consider the roots $\gamma' = \gamma + 2\beta$ and $\beta' = -\beta$.

We have

$$n_{\gamma'} x^{t,u}_{\beta',\gamma'} n_{\gamma'}^{-1} = x_{\beta'(C_{11}\beta'\gamma' tu)}.$$

On the other hand,

$$n_{\gamma'} x^{t,u}_{\beta',\gamma'} n_{\gamma'}^{-1} = \left[ n_{\gamma'} x_{\beta'(C_{11}\beta'\gamma' t)n_{\gamma'}^{-1}}, x_{\gamma(u)} \right] \cdot x_{-\gamma'} (C_{21}\beta\gamma t^2 u)$$
and by Lemma 4.2 we get
\[ n_{\gamma'}x_{\beta}(C_{11}\beta\gamma t)n_{\gamma'}^{-1} = x_{-\gamma'}(-C_{12}(-\gamma')\beta t^2)[x_{-\gamma'}(-1), x_{\beta}(C_{11}\beta\gamma t)]. \]

Hence
\[
\begin{align*}
n_{\gamma'}&x_{\beta',\gamma'}n_{\gamma'}^{-1}x_{-\gamma'}(-C_{21}\beta\gamma t^2u) \\
&= \left[x_{-\gamma'}(-1), x_{\beta}(C_{11}\beta\gamma t)\right]x_{-\gamma'}(-C_{12}(-\gamma')\beta t^2)\cdot x_{\gamma}(u) \\
&= x_{-\beta}(-C_{11}(-\gamma-\beta)\gamma C_{11}(-\gamma')\beta C_{11}\beta\gamma tu) \cdot x_{-\gamma'}(C_{21}(-\gamma-\beta)\gamma (-C_{11}\beta\gamma t)^2u) \\
&= x_{-\beta}(-C_{11}(-\gamma-\beta)\gamma C_{11}(-\gamma')\beta C_{11}\beta\gamma tu) x_{-\gamma'}(C_{21}(-\gamma-\beta)\gamma t^2u) \\
&= x_{\beta'}(C_{11}\beta'\gamma tu) x_{-\gamma'}(-C_{21}\beta\gamma t^2u).
\end{align*}
\]

Thus
\[ x_{\beta',\gamma'}^{t,u} = x_{\beta,\gamma}^{t,u}. \]

and, in view of the above,
\[ x_{\gamma',\beta'}^{t,u} = x_{\beta',\gamma'}^{t,u} = x_{\beta,\gamma}^{t,u} = x_{\gamma,\beta'}^{t,u}. \]

This concludes the proof of the lemma for $B_n$.

(2) Suppose now that $\Phi$ is of type $C_n$. Recall, its 3-graded root system is:
\[
\begin{align*}
\Phi_0 &= \{e_i - e_j \mid 1 \leq i, j \leq n, i \neq j\}, \\
\Phi_1 &= \Phi_n^+ = \{e_i + e_j \mid 1 \leq i < j \leq n\} \cup \{2e_i \mid 1 \leq i \leq n\}, \\
\Phi_{-1} &= \Phi_n^- = \{-e_i - e_j \mid 1 \leq i < j \leq n\} \cup \{-2e_i \mid 1 \leq i \leq n\}.
\end{align*}
\]

If $\alpha = \beta + \gamma = e_i - e_j$, with $\beta \in \Phi_n^+$ and $\gamma \in \Phi_n^-$, then there are the following possibilities for $\beta, \gamma$:
\[
\begin{align*}
\beta &= e_i + e_j, & \gamma &= -2e_j, \\
\beta &= 2e_i, & \gamma &= -e_i - e_j, \\
\beta &= e_i + e_k, & \gamma &= -e_k - e_j, & k \neq i, j,
\end{align*}
\]

which can be summarized as
\[ \beta_k = e_i + e_k, \quad \gamma_k = -e_k - e_j, \quad 1 \leq k \leq n. \] (4.2)

Hence, for $\beta \in \Phi_n^+$ and $\gamma \in \Phi_n^-$, the pairs of solutions of $\beta + \gamma = \alpha$ are:
\((i)\) \((\beta_i, \gamma_i), (\beta_j, \gamma_j)\);
\((ii)\) \((\beta_k, \gamma_k), (\beta_l, \gamma_l)\), \(k, l \neq i, j\);
\((iii)\) \((\beta_i, \gamma_i), (\beta_k, \gamma_k)\) or \((\beta_j, \gamma_j), (\beta_k, \gamma_k)\), \(k \neq i, j\).

They generate subsystems of type \(B_2\), \(A_3\), and \(C_3\), respectively. We have already proved the lemma in the first two cases. Now we consider the case \(C_3\).

We can repeat the above development for \(C_3\) and we are left with \((iii)\) to be examined. Now, \(\{\beta_i, \gamma_i, -\gamma_k\} \) and \(\{\gamma_j, \beta_j, -\beta_k\} \) are fundamental systems of noncompact roots of type \(C_3\). Hence either situation in \((iii)\) can be characterized as follows: Suppose \(\{\gamma, \beta, -\beta'\} \) is a system of fundamental roots for \(\Phi(\gamma, \beta, -\beta')\) of type \(C_3\) consisting of noncompact roots, where \(\gamma\) is the long root and \(\beta\) is not orthogonal to \(\gamma\). Let

\[ \alpha = \beta + \gamma = \beta' + \beta'', \]

where \(\beta'' = \gamma + \beta - \beta'\). We want to confirm that

\[ x_{\beta, \gamma}^{t, u} = x_{\gamma, \beta}^{t, u} = x_{\gamma}^{t, u} = x_{\beta''}^{t, u}, \beta' = x_{\beta''}^{t, u}, \beta' \quad \text{for} \quad t, u \in K. \]

We already know that

\[ x_{\beta, \gamma}^{t, u} = x_{\gamma, \beta}^{t, u} \quad \text{and} \quad x_{\beta''}^{t, u}, \beta' = x_{\beta''}^{t, u}, \beta' \]

and that these symbols depend only on the product \(tu\), for fixed roots, as a result of our investigations of \(B_2\) and \(A_2\), respectively. Therefore, it is sufficient to show that

\[ x_{\beta, \gamma}^{t, u} = x_{\beta''}^{t, u}, \beta'. \]

Lemma 4.4(a) yields

\[ n_{\gamma} x_{\beta, \gamma}^{t, u} n_{\gamma}^{-1} = x_{\beta}(C_{11\beta, \gamma} tu). \]

We obtain

\[ n_{\gamma} x_{\beta''}(u) n_{\gamma}^{-1} = x_{-\gamma + 2\beta''}(-C_{12(\gamma)} \beta'' u^2)[x_{-\gamma}(-1), x_{\beta''}(u)], \]

\[ n_{\gamma} x_{\pm \beta'}(u) n_{\gamma}^{-1} = x_{\pm \beta'}(u), \quad \text{where} \quad u \in K, \]

and

\[ x_{\beta''}^{t, u} = x_{\beta''}(N_{\beta''}(N_{\beta'}, t), x_{\beta'}(u)). \]

Hence,
In order to obtain the last equality, we substitute \(-\gamma, \beta'',\) and \(\beta'\) for \(\alpha, \beta,\) and \(\gamma,\) respectively, in Lemma 2.1(b) (note that then \(\beta'' + \beta'\) is replaced by \(\alpha'\)), and apply (b) and (c) of Section 1. Then

\[
N_{\beta'',\beta'}N_{-\gamma,\beta'} = N_{-\gamma,\beta''+\beta'} = N_{-\gamma,\beta'+\gamma} = -N_{\beta'+\gamma, -\gamma} = -N_{\beta,\gamma}
\]

and consequently

\[
x_{\beta,\gamma} = x_{\beta'',\beta'}.
\]

This concludes the proof. \(\square\)

5. Chevalley’s commutator formula

Recall that, for \(\beta, \gamma \in \Phi_n\) such that \(\beta + \gamma \in \Phi\) and \(t, u \in K,\) we have defined

\[
x_{\beta,\gamma}(t) = [x_\beta(C_{11}\beta\gamma t), x_\gamma(u)]x_{\beta+2\gamma}(C_{12}\beta\gamma C_{11}\beta\gamma tu^2)x_{2\beta+\gamma}(-C_{21}\beta\gamma t^2 u),
\]

where any of the last two factors represents the identity if its index is not a root. We adopt the convention of calling all roots long if only one root length occurs in \(\Phi.\)

Now we restate Lemma 4.2 in full generality:

**Lemma 5.1.** Let \(\beta, \gamma \in \Phi_n,\) with \(|\beta| \leq |\gamma|\). Let \(t, u \in K\) with \(u \neq 0\). Then

\[
n_\gamma(u)x_\beta(t)n_\gamma(u)^{-1} = \begin{cases} 
  x_{\beta+\gamma}(C_{11}\beta\gamma tu) & \text{if } \beta + \gamma \in \Phi, \\
  x_{\beta-\gamma}(-C_{11}(-\gamma)\beta tu^{-1}) & \text{if } \beta - \gamma \in \Phi, \\
  x_\beta(t) & \text{if } 0 \neq \beta \pm \gamma \notin \Phi.
\end{cases}
\]

**Proof.** Remark 4.5 and Lemma 4.2 assert, besides (c), that

\[
n_\gamma(u)x_\beta(t)n_\gamma(u)^{-1} = x_{\gamma,\beta}C_{11}\betau^t, t
\]

if \(\beta + \gamma \in \Phi\) and

\[
n_\gamma(u)x_\beta(t)n_\gamma(u)^{-1} = x_{-\gamma,\beta}^{-C_{11}(-\gamma)\betau^{-1}}, t
\]

if \(\beta - \gamma \in \Phi.\) \(\square\)
Lemma 5.2. For every compact root \( \alpha \) there are noncompact roots \( \beta, \beta', \gamma, \gamma' \) such that \( \alpha = \beta + \gamma = \beta' + \gamma' \), where \( \beta, \beta' \) are long, \( \beta \in \Phi_1 \), and \( \beta' \in \Phi_{-1} \).

Proof. Let \( \alpha \) be a compact root. If all roots have the same length, any decomposition of \( \alpha \) into the sum of two noncompact roots will suffice. Otherwise, we make use of the classification of the 3-graded root systems: For the type \( B_n \), one has, for \( 2 \leq i, j \leq n \) and \( i \neq j \), that
\[
{s_i e_i + s_j e_j = (e_1 + s_i e_i) + (-e_1 + s_j e_j) = (-e_1 + s_i e_i) + (e_1 + s_j e_j),}
\]
and also
\[
s_i e_i = (e_1 + s_i e_i) + (-e_1) = (-e_1 + s_i e_i) + e_1,
\]
where \( s_i, s_j = \pm 1 \).

For the type \( C_n \), one has, for \( 1 \leq i, j \leq n \), where \( i \neq j \), that
\[
e_i - e_j = 2e_i + (-e_i - e_j) = -2e_j + (e_i + e_j).
\]

Definition 5.3. Let \( \Phi \) be a root system of type \( A_n \). Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be the simple roots in \( \Phi \). If \( \alpha_1 \) is the noncompact simple root, then as a 3-graded root system, \( \Phi \) is of type \( A_1 \).

The next two results describe the existence of a special decomposition of a compact root into the sum of two noncompact ones, in the presence of a second, not larger, compact root, whose addition to the first gives also a root.

Lemma 5.4. Let \( \alpha, \alpha' \) be compact roots such that \( |\alpha| \leq |\alpha'| \) and \( \alpha + \alpha' \) is a root. Then one of the following occurs:

(a) \( \alpha \) is long. If \( \beta, \gamma \in \Phi_n \) such that \( \alpha' = \beta + \gamma \), then \( \{\alpha, \beta, \gamma\} \) is a system of simple roots of \( \Phi(\alpha, \beta, \gamma) \) of type \( A_3 \). As a 3-graded root system, \( \Phi(\alpha, \beta, \gamma) \) has type \( A_3 \).
(b) \( \alpha \) is short and \( \alpha' \) is long. If \( \beta, \gamma \in \Phi_n \) such that \( \alpha' = \beta + \gamma \), then \( \{\alpha, \beta, \gamma\} \) is a system of simple roots of \( \Phi(\alpha, \beta, \gamma) \) of type \( B_3 \).
(c) \( \alpha' \) is short and \( \alpha, \alpha' \) are orthogonal. If \( \beta, \gamma \in \Phi_n \) such that \( |\beta| \leq |\gamma| \) and \( \alpha' = \beta + \gamma \), then \( \{\alpha, \beta - \alpha, \gamma\} \) is a system of simple roots of \( \Phi(\alpha, \beta, \gamma) \) of type \( B_3 \).
(d) \( \alpha' \) is short and \( \alpha, \alpha' \) are not orthogonal. Then there exists exactly one pair of roots \( \beta, \gamma \in \Phi_n \) such that \( \beta \) is short, \( \gamma \) is long, \( \alpha' = \beta + \gamma \), and \( \{\alpha, \beta, \gamma\} \) is a system of simple roots of \( \Phi(\alpha, \beta, \gamma) \) of type \( C_3 \).

Proof. Let us write \( \alpha' = \beta + \gamma \) with \( \beta, \gamma \in \Phi_n \). Then \( \Phi(\alpha, \beta, \gamma) \) is an irreducible 3-graded root system of rank \( d \geq 2 \). If \( d = 2 \), then \( \Phi(\alpha, \beta, \gamma) \) has type \( A_2 \) or \( B_2 \). This is a contradiction since the subsystem formed by the compact roots in \( \Phi(\alpha, \beta, \gamma) \) has rank at least 2. We conclude that \( d = 3 \). The case \( A_2 \) is also excluded since the sum of two compact roots is never a root for such a 3-graded root system. Hence \( \Phi(\alpha, \beta, \gamma) \) has type \( A_3, B_3, \) or \( C_3 \).
(a) Since $\alpha$ and $\alpha'$ are long, we can exclude the types $B_3$ and $C_3$ from our analysis, because in the first case the sum of two long compact roots is never a root and in the second one there are no long compact roots at all. Hence $\Phi(\alpha, \beta, \gamma)$ is of type $A_3^1$. If necessary, we can make use of the automorphisms of $A_3^1$ given by permutations of the indexes $2, 3, 4$ and also of the (degree reversing) anti-automorphism $-I_d$, where $I_d$ is the identity map, and write $\alpha = e_2 - e_3$ and $\alpha' = e_3 - e_4$. Notice that such an identification map still preserves the compact/noncompact attributes of the roots. Hence

$$\alpha' = (e_3 - e_1) + (e_1 - e_4)$$

is the only decomposition of $\alpha'$ as a sum of noncompact roots in $\Phi(\alpha, \beta, \gamma)$, up to the order of the summands, and therefore one of them must be $\beta$ and the other $\gamma$. Hence $\{\alpha, \beta, \gamma\}$ is a system of simple roots of $\Phi(\alpha, \beta, \gamma)$.

(b) In this case $\Phi(\alpha, \beta, \gamma)$ has type $B_3$ since there are compact roots of different lengths. Since $\alpha'$ is long so are $\beta$, $\gamma$. One can assume $\beta = e_1 - e_2$, by transposing the indexes $2, 3$ and reversing some of the $e_i$, if necessary. Moreover, one can assume $\gamma = -e_1 + e_3$ by reversing the vector $e_3$, if necessary. Hence $\alpha' = e_3 - e_2$. Notice that these automorphisms of root systems preserve the sets of compact and noncompact roots and then $\alpha = e_2$ or $\alpha = -e_3$. In either case, $\{\alpha, \beta, \gamma\}$ is a system of simple roots of $\Phi(\alpha, \beta, \gamma)$.

(c) A 3-graded root system of type $A_3^1$ or $C_3$ has no orthogonal compact roots and thus $\Phi(\alpha, \beta, \gamma)$ has type $B_3$. From the hypothesis $\alpha' = \beta + \gamma$, where $\alpha'$ is short and $|\beta| \leq |\gamma|$, we conclude that $\beta$ is short and $\gamma$ is long. Reversing $e_1$, if necessary, we can assume that $\beta = e_1$. Transposing the vectors $e_2, e_3$, and reversing some of them, if necessary, we can write $\gamma = -e_1 + e_2$. Then $\alpha' = \beta + \gamma = e_2$ and $\alpha = \pm e_3$. Now, clearly $\{\alpha, \beta - \alpha, \gamma\}$ is a set of simple roots of $\Phi(\alpha, \beta, \gamma)$.

(d) Here $\Phi$ is of type $C_n$, $n \geq 3$, given the existence of two linearly independent, nonorthogonal short roots. We can make use of the ordinary automorphisms of $C_n$ given by permutations of the $e_i$, and also $-I_d$, to identify $\alpha = e_1 - e_2$ and $\alpha' = e_2 - e_3$ (notice that these automorphisms preserve the sets of compact and noncompact roots). Then $\beta = e_2 + e_3$ and $\gamma = -2e_3$ is the (unique) solution to the requirements in the statement of (d).

Corollary 5.5. Let $\alpha, \alpha'$ be compact roots such that $|\alpha| \leq |\alpha'|$ and $\alpha + \alpha'$ is a root. Then one can decompose $\alpha'$ into the sum of two noncompact roots $\beta, \gamma$ such that

(i) $\alpha + \beta$ is a root,

(ii) $\gamma$ is long and strongly orthogonal to $\alpha$ (that is $\alpha \pm \gamma \notin \Phi$).

Proof. We follow the cases in the statement of Lemma 5.4. If (a) holds, either $\beta$ or $\gamma$ lies in the middle of the Dynkin diagram of $\{\alpha, \beta, \gamma\}$, since $\beta + \gamma$ is a root. We can suppose, without loss of generality, that $\beta$ does and hence $\alpha, \beta, \gamma$ satisfy (i) and (ii) above. For (b), the same reasoning works. If (c) holds, then $\alpha, \beta$ are short and $\alpha + \beta \in \Phi$. Moreover, since $\beta - \alpha$ is in the middle of the Dynkin diagram of $\{\alpha, \beta - \alpha, \gamma\}$, the root $\gamma$ is strongly orthogonal to $\alpha$. Finally, for (d), the proof is immediate.
Lemma 5.6. Let \( \alpha, \alpha' \) be linearly independent compact roots such that \( |\alpha| \leq |\alpha'| \) and \( \alpha + \alpha' \notin \Phi \). Then either one of the following occurs:

(a) there exists a long root \( \beta \in \Phi_n \) such that \( \alpha + \beta, \alpha' - \beta \in \Phi \),
(b) there exists a long root \( \gamma \in \Phi_n \) such that \( \alpha \pm \gamma \notin \Phi \) and \( \alpha' - \gamma \in \Phi \).

If \( \Phi \) contains roots of different lengths, then (b) occurs.

Proof. Let us write \( \alpha', \delta + \epsilon \), with \( \delta, \epsilon \in \Phi_n \), where \( \delta \) is long. The root subsystem \( \Phi(\alpha, \delta, \epsilon) \) of \( \Phi \) has rank \( d \leq 3 \); since \( \alpha \) and \( \alpha' \) are linearly independent (or \( \delta \) and \( \epsilon \)), we have \( d \geq 2 \). Further, \( d = 2 \) is not possible; indeed, the cardinality of a root system of type \((A_1 \times A_1)\) is distinct from that of \( \Phi(\alpha, \delta, \epsilon) \), and the compact root subsystem of a 3-graded root system of type \( B_2 \) or \( A_2 \) has rank 1. Hence \( d = 3 \).

It is sufficient to verify (a) or (b) among the noncompact roots inside the root subsystem \( \Phi(\alpha, \delta, \epsilon) \). Consider first the case where all the roots in each irreducible component of \( \Phi(\alpha, \delta, \epsilon) \) have the same length. Then \( \Phi(\alpha, \delta, \epsilon) \) has type \((A_1 \times A_2)\) or \( A_3 \). In the first case, \( \Phi(\alpha, \delta, \epsilon) = \Psi_1 \times \Psi_2 \), where \( \alpha \in \Psi_1 \), \( \delta, \epsilon \in \Psi_2 \), (a) does not hold for any noncompact root in the subsystem, and (b) holds for \( \gamma = \delta \) or \( \epsilon \). In the second case, we can realize \( \Phi(\alpha, \delta, \epsilon) \) by \( \{ e_i - e_j | i \neq j \} \), where the \( e_i \) form an orthonormal basis of \( \mathbb{R}^4 \), such that \( \delta = e_1 - e_2 \) and \( \epsilon = e_2 - e_3 \). Hence

\[
\Delta = \{ \delta, \epsilon, \rho = e_3 - e_4 \}
\]

is a system of fundamental roots. Using this realization, we see that the possibilities for \( \alpha \) are: \( e_1 - e_4, e_2 - e_4, e_4 - e_2, e_4 - e_3 \). In the first case, \( \alpha \) is the highest root of the root subsystem and \( \rho \) must be compact. Then (a) does not hold for any noncompact \( \beta \) in \( \Phi(\alpha, \delta, \epsilon) \) and (b) holds for \( \gamma = \epsilon \). If \( \alpha = \epsilon + \rho \), then \( \rho \) is noncompact, \( \beta = \delta - \rho \) satisfies (a) and no noncompact \( \gamma \) in \( \Phi(\alpha, \delta, \epsilon) \) satisfies (b). If \( \alpha = \epsilon - \rho \), then \( \rho \) is noncompact, (a) holds for \( \beta = \epsilon, \delta + \epsilon + \rho \) and no noncompact \( \gamma \) in \( \Phi(\alpha, \delta, \epsilon) \) satisfies (b). Finally, if \( \alpha = \epsilon \), \( \rho \) is compact, (a) does not hold for any noncompact \( \beta \) in \( \Phi(\alpha, \delta, \epsilon) \) and (b) holds for \( \gamma = \delta \).

Now suppose that two root lengths occur in at least one irreducible component of \( \Phi(\alpha, \delta, \epsilon) \). Then, that subsystem must have type \((A_1 \times B_2), B_3, \) or \( C_3 \). In the first case, \( \Phi(\alpha, \delta, \epsilon) = \Psi_1 \times \Psi_2 \), where \( \alpha \in \Psi_1 \) and \( \delta, \epsilon \) form a system of simple roots of \( \Psi_2 \) since \( \delta \) is long, \( \epsilon \) is short here. Consequently, no long noncompact root \( \beta \) in \( \Phi(\alpha, \delta, \epsilon) \) satisfies (a), and (b) holds for \( \gamma = \delta, \delta + 2\epsilon \).

Second, suppose \( \Phi(\alpha, \delta, \epsilon) \) has type \( B_3 \). If \( \alpha' \) were short, so would be \( \alpha \) and their sum would be a root. Therefore, \( \alpha' \) is long and so are \( \delta \) and \( \epsilon \), and we can assume that \( \delta = e_1 - e_2, \epsilon = e_2 - e_3 \) in the usual realization of \( \Phi(\alpha, \delta, \epsilon) \) in \( \mathbb{R}^3 \), with the \( \{ e_i \} \) being an orthonormal basis of \( \mathbb{R}^3 \). Hence

\[
\Delta = \{ \delta, \epsilon, \rho = e_3 \}
\]

is a system of simple roots of our subsystem. As \( (\delta + \epsilon) + 2\rho \in \Phi \), the root \( \rho \) must be compact. If \( \alpha = \delta + \epsilon + \rho \), then (b) is satisfied for \( \gamma = \epsilon, -\epsilon - 2\rho \); if \( \alpha = \epsilon, \gamma = \delta \),
\[ \delta + 2\varepsilon + 2\rho \text{ satisfies } (b). \] In either case, no long noncompact \( \beta \) in \( \Phi(\alpha, \delta, \varepsilon) \) satisfies \((a)\). The compact roots \( \pm(\delta + \varepsilon + 2\rho) \), along with \( \delta, \varepsilon \), do not generate \( \Phi(\alpha, \delta, \varepsilon) \) and hence they are not candidates for \( \alpha \).

Finally, suppose \( \Phi(\alpha, \delta, \varepsilon) \) is of type \( C_3 \). Then both \( \alpha, \alpha' \) are short and we can realize \( \Phi(\alpha, \delta, \varepsilon) \) by \( \{\pm e_i \pm e_j \mid 1 \leq i, j \leq 3\} \) in such a way that \( \alpha = e_1 - e_2 \) and \( \alpha' = e_3 - e_2 \). Hence

\[ \Delta = \{\alpha, -\alpha', \rho = 2e_3\} \]

is a system of fundamental roots of \( \Phi(\alpha, \delta, \varepsilon) \). Since not all of these roots are compact, we must have \( \rho \) noncompact. No long noncompact \( \beta \) in \( \Phi(\alpha, \delta, \varepsilon) \) satisfies \((a)\) and \( \gamma = \rho \) satisfies \((b)\).

**Lemma 5.7.** Chevalley's commutator formula holds if one of the symbols in the commutator is indexed by a noncompact root.

**Proof.** Suppose \( \beta, \gamma, \delta \in \Phi_n \), \( \alpha = \beta + \gamma \), \( \alpha' \in \Phi_0 \) and \( t, u \in K \). Recall that

\[ x_\alpha(tu) = x^t,u_\beta,\gamma = [x_\beta(C_{11}^{\beta\gamma}t), x_\gamma(u)]x_{\beta+2\gamma}(C_{12}^{\beta\gamma}C_{11}^{\beta\gamma}tu^2)x_{\beta+\gamma}(-C_{21}^{\beta\gamma}t^2u). \]

Then

\[ [x_\alpha(t), x_\delta(u)] = x_{\alpha+\delta}(C_{11}^{\alpha\delta}tu)x_{2\alpha+\delta}(C_{21}^{\alpha\delta}t^2u). \quad (5.1) \]

Indeed, \((B^*_2)\) yields

\[ [x_\alpha(t), x_\delta(u)] = \left[[x_\beta(C_{11}^{\beta\gamma}t), x_\gamma(1)]x_{\beta+2\gamma}(C_{12}^{\beta\gamma}C_{11}^{\beta\gamma}tu^2)x_{\beta+\gamma}(-C_{21}^{\beta\gamma}t^2u), x_\delta(u)\right] \]

\[ = x_{\alpha+\delta}(C_{11}^{\alpha\delta}tu)x_{2\alpha+\delta}(C_{21}^{\alpha\delta}t^2u). \]

Moreover, \((5.1)\) yields

\[ [x_\delta(u), x_\alpha(t)] = x_{\delta+\alpha}(C_{11}^{\delta\alpha}ut)x_{\delta+2\alpha}(-C_{21}^{\delta\alpha}ut^2). \quad (5.2) \]

Now

\[ [x_\beta(t), x_\gamma(u)] = x_\alpha(C_{11}^{\beta\gamma}tu)x_{\beta+2\gamma}(-C_{12}^{\beta\gamma}tu^2)x_{\beta+\gamma}(-C_{21}^{\beta\gamma}t^2u). \quad (5.3) \]

Indeed,

\[ x_\alpha(C_{11}^{\beta\gamma}tu) = x_{\beta+\gamma}^{C_{11}^{\beta\gamma}tu} = [x_\beta(t), x_\gamma(u)]x_{\beta+2\gamma}(C_{12}^{\beta\gamma}tu^2)x_{\beta+\gamma}(-C_{21}^{\beta\gamma}t^2u). \]

Finally, for \( \beta, \gamma \in \Phi_n \) such that \( 0 \neq \beta + \gamma \notin \Phi \) and \( t, u \in K \), one has

\[ [x_\beta(t), x_\gamma(u)] = 1 \quad (5.4) \]

by \((B^*_1)\). \( \square \)
Lemma 5.8. Let $\alpha, \alpha' \in \Phi_0$ such that $\alpha'$ is long and $\alpha + \alpha' \in \Phi$. Then

$$[x_\alpha(t), x_{\alpha'}(u)] = x_{\alpha + \alpha'}(C_{11\alpha\alpha'}tu) \cdot x_{2\alpha + \alpha'}(C_{21\alpha\alpha'}t^2u).$$

Proof. As before, the second factor on the right-hand side is equal to the identity if $2\alpha + \alpha' \notin \Phi$.

First, suppose $\alpha$ is short. Let $\beta, \gamma \in \Phi_n$ such that $\alpha' = \beta + \gamma$. Then the subsystem $\Phi(\alpha, \beta, \gamma)$ has type $B_3$ with simple roots $\alpha, \beta, \gamma$, by Lemma 5.4(b). (Notice that, in particular, $\beta, \gamma$ are long roots.) Without loss of generality, we can assume that $\beta$ lies in the middle of the corresponding Dynkin diagram. We have

$$x_{\alpha'}(u) = [x_\beta(C_{11\beta\gamma}u), x_\gamma(1)]$$

and then

$$[x_\alpha(t), x_{\alpha'}(u)] = x_\alpha(t)x_{\alpha'}(u)x_\alpha(t)^{-1}x_{\alpha'}(u)^{-1} = x_\alpha(t)[x_\beta(C_{11\beta\gamma}u), x_\gamma(1)]x_\alpha(t)^{-1}x_{\alpha'}(u)^{-1} = [x_\alpha(t), x_\beta(C_{11\beta\gamma}u)x_\alpha(t)^{-1}, x_\alpha(t)x_\gamma(1)x_\alpha(t)^{-1}]x_{\alpha'}(u)^{-1} = [x_\alpha(t), x_\beta(C_{11\beta\gamma}u), [x_\alpha(t), x_\gamma(1)]x_{\alpha'}(u)^{-1} = [x_{\alpha + \beta}(C_{11\alpha\beta}C_{11\beta\gamma}tu)x_{2\alpha + \beta}(C_{21\alpha\beta}C_{11\beta\gamma}t^2u)x_\beta(C_{11\beta\gamma}u), x_\gamma(1)]x_{\alpha'}(u)^{-1} = [x_\beta(C_{11\beta\gamma}u), [x_{\alpha + \beta}(C_{11\alpha\beta}C_{11\beta\gamma}tu)x_{2\alpha + \beta}(C_{21\alpha\beta}C_{11\beta\gamma}t^2u), x_\gamma(1)] \cdot [x_{\alpha + \beta}(C_{11\alpha\beta}C_{11\beta\gamma}tu)x_{2\alpha + \beta}(C_{21\alpha\beta}C_{11\beta\gamma}t^2u), x_\gamma(1)] = [x_\beta(C_{11\beta\gamma}u), x_{\alpha + \alpha'}(C_{11(\alpha + \beta)}C_{11\beta\gamma}tu) \cdot x_{2\alpha + \alpha'}(C_{21\alpha\beta}C_{11\beta\gamma}C_{11(2\alpha + \beta)}C_{11\beta\gamma}t^2u)x_{\alpha + \alpha'}(C_{21(\alpha + \beta)}C_{11\beta\gamma}(tu)^2)] \cdot x_{2\alpha + \alpha'}(C_{21(\alpha + \beta)}C_{11\beta\gamma}(tu)^2)x_{\alpha + \alpha'}(C_{11(\alpha + \beta)}C_{11\beta\gamma}C_{11(2\alpha + \beta)}C_{11\beta\gamma}tu) \cdot x_{2\alpha + \alpha'}(C_{21\alpha\beta}C_{11\beta\gamma}C_{11(2\alpha + \beta)}C_{11\beta\gamma}t^2u)
$$

(the last 3 of the symbols above commute by Lemmas 5.1, 5.7)

$$= [x_\beta(C_{11\beta\gamma}u), x_{\alpha + \alpha'}(C_{11(\alpha + \beta)}C_{11\beta\gamma}C_{11\beta\gamma}tu) \cdot x_{2\alpha + \alpha'}(C_{21(\alpha + \beta)}C_{11\beta\gamma}(tu)^2)] \cdot x_{\alpha + \alpha'}(C_{11(\alpha + \beta)}C_{11\beta\gamma}C_{11(2\alpha + \beta)}C_{11\beta\gamma}tu)x_{\alpha + \alpha'}(C_{21\alpha\beta}C_{11\beta\gamma}C_{11(2\alpha + \beta)}C_{11\beta\gamma}t^2u) = x_{\alpha + \alpha'}(C_{11(\alpha + \beta)}C_{11\beta\gamma}C_{11\beta\gamma}tu)x_{2\alpha + \alpha'}(C_{21\alpha\beta}C_{11\beta\gamma}C_{11(2\alpha + \beta)}C_{11\beta\gamma}t^2u) = x_{\alpha + \alpha'}(C_{11\alpha\alpha'}tu)x_{2\alpha + \alpha'}(C_{21\alpha\alpha'}t^2u).$$
Indeed, since $\alpha + \gamma$ is not a root, we get

$$N_{\alpha+\beta,\gamma}N_{\alpha,\beta}N_{\beta,\gamma} = N_{\alpha,\alpha'}$$

by Lemma 2.1(b). We also have

$$C_{11(2\alpha+\alpha')}\beta C_{21}\alpha\beta C_{11(2\alpha+\beta)}\gamma = C_{21(\alpha+\beta)}\gamma$$

and

$$C_{21}\alpha\beta C_{11}\beta\gamma C_{11(2\alpha+\beta)}\gamma = C_{21}\alpha\alpha'$$

by Lemma 3.1(e) and (f), respectively.

The case where $\alpha$ and $\alpha'$ are both long can be proved as above, with $B_3$ replaced by $A_3$. As a matter of fact, since some factors reduce to the identity, the proof in this case becomes even simpler.

It is possible to modify the above computation to cover the more general case involving two compact roots $\alpha, \alpha'$ such that $|\alpha| \leq |\alpha'|$ and $\alpha + \alpha' \in \Phi$. However, a different approach is adopted here: we change root compactness by using conjugation, a technique that can be applied to the case $\alpha + \alpha' \in \Phi$ as well.

**Lemma 5.9.** Let $\alpha \in \Phi_0$ and $\beta \in \Phi_n$ such that $\beta$ is long and $\alpha \pm \beta \notin \Phi$. Then

$$n_\beta x_\alpha(t)n_\beta^{-1} = x_\alpha(t), \quad t \in K.$$

**Proof.** By Lemma 5.2, one can choose a decomposition of $\alpha$ into the sum of two noncompact roots

$$\alpha = \beta_1 + \beta_2,$$

such that $\beta_2$ is long and has the same sign as $\beta$. Then we can write

$$x_\alpha(t) = x_{\beta_1, \beta_2}^{t,1} = \left[x_{\beta_1}(C_{11}\beta_1, \beta_2 t), x_{\beta_2}(1)\right]x_{2\beta_1+\beta_2}(-C_{21}\beta_1, \beta_2 t^2). \quad (5.5)$$

Because $\beta \pm \alpha \notin \Phi$, one has $(\beta, \alpha) = 0$, that is

$$(\beta, \beta_1) = -(\beta, \beta_2).$$

(a) If $(\beta, \beta_1) = (\beta, \beta_2) = 0$ then, since $\beta - \beta_1, \beta + \beta_2 \notin \Phi$, one has

$$\beta \pm \beta_1, \beta \pm \beta_2 \notin \Phi$$

and
\[
\begin{align*}
\beta \in (2\beta_1 + \beta_2) & \notin \Phi,
\end{align*}
\]
by Lemma 5.1(c), for instance. Finally, if \(2\beta_1 + \beta_2 \in \Phi\) then
\[
\beta \pm (2\beta_1 + \beta_2) \notin \Phi,
\]
since \((\beta, 2\beta_1 + \beta_2) = 0\) and \(\beta - (2\beta_1 + \beta_2) \notin \Phi\). Thus, by Lemma 5.1(c) again, conjugation by \(n_\beta\) fixes \(x_{2\beta_1 + \beta_2}(-C_{21}\beta_1\beta_2t^2)\). Using (5.5) confirms the assertion.

(b) Suppose now that \((\beta, \beta_1) = (\beta, \beta_2) \neq 0\). Here it is convenient to describe \(\Phi(\beta, \beta_1, \beta_2)\) in order to guarantee that all pairs of roots considered next are linearly independent.

First, we notice that \(\Phi(\beta, \beta_1, \beta_2)\) is an irreducible 3-graded subsystem of \(\Phi\). It has dimension \(d = 2\) or \(3\); if \(d = 2\) then the subsystem’s type is either \(A_2\) or \(B_2\); this is a contradiction since a compact root cannot be orthogonal to a long noncompact root in those cases. Therefore \(d = 3\); in particular \(\beta, \beta_1, \beta_2\) are linearly independent. Thus the type of \(\Phi(\beta, \beta_1, \beta_2)\) is not \(C_3\) because, in that case, we would have \((\beta, \beta_1) = (\beta, \beta_2) = 0\). Also, the type of \(\Phi(\beta, \beta_1, \beta_2)\) is not \(A_3^2\) because for type \(A_3^2\) we would get \((\beta, \alpha) \neq 0\). We conclude that \(\Phi(\beta, \beta_1, \beta_2)\) is either of type \(B_3\) or \(A_3^1\).

Since \(\beta + \beta_2 \notin \Phi\), one has \((\beta, \beta_2) > 0\) and hence \(\beta - \beta_2 \in \Phi\). Similarly, because \(\beta - \beta_1 \notin \Phi\) (and these roots are linearly independent), it follows that \((\beta, \beta_1) < 0\) and then \(\beta + \beta_1 \in \Phi\). Now, by Lemma 5.4 applied to \(\alpha, \alpha' = \beta - \beta_2\) (or by direct calculation), we have, in either case, that \(\{\beta, -\beta_2, \alpha\}\) is a system of simple roots for \(\Phi(\beta, \beta_1, \beta_2)\), with \(-\beta_2\) in the middle of the corresponding Dynkin diagram.

Let us examine first the case \(B_3\). Then \(\alpha\) is the short root in \(\{\beta, -\beta_2, \alpha\}\), since the others are long. Moreover, we have the roots
\[
\begin{align*}
\beta_1 &= -\beta_2 + \alpha, & \beta + \beta_1 &= \beta - \beta_2 + \alpha, \\
2\beta_1 + \beta_2 &= -\beta_2 + 2\alpha, & \beta + 2\beta_1 + \beta_2 &= \beta - \beta_2 + 2\alpha.
\end{align*}
\]
By Lemma 5.1(a), (b),
\[
\begin{align*}
\beta \in (2\beta_1 + \beta_2) & \notin \Phi,
\end{align*}
\]
by Lemma 5.1(c), for instance. Finally, if \(2\beta_1 + \beta_2 \in \Phi\) then
\[
\beta \pm (2\beta_1 + \beta_2) \notin \Phi,
\]
since \((\beta, 2\beta_1 + \beta_2) = 0\) and \(\beta - (2\beta_1 + \beta_2) \notin \Phi\). Thus, by Lemma 5.1(c) again, conjugation by \(n_\beta\) fixes \(x_{2\beta_1 + \beta_2}(-C_{21}\beta_1\beta_2t^2)\). Using (5.5) confirms the assertion.

(b) Suppose now that \((\beta, \beta_1) = (\beta, \beta_2) \neq 0\). Here it is convenient to describe \(\Phi(\beta, \beta_1, \beta_2)\) in order to guarantee that all pairs of roots considered next are linearly independent.

First, we notice that \(\Phi(\beta, \beta_1, \beta_2)\) is an irreducible 3-graded subsystem of \(\Phi\). It has dimension \(d = 2\) or \(3\); if \(d = 2\) then the subsystem’s type is either \(A_2\) or \(B_2\); this is a contradiction since a compact root cannot be orthogonal to a long noncompact root in those cases. Therefore \(d = 3\); in particular \(\beta, \beta_1, \beta_2\) are linearly independent. Thus the type of \(\Phi(\beta, \beta_1, \beta_2)\) is not \(C_3\) because, in that case, we would have \((\beta, \beta_1) = (\beta, \beta_2) = 0\). Also, the type of \(\Phi(\beta, \beta_1, \beta_2)\) is not \(A_3^2\) because for type \(A_3^2\) we would get \((\beta, \alpha) \neq 0\). We conclude that \(\Phi(\beta, \beta_1, \beta_2)\) is either of type \(B_3\) or \(A_3^1\).

Since \(\beta + \beta_2 \notin \Phi\), one has \((\beta, \beta_2) > 0\) and hence \(\beta - \beta_2 \in \Phi\). Similarly, because \(\beta - \beta_1 \notin \Phi\) (and these roots are linearly independent), it follows that \((\beta, \beta_1) < 0\) and then \(\beta + \beta_1 \in \Phi\). Now, by Lemma 5.4 applied to \(\alpha, \alpha' = \beta - \beta_2\) (or by direct calculation), we have, in either case, that \(\{\beta, -\beta_2, \alpha\}\) is a system of simple roots for \(\Phi(\beta, \beta_1, \beta_2)\), with \(-\beta_2\) in the middle of the corresponding Dynkin diagram.

Let us examine first the case \(B_3\). Then \(\alpha\) is the short root in \(\{\beta, -\beta_2, \alpha\}\), since the others are long. Moreover, we have the roots
\[
\begin{align*}
\beta_1 &= -\beta_2 + \alpha, & \beta + \beta_1 &= \beta - \beta_2 + \alpha, \\
2\beta_1 + \beta_2 &= -\beta_2 + 2\alpha, & \beta + 2\beta_1 + \beta_2 &= \beta - \beta_2 + 2\alpha.
\end{align*}
\]
\[
\begin{align*}
\left[ x_{\beta + \beta_1}(C_{11}\beta\beta_1 C_{11}\beta_1\beta_2 t), x_{\beta_2 - \beta}(-C_{11}(-\beta)\beta_2) \right] \\
= x_\alpha(-C_{11}(\beta + \beta_1)(\beta_2 - \beta) C_{11}\beta\beta_1 C_{11}\beta_1\beta_2 C_{11}(-\beta)\beta_2 t) \\
\cdot x_{\beta_2 + 2\beta_1 + \beta}(-C_{21}(\beta + \beta_1)(\beta_2 - \beta) C_{11}(-\beta)\beta_2 t^2) \\
= x_\alpha(t)x_{\beta_2 + 2\beta_1 + \beta}(C_{11}\beta(2\beta_1 + \beta_2) C_{21}\beta_1\beta_2 t^2),
\end{align*}
\]

since

\[
C_{11}(\beta + \beta_1)(\beta_2 - \beta) C_{11}\beta\beta_1 C_{11}\beta_1\beta_2 C_{11}(-\beta)\beta_2 = -1
\]

and

\[
C_{21}(\beta + \beta_1)(\beta_2 - \beta) C_{11}(-\beta)\beta_2 = -C_{11}(\beta + \beta_2) C_{21}\beta_1\beta_2.
\]

In fact, the first equation follows easily from Lemma 2.1(b) and the second from Lemma 3.1(f) applied to \(C_{21}(\beta + \beta_1)(\beta_2 - \beta) = -C_{21}(-\beta_2 + \beta)\).

Finally, we notice that the proof for the case \(A_1^1\) is similar to the one above, the only difference being the fact that there are no short roots involved. This makes the above development simpler.

**Lemma 5.10.** Under the assumptions of Theorem 1.4, the relations \((B)\) hold.

**Proof.** In view of Lemma 5.7, it suffices to show that

\[
\left[ x_\alpha(t), x_{\alpha'}(u) \right] = x_{|\alpha|+|\alpha'|}(C_{11}|\alpha|\alpha' t u) \cdot x_{2|\alpha|+|\alpha'|}(C_{21}|\alpha|\alpha' t^2 u)
\]

for linearly independent \(\alpha, \alpha' \in \Phi_0\) such that \(|\alpha| \leq |\alpha'|\) (and then the case \(|\alpha'| \leq |\alpha|\) follows trivially). We have already proved this case in the special situation where \(\alpha'\) is long and \(\alpha + \alpha' \in \Phi\) (see Lemma 5.8). Now we shall prove it in full generality. Let \(\alpha, \alpha' \in \Phi_0\) be linearly independent roots with \(|\alpha| \leq |\alpha'|\).

(i) Suppose \(\alpha + \alpha' \in \Phi\). By Corollary 5.5, we can write \(\alpha' = \beta + \gamma\), with \(\beta, \gamma \in \Phi_n\) such that \(\alpha + \beta\) is a root and \(\gamma\) is long and strongly orthogonal to \(\alpha\). Then

\[
n_\gamma \left[ x_\alpha(t), x_{\alpha'}(u) \right] n_\gamma^{-1} = \left[ n_\gamma x_\alpha(t) n_\gamma^{-1}, n_\gamma x_{\alpha'}(u) n_\gamma^{-1} \right] = \left[ x_\alpha(t), x_\beta(N_\beta, \gamma u) \right]
\]

\[
= x_{\alpha + \beta}(C_{11}\alpha\beta N_{\beta, \gamma} t u) x_{2\alpha + \beta}(C_{21}\alpha\beta N_{\beta, \gamma} t^2 u)
\]

by Lemmas 4.4(a) (or 5.1(a)), 5.7 and 5.9.

Since

\[
2(2\alpha + \beta, \gamma)/(\gamma, \gamma) = -1
\]

and

\[
2(2\alpha + \beta + \gamma, \gamma)/(\gamma, \gamma) = 1,
\]

\[
\left[ x_{\beta + \beta_1}(C_{11}\beta\beta_1 C_{11}\beta_1\beta_2 t), x_{\beta_2 - \beta}(-C_{11}(-\beta)\beta_2) \right]
\]

\[
= x_\alpha(-C_{11}(\beta + \beta_1)(\beta_2 - \beta) C_{11}\beta\beta_1 C_{11}\beta_1\beta_2 C_{11}(-\beta)\beta_2 t) \\
\cdot x_{\beta_2 + 2\beta_1 + \beta}(-C_{21}(\beta + \beta_1)(\beta_2 - \beta) C_{11}(-\beta)\beta_2 t^2) \\
= x_\alpha(t)x_{\beta_2 + 2\beta_1 + \beta}(C_{11}\beta(2\beta_1 + \beta_2) C_{21}\beta_1\beta_2 t^2),
\]

since

\[
C_{11}(\beta + \beta_1)(\beta_2 - \beta) C_{11}\beta\beta_1 C_{11}\beta_1\beta_2 C_{11}(-\beta)\beta_2 = -1
\]

and

\[
C_{21}(\beta + \beta_1)(\beta_2 - \beta) C_{11}(-\beta)\beta_2 = -C_{11}(\beta + \beta_2) C_{21}\beta_1\beta_2.
\]

In fact, the first equation follows easily from Lemma 2.1(b) and the second from Lemma 3.1(f) applied to \(C_{21}(\beta + \beta_1)(\beta_2 - \beta) = -C_{21}(-\beta_2 + \beta)\).

Finally, we notice that the proof for the case \(A_1^1\) is similar to the one above, the only difference being the fact that there are no short roots involved. This makes the above development simpler.

**Lemma 5.10.** Under the assumptions of Theorem 1.4, the relations \((B)\) hold.

**Proof.** In view of Lemma 5.7, it suffices to show that

\[
\left[ x_\alpha(t), x_\alpha'(u) \right] = x_{|\alpha|+|\alpha'|}(C_{11}|\alpha|\alpha' t u) \cdot x_{2|\alpha|+|\alpha'|}(C_{21}|\alpha|\alpha' t^2 u)
\]

for linearly independent \(\alpha, \alpha' \in \Phi_0\) such that \(|\alpha| \leq |\alpha'|\) (and then the case \(|\alpha'| \leq |\alpha|\) follows trivially). We have already proved this case in the special situation where \(\alpha'\) is long and \(\alpha + \alpha' \in \Phi\) (see Lemma 5.8). Now we shall prove it in full generality. Let \(\alpha, \alpha' \in \Phi_0\) be linearly independent roots with \(|\alpha| \leq |\alpha'|\).

(i) Suppose \(\alpha + \alpha' \in \Phi\). By Corollary 5.5, we can write \(\alpha' = \beta + \gamma\), with \(\beta, \gamma \in \Phi_n\) such that \(\alpha + \beta\) is a root and \(\gamma\) is long and strongly orthogonal to \(\alpha\). Then

\[
n_\gamma \left[ x_\alpha(t), x_\alpha'(u) \right] n_\gamma^{-1} = \left[ n_\gamma x_\alpha(t) n_\gamma^{-1}, n_\gamma x_\alpha'(u) n_\gamma^{-1} \right] = \left[ x_\alpha(t), x_\beta(N_\beta, \gamma u) \right]
\]

\[
= x_{\alpha + \beta}(C_{11}\alpha\beta N_{\beta, \gamma} t u) x_{2\alpha + \beta}(C_{21}\alpha\beta N_{\beta, \gamma} t^2 u)
\]

by Lemmas 4.4(a) (or 5.1(a)), 5.7 and 5.9.

Since

\[
2(2\alpha + \beta, \gamma)/(\gamma, \gamma) = -1
\]

and

\[
2(2\alpha + \beta + \gamma, \gamma)/(\gamma, \gamma) = 1,
\]
hold, it is clear that $2\alpha + \beta \in \Phi$ if and only if $2\alpha + \alpha' \in \Phi$.

Hence

$$\begin{align*}
[x_\alpha(t), x_{\alpha'}(u)] &= n^{-1}_y x_{\alpha+\beta}(C_{11\alpha\beta}N_{\beta,\gamma}t u) x_{2\alpha+\beta}(C_{21\alpha\beta}N_{\beta,\gamma}t^2 u) n_y \\
&= (n^{-1}_y x_{\alpha+\beta}(C_{11\alpha\beta}N_{\beta,\gamma}t u) n_y) (n^{-1}_y x_{2\alpha+\beta}(C_{21\alpha\beta}N_{\beta,\gamma}t^2 u) n_y) \\
&= x_{\alpha+\alpha'}(C_{11(\alpha+\beta)}\gamma C_{11\alpha\beta}N_{\beta,\gamma}t u) x_{2\alpha+\alpha'}(C_{11(2\alpha+\beta)}\gamma C_{21\alpha\beta}N_{\beta,\gamma}t^2 u) \\
&= x_{\alpha+\alpha'}(C_{11\alpha\alpha'\gamma}t u) x_{2\alpha+\alpha'}(C_{21\alpha\alpha'\gamma}t^2 u),
\end{align*}$$

since

$$C_{11(\alpha+\beta)}\gamma C_{11\alpha\beta}C_{11\beta\gamma} = C_{11\alpha\alpha'},$$

by Lemma 2.1(b) and, in case $2\alpha + \alpha' \in \Phi$, one has that $\{\alpha, \beta, \gamma\}$ is a system of fundamental roots of $\Phi(\alpha, \beta, \gamma)$ of type $B_3$, by Lemma 5.4(b), and then

$$C_{11(2\alpha+\beta)}\gamma C_{21\alpha\beta}C_{11\beta\gamma} = C_{21\alpha\alpha'},$$

by Lemma 3.1(f).

(ii) Now, suppose $\alpha + \alpha' \notin \Phi$. We prove that

$$[x_\alpha(t), x_{\alpha'}(u)] = 1.$$

By Lemma 5.6, one can write

$$\alpha' = \beta + \gamma,$$

with $\beta, \gamma \in \Phi_n$, $\gamma$ long, such that either $\alpha + \gamma \in \Phi$ or $\alpha \pm \gamma \notin \Phi$.

If $\alpha + \gamma$ is a root, then

$$n_y[x_\alpha(t), x_{\alpha'}(u)] n_y^{-1} = [x_{\alpha+\gamma}(N_{-\gamma,\alpha+\gamma}t), x_\beta(N_{\beta,\gamma}u)] = 1,$$

as $\alpha + \beta + \gamma = \alpha + \alpha' \notin \Phi$.

Finally, if $\alpha \pm \gamma \notin \Phi$, then $\alpha$ is orthogonal to $\gamma$. Because $\Phi(\beta, \gamma)$ is of type $A_2$ or $B_2$ and $\gamma$ is long, we have

$$2(\beta, \gamma)/(\gamma, \gamma) = -1.$$

From

$$2(\alpha + \beta, \gamma)/(\gamma, \gamma) = -1,$$

we see that $\alpha + \beta \notin \Phi$ (otherwise $\alpha + \alpha'$ would be a root). Hence

$$n_y[x_\alpha(t), x_{\alpha'}(u)] n_y^{-1} = [x_\alpha(t), x_\beta(N_{\beta,\gamma}u)] = 1$$

by Lemmas 5.1(a), 5.7, and 5.9. □
Finally, we complete the proof of Theorem 1.4.

**Proof (Conclusion of Theorem 1.4).** Put together, compact and noncompact symbols satisfy the relations (A) and (B), as shown above. Moreover, it is well known (see [3], for instance) that these imply

\[ n_\alpha h_\beta(t)n_{\alpha}^{-1} = h_{w_\alpha(\beta)}(t), \]

for any \( \alpha, \beta \in \Phi \), where \( n_\alpha = n_\alpha(1) \). This formula, together with \((C^\star)\), implies \((C)\), since the action of the Weyl group of \( \Phi \) is transitive on the set of long roots and also on the set of short roots. \( \square \)

### 6. Interpretation of constants in Hermitian presentations

#### 6.1. Lie triple systems

We shall use the same notation as in Section 1.

Given an irreducible root system \( \Phi \), we consider its realization as the root system of a simple Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \) with respect to a Cartan subalgebra \( \mathfrak{h} \). Let us fix a Chevalley basis of \( \mathfrak{g} \):

\[ \{ e_\alpha \mid \alpha \in \Phi \} \cup \{ h_i = h_{\alpha_i} \mid 1 \leq i \leq l \}. \]

As we know, the structure constants associated with such a basis can be used to provide presentations of a Chevalley group associated with \( \Phi \).

We recall that a **Lie triple system** in \( \mathfrak{g} \) consists of any vector subspace \( V \) of \( \mathfrak{g} \) closed under double brackets, that is,

\[ \{ u, v, w \} = \left[ [u, v], w \right] \in V \]  

(6.1)

for any \( u, v, w \in V \). In other words, the expression above defines a trilinear operator in \( V \).

First, let us examine the structure of the trivial Lie triple system \( \langle e_\alpha \mid \alpha \in \Phi \rangle = V \subset \mathfrak{g} \). Obviously, \( V \) is closed under triple products. The triple system constants \( N_{\alpha, \beta, \gamma} \) are defined by

\[ \{ e_\alpha, e_\beta, e_\gamma \} = N_{\alpha, \beta, \gamma} e_{\alpha + \beta + \gamma} \]  

(6.2)

whenever \( \alpha, \beta, \gamma, \alpha + \beta + \gamma \in \Phi \).

With the purpose of extracting information on these structure constants from iterations of the triple product, one could extend the range of the indexes to include certain cases where \( \alpha + \beta + \gamma \notin \Phi \) (and even allow indexing by linear combinations of roots), as we did with the structure constants of simple Lie algebras, in order to deal with the Jacobi identity. We adopt here a different approach instead, namely, the use of higher order constants,
which encode, in their own definition, all the cases of different configurations of roots that are of interest. Hence, to obtain a scalar identity for the triple system constants from

\[
\{x, y, \{u, v, w\}\} = \{x, y, u\}, v, w\} + \{u, x, y, v\} + \{u, v, x, y, w\},
\]

where \(x, y, u, v, w \in V\), we use the quintuple constants \(N_{\alpha, \beta, \gamma, \delta, \epsilon}\) defined by

\[
\{\{e_{\alpha}, e_{\beta}, e_{\gamma}\}, e_{\delta}, e_{\epsilon}\} = N_{\alpha, \beta, \gamma, \delta, \epsilon} e_{\alpha + \beta + \gamma + \delta + \epsilon},
\]  

(6.3)

where \(\alpha, \beta, \gamma, \delta, \epsilon, \alpha + \beta + \gamma + \delta + \epsilon \in \Phi\). Finally, describing the higher order constants in terms of lower ones leads to the identity (f) in Lemma 6.1 below.

Let \(\alpha, \beta, \gamma \in \Phi\) such that \(\alpha + \beta + \gamma \in \Phi\) and then exactly one of the following occurs:

(i) \(\alpha + \beta \in \Phi\),
(ii) \(\alpha = -\beta\),
(iii) \(\alpha = \beta\) or \(\alpha\) and \(\beta\) are linearly independent with \(\alpha + \beta \notin \Phi\).

If \(\gamma = \gamma\) then \(\alpha + 2\beta \in \Phi\) and only (i) or (ii) can happen. Below we list some elementary properties of the triple system constants \(N_{\alpha, \beta, \gamma}\).

**Lemma 6.1.** As above, let \(\alpha, \beta, \gamma \in \Phi\) such that \(\alpha + \beta + \gamma \in \Phi\), then the triple system constants \(N_{\alpha, \beta, \gamma}\) of \(V\) with respect to the restricted Chevalley basis \(\{e_\alpha \mid \alpha \in \Phi\}\) of \(V\), have the following properties:

(a) \(N_{\alpha, \beta, \gamma} = \begin{cases} N_{\alpha, \beta}N_{\alpha + \beta, \gamma} & \text{if } \alpha + \beta \in \Phi, \\ 2(\alpha, \gamma)/(\alpha, \alpha) & \text{if } \alpha + \beta = 0, \\ 0 & \text{otherwise}, \end{cases}\)

(b) \(N_{\alpha, \beta, \gamma} = -N_{\beta, \alpha, \gamma}\),

(c) \(N_{\alpha, \beta, -\beta} = r(1 + p)\), where \(p\) and \(r\) are the largest integers such that \(-p\beta + \alpha, r\beta + \alpha \in \Phi\),

(d) \(N_{-\alpha, -\beta, -\gamma} = N_{\alpha, \beta, \gamma}\),

(e) \(N_{\alpha, \beta, \gamma} + N_{\beta, \gamma, \alpha} + N_{\gamma, \alpha, \beta} = 0\),

(f) \(N_{\alpha, \beta, \gamma, \delta, \epsilon} - N_{\alpha, \beta, \gamma, \epsilon, \delta} + N_{\delta, \epsilon, \alpha, \beta, \gamma} - N_{\delta, \epsilon, \beta, \alpha, \gamma} + N_{\delta, \epsilon, \gamma, \alpha, \beta} - N_{\delta, \epsilon, \gamma, \alpha, \beta} = 0\), where \(\alpha, \beta, \gamma, \delta, \epsilon, \alpha + \beta + \gamma + \delta + \epsilon \in \Phi\), and

\[
N_{\alpha, \beta, \gamma, \delta, \epsilon} = \begin{cases} N_{\alpha, \beta, \gamma}N_{\alpha + \beta + \gamma, \delta, \epsilon} & \text{if } \alpha + \beta + \gamma \in \Phi, \\ -(2(\delta, \gamma)/(\gamma, \gamma))N_{\alpha, \beta}N_{\delta, \epsilon} & \text{if } \alpha + \beta + \gamma = 0, \\ 0 & \text{otherwise}, \end{cases}
\]

(g) \(N_{\alpha, \beta, \gamma} = \begin{cases} 2C_{12\alpha \beta} & \text{if } \alpha + \beta \in \Phi, \\ -2 & \text{if } \alpha + \beta = 0. \end{cases}\)

**Proof.** Immediate from previous results. \(\Box\)

### 6.2. Hermitian presentations and Lie triple systems

Let us specialize the situation laid out in Section 6.1 to the case where \(\Phi\) is a 3-graded root system. The 3-grading of \(\Phi\) induces an obvious 3-grading on \(\mathfrak{g}\). We (can) fix here a positive system of \(\Phi\) such that all the roots in \(\Phi_1\) are positive. Now we consider the Lie
triple system on $V_n = g_{-1} \oplus g_1$, the (canonical) maximal noncompact Lie triple subsystem of $g$.

The vectors $e_\beta$, where $\beta \in \Phi_n = \Phi_{-1} \cup \Phi_1$, form a restricted Chevalley basis of the Lie triple system $V_n = g_{-1} \oplus g_1$, so the corresponding structure constants are integral.

Examining the relations $\left(B^{**}_2\right)$ closely, one observes that the constants appearing there depend only on the triple system constants of the Chevalley triple system $V_n = g_{-1} \oplus g_1$ with respect to a restricted Chevalley basis as above, which is not necessarily the full information provided by all structure constants $N_{\alpha,\beta}$. In fact, we can rewrite the relations $\left(B^{**}_2\right)$ as

$$\left[ x_\beta(t), x_\gamma(u) \right] x_\beta + 2u \gamma + 2 (N_{\beta,\gamma,\gamma} N_2 t^2 u^2 s),$$

for $\beta, \gamma, \delta \in \Phi_n; \beta + \gamma \in \Phi; \ s, t, u \in K,$

where again any factor whose index is not a root is interpreted as 1. (If the characteristic of the underlying field $K$ is 2, then it is understood that the coefficients above involving a division by 2 and appearing in the symbols indexed by a root are first evaluated in characteristic zero before being interpreted as elements of $K$; anyway, in this case, the value of all these coefficients is $1 \in K$.)

The constants appearing in the last factor on the right-hand side of $\left(B^{**}_2\right)$ above come from the identity

$$\left[ e_\delta, [e_\beta, e_\gamma] / N_{\beta,\gamma} \right], \left[ e_\beta, e_\gamma / N_{\beta,\gamma} \right] = \left[ [[e_\delta, \left[ e_\beta, e_\gamma \right]], \left[ e_\beta, e_\gamma \right]] - [[e_\delta, \left[ e_\beta, e_\gamma \right]], \left[ \left[ e_\delta, \left[ e_\beta, e_\gamma \right] \right], e_\gamma \right], e_\beta \right]. \tag{6.4}$$

Conversely, one can recover the triple system constants $N_{\beta,\gamma,\delta}$ of the triple system $V_n = g_{-1} \oplus g_1$ from the coefficients appearing in the corresponding Hermitian presentation. Indeed, the coefficients appearing in the first factor on the right-hand side of $\left(B^{**}_2\right)$ that are indexed by a valid root are exactly the triple system constants $N_{\beta,\gamma,\delta}$ for which $\beta + \gamma \in \Phi$. According to Lemma 6.1(a), those are the only triple system constants that can change in sign; they are called nontrivial and, of course, suffice to describe the whole set of triple system constants of $V_n$ with respect to the same Chevalley basis of $g$ underlying the presentation. In summary, we have

**Proposition 6.2.** Using the previous notation, the triple system constants $N_{\beta,\gamma,\delta}$ of the Lie triple system $g_{-1} \oplus g_1$, with respect to some Chevalley basis of $g$, determine uniquely the constants $C_{ij\alpha\beta}$, with respect to the same basis, that appear in the Hermitian ‘*’ relations (or, equivalently, in the double commutator relations $\left(B^{**}_2\right)$), and vice versa, as described above.

**Example 6.3.** Consider $\Phi = A_n^1, n \geq 3$. The choice of signs for the triple system constants $N_{\beta,\gamma,\delta}$, where $\beta, \gamma, \delta \in \Phi_n$, is completely determined by the signs of the nontrivial ones
among them. In view of Lemma 6.1(b), (d), it is sufficient to specify them for \( \beta, \delta \in \Phi_n^+ \) and \( \gamma \in \Phi_n^- \). The nonzero ones are exactly

\[ N_{\beta, \gamma}, -\gamma = 1, \]

where \( \beta \in \Phi_n^+ \), \( \gamma \in \Phi_n^- \), and \( \beta + \gamma \in \Phi \), and hence uniquely determined. This agrees with the fact that the constants in the Hermitian presentation of \( SL_{n+1}(K) \), given in [5], are uniquely determined, regardless of the choice of the signs of the structure constants \( N_{\alpha, \beta} \).

We close this section stating some additional elementary properties of those triple system constants that are indexed by noncompact roots:

**Lemma 6.4.** Using the above notation, the triple system constants \( N_{\beta, \gamma, \delta} \) (\( \beta, \gamma, \delta \in \Phi_n \), \( \beta + \gamma + \delta \in \Phi \)) of \( V_n = g_{-1} \oplus g_1 \) with respect to the restricted Chevalley basis \( \{ e_\beta \mid \beta \in \Phi_n \} \) satisfy the following properties:

(a) \( N_{\beta, \gamma, -\gamma} = r \), where \( r \) is the largest integer such that \( \beta + r\gamma \in \Phi \) (recall that \( r = (\beta, \beta)/(\beta + \gamma, \beta + \gamma) \) if \( \beta + \gamma \in \Phi \)),

(b) \( N_{\beta, \gamma, \delta} = N_{\delta, \gamma, \beta} \) if \( \beta \) and \( \delta \) have the same sign,

(c) \( N_{\beta, \gamma, \delta}(N_{\beta + \gamma + \delta, \gamma, \beta} - N_{\beta + \gamma + \delta, \beta, \gamma}) = N_{\delta, \beta, \gamma, \beta + \gamma}. \)

**Proof.** Immediate from previous results. \( \square \)

**7. Examples**

The spin group \( \operatorname{Spin}_{2n+1}(K) \) and the symplectic group \( \operatorname{Sp}_n(K) \) correspond to the universal Chevalley groups of type \( B_n \) and \( C_n \), respectively. In each case we shall introduce a notation for the root groups which allows us to give a more transparent version of Corollary 1.5.

**7.1. The case \( B_{n+1} \)**

Here

\[ \Phi_n^+ = \{ e_0 \pm e_i \mid 1 \leq i \leq n \} \cup \{ e_0 \}, \]
\[ \Phi_n^- = \{ -e_0 \pm e_i \mid 1 \leq i \leq n \} \cup \{ -e_0 \}. \]

In [3, pp. 179–181], a Chevalley basis for the simple complex Lie algebra of type \( B_{n+1} \) (i.e. \( \mathfrak{so}_{2n+3}(\mathbb{C}) \)) is exhibited, which produces a set of structure constants such that

\[ N_{e_0 + v e_i, -e_0} = v, \quad (7.1) \]

where \( i > 0 \) and \( v = \pm 1 \), and

\[ N_{e_0 + v e_i, -e_0 + \rho e_j} = v, \quad (7.2) \]
where $0 < i < j$ and $\nu, \rho = \pm 1$. From these coefficients (or by direct computation using the above Chevalley basis), we obtain

$$N_{\sigma e_0 + \nu e_i} = \nu,$$

(7.3)

where $i > 0$ and $\sigma, \nu = \pm 1$, and

$$N_{\sigma e_0 + \nu e_i} = \nu,$$

(7.4)

where $0 < i < j$ and $\sigma, \rho, \nu = \pm 1$. Moreover

$$N_{\nu e_i, \sigma e_0} = -\nu,$$

(7.5)

for $0 < i < j$, $\sigma, \rho, \nu = \pm 1$, and

$$N_{\nu e_i, \sigma e_0} = 2\nu,$$

(7.6)

for $i > 0$ and $\sigma, \nu = \pm 1$.

Let $V_n$ be the Lie triple system generated by the root vectors corresponding to the noncompact roots. The nontrivial triple system constants for $V_n$ are given by

$$N_{\nu e_i, \rho e_j, \tau e_0} = -1, N_{\nu e_i, \rho e_j, \tau e_0} = 1$$

for $0 < i < j$ and $\sigma, \rho, \nu, \tau = \pm 1$,

$$N_{\nu e_i, \rho e_j, \tau e_0} = 2, N_{\nu e_i, \rho e_j, \tau e_0} = -1,$$

(7.7)

for $i > 0$ and $\sigma, \nu, \tau = \pm 1$. The remaining constants are obtained by interchanging the first two indexes of the ones given above.

Finally

$$N_{\nu e_i, \nu e_i, \nu e_i} = -2$$

(7.9)

(see Lemma 6.4(c)).

Recall the sign function $s: \mathbb{Z} \rightarrow \{ -1, 0, 1 \}$ given by

$$s(0) = 0, \quad s(i) = 1, \quad s(-i) = -1, \quad i > 0.$$

We write

$$x^\sigma_i(t) = x_{\sigma e_0 + s(i)e_i}(t),$$

for $\sigma = \pm, i = 0, \pm 1, \ldots, \pm n, t \in K$, i.e.

$$x^+_k(t) = x_{e_0 + e_k}(t), \quad x^-_k(t) = x_{e_0 - e_k}(t), \quad x^+_0(t) = x_{e_0}(t),$$

$$x^-_k(t) = x_{-e_0 + e_k}(t), \quad x^-_k(t) = x_{-e_0 - e_k}(t), \quad x^-_0(t) = x_{-e_0}(t),$$

for $k$. 
where \( k = 1, \ldots, n \).

**Presentation of \( G_u \):** The group \( G_u \cong \text{Spin}_{2n+3}(K) \) is generated by the symbols

\[ x_{-n}^\sigma(t), \ldots, x_n^\sigma(t), \]

where \( \sigma = \pm \) and \( t \in K \), subject to the relations

**(A*)** \( x_i^\sigma(t)x_i^\sigma(u) = x_i^\sigma(t + u) \).

**(B_1^*)** If \( i, j \neq 0 \), then

\[ [x_i^\sigma(t), x_j^\sigma(u)] = [x_i^\sigma(t), x_0^\sigma(u)] = [x_i^\sigma(t), x_i^{-\sigma}(u)] = 1. \]

**(B_2^*)** If \( 0 < i < j \); \( v, \rho = \pm 1 \) and \( \sigma, \tau = \pm \), then

\[
[[x_i^\sigma(t), x_0^{-\sigma}(u)], x_k^\sigma(s)] = [[x_0^{-\sigma}(-u), x_{vi}^\sigma(t)], x_k^\sigma(s)] =
\begin{cases}
  x_{vi}^\sigma(tus), & k = -\rho j, \\
  x_{\rho j}^\sigma(-tus), & k = -vi, \\
  1, & k \neq -vi, -\rho j,
\end{cases}
\]

\[
[[x_{vi}^\sigma(t), x_0^{-\sigma}(u)], x_{vi}^{-\sigma}(tu^2), x_k^\sigma(s)] = [[x_0^{-\sigma}(-u), x_{vi}^\sigma(t)], x_{vi}^{-\sigma}(-tu^2), x_k^\sigma(s)] =
\begin{cases}
  x_0^{-\sigma}(-tus)x_{vi}^{-\sigma}(-t^2u^2s), & k = -vi, \\
  x_{vi}^{-\sigma}(2tus), & k = 0, \\
  1, & k \neq 0, -vi,
\end{cases}
\]

**(W*)** \( n_i^\sigma(t)x_i^\sigma(u)n_i^\sigma(t)^{-1} = x_{-i}^{-\sigma}(-t^{-2}u), i = 0, \pm 1, \ldots, \pm n; t \neq 0, \)

**(C*)** \( h_i^\sigma(t)h_i^\sigma(u) = h_i^\sigma(tu), i = 0, 1; tu \neq 0, \)

with

\[ n_i^\sigma(t) = x_i^\sigma(t)x_{-i}^{-\sigma}(-t^{-1})x_i^\sigma(t), \]

\[ h_i^\sigma(t) = n_i^\sigma(t)n_i^\sigma(1)^{-1}, \quad t \neq 0. \]

If \( n \geq 2 \), we can omit relations \( (W*) \) for the long noncompact roots since

\[ (-\sigma e_0 + ve_i) + (\sigma e_0 + e_j) + (\sigma e_0 - e_j) = \sigma e_0 + ve_i, \]

where \( i, j > 0 \) are distinct and \( \sigma, v = \pm 1 \). The reason for that is the same as the one in the case \( C_n \); an explanation is given in the next example. Moreover, it is sufficient to require the relations \( (C*) \) for \( i = 0, 1 \) (or just for one long and one short root; this can be seen from the conclusion of the proof of Theorem 1.4).
Let $\infty, 0, 1, \ldots, n, -0, -1, \ldots, -n$ index the rows and columns of $M_{2n+3}(K)$ in this order; observe $-0 \neq 0$. Let us denote by $u_{a,b}$ the matrix units in $M_{2n+3}(K)$. We suppose that $\text{char } K \neq 2$ and we use [3, p. 186] (or [6, p. 69]). Then the correspondence

$$x^+_k(t) = x_{e_0+e_k}(t) \mapsto 1 + t(u_{0,-k} - u_{k,0}),$$

$$x^-_{-k}(t) = x_{e_0-e_k}(t) \mapsto 1 + t(u_{0,k} - u_{-k,0}),$$

$$x^+_0(t) = x_{e_0}(t) \mapsto 1 + t(2u_{0,\infty} - u_{\infty,-0}) - t^2u_{0,-0},$$

$$x^-_0(t) = x_{e_0}(t) \mapsto 1 + t(-2u_{-0,\infty} + u_{\infty,0}) - t^2u_{0,0},$$

where $k = 1, \ldots, n$, extends to an explicit homomorphism from $G_\alpha$ onto $\Omega_{2n+3}(K)$, the commutator subgroup of the isometry group $O_{2n+3}(K)$ of the quadratic form

$$x_\infty^2 + x_0x_{-0} + x_1x_{-1} + \cdots + x_nx_{-n}.$$

Because this is a central homomorphism, the group $\Omega_{2n+3}(K)$ is also a Chevalley group of type $B_{n+1}$.

### 7.2. The case $C_n$

If $\Phi$ is of type $C_n$, then the noncompact roots are

$$\Phi^+_n = \{e_i + e_j \mid 1 \leq i \leq j \leq n\},$$

$$\Phi^-_n = \{-e_i - e_j \mid 1 \leq i \leq j \leq n\}.$$

In [3, pp. 182–183], a Chevalley basis for the simple complex Lie algebra of type $C_n$ (i.e. $sp_{2n}(\mathbb{C})$) is constructed, which produces the (unique) set of structure constants such that

$$N_{e_i+e_j, -e_j - e_k} = N_{e_i+e_k, -2e_k} = N_{2e_i, -e_i - e_k} = 1,$$

for $i, j, k$ all distinct. From these coefficients or by direct computation on the cited Chevalley basis, one has

$$N_{e_i-e_k, e_j+e_k} = 1, \quad N_{e_i-e_k, e_i+e_k} = 2, \quad N_{e_i-e_k, 2e_k} = 1,$$

and hence

$$N_{e_i-e_k, -e_i-e_j} = -1, \quad N_{e_i-e_k, -e_i-e_k} = -2, \quad N_{e_i-e_k, -2e_i} = -1,$$

where $i, j, k$ are all distinct.
Let $V_n$ be the Lie triple system given by the subspace spanned by the root spaces corresponding to the noncompact roots of $\Phi$. We exhibit Tables 1 and 2 for the triple system constants $N_{\beta, \gamma, \delta}$ of $V_n$ in the cases where $\beta > 0$, $\gamma < 0$, $\beta + \gamma$, $\beta + \gamma + \delta \in \Phi$.

From Tables 1 and 2, we conclude the following: Let $\beta, \gamma, \delta \in \Phi_n$ such that $\beta > 0$, $\gamma < 0$, $\beta + \gamma$, $\beta + \gamma + \delta \in \Phi$ and let $\theta$ be the angle between $\beta + \gamma$ and $\delta$. Then

$$N_{\beta, \gamma, \delta} = \varphi(\theta) \text{sign}(\beta) \text{sign}(\delta),$$

(7.13)

where $\varphi(\pi/2) = 2$, $\varphi(2\pi/3) = \varphi(3\pi/4) = 1$.

Consider now $\alpha = e_i - e_j$ and $\delta = 2e_j$. We write $\alpha = \beta + \gamma$, where $\beta = e_i + e_j$ and $\gamma = -2e_j$. Then by Lemma 6.4(c) we get

$$N_{\delta, \beta + \gamma, \beta + \gamma} = N_{\beta, \gamma, \delta} N_{\beta + \gamma, \delta, \beta} = N_{\beta, \gamma, -\gamma} N_{\beta, \gamma, \beta} = 2.$$

By Lemma 6.1(d), this holds in general, i.e.

$$N_{\delta, \alpha, \alpha} = 2$$

(7.14)

if $\alpha$ is a compact root, $\delta$ is a noncompact root and $\alpha + \delta, 2\alpha + \delta \in \Phi$.

Next, we write $x_{ij}^\sigma(t)$ for $x_{\sigma(e_i + e_j)}(t)$ where $1 \leq i, j \leq n, \sigma = \pm$ and $t \in K$. This introduces extra symbols used below; they satisfy the (extra) symmetry relations

$$(S)\quad x_{ij}^\sigma(t) = x_{ji}^\sigma(t).$$

For the symplectic group, for instance, we obtain the following presentation:

**Group:** $G_u \cong \text{Sp}_{2n}(K), n \geq 3$.

**Generators:** $x_{ij}^\sigma(t)$, where $t \in K, \sigma = \pm$, and $1 \leq i, j \leq n$. 
Relations:

\((S)\) \( x_{ij}^\sigma(t) = x_{ji}^\sigma(t), \)
\((A^*)\) \( x_{ij}^\sigma(t)x_{ij}^\sigma(u) = x_{ij}^\sigma(t + u), \)
\((B_1^*)\) \( [x_{ij}^\sigma(t), x_{kl}^\sigma(u)] = 1 \text{ if } \sigma i, \sigma j \neq \rho k, -\rho l.\)
\((B_2^*)\) For \(i, j, k\) all distinct:

\[
[[x_{ij}^\sigma(t), x_{-ij}^\sigma(u)], x_{lm}^\sigma(s)] = [[x_{ii}^\sigma(t), x_{-ii}^\sigma(u) \cdot x_{-ii}^\sigma(-(t^2u)^2), x_{lm}^\sigma(s)]] = \begin{cases} x_{im}^\sigma(tus), & k = l \neq m \neq i, \rho = \sigma, \\ x_{km}^\sigma(-tus), & i = l \neq m \neq k, \rho = -\sigma, \\ x_{il}^\sigma(2tus), & k = l, m = i, \rho = \sigma, \\ x_{kk}^\sigma(-2tus), & k = l, m = i, \rho = -\sigma, \\ x_{ik}^\sigma(tus)x_{ii}^\sigma(t^2u^2s), & k = l = m, \rho = \sigma, \\ x_{ik}^\sigma(-tus)x_{kk}^\sigma(t^2u^2s), & i = l = m, \rho = -\sigma, \\ 1, & \rho \neq \rho m \neq -\sigma i, \rho k. \end{cases}
\]
\((C^*)\) \( h_{i1}^\sigma(t)h_{i1}^\sigma(u) = h_{i1}^\sigma(tu), \quad i = 1, 2; tu \neq 0, \) where

\[
h_{i1}^\sigma(t) = n_{i1}^\sigma(t)n_{i1}^\sigma(1)^{-1}, \quad i = 1, 2, t \neq 0, \text{ and } n_{ij}^\sigma(t) = x_{ij}^\sigma(t)x_{-ij}^\sigma(-t^{-1})x_{ij}^\sigma(t).
\]

We can omit relations \((W^*)\) since

\[
(e_i + e_j) - (e_j + e_k) + (e_i + e_k) = 2e_i,
\]
for \(i, j, k\), all distinct. Hence, using the double commutator formula, one can express \(x_\beta(t)\), where \(\beta\) is any noncompact root, as the product of symbols associated with non-compact roots that are distinct from \(\pm \beta\). By means of such an expression, one can prove \((W^*)\) from \((S), (A^*), (B_1^*)\) and \((B_2^*)\), as one does with the usual Steinberg relations; see [3, pp. 191–192], for instance.

The addition of new generators increase the number of “one-parameter” symbols to a total of \(|\Phi|\), that is, the same as the number of one-parameter symbols appearing in the original Steinberg relations for \(C_n\). Finally, we mention that the correspondence

\[
x_{ij}^+(t) \mapsto 1 + t(u_{i,n+j} + u_{j,n+i}), \quad x_{ij}^-(t) \mapsto 1 + t(u_{n+i,j} + u_{n+j,i}) \quad (i \neq j), \\
x_{ii}^+(t) \mapsto 1 + tu_{i,n+i}, \quad x_{ii}^-(t) \mapsto 1 + tu_{n+i,i},
\]

where the \(u_{a,b}\) are the elementary matrices in \(M_{2n}(K)\), extends to an explicit isomorphism from \(G_u\) onto \(\text{Sp}_{2n}(K)\) (see [6, p. 70], and also [3, pp. 185–186]).
Appendix A

Definition A.1 (3-graded root systems). An irreducible root system $\Phi$, together with a nonempty subset $\Phi_1$, is called a 3-graded root system if $\Phi$ is the disjoint union of subsets

$$\Phi = \Phi_{-1} \sqcup \Phi_0 \sqcup \Phi_1,$$

with $\Phi_{-1} = -\Phi_1$, and the following property holds:

$$\Phi \cap (\Phi_i + \Phi_j) \subset \Phi_{i+j}, \quad i, j \in \mathbb{Z},$$

where $\Phi_i = \emptyset$ in case $|i| \geq 2$.

Finally, a general root system $\Phi$ together with a subset $\Phi_1$ is a 3-graded root system if each of its irreducible components $\Psi$ along with $\Psi \cap \Phi_1$ is 3-graded.

A root $\alpha \in \Phi$ is said to have degree $i$ if $\alpha \in \Phi_i$. From the classification of Hermitian symmetric spaces and the developments below in this section, we know that there is a one-to-one correspondence between (classes of equivalence of) 3-graded root systems and classes of equivalence of Hermitian symmetric spaces of noncompact (or compact) type. Given a 3-graded root system $\Phi$, let $\Omega_\Phi$ be a Hermitian symmetric space of noncompact type associated with $\Phi$ and $G_\Phi$ the connected component of its full group of isometries. Then $\Phi_0$ can be seen as the root system of the complexification of any maximal compact subgroup of $G_\Phi$.

This interpretation justifies the following terminology: a root of degree 0 in $\Phi$ is also called compact. All the others are noncompact roots which form a set denoted by $\Phi_n$, i.e. $\Phi_n = \Phi_{-1} \sqcup \Phi_1$. The complex semisimple Lie algebras corresponding to such root systems are called Hermitian.

A 3-graded root system is symmetric in the sense that

$$\Phi_{-i} = -\Phi_i, \quad i \in \mathbb{Z}.$$

Let $\Phi$ be an irreducible root system. Suppose $\Phi$ is 3-graded. Let $\alpha_1, \alpha_2 \in \Phi_1$ be two roots. Then they are equal or they are linearly independent and since their sum is never a root, we have

$$(\alpha_1, \alpha_2) \geq 0. \quad (A.1)$$

Let

$$\sigma = \sum_{\alpha \in \Phi_1} \alpha.$$

Then (A.1) implies that

$$(\sigma, \alpha) > 0, \quad \alpha \in \Phi_1,$$
and hence all roots in $\Phi_1$ ($\Phi_{-1}$) are positive (negative) with respect to the partial order defined by $\sigma$ on $\Phi$.

Now, let $\Pi$ be a system of fundamental roots for $\Phi$ with respect to which all roots in $\Phi_1$ are positive (such a system exists in view of the above consideration). Let $\beta_m$ be the highest root with respect to the partial order defined by $\Pi$. At least one root in $\Pi$ has degree 1; otherwise all simple roots and consequently all roots would have degree zero. Moreover, this degree 1 simple root is unique. In fact, all the fundamental roots appear in the decomposition of the highest root $\beta_m$ with respect to them:

$$\beta_m = \alpha_{i_1} + \cdots + \alpha_{i_n}, \quad \alpha_{i_j} \in \Pi,$$

which can be rewritten in such a way that all partial sums

$$\alpha_{i_1} + \cdots + \alpha_{i_k}, \quad 1 \leq k \leq n,$$

are roots. If two simple roots had degree 1, a certain partial sum above would be the sum of two roots of degree 1, which, of course, is not possible.

Let $\beta_1$ be the unique root of degree 1 in $\Pi$ or, in other words, the unique noncompact simple root of $\Phi$ with respect to $\Pi$.

Actually we have proved above a stronger fact, namely that, in the decomposition of $\beta_m$, the root $\beta_1$ appears just once:

$$\beta_m = \beta_1 + \sum_{i=2}^{n} k_i \alpha_i, \quad k_i \in \mathbb{N}, \alpha_i \in \Pi.$$

(A.2)

Given two systems of fundamental roots, there exists an element of its Weyl group taking one basis onto the other. Hence the coefficients of the highest root in the decomposition into simple roots are independent, up to permutation, of the chosen basis.

In our case, we have shown that at least one of the coefficients of the highest root must equal 1.

Conversely, suppose $\Phi$ is such that, with respect to a basis $\Pi$ (and hence to any basis), there is a single occurrence of $\beta_1 \in \Pi$ in the decomposition of the highest root into a sum of simple roots. Then, in the decomposition of any root of $\Phi$, $\beta_1$ appears with coefficient 1, 0 or $-1$. Defining $\Phi_1$ as the set of roots of $\Phi$ whose coefficient of $\beta_1$ is 1 in such a decomposition, one easily sees that the pair $(\Phi, \Phi_1)$ forms a 3-graded root system.

We conclude that the irreducible root systems admitting a 3-gradation are precisely those having 1 among the coefficients of the maximal root written as a linear combination of simple roots.

We use this criterion to describe the irreducible 3-graded root systems up to isomorphism. Let $\Pi$ be a basis of such a root system $\Phi$ with respect to which the roots of $\Phi_1$ are positive and $\beta_1$ is the only noncompact simple root.

Let

$$\Pi = \{\alpha_1 = \beta_1, \alpha_2, \ldots, \alpha_n\},$$
where the simple roots $\alpha_i$ are indexed as in [2,7]. Then the type of $\Phi$ is one of (1) to (6) in Table 3. These are exactly the roots that occur with coefficient 1 when the positive roots are expressed as linear combinations of simple roots (see [2]).

Root systems of type $E_8$, $F_4$, or $G_2$ do not have a 3-graded structure. There is some redundancy in Table 3. Any two 3-graded root systems in different rows are obviously non-isomorphic and any two of them in the same row, i.e., with the same underlying ordinary root system, are isomorphic if and only if there is a symmetry of their Dynkin diagram taking one simple noncompact root to the other.

For the rest of this section, $\Phi$ denotes a general 3-graded root system. Let $\Phi^+_n$ and $\Phi^-_n$ be the sets of positive noncompact and negative noncompact roots, respectively.

Writing $\alpha = \sum_{i=1}^{n} c_i(\alpha)\alpha_i \in \Phi$, one has

$$\Phi_0 = \{ \alpha \in \Phi \mid c_1(\alpha) = 0 \},$$

$$\Phi^+_n = \Phi_1 = \{ \alpha \in \Phi \mid c_1(\alpha) = 1 \},$$

$$\Phi^-_n = \Phi_{-1} = \{ \alpha \in \Phi \mid c_1(\alpha) = -1 \}.$$

Let $\Gamma \subset \Phi$ be nonempty. It is well known that the set $\Phi(\Gamma)$ consisting of those roots that are integral linear combinations of elements of $\Gamma$, along with the real vector subspace spanned by them, form a root system.

Now, suppose $\Gamma \subset \Phi_n$, $\Gamma \neq \emptyset$. We claim that the pair

$$\Phi(\Gamma), \Phi(\Gamma)_1 = \Phi_1 \cap \Phi(\Gamma)$$

forms a 3-graded root system.

In fact, writing

$$\Phi(\Gamma)_{-1} = -\Phi(\Gamma)_1,$$

$$\Phi(\Gamma)_0 = \Phi(\Gamma) - (\Phi(\Gamma)_{-1} \sqcup \Phi(\Gamma)_1),$$

$$\Phi(\Gamma)_i = \emptyset, \quad |i| > 1,$$

one has immediately

$$\Phi(\Gamma) = \Phi(\Gamma)_{-1} \sqcup \Phi(\Gamma)_0 \sqcup \Phi(\Gamma)_1.$$
In order to prove

\[ \Phi(\Gamma) \cap (\Phi(\Gamma)_i + \Phi(\Gamma)_j) \subset \Phi(\Gamma)_{i+j}, \]

it is sufficient to verify that

\[ \Phi(\Gamma)_i = \Phi_i \cap \Phi(\Gamma). \quad (A.3) \]

This is obvious for \( i \neq 0 \). For \( i = 0 \) one has

\[
\begin{align*}
\Phi_0 \cap \Phi(\Gamma) &= (\Phi - (\Phi_{-1} \sqcup \Phi_1)) \cap \Phi(\Gamma) \\
&= \Phi \cap \Phi(\Gamma) - ((\Phi_{-1} \sqcup \Phi_1) \cap \Phi(\Gamma)) \\
&= \Phi(\Gamma) - ( (\Phi_{-1} \cap \Phi(\Gamma)) \sqcup (\Phi_1 \cap \Phi(\Gamma)) ) \\
&= \Phi(\Gamma) - (\Phi_{-1}(\Gamma) \sqcup \Phi_1(\Gamma)) = \Phi_0(\Gamma).
\end{align*}
\]

References