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Multi-reggeon compound states and resummed anomalous dimensions in QCD

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Abstract

We perform the OPE analysis of the contribution of colour-singlet compound states of reggeized gluons to a generic hard process in QCD and calculate the spectrum of the corresponding higher twist anomalous dimensions in multi-colour limit. These states govern high energy asymptotics of the structure functions and their energies define the intercept of the Regge singularities both in the pomeron and the odderon sectors. We argue that due to nontrivial analytical properties of the energy spectrum, the twist expansion does not hold for the gluonic states with the minimal energy generating the leading Regge singularities. It is restored however after one takes into account the states with larger energies whose contribution to the Regge asymptotics is subleading.

1. Introduction

Scale dependence of hadronic cross-sections is driven by anomalous dimensions calculable in perturbative QCD as series in the coupling constant. The classical example is provided by the deeply inelastic scattering of a virtual photon $\gamma^*(q)$ off a (polarized) hadron with momentum p_μ. In that case, the operator product expansion (OPE) allows one to expand the moments of the structure function $F(x, Q^2)$ (with $x = Q^2/2(p\cdot q)$ and $Q^2 = -q_\mu^2$) in inverse powers of hard scale Q and identify the expansion coefficients as the forward matrix elements of Wilson operators $\mathcal{O}^a_{n,j}(0)$ of increasing twist $n \geqslant 2$ and Lorentz spin $(j-1) \geqslant 1$

$$\tilde F(j, Q^2) \equiv \int_0^1 dx\, x^{j-2} F(x, Q^2) = \sum_{n=2}^{\infty} \frac{1}{Q^n} \sum_a C_n^a\big(j, \alpha_s(Q^2)\big)\langle p|\mathcal{O}^a_{n,j}(0)|p\rangle, \quad (1.1)$$

where $C_n^a(j, \alpha_s)$ are the corresponding coefficient functions and the superscript a is introduced to enumerate operators of the same twist n. Their total num-

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ber depends on n and rapidly grows for $n \geqslant 3$. The Q^2-dependence of $\tilde{F}(j, Q^2)$ follows from the dependence of the twist-n operators on the normalization scale $\mu^2 = Q^2$. In general, these operators mix under renormalization. Diagonalizing the corresponding mixing matrix, one can construct multiplicatively renormalizable operators

$$Q^2 \frac{d}{dQ^2} \langle p | \mathcal{O}^a_{n,j}(0) | p \rangle = \gamma^a_n(j) \langle p | \mathcal{O}^a_{n,j}(0) | p \rangle, \quad (1.2)$$

where the anomalous dimension has a perturbative expansion $\gamma^a_n(j) = \sum_{k=1}^{\infty} \gamma^a_{k,n}(j)(\alpha_s(Q^2)/\pi)^k$ with the expansion coefficients $\gamma^a_{k,n}(j)$ having a nontrivial j-dependence.

For $j \to 1$, the moments (1.1) receive the dominant contribution from the small-x region, in which the structure function has a Regge behaviour $F(x, Q^2) \sim (1/x)^{\alpha-1}$, with α close to unity. To find the scale dependence of the structure function $F(x, Q^2)$ at small x, one has to analytically continue the anomalous dimensions $\gamma^a_n(j)$ from positive integer $j \geqslant 2$ to "unphysical" $j \sim 1$ and invert the moments (1.1). For twist two this can be done using the well-known DGLAP expressions for $\gamma^a_{n=2}(j)$. For twist $n \geqslant 3$, the calculation of $\gamma^a_n(j)$ is much more involved already for positive integer $j \geqslant 2$ mainly because the number of operators increases with the twist and the size of the corresponding mixing matrices depends on j. As a consequence, analytical continuation of anomalous dimensions of the operators of twist $n \geqslant 3$ within the conventional OPE approach turns out to be an extremely difficult (if not impossible) task.

Another approach to finding the asymptotic behaviour of the anomalous dimensions $\gamma^a_n(j)$ for $j \to 1$ has been proposed in Ref. [1]. It relies on the relation between the twist expansion of the moments (1.1) and the small-x behaviour of the structure functions $F(x, Q^2)$ obtained within the framework of the BFKL approach [2]. Since the anomalous dimensions do not depend on the choice of the scattered particles, one can simplify the analysis by considering the deeply inelastic scattering of a virtual photon $\gamma^*(Q^2)$ off a perturbative "onium" state of the mass $p^2_\mu = M^2$, such that $\Lambda^2_{\text{QCD}} \ll M^2 \ll Q^2$. In the BFKL approach, one obtains the small-x behaviour in this process by resumming a special class of perturbative corrections enhanced by powers of $\alpha_s \ln(1/x)$. Among them there are $\alpha_s \ln(1/x) \ln(Q^2/M^2)$-terms responsible for anomalous Q^2-dependence of $F(x, Q^2)$. Expanding the resulting expression for the moments $\tilde{F}(j, Q^2)$ in powers of $1/Q$, one can separate the twist-n contribution and calculate the corresponding anomalous dimensions $\gamma^a_n(j)$ for $j \to 1$.[3] Still, the actual form of the underlying twist-n operators remains to be found. It has been conjectured [4,5] that they belong to the class of quasipartonic operators.

To begin with, we recall that in the BFKL approach, in the generalized leading logarithmic approximation (GLLA) [6], the structure function takes the following form at small-x

$$F(x, Q^2) = \sum_{N \geqslant 2} \bar{\alpha}_s^{N-2} F_N(x, Q^2),$$
$$F_N(x, Q^2) = \sum_{\boldsymbol{q}} (1/x)^{-\bar{\alpha}_s E_N(\boldsymbol{q})} \beta^{\boldsymbol{q}}_{\gamma^*}(Q) \beta^{\boldsymbol{q}}_p(M), \quad (1.3)$$

with $\bar{\alpha}_s = \alpha_s N_c/\pi$. Here $F_N(x, Q^2)$ describes the contribution of color-singlet compound states built from $N = 2, 3, \ldots$ reggeized gluons, or briefly the N-reggeon states. These states satisfy a Schrödinger-like equation for the system of N gluons interacting on the two-dimensional plane of transverse coordinates [6]. Their spectrum is labelled in (1.3) by the set of quantum numbers $\boldsymbol{q}$. For $x \to 0$, the sum over $\boldsymbol{q}$ in (1.3) is dominated by the state with the minimal energy $E_N(\boldsymbol{q})$. The contribution of this state to $F_N(x, Q^2)$ has the Regge form $F_N(x, Q^2) \sim (1/x)^{-\bar{\alpha}_s E_N(\boldsymbol{q})}$ and its Q-dependence is carried by the residue factors $\beta^{\boldsymbol{q}}_{\gamma^*}(Q)\beta^{\boldsymbol{q}}_p(M)$ which measure the overlap of the wave function of the state with the wave functions of scattered particles.

For $\bar{\alpha}_s \ln x \sim 1$ and $x \to 0$, the sum over N in (1.3) is dominated by the $N = 2$ term and gives rise to the BFKL pomeron [2]. Expanding its contribution to (1.1) in powers of $1/Q$, one finds that the $N = 2$ reggeon states generate an infinite tower of composite gluonic operators of increasing twist $n \geqslant 2$ [4]. At $n = 2$, the anomalous dimensions of twist-two (gluonic) operators, $\gamma_2(j)$, satisfy in the leading logarithmic approximation, $\bar{\alpha}_s/(j-1) = \text{fixed}$ and $j \to 1$, the

[3] It is known [3], that the OPE expansion breaks down for very small values of x, away from the region $\ln(1/x) \ll \ln^2(Q^2/\Lambda^2_{\text{QCD}})$. We shall assume that Q^2 is large enough so that this condition is fulfilled.

master equation [1]

$$\left(\frac{\bar{\alpha}_s}{j-1}\right)^{-1} = 2\psi(1) - \psi\big(\gamma_2(j)\big) - \psi\big(1-\gamma_2(j)\big), \tag{1.4}$$

where $\psi(x) = d\ln\Gamma(x)/dx$ is the Euler digamma function. Its solution looks like

$$\gamma_2(j) = \frac{\bar{\alpha}_s}{j-1} + 2\zeta(3)\left(\frac{\bar{\alpha}_s}{j-1}\right)^4 + 2\zeta(6)\left(\frac{\bar{\alpha}_s}{j-1}\right)^6 + \mathcal{O}\big(\bar{\alpha}_s^8\big), \tag{1.5}$$

with $\zeta(k)$ the Riemann zeta-function. Eq. (1.5) is in agreement with the well-known two-loop expression for the twist-two anomalous dimension γ_j^{GG}. The series (1.5) diverges for $j-1=\mathcal{O}(\alpha_s)$ and its radius of convergence is determined by the right-most Regge singularity, $\gamma_2(j) \sim \sqrt{j-j_2}$ with $j_2 = 1+4\bar{\alpha}_s\ln 2$ being the intercept of the BFKL pomeron [2].

The contribution of the $N \geqslant 3$ reggeon states to (1.3) is suppressed by a power of the coupling constant and it serves to unitarize the asymptotic behaviour of the structure function $F(x,Q^2)$ at very high energy. To perform the OPE analysis of these states, one has to expand the moments of $F_N(x,Q^2)$ in inverse powers of Q and match the resulting expression for $\tilde{F}_N(j,Q^2)$ into (1.1). Similar to the $N=2$ case, the resummed anomalous dimensions of these operators can be deduced from the spectrum of the underlying N-reggeon states. This program has been carried out in Refs. [1,4] only for the $N=2$ states while for $N \geqslant 3$ the main difficulty was a poor understanding of the properties of the N-reggeon states. Later, the anomalous dimensions of twist-four gluonic operators have been calculated in the double-logarithmic approximation in Refs. [7,8]. These studies revealed that due to the presence of Regge cuts, the anomalous dimension of N-reggeon states has a complicated analytical structure which get simplified however in the multi-color limit. Inspired by this observation, we perform in the present Letter the OPE analysis to the $N \geqslant 3$ compound states in the limit $N_c \to \infty$. Namely, we identify the leading twist contribution of these states to the moments (1.1) and calculate the corresponding anomalous dimensions for $j \to 1$ in the multi-color limit. Our analysis relies on the remarkable integrability properties of the N-reggeon states [9,10] and on the exact expressions for the energy spectrum of these states found in [11, 12].

The Letter is organized as follows. In Section 2 we summarize the properties of the multi-reggeon states, perform the OPE expansion of their contribution to the structure function and establish the master equation for anomalous dimensions of high twist operators. This equation involves analytical continuation of the energy spectrum of the N-reggeon states in multi-colour QCD, which is performed in Section 3. In Section 4 we present our results for the anomalous dimensions. Section 5 contains concluding remarks.

2. Reggeon compound states in multi-colour QCD

Let us perform the OPE expansion of the contribution of the N reggeon compound state to the structure function (1.1). Going over to the moment space we find from (1.3)

$$\tilde{F}_N\big(j,Q^2\big) = \sum_{\boldsymbol{q}} \frac{1}{j-1+\bar{\alpha}_s E_N(\boldsymbol{q})} \beta_{\gamma^*}^{\boldsymbol{q}}(Q)\beta_p^{\boldsymbol{q}}(M), \tag{2.1}$$

where the sum runs over the spectrum of the N-reggeon states. These states satisfy the Schrödinger equation [6] which possesses, in the multi-color limit, "hidden" conserved charges $\boldsymbol{q}$ and, as a consequence, is completely integrable [9,10].

Due to complete integrability, the spectrum of the N-reggeon states in multi-colour QCD is uniquely specified by a complete set of the quantum numbers $\boldsymbol{q} = (q_2,\bar{q}_2,\ldots,q_N,\bar{q}_N)$ defined as eigenvalues of the corresponding integrals of motion. The wave function of the state, $\Psi_{\boldsymbol{q}}(\{\vec{z}\};\vec{z}_0)$, depends on two-dimensional impact parameters of N reggeons, $\{\vec{z}\} \equiv (\vec{z}_1,\ldots,\vec{z}_N)$, and the impact parameter of the center-of-mass of the state, $\vec{z}_0$. Introducing auxiliary complex (anti)holomorphic coordinates, $z = x+iy$ and $\bar{z} = x-iy$, one finds [4] that the effective QCD Hamiltonian for the system of N-reggeons is invariant under conformal, $SL(2,\mathbb{C})$ transformations on the plane $z \to (az+b)/(cz+d)$ and $\bar{z} \to (\bar{a}\bar{z}+\bar{b})/(\bar{c}\bar{z}+\bar{d})$ with $ad-bc = \bar{a}\bar{d}-\bar{b}\bar{c} = 1$. As a consequence, its eigenstates $\Psi_{\boldsymbol{q}}(\{\vec{z}\};\vec{z}_0)$ belong to irreducible representation of the $SL(2,\mathbb{C})$ group:

$$\Psi_{\boldsymbol q}\big(\{\vec z\};\vec z_0\big) \to (cz_0+d)^{2h}(\bar c\bar z_0+\bar d)^{2\bar h} \times \prod_{k=1}^{N}(cz_k+d)^{2s}(\bar c\bar z_k+\bar d)^{2\bar s}\Psi_{\boldsymbol q}\big(\{\vec z\};\vec z_0\big), \quad (2.2)$$

where $(s=0,\bar s=1)$ is the $SL(2,\mathbb{C})$ spin of a single reggeon and the parameters $(h,\bar h)$ define the $SL(2,\mathbb{C})$ spin of the N-reggeon state. For the principal series of the $SL(2,\mathbb{C})$, their possible values are parameterized by nonnegative integer n_h and real ν_h

$$h=\frac{1+n_h}{2}+i\nu_h, \qquad \bar h=\frac{1-n_h}{2}+i\nu_h. \quad (2.3)$$

By the definition, the wave function $\Psi_{\boldsymbol q}(\{\vec z\};\vec z_0)$ has to diagonalize the integrals of motion in the z- and $\bar z$-sectors, $\boldsymbol q=(q_2,\bar q_2,\dots,q_N,\bar q_N)$. Also, it has to be a single-valued function on the two-dimensional $\vec z$-plane and be normalizable with respect to the $SL(2,\mathbb{C})$ scalar product. These requirements lead to the quantization conditions for the integrals of motion.

As was shown in [12], the quantized values of $\boldsymbol q$ depend on real ν_h and integer $n_h\geqslant 0$ defined in (2.3) and on the set of integers $\boldsymbol\ell=(\ell_1,\dots,\ell_{2N-4})$. Then, the sum over the N-reggeon states in (2.1) looks like $\sum_{\boldsymbol q}\equiv\sum_{\boldsymbol\ell}\int_{-\infty}^{\infty}d\nu_h\sum_{n_h=0}^{\infty}$. The eigenstates with different quantum numbers $\boldsymbol q=\boldsymbol q(\nu_h;n_h,\boldsymbol\ell)$ are orthogonal with respect to the $SL(2,\mathbb{C})$ invariant scalar product

$$\big\langle\Psi_{\boldsymbol q}(\vec z_0)\big|\Psi_{\boldsymbol q'}(\vec z_0')\big\rangle \equiv \int\prod_{k=1}^{N}d^2z_k\,\Psi_{\boldsymbol q}\big(\{\vec z\};\vec z_0\big)\big(\Psi_{\boldsymbol q'}\big(\{\vec z\};\vec z_0'\big)\big)^* = \delta^{(2)}(z_0-z_0')\delta_{\boldsymbol{qq}'}, \quad (2.4)$$

where $\delta_{\boldsymbol{qq}'}\equiv\delta(\nu_h-\nu_h')\delta_{n_hn_h'}\delta_{\boldsymbol\ell\boldsymbol\ell'}$. The energy of the N-reggeon state, $E_N=E_N(\nu_h;n_h,\boldsymbol\ell)$, is a real continuous function of ν_h. For different integer n_h and $\boldsymbol\ell$, these functions form an infinite set of “trajectories” (see Fig. 1 on the left).

As we will see in a moment, to calculate the anomalous dimensions we will have to perform an analytical continuation of the energy spectrum of the N-reggeon state from “physical” values of the conformal spin h, Eq. (2.3), to arbitrary complex h. Since h depends on two parameters, n_h and ν_h, one has to decide which of these parameters (if not both) can be made complex. To this end, we note that n_h and ν_h have a different physical meaning. According to (2.2), they define the two-dimensional Lorentz spin, $h-\bar h=n_h$, and the scaling dimension, $h+\bar h=1+2i\nu_h$, of the N-reggeon state, respectively. For the wave function $\Psi_{\boldsymbol q}(\{\vec z\};\vec z_0)$ to be single-valued, n_h has to be integer while reality condition for ν_h follows from the requirement that $\Psi_{\boldsymbol q}(\{\vec z\};\vec z_0)$ has to be normalizable with respect to the scalar product (2.4). Performing analytical continuation, we shall require that $\Psi_{\boldsymbol q}(\{\vec z\};\vec z_0)$ has to be single-valued on the $\vec z$-plane for arbitrary complex h and lift the

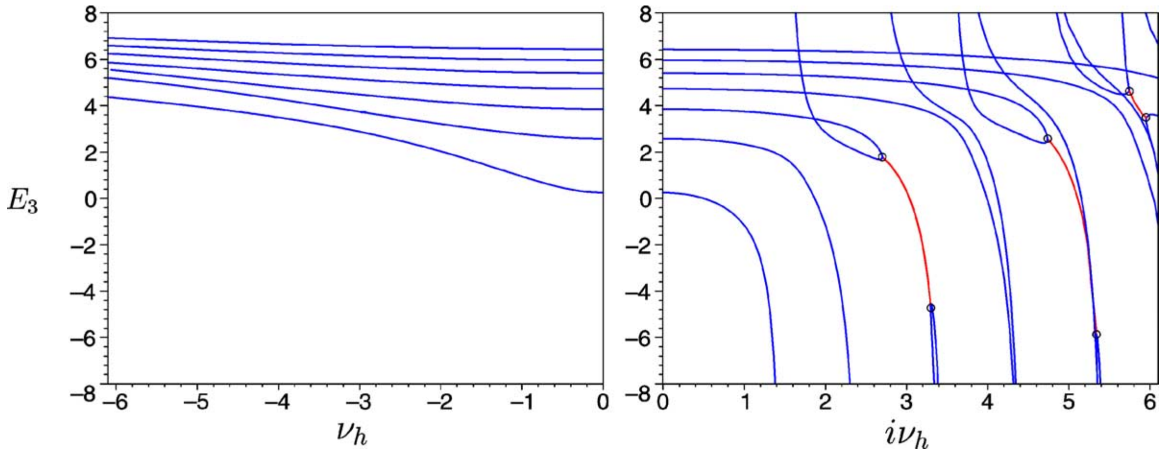

Fig. 1. The energy spectrum of the $N=3$ reggeon states $E_3(\nu_h;n_h,\boldsymbol\ell)$ for $n_h=0$ and $\boldsymbol\ell=(0,\ell_2)$, with $\ell_2=2,4,\dots,14$ from the bottom to the top (on the left). Analytical continuation of the energy along the imaginary ν_h-axis (on the right). The branching points are indicated by open circles. The lines connecting the branching points represent Re E_3.

normalizability condition. This implies that n_h has to be integer and analytical continuation goes in ν_h. The function $\Psi_{\boldsymbol{q}}(\{\vec{z}\};\vec{z}_0)$ defined in this way obeys (2.2), diagonalizes the integrals of motion $\boldsymbol{q}$ but it does not satisfy (2.4) for arbitrary complex h.

To expand (2.1) in inverse powers of Q, one examines the expression for the residue factors. They describe the coupling of the N-reggeon states to the scattered particles and are given by

$$\beta^{\boldsymbol{q}}_{\gamma^*}(Q)=\int d^2z_0\,\langle\Psi_{\gamma^*}|\Psi_{\boldsymbol{q}}(\vec{z}_0)\rangle,$$
$$\beta^{\boldsymbol{q}}_{p}(M)=\int d^2z_0\,\langle\Psi_{\boldsymbol{q}}(\vec{z}_0)|\Psi_{p}\rangle. \tag{2.5}$$

Here the scalar product is taken with respect to (2.4) while $\Psi_{\gamma^*}(\{\vec{z}\})$ and $\Psi_p(\{\vec{z}\})$ are the wave functions of the photon and the onium state, respectively, in the impact parameter representation. The $\vec{z}_0$-integration in (2.5) ensures that the total momentum transferred in the t-channel equals zero. Neglecting the running of the coupling constant, one finds that the impact factors depend on a single scale—the invariant mass of the particle. From dimensional counting, the dependence is fixed by the scaling dimension of the N-reggeon state

$$\beta^{\boldsymbol{q}}_{\gamma^*}(Q)=C^{\boldsymbol{q}}_{\gamma^*}Q^{-1-2i\nu_h},$$
$$\beta^{\boldsymbol{q}}_{p}(M)=C^{\boldsymbol{q}}_{p}M^{-1+2i\nu_h}, \tag{2.6}$$

where $C^{\boldsymbol{q}}_{\gamma^*}$ and $C^{\boldsymbol{q}}_{p}$ are dimensionless coefficients depending on the charges $\boldsymbol{q}=\boldsymbol{q}(\nu_h;n_h,\boldsymbol{\ell})$. They are different from zero provided that the reggeon state has the same quantum numbers (Bose symmetry, C-parity, two-dimensional angular momentum) as hadronic states to which it couples. In particular, $C^{\boldsymbol{q}}_{\gamma^*}$ vanishes for the reggeon states with the odderon quantum numbers. Our subsequent consideration does not rely on the properties of $C^{\boldsymbol{q}}_{\gamma^*}$ and it is valid for a generic short-distance dominated process which receives a nonvanishing contribution from the reggeon states both in the odderon and the pomeron sectors.

Substituting (2.6) into (2.1), one gets

$$\tilde{F}_N(j,Q^2)=\frac{1}{Q^2}\sum_{\boldsymbol{\ell}}\sum_{n_h\geqslant 0}\int_{-\infty}^{\infty}d\nu_h\,\frac{C^{\boldsymbol{q}}_{\gamma^*}C^{\boldsymbol{q}}_{p}}{j-1+\bar{\alpha}_s E_N(\boldsymbol{q})}\times\left(\frac{M}{Q}\right)^{-1+2i\nu_h}. \tag{2.7}$$

Let us examine (2.7) in two limits: (i) $Q^2/M^2=\text{fixed}$, $j\to 1$ and (ii) $Q^2/M^2\to\infty$, $j=\text{fixed}$.

In the limit (i), one obtains from (2.7) that the rightmost Regge singularity of $\tilde{F}_N(j,Q^2)$ is located at $j_N=1-\bar{\alpha}_s\min_q E_N(\boldsymbol{q})$

$$\tilde{F}^{(\mathrm{i})}_N(j,Q^2)\sim\frac{1}{Q^2}\int\frac{d\nu_h}{j-j_N+\sigma_N\nu_h^2}\left(\frac{M}{Q}\right)^{-1+2i\nu_h}\sim\frac{1}{Q}\frac{1}{\sqrt{j-j_N}}. \tag{2.8}$$

Here the dispersion parameter σ_N describes the ν_h-dependence of the energy $E_N(\boldsymbol{q})$ around its minimal value. Eq. (2.8) leads to a power rise of the structure function at small-x, $F_N(x,Q^2)\sim Q^{-1}(1/x)^{j_N}/(\ln 1/x)^{1/2}$, in agreement with the properties of the BFKL pomeron ($N=2$) [2] and higher $N\geqslant 3$ reggeon states [12],

In the limit (ii), we choose $j>j_N$ and expand the r.h.s. of (2.7) in powers of M/Q. For $M/Q\to 0$, the ν_h-integration in (2.7) can be performed by deforming the integration contour into the lower half-plane and picking up the contribution from singularities of the integrand in (2.7). The latter could come from singularities of the impact factors $C^{\boldsymbol{q}}_{\gamma^*}C^{\boldsymbol{q}}_{p}$, possible branch cuts of the energy $E_N(\boldsymbol{q})$ and zeros of the denominator. Let us examine these three possibilities one after another.

It is known [1] that the impact factors may have poles in ν_h due to mixing between gluonic and quark operators. Since the mixing does not affect the leading $j\to 1$ asymptotics of gluonic operators, we can safely neglect it. Next, one has to examine the possibility that the energy $E_N(\boldsymbol{q})$ has cuts on the complex ν_h-plane [7,8]. In that case, as we will show below, the integration around the cut in (2.7) provides a nontrivial contribution to $\tilde{F}_N(j,Q^2)$ which contradicts (1.1). For the OPE expansion (1.1) to be valid, the contribution of cuts should cancel in the sum (2.7) over the N-reggeon states. We will argue in Section 4 that this is exactly what happens for $N\geqslant 3$ reggeon states. Finally, the poles of (2.7) originating from zeros of the denominator can be defined as solutions to the master equation

$$j-1+\bar{\alpha}_s E_N(\boldsymbol{q})=0, \tag{2.9}$$

where $\boldsymbol{q} = \boldsymbol{q}(\nu_h; n_h, \boldsymbol{\ell})$. At $N = 2$ this equation matches (1.4) for $\gamma_2(j) = 1/2 - i\nu_h$. For $N \geqslant 3$, the general solution to (2.9) takes the form $\nu_h = \nu_h(\bar{\alpha}_s/(j-1); n_h, \boldsymbol{\ell})$. As follows from (2.7), its contribution to the moments of structure function scales as $\tilde{F}_N(j, Q^2) \sim Q^{-2}(M/Q)^{-1+2i\nu_h}$. It matches the OPE expansion, Eqs. (1.1) and (1.2), provided that $n - 2\gamma_n(j) = 1 + 2i\nu_h$, or equivalently

$$\gamma_n(j) = (n-1)/2 - i\nu_h = \big[n - (h + \bar{h})\big]/2, \qquad (2.10)$$

where h and $\bar{h}$ are the $SL(2, \mathbb{C})$ spins defined in (2.3). This equation establishes the relation between the anomalous dimension of the twist-n operator $\gamma_n(j)$, Eq. (1.2), and solutions to (2.9). Since this operator is built from N gluon strength tensors and carries the two-dimensional Lorentz spin $n_h \geqslant 0$, its twists satisfies $n \geqslant N + n_h$. In addition, writing the anomalous dimension as $\gamma_n(j) = \gamma_n^{(0)}\bar{\alpha}_s/(j-1) + \mathcal{O}(\bar{\alpha}_s^2)$, one expects that solutions to (2.9) should look like $i\nu_h = (n-1)/2 - \gamma_n^{(0)}\bar{\alpha}_s/(j-1) + \mathcal{O}(\bar{\alpha}_s^2)$ with $n \geqslant N + n_h$. Combining this expression together with (2.9) one finds

$$E_N(\boldsymbol{q}) \sim \frac{\gamma_n^{(0)}}{i\nu_h - (n-1)/2}. \qquad (2.11)$$

Thus, in order for the moments (2.7) to admit the OPE expansion (1.1), the energy of the N-reggeon state has to contain an infinite set of poles located along imaginary ν_h-axis at (half)integer points $\nu_h = -i(n-1)/2$ with $n \geqslant N + n_h$. For $i\nu_h = (n-1)/2 + \epsilon$ the Laurent expansion of the energy around the pole at $\epsilon = 0$ can be written as

$$E_N(\boldsymbol{q}) = \frac{c_{-1}}{\epsilon} - c_0 - c_1\epsilon + \cdots. \qquad (2.12)$$

Its substitution into (2.9) yields the following expression for the anomalous dimension of the twist-n operators, $\gamma_n(j) = -\epsilon$,

$$\gamma_n(j) = c_{-1}\left[\frac{\bar{\alpha}_s}{\omega} + c_0\left(\frac{\bar{\alpha}_s}{\omega}\right)^2 + \left(c_0^2 - c_1 c_{-1}\right)\left(\frac{\bar{\alpha}_s}{\omega}\right)^3 + \cdots\right] \qquad (2.13)$$

with $\omega = j - 1$. The coefficients c_k depend on the twist of the operator and on the quantum numbers specifying the N-reggeon state, $c_k = c_k(n, n_h, \boldsymbol{\ell})$. The $\mathcal{O}(\alpha_s)$-term in the r.h.s. of (2.13) defines the anomalous dimension in the double-logarithmic approximation [7, 8,13] and leads to the following scaling behaviour of the structure function (1.1) at small-x

$$F^{(\mathrm{ii})}\big(x, Q^2\big) \sim \frac{1}{Q^n}\exp\Big(2\sqrt{c_{-1}\ln(1/x)\ln Q^2}\Big). \qquad (2.14)$$

We conclude that in order to calculate the twist-n anomalous dimensions (2.13) one has to analytically continue the energy of the N-reggeon compound states $E_N(\boldsymbol{q})$ from "physical" values of the conformal $SL(2, \mathbb{C})$ spins (2.3) to the complex ν_h-plane and calculate the coefficients of its Laurent expansion around the poles located at $i\nu_h = (n-1)/2$ with $n \geqslant N + n_h$, or equivalently $h = (n + n_h)/2$ and $\bar{h} = (n - n_h)/2$.

3. Analytical continuation

The N-reggeon states are the eigenstates of the effective QCD Hamiltonian, which is a Hermitian operator on the space of functions satisfying (2.2), (2.3) and (2.4) with ν_h real and n_h integer. The energies of these states, $E_N(\boldsymbol{q})$, are smooth real functions on the real ν_h-axis and their wave functions $\Psi_{\boldsymbol{q}}(\{\vec{z}\}; \vec{z}_0)$ are orthogonal to each other with respect to the scalar product (2.4). Performing analytical continuation of the energy spectrum from the real ν_h-axis to the complex ν_h-plane, one replaces the normalization condition (2.4) by a weaker condition for $\Psi_{\boldsymbol{q}}(\{\vec{z}\}; \vec{z}_0)$ to be a single-valued function on the $\vec{z}$-plane. This condition ensures that the two-dimensional integrals entering (2.5) are well-defined for complex ν_h.

From point of view of quantum mechanics, the problem amounts to finding analytical properties of the energy $E = E(g)$ as a function of the coupling constant $g = \nu_h$. In past, this fundamental problem has been thoroughly studied in various models [14–17]. It was found that, in general, $E(g)$ is a *multi-valued* function of complex g and the number of its different branches is equal to the number of the energy levels at real g when the Hamiltonian is Hermitian. To study a global behaviour of the function $E(g)$, it proves convenient to glue together different "sheets" corresponding to its branches and consider $E(g)$ as a single-valued function on the resulting Riemann surface. Depending on the model, this surface may have a rather complicated structure and consist of a few disconnected parts due to some additional symmetry [15]. A remarkable property of the Riemann surface is that it encodes the *entire* spectrum of the model. Namely, knowing the energy of the ground state at real g, one can reconstruct the whole spectrum of the model by going around the branching points to other sheets of the Riemann surface corresponding to excited energy levels. We shall demonstrate below that upon analytical continuation to the complex ν_h the energy of the N-reggeon compound states E_N shares the same properties (for $N \geqslant 3$).

To begin with, let us consider the well-known expression for the energy of the $N = 2$ states [2]

$$E_2(\nu_h, n_h) = \psi\left(\frac{1+n_h}{2} + i\nu_h\right) + \psi\left(\frac{1+n_h}{2} - i\nu_h\right) - 2\psi(1), \tag{3.1}$$

with n_h nonnegative integer. $E_2(\nu_h, n_h)$ is a smooth even function on the real ν_h-axis. It takes its minimal value at $\nu_h = 0$ and increases monotonically for $\nu_h \to \pm\infty$. After analytical continuation, $E_2(\nu_h, n_h)$ becomes a meromorphic function of ν_h. It has an infinite set of poles located along imaginary ν_h-axis at $i\nu_h = \pm(n-1)/2$ with $n \geqslant 2 + n_h$. The leading twist contribution, $n = 2$, corresponds to $n_h = 0$. To obtain the anomalous dimension of twist two, Eq. (1.5), one matches the Laurent expansion of $E_2(\nu_h, 0)$ around the pole at $\nu_h = -i/2$ into (2.12) and applies (2.13). The BFKL pomeron is located on the same complex curve $E_2(\nu_h, 0)$ at $\nu_h = 0$ so that $E_2(0, 0) = -4\ln 2$. For $n_h \geqslant 1$ the complex curve $E_2(\nu_h, n_h)$ describes subleading Regge singularities and, at the same time, it generates contribution of higher twist.

For $N \geqslant 3$ reggeon states, analytical continuation of the energy spectrum is more subtle. Firstly, the energy of the N-reggeon states does not admit a simple representation like (3.1) and, secondly, $E_N = E_N(\nu_h; n_h, \boldsymbol{\ell})$ depends on the set of $2(N-2)$ integers $\boldsymbol{\ell}$, which parameterize eigenvalues of the integrals of motion. Thus, performing analytical continuation of E_N we have to deal with an infinite set of complex curves $E_N(\nu_h; n_h, \boldsymbol{\ell})$ labelled by $\boldsymbol{\ell}$-integers. For ν_h real, they define the energy of "physical" N-reggeon states, whose wave functions diagonalize the integrals of motion $\boldsymbol{q}$ and satisfy the normalization condition (2.4). Going over to complex ν_h, one preserves the former condition and relaxes the latter one.

For ν_h real, the integrals of motion in the holomorphic and antiholomorphic sector, q_k and $\bar{q}_k$, respectively, are conjugated to each other with respect to the $SL(2, \mathbb{C})$ scalar product (2.4) so that $\bar{q}_k = q_k^*$ (with $k = 2, \ldots, N$). Going over to complex ν_h, one finds that $\bar{q}_k \neq q_k^*$ and, in general, the quantum numbers in the two sectors are independent on each other. Nevertheless, the N-reggeon spectrum contains the states, whose integrals of motion take either real, or imaginary values on the real ν_h-axis, that is $\bar{q}_k(\nu_h; n_h, \boldsymbol{\ell}) = \pm q_k(\nu_h; n_h, \boldsymbol{\ell})$. For such states, the same relation between q_k and $\bar{q}_k$ also holds for complex ν_h but the charges take complex values.

Due to complete integrability, the energy of the N-reggeon state is a function of the integrals of motion $E_N = E_N(q_2, \bar{q}_2, \ldots, q_N, \bar{q}_N)$ with $q_k = q_k(\nu_h; n_h, \boldsymbol{\ell})$ and similar for $\bar{q}_k$. For ν_h real, the energy spectrum is described by infinite set of smooth real functions $E_N(\nu_h; n_h, \boldsymbol{\ell})$ labelled by integers n_h and $\boldsymbol{\ell}$. The minimal value of E_N on the real ν_h-axis determines the position of the dominant Regge singularity j_N, Eq. (2.8). For ν_h complex, one expects that, similar to the $N = 2$ case, the complex curve $E_N(\nu_h; n_h, \boldsymbol{\ell})$ has an infinite number of poles located at $i\nu_h = \pm(n-1)/2$ with $n \geqslant N + n_h$. As we will show below, this turns out to be the case but in comparison with the $N = 2$ case one encounters a novel phenomenon. For $N \geqslant 3$, in addition to the poles, the complex curve $E_N(\nu_h; n_h, \boldsymbol{\ell})$ contains (an infinite number of) square-root branching points on the complex ν_h-plane. Therefore, E_N is a *multi-valued* function on the complex ν_h-plane and its different branches are enumerated by integer n_h and $\boldsymbol{\ell}$. This property is in agreement with the previous findings of Refs. [7,8].

To construct the complex curve $E_N(\nu_h; n_h, \boldsymbol{\ell})$, we apply the approach developed in Ref. [12]. Namely, we start with the expression for the energy E_N obtained there for real ν_h and analytically continue it to complex ν_h following the procedure described above. In this way, one gets

$$E_N = \frac{1}{4}\Big[\varepsilon(h, q) + \varepsilon(h, -q) + \big(\varepsilon\big(1 - \bar{h}^*, \bar{q}^*\big)\big)^* + \big(\varepsilon\big(1 - \bar{h}^*, -\bar{q}^*\big)\big)^*\Big], \tag{3.2}$$

where the $SL(2,\mathbb{C})$ spins h and $\bar{h}$ are given by (2.3) and the notation was introduced for the quantum numbers in the two sectors, $q = \{q_k\}$ and $\bar{q} = \{\bar{q}_k\}$ (with $k = 2,\dots,N$). In similar manner, $-q \equiv \{(-1)^k q_k\}$, $\bar{q}^* \equiv \{\bar{q}_k^*\}$ and $-\bar{q}^* \equiv \{(-1)^k \bar{q}_k^*\}$, so that $E_N(-q,-\bar{q}) = E_N(q,\bar{q})$. For real ν_h one has $1 - \bar{h}^* = h$ and $\bar{q}^* = q$, so that (3.2) produces real values for $E_N(q,\bar{q})$. The function $\varepsilon(h,q)$ entering (3.2) is defined for arbitrary complex h and q as

$$\varepsilon(h,q) = i\frac{d}{d\epsilon}\ln\bigl[\epsilon^N Q(i+\epsilon;h,q)\bigr]\Big|_{\epsilon=0}, \tag{3.3}$$

where $Q(u;h,q)$, the so-called chiral Baxter block, has the following integral representation

$$Q(u;h,q) = \int_0^1 dz\, z^{iu-1} Q_1(z). \tag{3.4}$$

Here the function $Q_1(z)$ is the solution to the Nth order Fuchsian differential equation

$$\Biggl[(z\partial_z)^N z + (z\partial_z)^N z^{-1} - 2(z\partial_z)^N - \sum_{k=2}^{N} i^k q_k (z\partial_z)^{N-k}\Biggr] Q_1(z) = 0, \tag{3.5}$$

with the prescribed asymptotic behaviour at regular singular point $z = 1$, $Q_1(z) \sim (1-z)^{-h-1}$. Since $Q_1(z) \sim \ln^{N-1} z$ for $z \to 0$, the chiral block $Q(u;h,q)$ defined in (3.4) is a meromorphic function of u with the Nth order poles located at $u = ik$ for k positive integer. It is convenient to normalize $Q_1(z)$ in (3.4) in such a way that the residue at the Nth order pole $u = i$ equals unity. Together with (3.3) this leads to

$$Q(i+\epsilon;h,q) = \frac{1}{\epsilon^N} - i\frac{\varepsilon(h,q)}{\epsilon^{N-1}} + \mathcal{O}\biggl(\frac{1}{\epsilon^{N-2}}\biggr). \tag{3.6}$$

To evaluate $\varepsilon(h,q)$, one has to solve the differential equation (3.5) and match the resulting expression for $Q(u;h,q)$, Eq. (3.4), into (3.6). This allows one to determine (3.2) for arbitrary complex q and $\bar{q}$.

The quantization conditions for the integrals of motion $\boldsymbol{q} = (q_2,\bar{q}_2,\dots,q_N,\bar{q}_N)$ follow from the requirement for their common eigenfunctions $\Psi_{\boldsymbol{q}}(\{\vec{z}\};\vec{z}_0)$ to be single-valued functions on the $\vec{z}$-plane. These conditions take the form [12]

$$Q(i+\epsilon;h,q)Q(i-\epsilon;\bar{h},-\bar{q}) - Q(i+\epsilon;1-h,q)Q(i-\epsilon;1-\bar{h},-\bar{q}) = \mathcal{O}\bigl(\epsilon^0\bigr). \tag{3.7}$$

Substituting (3.6) into this relation, one finds that the l.h.s. scales as $1/\epsilon^{2N-1}$ for $\epsilon \to 0$. Requirement for the poles at $\epsilon = 0$ to vanish leads to the overdetermined system of $2N-1$ equations for the $2(N-2)$ charges q_k and $\bar{q}_k$ ($k = 3,\dots,N$). The two remaining charges are given by $q_2 = -h(h-1)$ and $\bar{q}_2 = -\bar{h}(\bar{h}-1)$. Solving this system, one obtains the quantized values of the integrals of motion and verifies that the obtained expressions satisfy three consistency conditions.

Applying Eqs. (3.2)–(3.7) one can calculate the energy of the N-reggeon state for arbitrary complex ν_h. For ν_h real, they lead to the results for the energy $E_N(\nu_h;n_h,\boldsymbol{\ell})$ obtained before in Ref. [12]. For ν_h complex, they define analytical continuation of the function $E_N(\nu_h;n_h,\boldsymbol{\ell})$. At $N = 2$ the differential equation (3.5) can be solved in terms of Legendre functions and the resulting expression for the energy (3.2) coincides with the known expression (3.1). For $N \geqslant 3$ the analysis is more involved since (3.5) cannot be solved exactly and one has to rely on a power-series solutions to (3.5) described at length in Ref. [12]. In the next section, we summarize the results for $N = 3,4,5,6$ reggeon states.

4. Spectral surface

At $N = 2$, the energy spectrum $E_2(\nu_h;n_h)$ is described on the complex ν_h-plane by the family of *meromorphic* functions (3.1) labelled by a nonnegative integer n_h. For $N \geqslant 3$, an analytical expression for the energy is not available but the value of E_N can be calculated for arbitrary complex ν_h using Eqs. (3.2)–(3.7) as follows. Let us consider an arbitrary contour on the complex ν_h-plane that starts on the real axis and terminates at some complex ν_h. For given real ν_h, the energy $E_N(\nu_h;n_h,\boldsymbol{\ell})$ takes a discrete set of real "physical" values labelled by integer n_h and $\boldsymbol{\ell} = (\ell_1,\dots,\ell_{2N-4})$ (see the left panel in Fig. 1) [11,12]. Calculating the energy along this contour point-by-point from Eqs. (3.2)–(3.7), one analytically continues the function $E_N(\nu_h; n_h, \boldsymbol{\ell})$ to complex ν_h.

This procedure allows us to determine the global properties of the complex curve $E_N(\nu_h; n_h, \boldsymbol{\ell})$ on the complex ν_h-plane. If E_N were a single-valued function of complex ν_h, it would resume its original value after going around arbitrary closed contour on the ν_h-plane. This is the case at $N = 2$, Eq. (3.1), whereas for $N \geqslant 3$ one finds that $E_N(\nu_h; n_h, \boldsymbol{\ell})$ is a *multi-valued* function of ν_h. Namely, E_N has (an infinite number of) branching points on the complex plane, $\nu_h = \nu_{\mathrm{br},k}$, such that E_N changes its value after encircling these points. We found however that E_N resumes its value if one encircles the branching point twice. This implies that E_N has square-root cuts

$$E_N^{\pm} \sim a_k \pm b_k \sqrt{\nu_{\mathrm{br},k} - \nu_h}, \tag{4.1}$$

where $E_N^{\pm}$ defines the energy on the upper and lower edges of the cut, respectively. Aside of the branch cuts, E_N has an infinite number of poles. In agreement with our expectations, Eq. (2.11), they are located along imaginary axis at $i\nu_h = \pm(n-1)/2$ with $n \geqslant N + n_h$.

Thus, for $N \geqslant 3$ the energy $E_N(\nu_h; n_h, \boldsymbol{\ell})$ is a meromorphic function on the complex ν_h-plane with the square-root cuts running between the branching points $\nu_{\mathrm{br},k}$. Another way to represent the family of multivalued functions $E_N(\nu_h; n_h, \boldsymbol{\ell})$ is to sew together their branches along the square-root cuts, Eq. (4.1), and define E_N as a single-valued meromorphic function on the resulting Riemann surface. Following [17], we shall call it the *spectral surface*. Its topology depends on the number of reggeons N. Different sheets of this surface can be enumerated by integer n_h and $\boldsymbol{\ell}$.

Let us consider a particular $(n_h, \boldsymbol{\ell})$th sheet of the spectral surface lying over the ν_h-plane and suppose that it is sewed with $(n'_h, \boldsymbol{\ell}')$th sheet along the square-root cut that starts at the branching point ν_{br}. For real ν_h, the value of function E_N on these two sheets defines the "physical" energy of two N-reggeon states, $E_N(\nu_h; n_h, \boldsymbol{\ell})$ and $E_N(\nu_h; n'_h, \boldsymbol{\ell}')$, respectively. If one analytically continues these functions along the contour that starts at real ν_h and terminates at the branching point, $\nu_h = \nu_{\mathrm{br}}$, then at the vicinity of the branching point the functions behave as (4.1) leading to

$$E_N(\nu_{\mathrm{br}}; n_h, \boldsymbol{\ell}) = E_N(\nu_{\mathrm{br}}; n'_h, \boldsymbol{\ell}'). \tag{4.2}$$

Thus, in a complete analogy with quantum mechanics [14], the branching cuts (4.1) arise due to collision of the energy levels at some complex $\nu_h = \nu_{\mathrm{br}}$ away from the real ν_h-axis. Since the wave functions of the two states have to coincide at the branching point [14], they have to have the same two-dimensional Lorentz spin, $n_h = n'_h$, and possess the same quantum numbers (quasimomentum, C-parity, Bose symmetry, etc.) leading to additional selection rules for $\boldsymbol{\ell}$ and $\boldsymbol{\ell}'$. This implies that the spectral surface cannot be simply connected and it should consist of (infinite number of) disconnected components enumerated by nonnegative integer n_h and the quantum numbers just mentioned.

To illustrate our findings, let us consider the $N = 3$ reggeon states with the Lorentz spin $n_h = 0$. Their spectrum is specified by two integers $\boldsymbol{\ell} = (\ell_1, \ell_2)$. Assigning the values of these integers to different energy levels, we follow the convention adopted in Ref. [12]. The energy $E_3(\nu_h; 0, \boldsymbol{\ell})$ is a smooth even function on the real ν_h-axis (see Fig. 1 on the left). It approaches the minimal value $\min_{\nu_h \boldsymbol{\ell}} E_3 = 0.24717$ at $\nu_h = 0$ along the "trajectory" with $\ell_1 = 0$ and $\ell_2 = 2$. The charges $q_3(\nu_h)$ and $\bar{q}_3(\nu_h)$ take pure imaginary values along this trajectory, so that $\bar{q}_3 + q_3 = 0$. After analytical continuation, the same relation holds for arbitrary complex ν_h including the branching point (4.2). Therefore, the energy level with $\ell_1 = 0$ and $\ell_2 = 2$ could only collide with the levels for which $\bar{q}_3 + q_3 = 0$. As was shown in [12], the latter condition is satisfied for $\ell_1 = 0$ and $\ell_2 =$ even positive. Examples of such energy levels with $\ell_2 = 2, 4, \ldots, 14$ are shown in Fig. 1. They define a connected component of the $N = 3$ spectral surface.

To elucidate analytical properties of the energy, we used Eqs. (3.2)–(3.7) to analytically continue eight functions $E_3(\nu_h; 0, \boldsymbol{\ell} = (0, \ell_2))$ with $\ell_2 = 2, 4, \ldots, 14$. These functions define eight different branches of the complex curve $E_3(\nu_h)$ which can be represented as a two-dimensional surface in $\mathbb{C}^2 = \mathbb{R}^4$ space with the (ν_h, E_3)-coordinates. The slices of this surface along $\operatorname{Im}\nu_h = 0$ and $\operatorname{Re}\nu_h = 0$ hyperplanes are shown in Fig. 1 on the left and the right panels, respectively. We observe that, firstly, the energy has poles at $i\nu_{\mathrm{pole}} = 3/2, 5/2, 7/2, 9/2, 11/2$ and, secondly, different energy levels collide at $i\nu_{\mathrm{br}} = 2.70, 3.29, 4.73, 5.34, 5.74, 5.94$. At the vicinity of the branching point the energies of two colliding levels behave as $E_3^{\pm}$, Eq. (4.1). As expected, the poles of E_3 are located along the imaginary ν_h-axis at $i\nu_h = (n-1)/2$ with $n = 4, 6, \ldots$. This is not the case, however, for the branching points. It turns out that some of the branching points are located away from the imaginary ν_h-axis and, therefore, they cannot be seen in Fig. 1. For instance, the branches with $\ell_2 = 2$ and $\ell_2 = 4$ collide at $i\nu_{\rm br} = 1.723 + i0.248$. In other words, the "ground state" branch, $\ell_2 = 2$, is sewed with the $\ell_2 = 4$ branch which in its turn is sewed with the $\ell_2 = 6$ branch and so on. Thus, going around the contour on the complex ν_h-plane which starts at some real ν_h, encircles the branching points and returns to the starting point one can reconstruct the energy spectrum of the $N = 3$ reggeon states with $\ell_1 = 0$ and $\ell_2 =$ even positive.

The fact that the branching points $\nu_{\rm br}$ take complex irrational values implies that the contribution of the corresponding square-root cuts to the integral (2.7) scales as $1/[Q^{1+2i\nu_{\rm br}} \ln^{3/2} Q]$ and, therefore, it breaks the OPE expansion (1.1). Although such corrections are present in (2.7) for given n_h and $\boldsymbol{\ell}$, they cancel against each other in the sum over all states. To see this let us examine two terms in the sum (2.7) corresponding to the energy levels colliding at $\nu_h = \nu_{\rm br}$. Each of them is associated with a particular sheet of the spectral surface. Denoting the energy on the upper $(+)$ and the lower $(-)$ edges of the square-root cut on these two sheets as $E_1^{\pm}(\nu_h)$ and $E_2^{\pm}(\nu_h)$ one finds that $E_1^{\pm}(\nu_h) = E_2^{\mp}(\nu_h)$. As a consequence, the contour integrals around the same cut on the two sheets differ by a sign and their sum equals zero. Thus, the contribution of the cuts to (2.7) cancels completely in the sum over *all* $N = 3$ reggeon states belonging to the same spectral surface. At the same time, if one had retained in (2.7) only contribution of the state with the minimal energy $E_3^{(\rm gr)}$—the one defining the leading Regge singularity (2.8), the OPE expansion would have been broken. To restore the OPE one has to keep in (2.7) the contribution of subleading Regge singularities.

As another important example, we consider analytical continuation of the $N = 3$ reggeon state with the Lorentz spin $n_h = 1$, the so-called descendant state [12]. This state has $q_3(\nu_h) = 0$ on the real ν_h-axis and its energy coincides with the energy of the $N = 2$ state, $E_{3,\rm d} = E_2(\nu_h; 1)$, Eq. (3.1). Both relations survive the analytical continuation and hold true for arbitrary complex ν_h. Thus, in distinction with the previous case, the energy of this state $E_{3,\rm d}(\nu_h)$ is a single-valued, meromorphic function on the complex ν_h-plane. At $\nu_h = 0$ it takes the value $E_3 = E_2(0; 1) = 0$ which defines the "physical" ground state energy for the system of $N = 3$ reggeons [18]. Going over to the imaginary ν_h-axis, one finds that E_3 has poles in the lower half-plane at $i\nu_h = 1, 2, \ldots$.

We would like to stress that the $N = 3$ states with $n_h = 0$ and $n_h = 1$ described above define two different components of the unifying $N = 3$ spectral surface. The reason why we have selected them is that they contain two special states $E_3(\nu_h = 0; n_h = 1) = 0$ and $E_3(\nu_h = 0; n_h = 0) = 0.24717$, which are currently considered as solutions for the odderon state in QCD with the intercept $j_{\mathbb{O}} = 1 - \bar\alpha_s E_3$ [18,19]. As was shown in Section 2, the poles of the energy in the lower half-plane $\operatorname{Im} \nu_h < 0$, induce the contribution to the OPE expansion (1.1) of the twist $n = h + \bar h = 1 + 2i\nu_h$. The minimal twist $n_{\rm min}$ corresponds to the pole closest to the origin. As follows from our analysis, $n_{\rm min} = 3$ for the descendant state with $n_h = 1$ [4] and $n_{\rm min} = 4$ for the state with $n_h = 0$. Thus, the leading twist contribution of the two above mentioned odderon states with the intercepts $j_{\mathbb{O}} = 1$ and $j_{\mathbb{O}} < 1$ scales at large Q^2 as $\sim 1/Q^3$ and $\sim 1/Q^4$, respectively.

5. Anomalous dimensions

To calculate the anomalous dimensions (2.13), one has to work out the Laurent expansion of the energy around its poles, Eq. (2.12). We recall that for the N-reggeon state the twist-n contribution comes from poles located at $i\nu_h = (n-1)/2$ with $n \geqslant N + n_h$, or equivalently $h = (n + n_h)/2$ and $\bar h = (n - n_h)/2$.

Let us first consider the $N = 3$ descendant state, $n_h = 1$. For complex ν_h its energy, $E_{3,\rm d}$, coincides with $E_2(\nu_h, 1)$ defined in (3.1). The closest to the origin pole is located at $i\nu_h = 1$ and the corresponding $SL(2, \mathbb{C})$ spins (2.3) are given by $h = 1 + i\nu_h = 2$ and $\bar h = i\nu_h = 1$ so that the twist equals $n = h + \bar h = 3$. Using (3.1), one finds the expansion of the energy $E_{3,\rm d} = E_{3,\rm d}(h + \epsilon)$ for $\epsilon \to 0$ as

$$E_{3,\rm d}(2 + \epsilon) = \frac{1}{\epsilon} + 1 - \epsilon - \bigl(2\zeta(3) - 1\bigr)\epsilon^2 + \cdots. \tag{5.1}$$

[4] We disagree on this point with Ref. [5], in which it was claimed that $n_{\rm min} = 4$ for the odderon state with the intercept 1.

Applying Eqs. (2.12) and (2.13), one finds from (5.1) the twist-3 anomalous dimension corresponding to the odderon state with the intercept $j_{\mathbb{O}} = 1$ as ($\omega = j - 1$)

$$\gamma_3^{(N=3)}(j) = \frac{\bar{\alpha}_s}{\omega} - \left(\frac{\bar{\alpha}_s}{\omega}\right)^2 + \left(2\zeta(3)+1\right)\left(\frac{\bar{\alpha}_s}{\omega}\right)^4 + \cdots, \tag{5.2}$$

where the subscript and the superscript indicate the twist and the number of reggeons entering the state, respectively.

Let us now consider the $N = 3$ states with $n_h = 0$. Their analytical properties are encoded in the spectral surface shown in Fig. 1. Applying Eqs. (3.2)–(3.7), we calculate the energy at the vicinity of the poles at $h = 1/2 + i\nu_h = 2, 3, 4, 5$ and obtain the following expressions for $E_3 = E_3(h + \epsilon)$

$$\begin{aligned}
E_3(2+\epsilon) &= \epsilon^{-1} + \frac{1}{2} - \frac{1}{2}\epsilon + 1.7021\epsilon^2 + \cdots, \\
E_3(3+\epsilon) &= 2\epsilon^{-1} + \frac{15}{8} - 1.6172\epsilon + 0.719\epsilon^2 + \cdots, \\
E_3^{(a)}(4+\epsilon) &= \epsilon^{-1} + \frac{11}{12} - 0.6806\epsilon - 1.966\epsilon^2 + \cdots, \\
E_3^{(b)}(4+\epsilon) &= 2\epsilon^{-1} + \frac{15}{4} - 3.2187\epsilon + 3.430\epsilon^2 + \cdots, \\
E_3^{(a)}(5+\epsilon) &= 2\epsilon^{-1} + \frac{125}{48} - 2.0687\epsilon \\
&\quad + 1.047\epsilon^2 + \cdots, \\
E_3^{(b)}(5+\epsilon) &= 2\epsilon^{-1} + \frac{53}{12} - 2.4225\epsilon \\
&\quad + 0.247\epsilon^2 + \cdots.
\end{aligned} \tag{5.3}$$

Here ellipses denote $\mathcal{O}(\epsilon^3)$ terms and the additional superscript was introduced to distinguish between different branches. We recall that all states in (5.3) have the same Lorentz spin $n_h = 0$, so that $\bar{h} = h = 1/2 + i\nu_h$ and the corresponding twist equals $n = 2h \geqslant 4$. Applying Eqs. (2.12) and (2.13), one finds from the first relation in (5.3) that the leading, twist-4 anomalous dimension corresponding to the odderon state with the intercept $j_{\mathbb{O}} < 1$ is given by

$$\gamma_4^{(N=3)}(j) = \frac{\bar{\alpha}_s}{\omega} - \frac{1}{2}\left(\frac{\bar{\alpha}_s}{\omega}\right)^2 - \frac{1}{4}\left(\frac{\bar{\alpha}_s}{\omega}\right)^3 - 1.0771\left(\frac{\bar{\alpha}_s}{\omega}\right)^4 + \cdots. \tag{5.4}$$

The remaining relations in (5.3) lead to similar expressions for higher-twist anomalous dimensions. To save space we do not present them here. The following comments are in order.

The first few terms of the Laurent expansion in (5.3) can be calculated exactly. The reason for this is that solutions to the differential equation (3.5) can be expanded in powers of ϵ and the first few terms can be obtained in a closed form. The coefficient in front of $1/\epsilon$ in (5.3) equals either 1 (only for h even), or 2. In the latter case, writing the energy as $E_3(h + \epsilon) = 2\epsilon^{-1} + \gamma(h) + \mathcal{O}(\epsilon)$ one finds that finite $\mathcal{O}(\epsilon^0)$-terms have the following remarkable property. It turns out that $\gamma(4) = 15/4$ and $\gamma(5) = 53/12$ coincide with the energy of the Heisenberg $SL(2,\mathbb{R})$ magnet model of spin $s = 1$. Most importantly, the energy spectrum of this model determines the anomalous dimensions of local composite *three-quark* (baryonic) operators of helicity-3/2 [20,21]. For such operators, integer $(h - 3)$ counts the number of covariant derivatives and $\gamma(h)$ defines their anomalous dimension. Similar relation holds for higher N between the energy of the N-reggeon states ($SL(2,\mathbb{C})$ Heisenberg spin magnet) and anomalous dimensions of N-particle operators ($SL(2,\mathbb{R})$ Heisenberg spin magnet). Its origin will be discussed in a forthcoming publication.

Going over to the $N \geqslant 4$ case, we shall concentrate on the N-reggeon states providing the contribution to the OPE expansion (1.1) of the minimal twist $n_{\min}$. As before, we shall denote the corresponding anomalous dimension as $\gamma_{n_{\min}}^{(N)}(j)$.

For $N =$ even we find that the minimal twist equals the number of reggeons involved $n_{\min} = N$. It corresponds to the pole of the complex curve E_N located at $i\nu_h = (N-1)/2$ and $n_h = 0$, or equivalently $h = \bar{h} = N/2$. This pole is situated on the same spectral surface as the "physical" ground N-reggeon state, that is the state with the minimal energy for real ν_h. Its energy, $E_N^{(\mathrm{gr})} = \min_{\nu_h n_h \boldsymbol{\ell}} E_N(\nu_h; n_h, \boldsymbol{\ell})$, defines the intercept of the N-reggeon states in the pomeron sector [11,12]: $E_4^{(\mathrm{gr})} = -0.67416$ and $E_6^{(\mathrm{gr})} = -0.39458$. The expansion of the energy $E_N(h + \epsilon)$ around $\epsilon = 0$ looks like

$$\begin{aligned}
E_4(2+\epsilon) &= \frac{2}{\epsilon} + 1 - \frac{1}{2}\epsilon - 1.2021\epsilon^2 + \cdots, \\
E_6(3+\epsilon) &= \frac{4}{\epsilon} + \frac{3}{2} - \frac{7}{16}\epsilon - 0.238\epsilon^2 + \cdots.
\end{aligned} \tag{5.5}$$

Applying Eqs. (2.12) and (2.13), one obtains the following expressions for the leading, twist-N anomalous dimension of the N-reggeon states

$$\gamma_4^{(N=4)}(j) = 2\frac{\bar\alpha_s}{\omega} + 2\left(\frac{\bar\alpha_s}{\omega}\right)^2 - 13.6168\left(\frac{\bar\alpha_s}{\omega}\right)^4 + \cdots,$$
$$\gamma_6^{(N=6)}(j) = 4\frac{\bar\alpha_s}{\omega} + 6\left(\frac{\bar\alpha_s}{\omega}\right)^2 + 2\left(\frac{\bar\alpha_s}{\omega}\right)^3 - 33.23\left(\frac{\bar\alpha_s}{\omega}\right)^4 + \cdots \quad (5.6)$$

with $\omega = j - 1$. We observe that higher order corrections to (5.6) are large indicating that the series have a finite radius of convergence and its value decreases with N. This is in a qualitative agreement with the fact that the intercept of the N-reggeon states scales at large N as $|j_N - 1| \sim 1/N$ [12]. Writing the $\mathcal{O}(\bar\alpha_s)$-correction to (5.6) as $\gamma_{2n}(\omega) = \varepsilon_n \bar\alpha_s/\omega$ (with $n = 2, 3$), one finds that ε_n verifies the condition $\varepsilon_n \leqslant 2(n-1)$ established in [13]. In addition, $\gamma_{2n}(\omega) < n\gamma_2(\omega/n)$ which means that the anomalous dimension of N-reggeon states is subleading in the multi-color limit as compared with the anomalous dimension of $N/2$ BFKL pomerons [7,8].

For $N =$ odd, similar to the $N = 3$ case, we consider separately the sectors with the Lorentz spin $n_h = 0$ and $n_h = 1$. In the first case, the minimal twist equals $n_{\min} = (N + 1)$ and it corresponds to the pole of the energy at $i\nu_h = N/2$ and $n_h = 0$, or equivalently $h = \bar h = (N + 1)/2$. For instance, at $N = 5$ the expansion of the energy $E_N(h + \epsilon)$ and the anomalous dimension look like

$$E_5(3+\epsilon) = \frac{3}{\epsilon} + \frac{7}{6} - \frac{101}{216}\epsilon - 0.1136\epsilon^2 + \cdots,$$
$$\gamma_6^{(N=5)}(j) = 3\frac{\bar\alpha_s}{\omega} + \frac{7}{2}\left(\frac{\bar\alpha_s}{\omega}\right)^2 - \frac{1}{8}\left(\frac{\bar\alpha_s}{\omega}\right)^3 - 13.032\left(\frac{\bar\alpha_s}{\omega}\right)^4 + \cdots. \quad (5.7)$$

The pole (5.7) is located on the same spectral surface as the physical ground state in this sector, $E_5^{(\mathrm{gr})}(n_h = 0) = 0.12751$. This state has the quantum numbers of the odderon. It has the intercept $1 - \bar\alpha_s E_5^{(\mathrm{br})}$ which is smaller then 1 and its leading twist equals 6.

In the second case, for $N =$ odd and $n_h = 1$, the minimal twist is smaller $n_{\min} = N$. It corresponds to the pole of the energy E_N at $i\nu_h = (N - 1)/2$, or equivalently $h = (N + 1)/2$ and $\bar h = (N - 1)/2$. At $N = 5$ the expansion of the energy $E_N(h + \epsilon)$ looks like

$$E_{5,\mathrm{d}}(3+\epsilon) = \frac{3+\sqrt5}{2\epsilon} + 1.36180 - 0.4349\epsilon - 0.315\epsilon^2 + \cdots,$$
$$\gamma_5^{(N=5)}(j) = \frac{3+\sqrt5}{2}\frac{\bar\alpha_s}{\omega} + 3.56524\left(\frac{\bar\alpha_s}{\omega}\right)^2 + 1.8743\left(\frac{\bar\alpha_s}{\omega}\right)^3 - 11.219\left(\frac{\bar\alpha_s}{\omega}\right)^4 + \cdots. \quad (5.8)$$

As in the $N = 3$ case, this reggeon state is descendant [12], that is its energy equals the energy of the $N = 4$ state with $n_h = 1$. The pole (5.8) belongs to the same spectral surface as the $N = 5$ ground state, $E_5^{(\mathrm{gr})}(n_h = 1) = 0$, which has the quantum numbers of the odderon and intercept equal to unity. The second relation in (5.8) defines the leading, twist-5 anomalous dimension of this state.

6. Conclusions

In this Letter, we performed the OPE analysis of the contribution of N-reggeon compound states to the moments of the structure function $\tilde F(j, Q^2)$ for $j \to 1$ and calculated the anomalous dimensions of the leading twist contribution in multi-color QCD. To this end, we analytically continued the energy of the N-reggeon states E_N from the "physical" region of parameters (real ν_h-axis) to complex ν_h and established the relation between the anomalous dimensions for $j \to 1$ and the Laurent expansion of E_N around its poles. At $N = 2$ the energy is a meromorphic function on the complex ν_h-plane [4] while for $N \geqslant 3$ analytical properties of the energy E_N are changed dramatically. Namely, we found that, in agreement with previous findings [7,8], the energy E_N is a multi-valued function on the complex ν_h-plane. The reason for this is that different energy levels in the spectrum of the N-reggeon states collide after analytical continuation to complex ν_h and, as a consequence, their energies develop square-root cuts.

Due to nonvanishing contribution of the cuts, each energy level (the sheet of the spectral surface) breaks the twist expansion of $\tilde{F}(j, Q^2)$, but it is restored in the sum over all N-reggeon states.

To summarize our main results, we found that the leading contribution of the N-reggeon states to the moments (1.1) has the twist N. For even and odd N it comes from the reggeon states with the two-dimensional angular momentum $n_h = 0$ and $n_h = 1$, respectively. For $N = 3, 4, 5, 6$ the corresponding anomalous dimensions $\gamma_N^{(N)}$ are given by Eqs. (5.2), (5.6), (5.8). The explicit form of the underlying twist-N operators need to be found. We demonstrated that two solutions for the odderon states with the intercepts $j_{\mathbb{O}} = 1$ and $j_{\mathbb{O}} < 1$, recently discussed in the literature, have the twist 3 and 4, respectively. Their anomalous dimensions are defined in Eqs. (5.2) and (5.4). Notice that one-loop corrections to the twist-N anomalous dimensions, Eqs. (5.6)–(5.8), have a rather simple form. It would be interesting to compare these expressions with those obtained by explicit calculation of the twist-N anomalous dimensions in the double-logarithmic approximation (see, e.g., [13]).

Finally, let us comment on the relation of our results to those obtained in Ref. [5]. In that paper, the anomalous dimension of the $N = 3$ and $N = 4$ reggeon states have been calculated in the multi-color limit and the expressions obtained there differ from Eqs. (5.2) and (5.6). The reason for disagreement is the following. The approach proposed in [5] is based on the assumption that for $n_h \neq 0$ the energy of the N-reggeon states as a function of conformal spins $h = (1 + n_h)/2$ and $\bar{h} = (1 - n_h)/2$ can be obtained by analytical continuation in h from the energy in the sector with the Lorentz spin $n_h = 0$. As follows from our analysis, this assumption is erroneous. For $N \geqslant 3$ the energy E_N is a multi-valued function of conformal spin (see Fig. 1) and it can only be analytically continued over the spectral surface described in Section 4. The states with different spins n_h belong to different *disconnected* components of this surface so that their energies are not related to each other by analytical continuation. If one ignored nontrivial analytical properties of the energy and applied the approach of Ref. [5], one would generate spurious states which do not belong to the physical spectrum of the reggeon states.

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