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Asymptotic Stability and Generalized Gelfand Spectral Radius Formula

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ABSTRACT

Let Σ be a set of $n \times n$ complex matrices. For $m = 1, 2, \dots,$ let Σ^m be the set of all products of matrices in Σ of length m. Denote by Σ' the multiplicative semigroup generated by Σ. Σ is said to be *asymptotically stable* (in the sense of dynamical systems) if there is $0 < \alpha < 1$ such that there are bounded neighborhoods $U, V \subset \mathbf{C}^n$ of the origin for which $AV \subset \alpha^m U$ for all $A \in \Sigma^m$, $m = 1, 2, \dots$. For a bounded set Σ of $n \times n$ complex matrices, it is shown that the following conditions are mutually equivalent:

(i) Σ is asymptotically stable; (ii) $\hat{\rho}(\Sigma) = \limsup_{m \to \infty}[\sup_{A \in \Sigma^m} \|A\|]^{1/m} < 1$; (iii) $\rho(\Sigma) = \limsup_{m \to \infty}[\sup_{A \in \Sigma^m} \rho(A)]^{1/m} < 1$, where $\rho(A)$ stands for the spectral radius of A; and (iv) there exists a positive number α such that $\rho(A) \leqslant \alpha < 1$ for all $A \in \Sigma'$. This fact answers an open question raised by Brayton and Tong. The generalized Gelfand spectral radius formula, that is, $\rho(\Sigma) = \hat{\rho}(\Sigma)$, conjectured by Daubechies and Lagarias and proved by Berger and Wang using advanced tools from ring theory and then by Elsner using analytic-geometric tools, follows immediately form the above asymptotic stability theorem. © *Elsevier Science Inc., 1997*

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1. INTRODUCTION

Let $\mathbf{C}^{n\times n}$ be the vector space of all $n \times n$ complex matrices and Σ denote a bounded set in $\mathbf{C}^{n\times n}$. For $m = 1, 2, \ldots,$ let Σ^m be the set of all products of matrices in Σ of length m, that is,

$$\Sigma^m = \{A_1 A_2 \ldots A_m : A_i \in \Sigma,\, i = 1, \ldots, m\}.$$

For $A \in \mathbf{C}^{n\times n}$, $\rho(A)$ stands for the spectral radius of A and $\|A\|$ the operator norm of A associated with a vector norm $\|x\|$ on $\mathbf{C}^n$. The generalized spectral radius of Σ, $\rho(\Sigma)$, is defined by

$$\rho(\Sigma) = \limsup_{m\to\infty}\left[\sup_{A\in\Sigma^m} \rho(A)\right]^{1/m}.$$

We remark here that $\rho(\Sigma)$ can be written also in the following form:

$$\rho(\Sigma) = \sup_{m\geqslant 1}\left[\sup_{A\in\Sigma^m} \rho(A)\right]^{1/m}.$$

The joint spectral radius of Σ, $\hat{\rho}(\Sigma)$, is defined by

$$\hat{\rho}(\Sigma) = \limsup_{m\to\infty}\left[\sup_{A\in\Sigma^m} \|A\|\right]^{1/m}.$$

The notion of joint spectral radius was introduced by Rota and Strang [1] and that of generalized spectral radius by Daubechies and Lagarias [2]. In studying the smoothness properties of compactly supported wavelets and solutions of two-scale dilation equations, Daubechies and Lagarias [2] conjectured that $\rho(\Sigma) = \hat{\rho}(\Sigma)$ if Σ is finite. This conjecture was proved by Berger and Wang [3] even if Σ is bounded. Berger–Wang's proof makes use of advanced tools from ring theory. Recently, however, Elsner [8] gave an analytic-geometric proof.

GENERALIZED GELFAND SPECTRAL RADIUS FORMULA. If Σ is a bounded set in $\mathbf{C}^{n\times n}$, then $\rho(\Sigma) = \hat{\rho}(\Sigma)$.

For $A \in \mathbf{C}^{n\times n}$, the Gelfand spectral radius formula asserts that $\rho(A) = \lim_{m\to\infty}\|A^m\|^{1/m}$; since $\rho(A) = [\rho(A^m)]^{1/m}$ for $m = 1, 2, \ldots,$ the above theorem may be called generalized Gelfand spectral radius formula.

The purpose of this paper is to prove an asymptotic stability theorem for matrices from which the generalized Gelfand spectral radius formula follows. Our asymptotic stability theorem answers an open question raised by Brayton and Tong [4, p. 1122].

2. AN ASYMPTOTIC STABILITY THEOREM

For a set Σ in $\mathbf{C}^{n\times n}$, let Σ' denote the multiplicative semigroup generated by Σ, that is, $\Sigma' = \bigcup_{m=1}^{\infty}\Sigma^m$. According to Brayton and Tong [4, 5], a set Σ of $n \times n$ complex matrices is said to be *asymptotically stable* (in the sense of dynamical systems) if there is $0 < \alpha < 1$ such that there are bounded neighborhoods $U, V \subset \mathbf{C}^n$ of the origin for which $AV \subset \alpha^m U$ for all $A \in \Sigma^m$, $m = 1, 2, \dots$. By a standard construction of the Minkowski functional form an absolutely convex open set, it is not hard to see that Σ is asymptotically stable if and only if there exists a norm $\|\cdot\|$ on $\mathbf{C}^n$ and a positive number α such that $\|A\| \leqslant \alpha < 1$ for all $A \in \Sigma$. (See also the remark after Lemma 1.) The novel central result of this article is the following asymptotic stability theorem.

THEOREM 1. *Let Σ be a bounded set in $\mathbf{C}^{n\times n}$. Then the following conditions are mutually equivalent.*

(i) *Σ is asymptotically stable; that is, there exists a norm $\|\cdot\|$ on $\mathbf{C}^n$ and a positive number α such that $\|A\| \leqslant \alpha < 1$ for all $A \in \Sigma$.*

(ii) $\hat{\rho}(\Sigma) < 1$.

(iii) $\rho(\Sigma) < 1$.

(iv) *There exists a positive number α such that $\rho(A) \leqslant \alpha < 1$ for all $A \in \Sigma'$.*

The equivalence of the two statements (i) and (iv) of Theorem 1 is a strengthening of a theorem of Brayton and Tong [4]. However, our proof of Theorem 1 makes use of Brayton–Tong's theorem. Brayton and Tong [4, p. 1122] conjectured that there is a bounded set Σ in $\mathbf{C}^{n\times n}$ such that $\rho(A) \leqslant \alpha < 1$ for all $A \in \Sigma'$, but Σ' is unbounded. Theorem 1 disproves this conjecture. We remark here that there is a set Σ in $\mathbf{C}^{n\times n}$ such that $\rho(A) \leqslant \alpha < 1$ for all $A \in \Sigma'$, but Σ is unbounded. For instance,

$$\Sigma = \left\{ \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} : n = 1, 2, \dots \right\}.$$

The following lemmas will be needed in the proof of Theorem 1.

LEMMA 1. *Let Σ be a bounded set in $\mathbf{C}^{n\times n}$. Then for any $\varepsilon > 0$ there exists a norm $\|\cdot\|$ on $\mathbf{C}^n$ such that*

$$\|A\| \leqslant \hat{\rho}(\Sigma) + \varepsilon \qquad \textit{for all } A \in \Sigma.$$

Proof. This result is due to Rota and Strang [1]. We provide here a much shorter proof. Let $\varepsilon > 0$ be given. Choose N so that

$$\sup_{A\in\Sigma^N} \|A\| \leqslant \alpha^N, \qquad \text{where } \alpha = \hat{\rho}(\Sigma) + \varepsilon.$$

Define

$$|||\, x \,||| = \|x\| + \frac{1}{\alpha} \sup_{B\in\Sigma^1} \|Bx\| + \cdots + \frac{1}{\alpha^{N-1}} \sup_{B\in\Sigma^{N-1}} \|Bx\|, \qquad x \in \mathbf{C}^n.$$

Then for any $A \in \Sigma$, we have

$$|||\, Ax \,||| \leqslant \alpha \,|||\, x \,||| \qquad \text{for all } x \in \mathbf{C}^n,$$

so that $|||\, A \,||| \leqslant \alpha = \hat{\rho}(\Sigma) + \varepsilon$ for all $A \in \Sigma$.

Remark. Lemma 1 can be used to give a useful formulation of the notion of asymptotic stability. For a set Σ in $\mathbf{C}^{n\times n}$, Σ is asymptotically stable if and only if there exists a positive number α such that $\|A\| \leqslant \alpha < 1$ for all $A \in \Sigma$. To see this, it is clear that if such a norm exists then Σ is asymptotically stable. To prove the converse, let $0 < \alpha < 1$ and U, V be bounded neighborhoods of the origin for which $AV \subset \alpha^m U$ for all $A \in \Sigma^m$, $m = 1, 2, \dots$. Choose $r > 0$ so that $rU \subset V$. Then

$$AU \subset r^{-1}\alpha^m U \quad \text{for all } A \in \Sigma^m, \qquad m = 1, 2, \dots .$$

Let $\|\cdot\|$ be a norm on $\mathbf{C}^n$ and let

$$W = \{x \in \mathbf{C}^n : \|x\| \leqslant 1\}.$$

Choose $t > 1$ so that

$$t^{-1}W \subset U \subset tW.$$

Then for each $m = 1, 2, \dots,$

$$AW = tA(t^{-1}W) \subset tAU \subset tr^{-1}\alpha^m U \subset t^2 r^{-1}\alpha^m W \qquad \text{for all } A \in \Sigma^m.$$

Hence

$$\left[\sup_{A \in \Sigma^m} \|A\|\right]^{1/m} \leqslant (t^2 r^{-1})^{1/m} \alpha, \qquad m = 1, 2, \dots,$$

so that $\hat{\rho}(\Sigma) < 1$. The conclusion now follows from Lemma 1.

LEMMA 2 (Brayton-Tong). *Let Σ be a set in $\mathbf{C}^{n \times n}$. Then Σ is asymptotically stable if and only if* (i) *Σ' is bounded and* (ii) *there exists a positive number α such that $\rho(A) \leqslant \alpha < 1$ for all $A \in \Sigma'$.*

Brayton–Tong's proof of Lemma 2 is of analytic-geometric nature. A purely analytic proof of Lemma 2 based on an explicit construction may be found in our recent article [7].

LEMMA 3 [8]. *Let Σ be a bounded set in $\mathbf{C}^{n \times n}$ and $\hat{\rho}(\Sigma) = 1$. If Σ' is unbounded, then there is a nonsingular S and $1 \leqslant n_1 < n$ such that for all $A \in \Sigma$,*

$$S^{-1}AS = \begin{pmatrix} A_{(2)} & * \\ 0 & A_{(1)} \end{pmatrix},$$

where $A_{(1)} \in \mathbf{C}^{n_1 \times n_1}$.

Proof. For $m = 1, 2, \dots,$ let $\alpha_m = 1 + 1/m$. Then $((1/a_m)\Sigma)'$ is bounded for each $m = 1, 2, \dots$. Thus for each $m = 1, 2, \dots,$ a standard result asserts that there exists a norm $\|\cdot\|_{(m)}$ on $\mathbf{C}^n$ such that $\|A\|_{(m)} \leqslant 1$ for all $A \in (1/a_m)\Sigma$, which we normalize by

$$\max_{x \neq 0} \frac{\|x\|_{(m)}}{\|x\|_2} = 1,$$

where, as usual, $\|\cdot\|_2$ is the Euclidean norm. As the family of functions $\{\|\cdot\|_{(m)}\}$ is pointwise bounded and equicontinuous on the compact set $\{x \in \mathbf{C}^n : \|x\|_2 \leqslant 1\}$, Arzela–Ascoli's theorem ensures that there exists a subsequence

$$\|x\|_{(m_i)} \to \|x\| \quad \text{as } i \to \infty, \qquad x \in \mathbf{C}^n.$$

The limit function $\|\cdot\|$ is a seminorm satisfying

$$\|Ax\| \leqslant \|x\| \quad \text{for all } A \in \Sigma \qquad \text{and} \qquad x \in \mathbf{C}^n.$$

By normalization for each m there is x_m such that $\|x_m\|_2 = \|x_m\|_{(m)} = 1$. We may assume that x_m converges to x. Then it is clear that $\|x\| = \|x\|_2 = 1$. Let

$$V = \{x \in \mathbf{C}^n : \|x\| = 0\}.$$

Let us show that V is a proper subspace of $\mathbf{C}^n$. Since there is x with $\|x\| \neq 0$, V does not coincide with $\mathbf{C}^n$. If $V = \{0\}$, then $\|\cdot\|$ becomes a norm on $\mathbf{C}^n$ such that $\|A\| \leqslant 1$ for all $A \in \Sigma'$, which contradicts unboundedness of Σ'. Therefore $V \neq \{0\}$. Thus V is a proper subspace of $\mathbf{C}^n$ and hence $\dim V = n - n_1 > 0$. With a suitable basis all $A \in \Sigma$ are reduced to the desired form.

We proceed now to prove Theorem 1.

(i) $\Rightarrow$ (ii) is immediate. For the norm $\|\cdot\|$ on $\mathbf{C}^{n \times n}$ is equivalent to the usual operator norm and

$$\|A\| \leqslant \alpha^m \qquad \text{for all } A \in \Sigma^m,$$

so that $\hat{\rho}(\Sigma) \leqslant \alpha < 1$.

(ii) $\Rightarrow$ (i) follows from Lemma 1.

(ii) $\Rightarrow$ (iii) is immediate because $\rho(\Sigma) \leqslant \hat{\rho}(\Sigma)$.

(iii) $\Rightarrow$ (iv) is immediate because $\rho(A) \leqslant \rho(\Sigma)^m \leqslant \rho(\Sigma) < 1$ for all $A \in \Sigma^m$, $m = 1, 2, \ldots$.

(iv) $\Rightarrow$ (ii). We proceed by induction on n, dimension. The assertion is true for $n = 1$. Assume the assertion is true for $n - 1$.

Let $\mathbf{B} = \Sigma / \hat{\rho}(\Sigma)$. Then $\hat{\rho}(\mathbf{B}) = 1$. First suppose, by contradiction, that $\hat{\rho}(\Sigma) \geqslant 1$. Then $\rho(B) \leqslant \alpha < 1$ for all $B \in \mathbf{B}'$. If $\mathbf{B}'$ is bounded, by Lemma

2, $\mathbf{B}$ is asymptotically stable so that $\hat{\rho}(\mathbf{B}) < 1$ by implication (i) $\Rightarrow$ (ii), a contradiction. Therefore $\mathbf{B}$ is unbounded. Then by Lemma 3 there exists a nonsingular S and $1 \leqslant n_1 < n$ such that

$$S^{-1}BS = \begin{pmatrix} B_{(2)} & * \\ 0 & B_{(1)} \end{pmatrix},$$

for all $B \in \mathbf{B}$, where $B_{(1)} \in \mathbf{C}^{n_1 \times n_1}$. Set

$$\mathbf{B}_{(i)} = \{B_{(i)} : B \in \mathbf{B}\}, \qquad i = 1, 2.$$

Then $\mathbf{B}_{(i)}$ is bounded for $i = 1, 2$ as $\mathbf{B}$ is, and

$$\rho(M) \leqslant \alpha < 1 \quad \text{for all } M \in \mathbf{B}'_{(i)} \qquad \text{for } i = 1, 2.$$

As the $B_{(i)}$ have dimensions less than n, by induction assumption

$$\hat{\rho}(\mathbf{B}_{(i)}) < 1 \qquad \text{for } i = 1, 2.$$

Therefore

$$\hat{\rho}(\mathbf{B}) = \max\{\hat{\rho}(\mathbf{B}_{(1)}), \hat{\rho}(\mathbf{B}_{(2)})\} < 1,$$

contradicting $\hat{\rho}(\mathbf{B}) = 1$. This contradiction shows that $\hat{\rho}(\Sigma) < 1$, completing the proof.

3. PROOF OF GENERALIZED GELFAND SPECTRAL RADIUS FORMULA

The proof of generalized Gelfand spectral radius formula now follows: Theorem 1 shows that for a bounded set Σ in $\mathbf{C}^{n \times n}$,

$$\rho(\Sigma) < \gamma \Leftrightarrow \hat{\rho}(\Sigma) < \gamma.$$

Then $\rho(\Sigma) = \hat{\rho}(\Sigma)$ is immediate.

4. FINITE CASE OF THEOREM 1

We note that condition (iv) of Theorem 1 cannot be weakened to be $\rho(A) < 1$ for all $A \in \Sigma'$, as the following simple example illustrates. Let $\Sigma = \{A_1, A_2, \dots\}$, where

$$A_k = \begin{pmatrix} 1 - \frac{1}{k} & 0 \\ 0 & 0 \end{pmatrix}, \qquad k = 1, 2, \dots .$$

When Σ is finite, we formulate the following:

Conjecture 1. If Σ is finite set in $\mathbf{C}^{n \times n}$ and $\rho(A) < 1$ for all $A \in \Sigma'$, then Σ is asymptotically stable.

Conjecture 1 is significant, since it is closely related to another conjecture of Daubechies and Lagarias [2].

Daubechies–Lagarias' conjecture. If Σ is a finite set in $\mathbf{C}^{n \times n}$, then there exists a positive integer k such that

$$\rho(\Sigma) = \left[\max_{A \in \Sigma^k} \rho(A) \right]^{1/k}.$$

Daubechies–Lagarias' conjecture remains unsolved. Recently, this conjecture is studied extensively by Lagarias and Wang [3]. We now show that Conjecture 1 is equivalent to Daubechies–Lagarias' (D–L) conjecture.

Proof of equivalence of conjecture 1 and D–L conjecture. Let Σ be a finite set in $\mathbf{C}^{n \times n}$. For notational simplicity, we write

$$\rho_k(\Sigma) = \sup_{A \in \Sigma^k} \rho(A), \qquad k = 1, 2, \dots .$$

First remark that since

$$\rho(\Sigma) = \sup_{k \geqslant 1} \rho_k(\Sigma)^{1/k},$$

D-L conjecture is true if $\rho(\Sigma) = 0$. Therefore we may assume that $\rho(\Sigma) > 0$. Since for each $\lambda > 0$

$$\rho_k(\lambda\Sigma)^{1/k} = \lambda \rho_k(\Sigma)^{1/k}, \qquad k = 1, 2, \dots$$

and

$$\rho(\lambda\Sigma) = \lambda\rho(\Sigma),$$

considering $\lambda\Sigma$ if necessary it is seen that D-L conjecture means that $\rho_k(\Sigma)^{1/k} < 1$ for $k = 1, 2, \dots$ implies $\rho(\Sigma) < 1$. On the other hand, by Theorem 1, Conjecture 1 means that $\rho(A) < 1$ for all $A \in \Sigma'$ implies $\rho(\Sigma) < 1$. Now the equivalence in question follows from the observation that $\rho_k(\Sigma)^{1/k} < 1$ for $k = 1, 2, \dots$ if and only if $\rho(A) < 1$ for all $A \in \Sigma'$.

5. INFINITE DIMENSIONAL PROBLEMS

Theorem 1 and the generalized Gelfand spectral radius formula do not hold, in general, in infinite-dimensional spaces, as the following example illustrates.

EXAMPLE. Let H be a Hilbert space with an orthonormal basis $\{e_1, e_2, \dots\}$. Define a family of bounded linear operators $\Sigma = \{A_1, A_2, \dots\}$ on H as follows:

$$A_k e_k = e_{k+1}, \qquad k = 1, 2, \dots, \tag{1}$$

$$A_k e_j = 0, \qquad j \neq k. \tag{2}$$

Then for any $k = 1, 2, \dots,$ and any $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots\}$, we have

$$\left(A_{i_1} A_{i_2} \dots A_{i_k}\right)^2 = 0.$$

Thus $\rho(A) = 0$ for any $A \in \Sigma'$. Clearly, Σ is bounded. Since

$$A_m A_{m-1} \dots A_1 e_1 = e_{m+1} \qquad \text{for } m = 1, 2, \dots,$$

$$\hat{\rho}(\Sigma) \geqslant \limsup_{m \to \infty} \| A_m A_{m-1} \dots A_1 \|^{1/m} = 1,$$

so that Σ is not asymptotically stable. As $\rho(\Sigma) = 0$, $\rho(\Sigma) \neq \hat{\rho}(\Sigma)$.

The questions of whether Theorem 1 and the generalized Gelfand spectral radius formula for families of compact operators on a Banach space are still unsettled.

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