A unified view of ostensibly disparate isoperimetric variational problems

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ABSTRACT

A class of isoperimetric variational calculus problems is solved. It is shown that the said class ties together seemingly dissimilar optimal control problems in economics and physics, to wit, the classical nonrenewable resource extracting model of the firm, the model of the patient worker, and the principle of maximum entropy. In particular, it is shown that a certain family of wealth maximizing nonrenewable resource extracting firms extract at the rate that maximizes the entropy of extraction, and that patient intertemporal utility maximizing individuals do not procrastinate and maximize the entropy of leisure.

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1. Introduction

A class of autonomous and concave isoperimetric variational calculus problems is studied in which the derivative function is absent and for which the integral constraint represents a normalization on the decision function. The solution to this class of problems is shown to be a constant function, dependent on the endpoints of the integral and the normalization (isoperimetric) parameter, but not on the integrand function and its parameters. In the special case in which the isoperimetric parameter is set equal to unity, the solution of the said class is the uniform density function and thus coincides with the solution of the classical maximum entropy problem, the latter being a special case of the class of problems contemplated here.

The class of isoperimetric variational problems under investigation also generates a classical intertemporal economic model in a special case, namely, the [7] nonrenewable resource extracting model of the firm. Moreover, it is shown that nonrenewable resource extracting firms with or without market power that maximize the present discounted value of an autonomous profit flow using a zero discount rate, and for which the cost of extraction is independent of the resource stock, extract at the rate that maximizes the entropy of extraction. A modern rendition of this statement is that patient intertemporal utility maximizing individuals do not procrastinate and maximize the entropy of leisure [see, e.g., [4]]. These results therefore permit an intuitive economic interpretation of the classical maximum entropy principle, and more generally, the aforementioned class of variational problems and its solution function.

2. A general result

The class of isoperimetric calculus of variations problem under consideration is given by

\[ V(\alpha, \beta, t_0, t_1) \stackrel{\text{def}}{=} \max_{x(t)} \left\{ \int_{t_0}^{t_1} F(x(t); \alpha) \, dt \text{ s.t. } \int_{t_0}^{t_1} x(t) \, dt = \beta \right\} , \]  

(1)

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where $V(\cdot)$ is the optimal value function, $\alpha \in \mathbb{R}^k$ is a constant parameter vector influencing the integrand function $F(\cdot)$, and $\beta \in \mathbb{R}$ is the constant isoperimetric constraint parameter. The augmented integrand corresponding to problem (P) is defined as

$$L(x, \lambda_0, \lambda; \alpha) \overset{\text{def}}{=} \lambda_0 F(x; \alpha) - \lambda x,$$

where $\lambda_0$ and $\lambda$ are the conjugate variables. Note that any piecewise smooth function $x(\cdot)$ such that $\int_{t_0}^{t_1} x(t) dt = \beta$ is defined as an admissible function.

In order to provide a straightforward proof of the main result, the following three assumptions are imposed on problem (P):

(A.1) $F(\cdot): X \times P \to \mathbb{R}, X \subset \mathbb{R}, P \subset \mathbb{R}^k$, and $F(\cdot) \in C^2(\mathbb{R})$ on its domain.

(A.2) $F_x(\alpha(t); \alpha) \neq 0$ for at least one nondegenerate interval, $[a, b] \subseteq [t_0, t_1]$, for all $\alpha \in P$ and for all admissible values of the function $x(\cdot)$.

(A.3) $F_x(\alpha(t); \alpha) \leq 0$ for all $t \in [t_0, t_1]$ and for all $\alpha \in P$ over an open convex set containing all the admissible values of the function $x(\cdot)$.

These three assumptions are self-explanatory and therefore do not require comment. With them in place, the central result of the paper can be established.

**Proposition 1.** Under assumptions (A.1)–(A.3), a unique and globally optimal solution to problem (P) exists, denoted by the function $x^*(\cdot)$ with the constant value $x^*(\beta, t_0, t_1) = \frac{\beta}{|t_1-t_0|}$ where $\lambda^*_0 = 1$ and $\lambda^*(\alpha, \beta, t_0, t_1) = F_x \left( \frac{\beta}{t_1-t_0}; \alpha \right)$ are the values of the corresponding conjugate variables, and $V(\alpha, \beta, t_0, t_1) = F \left( \frac{\beta}{t_1-t_0}; \alpha \right) \mid_{t_1-t_0}$ is the value of the optimal value function.

**Proof.** By Theorem 3.2.1 of [6], the necessary conditions for problem (P) are the Euler equation $\frac{\partial}{\partial \alpha} L_\alpha(x, \lambda_0, \lambda; \alpha) = L_\alpha(x, \lambda_0, \lambda; \alpha)$, the requirement that $\lambda_0$ and $\lambda$ do not both vanish, and that $\lambda_0$ and $\lambda$ are constant. Using Eq. (1), the Euler equation reduces to the algebraic equation

$$L_\alpha(x, \lambda_0, \lambda; \alpha) = \lambda_0 F_x(x; \alpha) - \lambda = 0.$$

If $\lambda_0 = 0$, then Eq. (2) implies that $\lambda = 0$ as well, thereby violating the necessary condition that $\lambda_0$ and $\lambda$ do not both vanish. Conversely, if $\lambda = 0$, then Eq. (2) implies that $\lambda_0 F_x(x; \alpha) = 0$. But by assumption (A.2), $F_x(\alpha(t); \alpha) \neq 0$ for $t \in [a, b]$, thereby implying that $\lambda_0 = 0$, which again violates the necessary condition that $\lambda_0$ and $\lambda$ do not both vanish, seeing as $\lambda_0$ is constant. Consequently neither conjugate variable vanishes at the optimum. Without loss of generality it is therefore assumed that $\lambda^*_0 = 1$. As a result, Eq. (2) becomes

$$F_x(x; \alpha) - \lambda = 0.$$

Recalling assumption (A.3), Theorem 7.4 of [3] implies that a solution of $\alpha$, and $\lambda$ is a constant, the solution function of $\alpha \in \mathbb{R}^k$ is a constant parameter vector and $\lambda$ is a constant. Therefore, in the special case in which $\beta = 1$, Proposition 1 demonstrates that the uniform density function is the unique and globally optimal solution to problem (P). But, as is well known [see, e.g., [5] Chapter 3], the uniform density $\beta^{-1}x^*(\beta, t_0, t_1) = \frac{1}{|t_1-t_0|} \in (0, 1)$ is also the unique and globally optimal solution to the classical maximum entropy problem, defined by

$$\max_{x(t)} \left\{ - \int_{t_0}^{t_1} x(t) \ln x(t) dt \ s.t. \ \int_{t_0}^{t_1} x(t) dt = 1 \right\}$$

This conclusion establishes one of the results alluded to in Section 1, namely, that the solution function to an autonomous and concave isoperimetric variational calculus problem in which the derivative function is absent and for which the integral constraint represents a normalization, maximizes the entropy of the distribution of that function.

Second, because $x^*(\beta, t_0, t_1) = \frac{\beta}{|t_1-t_0|}$ was found by solving the isoperimetric constraint and only made use of the necessary and sufficient condition in Eq. (3) to show that it is a constant function, it is independent of the integrand function $F(\cdot)$ and thus any parameters appearing in $F(\cdot)$. In other words, the solution function $x^*(\cdot)$ is invariant to the functional form of the integrand function $F(\cdot)$. This feature of the solution to problem (P) will be given an economic interpretation in Section 3.
Finally, observe that upon differentiating $V(\cdot)$ with respect to $\beta$ one obtains the result $V_\beta(\alpha, \beta, t_0, t_1) = F_x \left( \frac{\beta}{t_1-t_0} ; \alpha \right)$, which, because of $\lambda^*(\alpha, \beta, t_0, t_1) = F_x \left( \frac{\beta}{t_1-t_0} ; \alpha \right)$ and transitivity, yields $V_\beta(\alpha, \beta, t_0, t_1) = \lambda^*(\alpha, \beta, t_0, t_1)$, exactly as asserted by Theorem 7.3 of [3], a dynamic envelope result. With the central result established, the next two sections present the economic interpretations of Proposition 1 noted in Section 1.

3. The nonrenewable resource extracting firm

Define $C(\cdot)$ as the cost function of a nonrenewable resource extracting firm, that is,

$$C(w, q) \overset{\text{def}}{=} \min_v \{w \cdot v \text{ s.t. } q = f(v)\}, \quad (5)$$

where $f(\cdot)$ is a production function, $v \in \mathbb{R}^n_+$ is the vector of variable inputs used in extracting the natural resource and purchased at the market determined and time invariant prices $w \in \mathbb{R}^m_+$, and $q > 0$ is the extraction rate of the firm. The profit flow of the firm can then be defined as $\pi(q; p, w) \overset{\text{def}}{=} pq - C(w, q)$, where $p > 0$ is the market determined and time invariant price of the extracted resource. It is assumed that suppositions (A.1)–(A.3) hold for the function $\pi(\cdot)$.

Letting the initial stock of the nonrenewable resource be normalized at unity, thereby implying that the extraction rate is the proportion of the resource stock extracted, the classical model of the nonrenewable resource extracting firm without discounting is

$$\max_{q(\cdot)} \left\{ \int_{t_0}^{t_1} \pi(q(t); p, w) \, dt \text{ s.t. } \int_{t_0}^{t_1} q(t) \, dt = 1 \right\}. \quad (6)$$

This isoperimetric problem is formally identical to problem (P) when $\beta = 1$, as can be seen by identifying $\alpha$ with $(p, w)$, $F(\cdot)$ with $\pi(\cdot)$, and $x(\cdot)$ with $q(\cdot)$. Thus, by Proposition 1, the unique and globally optimal extraction rate in problem (6) is given by $q^* (t_0, t_1) = \frac{1}{t_1-t_0} \in (0, 1)$, which is constant. Given that this is the uniform density function, it is therefore also the solution to the classical maximum entropy problem (4), thereby establishing the economic result that wealth-maximizing, price-taking, nonrenewable resource extracting firms with an autonomous and concave profit flow, a zero discount rate, and extraction costs that are independent of the resource stock, extract at the rate that maximizes the entropy of extraction.

Note that problem (6) is stated for a nonrenewable resource extracting firm that takes input and output prices as given, i.e., for a firm that faces perfectly competitive input and output markets. It will now be shown that the conclusion in the preceding paragraph holds for more general market structures. This will be accomplished by showing that the solution to problem (6) is unaffected by the structure of the input or output markets. Key to this conclusion is the aforementioned fact that the solution function $x^*(\cdot)$ to problem (P) is independent of the integrand function $F(\cdot)$ and therefore the functional form of $F(\cdot)$.

If the firm in question is a monopsonist, then $w$ becomes a function of $v$. This implies that the cost function $C(\cdot)$, and in turn, the profit flow function $\pi(\cdot)$, are not functions of $w$. These facts, however, do not alter the mathematical properties of $C(\cdot)$ and $\pi(\cdot)$ with respect to $q$. Seeing as the solution of problem (P) is not a function of the integrand function, it follows that the solution to the monopsony version of the nonrenewable resource extraction problem (6) is given by $q^* (t_0, t_1) = \frac{1}{t_1-t_0}$ as well.

Similarly, if the firm is a monopolist, then $p$ would be a function of $q$ and the profit flow $\pi(\cdot)$ would not be a function of $p$. As long as $\pi(\cdot)$ still satisfied assumptions (A.1)–(A.3), the mathematical properties of $\pi(\cdot)$ with respect to $q$ would be unchanged. Hence, as above, and for the same reason, $q^* (t_0, t_1) = \frac{1}{t_1-t_0}$ is also the solution to the monopoly version of the nonrenewable resource extraction problem (6).

The important assumptions for reaching the above conclusions are (i) the firm does not discount, (ii) profit flow is autonomous, thereby ruling out technical change and time varying prices, and (iii) the extraction cost function is independent of the resource stock. Any violation of these three assumptions would break the link between the solution of the nonrenewable resource extraction problem (6), the maximum entropy problem (4), and the general isoperimetric problem (P), not the above changes in the market structure.

4. The patient worker

In this section a modern economic interpretation of problem (P) and Proposition 1 is given, to wit, that of an individual contemplating an intertemporal utility maximizing time path of work (or effort) $w(t)$ on a task, say a research paper, which requires a known number of hours of work $\beta \in \mathbb{R}_+$ to complete, and which must be completed by a given date $T > \beta, T \in \mathbb{R}_+$. Such considerations imply the integral constraint $\int_{t_0}^{t_1} w(t) \, dt = \beta$. Letting $U(\cdot) \in C^2$ be the strictly increasing and concave instantaneous utility function of leisure $\ell(t)$, and noting that $w(t) + \ell(t) = 1$ for all $t \in [0, T]$, the
value of \( U(\cdot) \) at any point in time is \( U(1 - w(t)) \). Assuming the individual is patient, so that the discount rate is zero, and then putting all of this information together, the isoperimetric problem to be solved by the individual is given by

\[
\max_{w(\cdot)} \left\{ \int_0^T U(1 - w(t)) \, dt \right\} \quad \text{s.t.} \quad \int_0^T w(t) \, dt = \beta.
\] (7)

The definition of procrastination used here is that the task is completed, as is indicated by the equality isoperimetric constraint, but that the proportion of one’s time spent working on the task increases as the deadline \( T \) approaches. Note that the aforementioned assumptions on \( U(\cdot) \) imply that it satisfies suppositions (A.1)–(A.3).

By Proposition 1, the unique solution to problem (7) is given by the constant \( w^*(\beta, T) = \frac{\beta}{T} \in (0, 1) \), thereby implying that the proportion of time spent working on the task is the same in each period. That is to say, the worker is not a procrastinator. Normalizing the amount of required effort by setting \( \beta = 1 \), it is seen that the resulting time path of work is the uniform density function, thereby demonstrating that patient intertemporal utility maximizing individuals do not procrastinate and maximize the entropy of leisure.

5. Conclusion

The closing remark amounts to an observation that enlarges the class of isoperimetric variational problems covered by Proposition 1 twofold, thereby further highlighting the rather ubiquitous nature of the class of isoperimetric problems studied here. In particular, if the conjugate variable of problem (P) is positive, then Theorem 2.1 of [1,2] shows that the solution of the reciprocal isoperimetric problem

\[
\min_{x(\cdot)} \left\{ \int_{t_0}^{t_1} x(t) \, dt \right\} \quad \text{s.t.} \quad \int_{t_0}^{t_1} F(x(t); \alpha) \, dt = V(\alpha, \beta, t_0, t_1),
\] (R)

is identical to the solution of the primal isoperimetric problem (P), namely, the uniform density function, and thus is the entropy maximizing solution.

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