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NOTE

ON BALANCED CLAW DESIGNS OF COMPLETE MULTI-PARTITE GRAPHS

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In this paper, it is shown that a necessary and sufficient condition for the existence of a balanced claw design BCD(m, n, c, λ) of a complete m-partite graph $\lambda K_m(n, n, \ldots, n)$ is $\lambda(m-1)n \equiv 0 \pmod{2c}$ and $(m-1)n \geq c$.

1. Introduction

A *complete m-partite graph* $\lambda K_m(n, n, \ldots, n)$ $(m \geq 2)$ is a *multigraph* with m independent sets of n points each, such that any two points in different subsets are joined exactly λ lines. A *complete graph* λK_m with m points may be regarded as a particular type of complete m-partite multigraph where $n = 1$. A *complete bipartite graph* $K(1, c)$ is called a *c-claw* $(c \geq 2)$. A point of degree c is called a *root* and each point of degree one is called a *leaf* of the c-claw.

A *claw design* CD(m, n, c, λ) is a line-disjoint decomposition of $\lambda K_m(n, n, \ldots, n)$ into subgraphs each isomorphic to a c-claw. The claw design is said to be *balanced* and is denoted by BCD(m, n, c, λ), if furthermore each point of $\lambda K_m(n, n, \ldots, n)$ belongs to exactly the same number of c-claws.

A necessary and sufficient condition for the existence of a particular type of CD(m, n, c, λ) or BCD(m, n, c, λ) has been obtained, such as, CD$(m, 1, c, 1)$ [5], CD$(m, 1, c, \lambda)$ [2], CD$(m, n, c, 1)$ [3], BCD$(m, 1, c, \lambda)$ [1] and BCD$(m, n, c, 1)$ [4].

2. Balanced Claw Designs

We use the following labeling scheme for $\lambda K_m(n, n, \ldots, n)$. Let the points of $\lambda K_m(n, n, \ldots, n)$ be $v_1, v_2, \ldots, v_{mn}$ and define the *length* of v_i and v_j by $\min\{|i-j|, mn-|i-j|\}$. Let v_i and v_j be *adjacent* if and only if the length of v_i and v_j is not divisible by m and let join v_i and v_j by exactly λ lines. Then the m disjoint independent sets of $\lambda K_m(n, n, \ldots, n)$ with this labeling are

$$V_i = \{v_i, v_{i+m}, v_{i+2m}, \ldots, v_{i+(n-1)m}\}, \quad i = 1, 2, \ldots, m.$$

By the *turning of a c-claw* we mean the simultaneous increasing of all indices of $c+1$ points of the c-claw by 1. The indices are reduced modulo mn to the set of residues $\{1, 2, \ldots, mn\}$.

Theorem. *There exists a* BCD(m, n, c, λ) *if and only if*
(i) $\lambda(m-1)n \equiv 0 \pmod{2c}$, *and*
(ii) $(m-1)n \geq c$.

Proof. (*Necessity*) Suppose that there exists a BCD(m, n, c, λ). Since $c+1$ points of a c-claw are all different, we have $(m-1) \geq c$. Therefore, condition (ii) is necessary. Let b be the number of the total c-claws and let r be the number of c-claws such that each point of $\lambda K_m(n, n, \ldots, n)$ belongs to exactly r c-claws. Then we have obviously

$$\lambda \binom{m}{2} n^2 = bc \quad \text{and} \quad mnr = b(c+1).$$

Therefore, we have

$$b = \lambda m(m-1)n^2/(2c) \quad \text{and} \quad r = \lambda(m-1)n(c+1)/(2c).$$

For a point v, let $r_1(v)$ be the number of c-claws in which v is a root point and let $r_2(v)$ be that of c-claws in which v is a leaf point. Then we have

$$r_1(v)c + r_2(v) = \lambda(m-1)n$$

and, of course,

$$r_1(v) + r_2(v) = r.$$

From these relations we have

$$r_1(v) = \lambda(m-1)n/(2c) \quad \text{and} \quad r_2(v) = \tfrac{1}{2}\lambda(m-1)n.$$

Thus r_1 and r_2 do not depend on the particular point v. Since b, r, r_1, r_2 are all integers, we have

$$\lambda(m-1)n \equiv 0 \pmod{2c}.$$

Therefore, condition (i) is also necessary.

(*Sufficiency*) Proof will be shown by a construction algorithm. For a set of parameters m, n, c, λ satisfying conditions (i) and (ii), we write $\lambda(m-1)n = 2cs$. Arrange $(m-1)n$ points adjacent to a point v_{mn} in increasing order of indices repeatedly λ times, i.e.,

$$v_1, v_2, \ldots, v_{m-1}, v_{m+1}, \ldots, v_{2m-1}, v_{2m+1}, \ldots, v_{nm-1},$$
$$\ldots$$
$$v_1, v_2, \ldots, v_{m-1}, v_{m+1}, \ldots, v_{2m-1}, v_{2m+1}, \ldots, v_{nm-1}.$$

Since $sc = \frac{1}{2}\lambda(m-1)n$, consider s c-claws C_i $(i = 1, 2, \ldots, s)$, where the root point of each C_i is v_{mn} and c leaf point of C_1 are first c points in the sequence and c leaf points of C_2 are next c points in the sequence, and so on. The turnings of these C_i $mn-1$ times yield, with C_i themselves, mns line-disjoint c-claws of $\lambda K_m(n, n, \ldots, n)$. Since $mnsc = \lambda\binom{m}{2}n^2$, we have a BCD$(m, n, c, \lambda)$ with $r_1 = s$ and $r_2 = \frac{1}{2}\lambda(m-1)n$. This completes the proof.

Corollary [1]. *There exists a* BCD$(m, 1, c, \lambda)$ *if and only if*

(i) $\lambda(m-1) \equiv 0 \pmod{2c}$, *and*

(ii) $m-1 \geqslant c$.

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