NOTE.

A DOUBLY DIVISIBLE NEARLY KIRKMAN SYSTEM

Paul SMITH

Department of Mathematics, University of Victoria, Victoria, B.C., Canada

Received 30 January 1976
Revised 3 May 1976

In this note we exhibit an example of a new class of combinatorial array. A Room square [9] is an \( r \times r \) square each of whose cells is either empty or contains a distinct unordered pair with entries from the set of integers \( \{1, 2, \ldots, r+1\} \) such that each integer appears exactly once in each row and exactly once in each column. We present an analogous square whose entries are unordered triples, none of whose two-element subsets is repeated.

By a doubly divisible design (DDD) we mean an \( m \times n \) rectangle each of whose cells is either empty or contains an unordered \( k \)-tuple of distinct positive integers chosen from the set \( S = \{1, 2, \ldots, kn\} \) with the conditions: (1) each integer appears exactly once in each row, (2) no integer appears more than once in each column, (3) no two distinct integers appear together in more than one \( k \)-tuple (block). Such designs have applications in statistics and particularly interesting applications in the construction of schedules for duplicate bridge tournaments. An extensive literature exists concerning DDD's with \( k = 2 \). (Room squares and Howell movements.)

We use the standard notation: for the parameters: \( v = kn \) is the cardinality of \( S \), \( b \) is the number of blocks (non-empty cells), \( r \) is the replication number for each of the \( v \) elements and \( k \) is the number of elements in each block.

If we replace condition (2) above by (2') each element appears exactly once in each column and then add condition (4) \( r = \lfloor (v-1)/(k-1) \rfloor \), we call the design a Wingo square. A Room square is a Wingo square with \( k = 2 \). The only example previously known of such a design with \( k > 2 \) is a duplicate bridge schedule composed by Laura Wingo [2] with parameters \( v = 24 \), \( b = 42 \), \( k = 4 \), \( r = 7 \).

A Kirkman system is a set of \( n(3n-1)/2 \) blocks of size 3 which is partitioned into \( (3n-1)/2 \) "parallel" classes each containing \( n \) blocks. The entries in each block are selected from the integers \( \{1, 2, \ldots, 3n\} \) in such a way that no two distinct elements appear together in two distinct blocks and each class contains all of the \( 3n \) elements. It has been shown [6] that Kirkman systems exist for all odd values of \( n \). It is clear that if we take a square with \( (3n-1)^2/4 \) cells and place the blocks of distinct parallel classes of a Kirkman system in distinct rows in such a way that the columns form a new partition, then the result is a Wingo square. No such construction is known to exist.
But a Wingo square does exist for \( k = 3, n = 8 \). The blocks in any row are chosen from the same parallel class of a nearly Kirkman system (NKS), and the columns yield a different partition into parallel classes. We constructed the square by first building cyclic NKS's on the first 24 integers by difference set methods. (Many such systems exist.) Since a cyclic NKS is completely determined by any one of its parallel classes, we treated the first class as a starter for a DDD. We then applied the appropriate criteria to permute the entries and produce a strong starter. A computer search yielded the strong starter: \( \{1, 12, 23, -2, 10, 18, 7, 12, 22, 6, 15, 19, -5, 9, 11, 13, 14, 16, 3, 4, 21, -4, 8, 9, 24\} \). The full square is shown in Fig. 1.

It may be remarked that in the language of block designs, the blocks of this array form a PBIBD with parameters \( v = 24, b = 88, k = 3, r = 11, \lambda_1 = 1, \lambda_2 = 0 \).

An analysis of the completed design will suggest the method of construction. The set of positive integers less than or equal to 24 was partitioned into three sets \( A = \{1, \ldots, 11\}, B = \{12, \ldots, 22\} \) and \( C = \{23, 24\} \). The assignment of elements to blocks and blocks to cells in the first row completely determines the design, since the elements in sets \( A \) and \( B \) cycle down the right pandiagonals, while the elements in set \( C \) remain constant. In order to find a set \( \{B_i\} \) of eight blocks for the initial row we assumed the form:

\[
B_1 = \{a_0, b_0, 23\}, \quad B_{12} = \{a_1, a_2, a_3\}, \quad B_{13} = \{b_1, b_2, b_3\}, \quad B_{14} = \{a_4, a_5, b_0\},
\]

\[
B_{15} = \{b_4, b_5, a_6\}, \quad B_{16} = \{a_6, a_7, b_9\}, \quad B_{17} = \{b_6, b_7, a_9\}, \quad B_{18} = \{a_{10}, b_{10}, 24\}
\]

where the \( a_i \) and \( b_i \) are distinct elements in \( A \) and \( B \) respectively. (1) For all pairs \( a_i \) and \( a_j \) (resp. \( b_i \), \( b_j \)) appearing together in a block, the ten differences \( a_i - a_j \) (resp. \( b_i - b_j \)) must be distinct mod 11. (2) For all pairs \( a_i, b_i \) appearing together in a block, the ten differences \( b_i - a_i \) must be distinct mod 11. Without loss we set \( a_0 = 11, b_0 = 22 \). It is easily verified that if these conditions are met, the set \( \{B_i\} \), \( i = 1, \ldots, 8, \) for fixed \( i \) comprising a parallel class. Here if \( a_k \) is in \( B_i \), then \( (a_k + i - 2, \text{mod} \ 11) + 1 \) is in \( B_j \) and a similar condition holds for \( b_k \). If \( c_k \) is in \( B_{1j} \), \( c_k \) is in \( B_{1i} \) for all \( i \).

The construction of NKS's was done empirically in two steps using backtrack algorithms. The first step was to find all partitions \( A^\perp \cup B \) of the form

\[
P_i(A) = \{(11), (a_1, a_2, a_3), (a_4, a_5), (a_6, a_7), (a_8, a_9), (a_{10})\}
\]

satisfying condition (1) of the previous paragraph. Corresponding partitions of \( B \) were obtained by setting \( b_i = a_k + 11 \). These partitions were stored.

The second step involved finding all possible sets of blocks satisfying condition (2), each formed by combining a \( P_i(A) \) and a \( P_i(B) \). It is possible to list all these sets, but since the C.P.U. time would be excessive, they were generated one at a time, and a third step algorithm was applied.

The third step, that of obtaining a doubly-divisible NKS from a set of initial blocks, is analogous to methods used for constructing Room squares of order \( 2n - 1 \) from a 1-factorization of the complete graph \( K_{2n} \) [10]. It is sufficient to place
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>16</td>
<td>24</td>
<td>32</td>
<td>40</td>
<td>48</td>
<td>56</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>18</td>
<td>27</td>
<td>36</td>
<td>45</td>
<td>54</td>
<td>63</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
<td>70</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>22</td>
<td>33</td>
<td>44</td>
<td>55</td>
<td>66</td>
<td>77</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>24</td>
<td>36</td>
<td>48</td>
<td>60</td>
<td>72</td>
<td>84</td>
</tr>
</tbody>
</table>

**Fig. 1.**
the eight blocks in the cells $C_i$, $i = 1, \ldots, 11$, of the first row in such a way that the following conditions are satisfied. (3) For each pair $s, t$ in $A$ appearing in cells $C_i$ and $C_j$, respectively, $s - t + i - j \equiv 0 \ mod \ 11$. A similar condition holds for $s, t$ in $B$.

It can easily be seen that this condition guarantees that each entry appears exactly once in each column of the $11 \times 11$ square.

Termination occurred with the discovery of the illustrated design, but it appears likely that many others could be obtained.

References

[8] Paul Smith, PBIB designs and the Kirkman problem, manuscript.