The weak type inequality for the maximal operator of the Marcinkiewicz–Fejér means of the two-dimensional Walsh–Fourier series

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Abstract

The main aim of this paper is to prove that the maximal operator $\sigma^*$ of the Marcinkiewicz–Fejér means of the two-dimensional Walsh–Fourier series is bounded from the Hardy space $H_{2/3}$ to the space weak-$L_{2/3}$.

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1. Introduction

The first result with respect to the a.e. convergence of the Walsh–Fejér means $\sigma_n f$ is due to Fine [2]. Later, Schipp [9] showed that the maximal operator $\sigma^* f$ is of weak type $(1, 1)$, from which the a. e. convergence follows by standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p (1 < p \leq \infty)$. This fails to hold for $p = 1$ but Fujii [3] proved that $\sigma^*$ is bounded from the dyadic Hardy space $H_1$ to the space $L_1$ (see also Simon [11]). Fujii's theorem was extended by Weisz [13]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh–Fourier series is bounded from the martingale Hardy space $H_p (G)$ to the space $L_p (G)$ for $p > 1/2$. Simon [12] gave a
counterexample, which shows that this boundedness does not hold for \( 0 < p < 1/2 \). In the endpoint case \( p = 1/2 \) Weisz [16] proved that \( \sigma^* \) is bounded from the Hardy space \( H_{1/2}(G) \) to the space weak-\( L_{1/2}(G) \).

For the two-dimensional Walsh–Fourier series Weisz [14] proved that the maximal operator

\[
\sigma^*f = \sup_{n \in \mathbf{P}} \frac{1}{n} \sum_{j=0}^{n-1} S_{j,j}(f)
\]

is bounded from the two-dimensional dyadic martingale Hardy space \( H_p(G \times G) \) to the space \( L_p(G \times G) \) for \( p > 2/3 \). In [7] the author proved that in the theorem of Weisz the assumption \( p > 2/3 \) is essential, in particular we showed that the maximal operator \( \sigma^* \) is not bounded from the Hardy space \( H_{2/3}(G \times G) \) to the space \( L_{2/3}(G \times G) \). By interpolation it follows that \( \sigma^*f \) is not bounded from the Hardy space \( H_p(G \times G) \) to the space weak-\( L_p(G \times G) \) for \( 0 < p < 2/3 \).

The main aim of this paper is to prove that in the endpoint case \( p = 2/3 \) the maximal operator of the Marcinkiewicz–Fejér means of the double Walsh–Fourier series is bounded from the dyadic Hardy space \( H_{2/3}(G \times G) \) to the space weak-\( L_{2/3}(G \times G) \).

2. Definitions and notation

Let \( \mathbf{P} \) denote the set of positive integers, \( \mathbb{N} := \mathbf{P} \cup \{0\} \). Denote \( Z_2 \) the discrete cyclic group of order 2, that is \( Z_2 = \{0, 1\} \), where the group operation is the modulo 2 addition and every subset is open. The Haar measure on \( Z_2 \) is given such that the measure of a singleton is 1/2. Let \( G \) be the complete direct product of the countable infinite copies of the compact groups \( Z_2 \). The elements of \( G \) are of the form \( x = (x_0, x_1, \ldots, x_k, \ldots) \) with \( x_k \in \{0, 1\} (k \in \mathbb{N}) \). The group operation on \( G \) is the coordinate-wise addition, the measure (denote by \( \mu \)) and the topology are the product measure and topology. The compact Abelian group \( G \) is called the Walsh group. A base for the neighborhoods of \( G \) can be given in the following way:

\[
I_0(x) := G, \quad I_n(x) := I_n(x_0, \ldots, x_{n-1}) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}, \quad (x \in G, n \in \mathbb{N}).
\]

These sets are called the dyadic intervals. Let \( 0 = (0 : i \in \mathbb{N}) \in G \) denote the null element of \( G \), \( I_n := I_n(0) \) \((n \in \mathbb{N}) \). Set \( e_n := (0, \ldots, 0, 1, 0, \ldots) \in G \) the \( n \)th coordinate of which is 1 and the rest are zeros \((n \in \mathbb{N}) \). Denote

\[
x_{i,j} := \sum_{s=i}^{j} x_s e_s, \quad x_{i,i-1} = 0,
\]

\[
2^N x := (x_N, x_{N+1}, \ldots),
\]

and \( T_n := G \setminus I_n \).

For \( k \in \mathbb{N} \) and \( x \in G \) denote

\[
r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbb{N}),
\]

the \( k \)th Rademacher function. If \( n \in \mathbb{N} \), then \( n = \sum_{i=0}^{\infty} n_i 2^i \), where \( n_i \in \{0, 1\} (i \in \mathbb{N}) \), i. e. \( n \) is expressed in the number system of base 2. Denote \( |n| := \max\{j \in \mathbb{N} : n_j \neq 0\} \), that is, \( 2^{|n|} \leq n < 2^{|n|+1} \).
The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

\[
\{ w_n(x) : \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \} (x \in G, n \in \mathbb{P}).
\]

The Walsh–Dirichlet kernel is defined by

\[
D_n(x) = \sum_{k=0}^{n-1} w_k(x).
\]

Recall that

\[
D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in I_n. \end{cases}
\] (1)

The one-dimensional Fejér kernel is introduced by

\[
K_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x), \quad x \in G
\]

and it is shown that in Schipp, Wade, Simon and Pál [10]

\[
|K_n(x)| \leq c \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} D_{2^i}(x + e_j), \quad 2^N \leq n < 2^{N+1}
\] (2)

and

\[
|K_{2N}(x)| \leq c \sum_{j=0}^{N} 2^{j-N} D_{2^N}(x + e_j).
\] (3)

The rectangular partial sums of the two-dimensional Walsh–Fourier series are defined as follows:

\[
S_{M,N}(f; x^1, x^2) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \hat{f}(i, j) w_i(x^1) w_j(x^2),
\]

where the number

\[
\hat{f}(i, j) = \int_{G \times G} f(x^1, x^2) w_i(x^1) w_j(x^2) d\mu(x^1, x^2)
\]

is said to be the \((i, j)\)th Walsh–Fourier coefficient of the function \(f\).

The norm (or quasinorm) of the space \(L_p(G \times G)\) is defined by

\[
\|f\|_p := \left( \int_{G \times G} |f(x^1, x^2)|^p d\mu(x^1, x^2) \right)^{1/p} \quad (0 < p < +\infty).
\]

The space \(\text{weak-}L_p(G \times G)\) consists of all measurable functions \(f\) for which

\[
\|f\|_{\text{weak-}L_p(G \times G)} := \sup_{\lambda > 0} \lambda \mu(|f| > \lambda)^{1/p} < +\infty.
\]

The \(\sigma\)-algebra generated by the dyadic two-dimensional \(I_k \times I_k\) cube of measure \(2^{-k} \times 2^{-k}\) will be denoted by \(F_k\) \((k \in \mathbb{N})\).
Denote by \( f = (f^n, n \in \mathbb{N}) \) one-parameter martingale with respect to \((F_n, n \in \mathbb{N})\) (for details see, e.g. [15]). The maximal function of a martingale \( f \) is defined by
\[
 f^* = \sup_{n \in \mathbb{N}} \left| f^n \right|.
\]
In case \( f \in L_1(G \times G) \), the maximal function can also be given by
\[
 f^*(x^1, x^2) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x^1) \times I_n(x^2))} \left| \int_{I_n(x^1) \times I_n(x^2)} f(u^1, u^2) \, d\mu(u^1, u^2) \right|,
\]
\( (x^1, x^2) \in G \times G \).

For \( 0 < p < \infty \) the Hardy martingale space \( H^p(G \times G) \) consists all martingales for which
\[
\| f \|_{H^p} := \left\| f^* \right\|_p < \infty.
\]

If \( f \in L_1(G \times G) \) then it is easy to show that the sequence \((S_{2^n, 2^n}(f) : n \in \mathbb{N})\) is a martingale. If \( f \) is a martingale, that is \( f = (f^{(0)}, f^{(1)}, \ldots) \) then the Walsh–Fourier coefficients must be defined in a little bit different way:
\[
 \hat{f}(i, j) = \lim_{k \to \infty} \int_{G \times G} f^{(k)}(x^1, x^2) w_i(x^1) w_j(x^2) \, d\mu(x^1, x^2).
\]

The Walsh–Fourier coefficients of \( f \in L_1(G \times G) \) are the same as the ones of the martingale \((S_{2^n, 2^n}(f) : n \in \mathbb{N})\) obtained from \( f \).

For \( n = 1, 2, \ldots \) and a martingale \( f \) the Marcinkiewicz–Fejér means of order \( 2^n \) of the two-dimensional Walsh–Fourier series of the function \( f \) is given by
\[
 \sigma_n(f; x^1, x^2) = \frac{1}{n} \sum_{j=0}^{n-1} S_{j, n}(f; x^1, x^2).
\]

For the martingale \( f \) we consider the maximal operator
\[
 \sigma^*_f = \sup_{n \in \mathbb{P}} \left| \sigma_n(f; x^1, x^2) \right|.
\]

The two-dimensional Marcinkiewicz–Fejér kernel of order \( n \) of the two-dimensional Walsh–Fourier series is defined by
\[
 K_n(x^1, x^2) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x^1) D_k(x^2).
\]
Let
\[
 K_{a, b}(x^1, x^2) := \sum_{k=a}^{a+b-1} D_k(x^1) D_k(x^2).
\]

By simple calculations we get
\[
 nK_n = \sum_{s=0}^{\lfloor n \rfloor} n_s K_{n(s+1), 2^n}, \quad n^{(s)} := \sum_{i=s}^{\infty} n_i 2^i \quad (n, s \in \mathbb{N}).
\]
A bounded measurable function $a$ is a $p$-atom, if there exists a dyadic two-dimensional cube $I \times I$, such that
(a) $\int_{I \times I} a \, d\mu = 0$;
(b) $\|a\|_\infty \leq \mu(I \times I)^{-1/p}$;
(c) $\supp a \subset I \times I$.

3. Formulation of main results

**Theorem 1.** The maximal operator $\sigma^*$ is bounded from the Hardy space $H_{2/3}^p (G \times G)$ to the space weak-$L_{2/3}^p (G \times G)$.

**Corollary 1 ([114]).** Let $p > 2/3$. Then the maximal operator $\sigma^*$ is bounded from the Hardy space $H_p (G \times G)$ to the space $L_p (G \times G)$.

4. Auxiliary propositions

We shall need the following lemmas (see [8,15]).

**Lemma 1** (Weisz). Suppose that an operator $V$ is sublinear and, for some $0 < p < 1$
\[ \sup_{\rho > 0} \rho^p \mu \{ x \in (G \times G) \setminus (I \times I) : |Va(x)| > \rho \} \leq c_p < \infty, \]
for every $p$-atom $a$, where $I \times I$ is the support of the atom. If $V$ is bounded from $L_{p_1}$ to $L_{p_1}$ for a fixed $1 < p_1 \leq \infty$, then
\[ \|Vf\|_{weak-L_p(G \times G)} \leq c_p \|f\|_{H_p(G \times G)}. \]

**Lemma 2** ([1]). Let $\alpha_1, \ldots, \alpha_n$ be real numbers. Then
\[ \frac{1}{n} \int_G \left| \sum_{k=1}^n \alpha_k D_k(x) \right| \, d\mu(x) \leq \frac{c}{\sqrt{n}} \left( \sum_{k=1}^n \alpha_k^2 \right)^{1/2}. \]

**Lemma 3** (Nagy). Let $n, a, b \in \mathbb{N}$, $(x^1, x^2) \in (I_a \setminus I_{a+1}) \times (I_b \setminus I_{b+1})$. Suppose that $s \leq a \leq b \leq |n|$, then
\[ K_{n[(s+1)/2]}(x^1, x^2) \leq c2^{a+b+s}. \]
If $a \leq b < s \leq |n|$ then
\[ K_{n[(s+1)/2]}(x^1, x^2) \begin{cases} 0 & \text{if } \exists i \in B_1, x_i^1 \neq x_i^2, \\ 0 & \text{if } \forall i \in B_1, x_i^1 = x_i^2, \exists m \in B_2, x^1 - e_a - e_m \not\in I_{b+1}, x^1_m = 1, \\ -w_{n[(s+1)/2]}(x^1 + x^2)2^{s+a+m-2} & \text{if } \forall i \in B_1, x_i^1 = x_i^2, \exists m \in B_2, x^1 - e_a - e_m \in I_{b+1}, x^1_m = 1, \\ w_{n[(s+1)/2]}(x^1 + x^2)2^{s+2a-1} & \text{if } x^1 - e_a \in I_{b+1} \forall i \in B_1, x_i^1 = x_i^2, \end{cases} \]
where $B_1 = \{b+1, \ldots, s-1\}$, $B_2 = \{a+1, \ldots, b\}$. 
Lemma 4 (Nagy). Let \( N, a, b \in \mathbb{N}, (x^1, x^2) \in (I_a \setminus I_{a+1}) \times (I_b \setminus I_{b+1}) \). Suppose that \( a \leq b < N \), then

\[
K_{2N}(x^1, x^2) = \begin{cases} 
0 & \text{if } \exists i \in B_1, x_i^1 \neq x_i^2, \\
0 & \text{if } \forall i \in B_1, x_i^1 = x_i^2, \forall m \in B_2, \\
x^1 - e_a - e_m \notin I_{b+1}, x_m^1 = 1, \\
2^{a+m-2} & \text{if } \forall i \in B_1, x_i^1 = x_i^2, \exists m \in B_2, \\
x^1 - e_a - e_m \in I_{b+1}, x_m^1 = 1, \\
2^{2a-1} & \text{if } x^1 - e_a \in I_{b+1} \left( \forall i \in B_1, x_i^1 = x_i^2 \right),
\end{cases}
\]

where \( B_1 = \{b + 1, \ldots, s - 1\} \), \( B_2 = \{a + 1, \ldots, b\} \).

For one-dimensional Walsh–Fejér means Lemmas 3 and 4 was proved by Gát [4].

Lemma 5. Let \((x^1, x^2) \in (I_a \setminus I_{a+1}) \times (I_b \setminus I_{b+1})\), where \(a < s \leq b\). Then

\[
\left| K_{n(s+1), 2s}(x^1, x^2) \right| \leq c2^{a+b} \sum_{m=a+1}^{s} D_{2^m}(x^1 + e_a + e_m).
\]

Proof. During the proof of Lemma 5 we apply some idea of Gát [5]. We can write

\[
K_{n(s+1), 2s}(x^1, x^2) = \sum_{j=0}^{2^s-1} D_{j+n(s+1)}(x^1) D_{j+n(s+1)}(x^2).
\]

It is evident that

\[
D_{j+n(s+1)}(x^1) = D_{n(s+1)}(x^1) + w_{n(s+1)}(x^1) D_{j}(x^1) = w_{n(s+1)}(x^1) D_{j+n(s+1)}(x^1),
\]

\[
D_{j+n(s+1)}(x^2) = D_{n(s+1)}(x^2) + w_{n(s+1)}(x^2) D_{j}(x^2).
\]

Then we obtain

\[
K_{n(s+1), 2s}(x^1, x^2) = w_{n(s+1)}(x^1) D_{n(s+1)}(x^2) 2^s K_{2s}(x^1) + w_{n(s+1)}(x^1 + x^2) \sum_{j=0}^{2^s-1} D_{j}(x^1) D_{j}(x^2) = I + II.
\]

Since

\[
|D_{n(s+1)}(x^2)| \leq c2^b
\]

from (1) and (3) we have the following estimation

\[
|I| \leq c2^b \sum_{j=0}^{s} 2^j D_{2^j}(x^1 + e_j) = c2^{a+b} D_{2^s}(x^1 + e_a).
\]

Since \( s \leq b \) and \( x^2 \in I_b \setminus I_{b+1} \) we can write that \( D_{j}(x^2) = j \). Consequently, for \( II \) we have

\[
|II| \leq \sum_{j=0}^{2^s-1} j D_{j}(x^1) \leq \sum_{j=0}^{2^s-1} j D_{j}(x^1).
\]
Since (see [10])

\[ D_j \left( x^1 \right) = w_j \left( x^1 \right) \left( \sum_{k=0}^{a-1} j k 2^k - j a 2^a \right), \quad (x^1 \in I_a \setminus I_{a+1}) \]

we have

\[ \sum_{j=0}^{2^s-1} j D_j \left( x^1 \right) = \sum_{j=0}^{2^s-1} j w_j \left( x^1 \right) \left( \sum_{k=0}^{a-1} j k 2^k - j a 2^a \right). \]

If there exist coordinates \( x_{a+l} \neq 0 \) and \( x_{a+q} \neq 0 \) \((a < a + l < a + q < s)\) then we have

\[ \sum_{j=0}^{2^s-1} j D_j \left( x^1 \right) = \sum_{j_i=0, i \neq a+l, a+q}^{1} j_{a+l=0}^{1} j_{a+q=0}^{1} \left( j_{a+l} 2^{a+l} + j_{a+q} 2^{a+q} + \Phi_1 \right) \times \left( -1 \right)^{j_{a+l}+j_{a+q}} \Phi_2, \]

where the functions \( \Phi_1 \) and \( \Phi_2 \) do not depend on \( j_{a+l} \) and \( j_{a+q} \). Consequently, we can write

\[ \sum_{j_{a+l}=0}^{1} \left( j_{a+q} 2^{a+q} + \Phi_1 \right) \times \left( -1 \right)^{j_{a+l}+j_{a+q}} \Phi_2 = 0 \]

and

\[ \sum_{j_{a+q}=0}^{1} j_{a+l} 2^{a+l} \times \left( -1 \right)^{j_{a+l}+j_{a+q}} \Phi_2 = 0. \]

The above equations give

\[ \sum_{j=0}^{2^s-1} j D_j \left( x^1 \right) = 0 \quad \text{for } x^1 - e_{a+l} x_{a+l} \notin I_s \text{ for some } l = 1, \ldots, s - a - 1. \quad (8) \]

Since

\[ |D_j(x^1)D_j(x^2)| \leq c 2^{a+b} \]

from (5)–(8) we complete the proof of Lemma 5.

\[ \square \]

Lemma 6. Let \((x^1, x^2) \in \mathcal{T}_N \times \mathcal{T}_N\) and \(n = 2^N A + B, 0 \leq B < 2^N\). Then

\[ \int_{I_N \times I_N} \left| K_n \left( x^1 + t^1, x^2 + t^2 \right) \right| d\mu \left( t^1, t^2 \right) \leq \frac{c}{23N} \left\{ \left| \sum_{j=1}^{2^N-1} D_j \left( x^1 \right) D_j \left( x^2 \right) \right| + \left| \sum_{j=1}^{B-1} D_j \left( x^1 \right) D_j \left( x^2 \right) \right| \right\}. \]

Proof. We write

\[ \int_{I_N \times I_N} \left| K_n \left( x^1 + t^1, x^2 + t^2 \right) \right| d\mu \left( t^1, t^2 \right) \]
\[ I \left( x^1, x^2, n \right) = I \left( x^1, x^2, n \right) + II \left( x^1, x^2, n \right). \]  
\[ \text{(9)} \]

Since
\[ D_{j+2N} (u) = w_l \left(2^N u\right) D_j (u) + D_{2N} (u) D_l \left(2^N u\right), \]
\[ x^1 + t^1, x^2 + t^2 \notin I_N, \]
\[ A \leq n2^{-N} \]
and
\[ w_j (u) = 1 \quad \text{for} \quad j = 0, \ldots, 2^N - 1 \quad \text{on} \quad I_N \quad \text{from (1)} \quad \text{we can write} \]
\[ I \left( x^1, x^2, n \right) = \frac{1}{n} \int_{I_N \times I_N} \left[ \sum_{l=0}^{A-1} w_{l} \left(2^N \left(x^1 + t^1\right)\right) w_{l} \left(2^N \left(x^2 + t^2\right)\right) \right] \]
\[ \times \left[ \sum_{j=1}^{2^{N-1}} D_{j} \left( x^1 + t^1 \right) D_{j} \left( x^2 + t^2 \right) \right] \mathrm{d}\mu \left( t^1, t^2 \right) \]
\[ \leq \frac{1}{n} \left[ \sum_{j=1}^{2^{N-1}} D_{j} \left( x^1 \right) D_{j} \left( x^2 \right) \right] \left[ \int_{I_N \times I_N} \left[ \sum_{l=0}^{A-1} w_{l} \left(2^N \left(x^1 + t^1\right)\right) w_{l} \left(2^N \left(x^2 + t^2\right)\right) \right] \right] \]
\[ \leq \frac{A}{n} \frac{1}{2^{2N}} \left[ \sum_{j=1}^{2^{N-1}} D_{j} \left( x^1 \right) D_{j} \left( x^2 \right) \right] \]
\[ \leq \frac{1}{2^{3N}} \left[ \sum_{j=1}^{2^{N-1}} D_{j} \left( x^1 \right) D_{j} \left( x^2 \right) \right]. \]  
\[ \text{(12)} \]

Analogously, we can prove that
\[ II \left( x^1, x^2, n \right) \leq \frac{1}{2^{3N}} \left[ \sum_{j=1}^{B-1} D_{j} \left( x^1 \right) D_{j} \left( x^2 \right) \right]. \]  
\[ \text{(13)} \]

Combining (9), (12) and (13) we complete the proof of Lemma 6. \( \square \)

Lemma 7. Let \((x^1, x^2) \in (I_{l_1} \setminus I_{l_1+1}) \times (I_{m_2} \setminus I_{m_2+1})\) and \(0 \leq t^1 \leq m^2 < N\). Then
\[ \int_{I_N \times I_N} \left| K_n \left( x^1 + t^1, x^2 + t^2 \right) \right| \mathrm{d}\mu \left( t^1, t^2 \right) \]
\[ \leq \frac{c}{2^{3N}} \left[ 2^{l_1-m^2} \sum_{r=l_1+1}^{m^2+1} 2^{r^1} D_{2m^2+1} \left( x^1 + e_{l_1} + e_{r^1} \right) \sum_{s=m^2+1}^{N} D_{2s} \left( x^2 + e_{m^2} + x^1_{m^2+1,s-1} \right) \right]. \]
\[+ 2^{l_1 + m^2 - s} \sum_{s = l_1 + 1}^{m^2} \sum_{r_1 = l_1 + 1}^{s} D_{2s} \left( x^1 + e_{l_1} + e_{r_1} \right)\], \quad n \geq 2N.

**Proof.** From Lemmas 3 and 4 and by (1) we can write the following estimations

\[
\left| K_{(s+1),2s} (x^1, x^2) \right|
\]

\[
\leq c 2^{l_1 - m^2} \sum_{m = l_1 + 1}^{m^2+1} 2^m D_{2m^2+1} \left( x^1 + e_{l_1} + e_m \right) D_{2s} \left( x^2 + e_{m^2} + x_{m^2+1,s-1}^1 \right) \tag{14}
\]

and

\[
\sum_{j=0}^{2N-1} D_j \left( x^1 \right) D_j \left( x^2 \right)
\]

\[
\leq c 2^{l_1 - m^2} \sum_{m = l_1 + 1}^{m^2+1} 2^m D_{2m^2+1} \left( x^1 + e_{l_1} + e_m \right) D_{2N} \left( x^2 + e_{m^2} + x_{m^2+1,N-1}^1 \right). \tag{15}
\]

From (4) we write

\[
\left| \sum_{j=1}^{B-1} D_j \left( x^1 \right) D_j \left( x^2 \right) \right| \leq \sum_{s=0}^{B-1} \left| K_{B(s+1),2s} \left( x^1, x^2 \right) \right|
\]

\[
\leq \sum_{s=0}^{N} \left| K_{B(s+1),2s} \left( x^1, x^2 \right) \right|
\]

\[
= \sum_{s=0}^{l_1} \left| K_{B(s+1),2s} \left( x^1, x^2 \right) \right| + \sum_{s = l_1 + 1}^{m^2} \left| K_{B(s+1),2s} \left( x^1, x^2 \right) \right|
\]

\[
+ \sum_{s = m^2+1}^{N} \left| K_{B(s+1),2s} \left( x^1, x^2 \right) \right|. \tag{16}
\]

Since (see Lemma 3)

\[
\left| K_{B(s+1),2s} \left( x^1, x^2 \right) \right| \leq c 2^{l_1 + m^2 + s}
\]

we have

\[
\sum_{s=0}^{l_1} \left| K_{B(s+1),2s} \left( x^1, x^2 \right) \right| \leq c \sum_{s=0}^{l_1} 2^{s + l_1 + m^2} \leq c 2^{l_1 + m^2}. \tag{17}
\]

Using Lemma 5 we can write

\[
\sum_{s = l_1 + 1}^{m^2} \left| K_{B(s+1),2s} \left( x^1, x^2 \right) \right| \leq c 2^{l_1 + m^2} \sum_{s = l_1 + 1}^{m^2} \sum_{m = l_1 + 1}^{s} D_{2s} \left( x^1 + e_{l_1} + e_m \right). \tag{18}
\]

Using the estimation (14) we obtain
Lemma 6

Using inequality (9) and Lemma 7 we have

\[
\sum_{s=m^2+1}^{N} \left| K_{B(s+1), 2s} (x^1, x^2) \right| \leq c 2^{l_1-m^2} \sum_{m=l^1+1}^{m^2+1} 2^m D_{2m^2+1} (x^1 + e_{l_1} + e_m) \]

\[
\times \sum_{s=m^2+1}^{N} D_{2s} (x^2 + e_{m^2} + x_{m^2+1,s-1}).
\]

Applying Lemma 6 from (15)–(19) we complete the proof of Lemma 7. □

Lemma 8. Let \((x^1, x^2) \in I_N \times I_N\) and \(n = 2^N A + B, 0 \leq B < 2^N\). Then

\[
\int_{I_N \times I_N} \left| K_n (x^1 + t^1, x^2 + t^2) \right| d\mu (t^1, t^2)
\]

\[
\leq \frac{c}{2^{3N}} \left\{ \sum_{j=1}^{2^{N-1}} D_j (x^1) D_j (x^2) \right\} + \left\{ \sum_{j=1}^{B-1} D_j (x^1) D_j (x^2) \right\} + 2^N \left| \sum_{j=1}^{B-1} D_j (x^2) \right|.
\]

Proof. Using inequality (9) from (1), (10) and (11) we have

\[
I (x^1, x^2, n) \leq \frac{1}{n} \int_{I_N \times I_N} \left| \sum_{l=0}^{A-1} w_l \left( 2^N (x^1 + t^1) \right) \right| w_l \left( 2^N (x^2 + t^2) \right)
\]

\[
\times \left| \sum_{j=1}^{2^{N-1}} D_j (x^1 + t^1) D_j (x^2 + t^2) \right| d\mu (t^1, t^2)
\]

\[
+ \frac{1}{n} \int_{I_N \times I_N} D_{2N} (x^1 + t^1) \left| \sum_{l=0}^{A-1} D_l \left( 2^N (x^1 + t^1) \right) \right| w_l \left( 2^N (x^2 + t^2) \right)
\]

\[
\times \left| \sum_{j=1}^{2^{N-1}} D_j (x^2 + t^2) \right| d\mu (t^1, t^2)
\]

\[
\leq \frac{c}{2^{3N}} \left| \sum_{j=1}^{2^{N-1}} D_j (x^1) D_j (x^2) \right| + \frac{2^N}{n} \left| \sum_{j=1}^{2^{N-1}} D_j (x^2) \right|
\]

\[
\times \int_{I_N \times I_N} \left| \sum_{l=0}^{A-1} D_l \left( 2^N (x^1 + t^1) \right) \right| w_l \left( 2^N (x^2 + t^2) \right) d\mu (t^1, t^2).
\]

Since (see Lemma 2 and (11))

\[
\int_{I_N \times I_N} \left| \sum_{l=0}^{A-1} D_l \left( 2^N (x^1 + t^1) \right) \right| w_l \left( 2^N (x^2 + t^2) \right) d\mu (t^1, t^2)
\]

\[
= \int_{I_N (x^1) \times I_N (x^2)} \left| \sum_{l=0}^{A-1} D_l \left( 2^N u^1 \right) \right| w_l \left( 2^N u^2 \right) d\mu (u^1, u^2)
\]
Lemma 8 and (22) we have

\[ \leq \frac{A}{2^{2N}} \leq \frac{n}{2^{3N}}. \]

from (20) we have

\[
I \left( x^1, x^2, n \right) \leq \frac{c}{2^{3N}} \left| \sum_{j=1}^{2N-1} D_j \left( x^1 \right) D_j \left( x^2 \right) \right| + \frac{c}{2^{2N}} \left| \sum_{j=1}^{2N-1} D_j \left( x^2 \right) \right|. \quad (21)
\]

Analogously, we can prove that

\[
II \left( x^1, x^2, n \right) \leq \frac{c}{2^{3N}} \left| \sum_{j=1}^{B-1} D_j \left( x^1 \right) D_j \left( x^2 \right) \right| + \frac{c}{2^{2N}} \left| \sum_{j=1}^{B-1} D_j \left( x^2 \right) \right|. \quad (22)
\]

Combining (9), (21) and (22) we complete the proof of Lemma 8. □

Lemma 9. Let \((x^1, x^2) \in I_N \times (I_{m^2} \setminus I_{m^2+1})\) and \(0 \leq m^2 < N\). Then

\[
\int_{I_N \times I_N} \left| K_n \left( x^1 + t^1, x^2 + t^2 \right) \right| \, d\mu \left( t^1, t^2 \right) \leq c \frac{2^{m^2}}{2^{2N}} \sum_{s=m^2}^{N-1} D_{2^s} \left( x^2 + e_{m^2} \right), \quad n > 2^N.
\]

Proof. Let \(0 \leq B \leq 2^N\). Since \(x^1 \in I_N\) and \(D_j \left( x^1 \right) = j, x^1 \in I_N, 1 \leq j < B \leq 2^N\) we have

\[
\sum_{j=1}^{B-1} D_j \left( x^1 \right) D_j \left( x^2 \right) = \sum_{j=1}^{B-1} j D_j \left( x^2 \right).
\]

Applying Abel’s transformation, from (2) we can write

\[
\left| \sum_{j=1}^{B-1} j D_j \left( x^2 \right) \right| \leq \left| \sum_{j=1}^{B-2} (j + 1) K_{j+1} \left( x^2 \right) \right| + (B)^2 \left| K_B \left( x^2 \right) \right| \leq \sum_{j=1}^{2^{N-1}} j \left| K_j \left( x^2 \right) \right| + (B)^2 \left| K_B \left( x^2 \right) \right| \leq c \sum_{m=1}^{N} 2^m \sum_{j=0}^{m-1} \sum_{s=j}^{m-1} D_{2^s} \left( x^2 + e_j \right) + c 2^{N-1} \sum_{j=0}^{N-1} \sum_{s=j}^{N-1} D_{2^s} \left( x^2 + e_j \right) \leq c 2^N \sum_{j=0}^{N-1} 2^j \sum_{s=j}^{N-1} D_{2^s} \left( x^2 + e_j \right). \quad (23)
\]
Let $0 \leq B \leq 2^N$. Then from (2) we have
\[
2^N \sum_{j=1}^{N-1} D_j \left( x^2 \right) \leq 2^N \sum_{j=0}^{N-1} 2^j \sum_{s=j}^{N-1} D_{2^s} \left( x^2 + e_j \right) .
\] (24)

Let $n = 2^N A + B$, $0 \leq B < 2^n$. Then using Lemma 8 and by (1), (23) and (24) we obtain
\[
\int_{I_N \times I_N} \left| K_n \left( x^1 + t^1, x^2 + t^2 \right) \right| \mathrm{d}\mu \left( t^1, t^2 \right) \leq \frac{c}{2^N} \sum_{j=0}^{N-1} \sum_{s=j}^{N-1} D_{2^s} \left( x^2 + e_j \right) .
\]
Analogously, we can prove that the following is true
\[
\int_{I_N \times I_N} \left| K_n \left( x^1 + t^1, x^2 + t^2 \right) \right| \mathrm{d}\mu \left( t^1, t^2 \right) \leq \frac{c}{2^N} \sum_{j=0}^{N-1} \sum_{s=j}^{N-1} D_{2^s} \left( x^2 + e_j \right) .
\]

Lemma 9 is proved. \[ \square \]

Analogously, we can prove that the following is true

**Lemma 10.** Let $\left( x^1, x^2 \right) \in \left( I_{l^1} \setminus I_{l^1+1} \right) \times I_N$ and $0 \leq l^1 < N$. Then
\[
\int_{I_N \times I_N} \left| K_n \left( x^1 + t^1, x^2 + t^2 \right) \right| \mathrm{d}\mu \left( t^1, t^2 \right) \leq \frac{c}{2^N} \sum_{s=l^1}^{N-1} D_{2^s} \left( x^1 + e_{l^1} \right) .
\]

## 5. Proofs of main results

**Proof of Theorem 1.** We shall apply Lemma 1, we may suppose that $a \in L_\infty$ is a $2/3$-atom with support $I_N \times I_N$. Since $\sigma_n a \left( x^1, x^2 \right) = 0$ if $n \leq 2^N$ we may assume that $n > 2^N$.

Suppose that $\rho = c 2^\lambda$ for some $\lambda \in \mathbb{N}$.

It is evident that
\[
\mu \left\{ \left( x^1, x^2 \right) \in I_N \times I_N : \left| \sigma^*_a \left( x^1, x^2 \right) \right| > c 2^\lambda \right\} = \mu \left\{ \left( x^1, x^2 \right) \in \overline{T}_N \times \overline{T}_N : \left| \sigma^*_a \left( x^1, x^2 \right) \right| > c 2^\lambda \right\} + \mu \left\{ \left( x^1, x^2 \right) \in I_N \times \overline{T}_N : \left| \sigma^*_a \left( x^1, x^2 \right) \right| > c 2^\lambda \right\} + \mu \left\{ \left( x^1, x^2 \right) \in \overline{T}_N \times I_N : \left| \sigma^*_a \left( x^1, x^2 \right) \right| > c 2^\lambda \right\} .
\] (25)

Let $\left( x^1, x^2 \right) \in \left( I_{l^1} \setminus I_{l^1+1} \right) \times \left( I_{m^2} \setminus I_{m^2+1} \right)$, $0 \leq l^1 \leq m^2 < N$. Then from Lemma 7 we have
Hence, we can suppose that we obtain
\[ q \text{ for some } \]

It is evident that (see (1)) \( \sigma^{*,1}(x^1, x^2) \neq 0 \) implies that
\[ x^1 \in I_N \left( 0, \ldots, 0, x_{f^1} = 1, 0, \ldots, 0, x_{r^1} = 1, 0, \ldots, 0, x_{m^2+1}, \ldots, x_{N-1} \right) \]
and
\[ x^2 \in I_N \left( 0, \ldots, 0, x_{m^2} = 1, x_{m^2+1}, \ldots, x_{q^2-1}, 1 - x_{q^2}, x_{q^2+1}, \ldots, x_{N-1} \right) \]
for some \( q^2 \) and \( r^1 \) for which \( l^1 \leq r^1 \leq m^2 < q^2 < N \).
Consequently,
\[ \sigma^{*,1}(x^1, x^2) \leq c 2^{l^1-m^2} 2^{l^1} 2^{m^2} 2^{q^2} \] \[ \leq c 2^{l^1+r^1+q^2}. \]
Let \( l^1 + r^1 + q^2 \leq \lambda \). Since
\[ \sigma^{*,1}(x^1, x^2) \leq c 2^{l^1+r^1+q^2} \leq c 2^\lambda \]
we obtain
\[ \mu \left\{ \sigma^{*,1} > c 2^\lambda \right\} = 0. \]
Hence, we can suppose that
\[ l^1 + r^1 + q^2 > \lambda. \]
It is evident that
\[ D := \sum_{l^1=0}^{N-1} \sum_{m^2=l^1+1}^{N-1} \mu \left\{ (x^1, x^2) \in (I_{l^1} \setminus I_{l^1+1}) \times (I_{m^2} \setminus I_{m^2+1}) : \sigma^{*,1}(x^1, x^2) > c 2^\lambda \right\} \]
\[ \leq c \sum_{q^2=0}^{N-1} \sum_{r^1=0}^{q^2} \sum_{l^1=0}^{r^1} \sum_{m^2=r^1}^{l^1+r^1+q^2} \sum_{x_{m^2+1}=0}^{1} \sum_{x_{N-1}=0}^{1} \sum_{x_{q^2+1}=0}^{1} \sum_{x_{N-1}}^{1} \mu \]
\[
\times \left\{ I_N \left( 0, \ldots, 0, x_{j_1} = 1, 0, \ldots, 0, x_{r_1} = 1, 0, \ldots, 0, x_{m_2+1}^1, \ldots, x_{N-1}^1 \right) \right\}
\]

Denote
\[
A_{q^2} := \left\{ \left( r^1, l^1 \right) : 0 \leq t^1 \leq r^1, r^1 \leq q^2 < N \right\}
\]

and
\[
B_{q^2} := \left\{ \left( r^1, l^1 \right) : t^1 + r^1 > \lambda - q^2 \right\}.
\]

Let \( \frac{\lambda - q^2}{2} > q^2 \) and \( \lambda - q^2 > 0 \). Then we have
\[A_{q^2} \cap B_{q^2} = \emptyset.\]

Let \( \frac{\lambda - q^2}{2} \leq q^2 \leq \lambda - q^2 \) and \( \lambda - q^2 > 0 \). Then it is evident that
\[A_{q^2} \cap B_{q^2} = \left\{ \left( r^1, l^1 \right) : \frac{\lambda - q^2}{2} \leq r^1 \leq q^2, \lambda - q^2 - r^1 \leq t^1 \leq r^1 \right\}.
\]

Let \( \lambda - q^2 < q^2 < N \) and \( \lambda - q^2 > 0 \). Then we can write
\[A_{q^2} \cap B_{q^2} = \left\{ \left( r^1, l^1 \right) : \frac{\lambda - q^2}{2} \leq r^1 \leq \lambda - q^2, \lambda - q^2 - r^1 \leq t^1 \leq r^1 \right\} \cup \left\{ \left( r^1, l^1 \right) : \lambda - q^2 \leq r^1 \leq q^2, 0 \leq t^1 \leq r^1 \right\}.
\]

Consequently,
\[
D \leq c \left\{ \sum_{q^2 = \lceil \lambda/3 \rceil}^{\lfloor \lambda/2 \rfloor} \left[ \sum_{r^1 = \lfloor (\lambda - q^2)/2 \rfloor}^{\lfloor \lambda \rfloor - q^2 - r^1} \sum_{m^2 = r^1}^{\lfloor \lambda \rfloor - q^2 - r^1} \sum_{r^1}^{\lfloor \lambda \rfloor} \left( \frac{2N - m^2}{2N} \right) \left( \frac{2N - q^2}{2N} \right) \right] - \sum_{q^2 = \lceil \lambda/2 \rceil}^{\lfloor \lambda \rfloor} \sum_{r^1 = \lfloor (\lambda - q^2)/2 \rfloor}^{\lfloor \lambda \rfloor - q^2 - r^1} \sum_{m^2 = r^1}^{\lfloor \lambda \rfloor - q^2 - r^1} \left( \frac{2N - m^2}{2N} \right) \left( \frac{2N - q^2}{2N} \right) \right\}
\]
\[
\leq c \left\{ \sum_{q^2 = \lceil \lambda/3 \rceil}^{\lfloor \lambda/2 \rfloor} \frac{1}{2q^2} \sum_{r^1 = \lfloor (\lambda - q^2)/2 \rfloor}^{\lfloor \lambda \rfloor - q^2 - r^1} \frac{r^1 - \frac{\lambda - q^2}{2}}{2r^1} \right\}
\]
\[
+ \sum_{q^2 = \lceil \lambda/2 \rceil}^{\lfloor \lambda \rfloor} \frac{1}{2q^2} \sum_{r^1 = \lfloor (\lambda - q^2)/2 \rfloor}^{\lfloor \lambda \rfloor - q^2 - r^1} \frac{r^1 - \frac{\lambda - q^2}{2}}{2r^1} + \sum_{q^2 = \lceil \lambda/2 \rceil}^{\lfloor \lambda \rfloor} \frac{1}{2q^2} \sum_{r^1 = \lfloor (\lambda - q^2)/2 \rfloor}^{\lfloor \lambda \rfloor - q^2 - r^1} \frac{r^1}{2r^1}\right\}
\leq \frac{c}{2^{(2/3)\lambda}}.
\]
Let $\lambda - q^2 \leq 0$. Then it is evident that

$$A_{q^2} \cap B_{q^2} = \left\{ (r^1, l^1) : 0 \leq r^1 \leq q^2, 0 \leq l^1 \leq r^1 \right\}.$$ 

Hence

$$D \leq c \sum_{q^2=[\lambda]} \sum_{r^1=0}^{2N-m^2} \sum_{l^1=0}^{2N-q^2} \sum_{m^2=r^1} \frac{2N-m^2}{2N} \frac{2N-q^2}{2N} \leq c \sum_{q^2=[\lambda]} \frac{1}{2q^2} \sum_{r^1=0}^{2^2} \frac{r^1}{2^r} \leq \frac{c}{2^{(2/3)\lambda}}.$$ 

Denote

$$\sigma^{*,2}(x^1, x^2) := 2^{l^1+m^2} \sum_{s=1}^{m^2} \sum_{r^1=l^1+1} D_2 (x^1 + e_{l^1} + e_{r^1})$$

$$= 3 \cdot 2^{l^1+m^2} + 2^{l^1+m^2} \sum_{s=1}^{m^2} \sum_{r^1=l^1+1} D_2 (x^1 + e_{l^1} + e_{r^1})$$

$$= 3\sigma^{*,2,1}(x^1, x^2) + \sigma^{*,2,2}(x^1, x^2).$$

(28)

It is simple to show that

$$\sum_{l^1=0}^{N-1} \sum_{m^2=l^1}^{N-1} \mu \left( \left\{ (x^1, x^2) \in (I_{l^1} \setminus I_{l^1+1}) \times (I_{m^2} \setminus I_{m^2+1}) : \sigma^{*,2,1}(x^1, x^2) > c2^l \right\} \right)$$

$$\leq c \left\{ \sum_{l^1=0}^{\lfloor \lambda/3 \rfloor} \sum_{m^2=[\lambda]-2^l}^{N-1} \frac{1}{2^{m^2+l^1}} + \sum_{l^1=\lfloor \lambda/3 \rfloor}^{N-1} \sum_{m^2=l^1}^{N-1} \frac{1}{2^{m^2+l^1}} \right\} \leq \frac{c}{2^{(2/3)\lambda}}.$$ 

(29)

Using (1) we can show that

$$\sigma^{*,2,2}(x^1, x^2) \neq 0$$

implies

$$x^1 \in I_N \left( 0, \ldots, 0, x^1_{l^1} = 1, 0, \ldots, 0, x^1_{r^1} = 1, 0, \ldots, x^1_{m^1} = 1, x^1_{m^1+1}, \ldots, x^1_{N-1} \right)$$

for some $r^1$ and $m^1$ for which $l^1 < r^1 \leq m^1 < m^2$. Then we have

$$\sigma^{*,2,2}(x^1, x^2) \leq c2^{l^1+m^2+m^1}.$$ 

Let $l^1 + m^2 + m^1 \leq \lambda$. Then we obtain that

$$\mu \left\{ \sigma^{*,2,2} > c2^\lambda \right\} = 0.$$ 

Hence we can suppose that

$$l^1 + m^2 + m^1 > \lambda.$$
Let
\[ G := \sum_{l^1=0}^{N-1} \sum_{m^2=l^1}^{N-1} \mu \left( \left\{ (x^1, x^2) \in (I_{l^1} \setminus I_{l^1+1}) \times (I_{m^2} \setminus I_{m^2+1}) : \sigma^{x^1,x^2} \left( x^1, x^2, \sigma^2 \right) > c^2 \right\} \right). \]

Denote
\[ E_{m^2} := \left\{ (m^1, l^1) : 0 \leq l^1 \leq m^1, 0 \leq m^1 \leq m^2 \right\} \]
and
\[ F_{m^2} := \left\{ (m^1, l^1) : l^1 + m^1 > \lambda - m^2 \right\}. \]

Let \( m^2 < \frac{\lambda - m^2}{2} \). Then it is evident that
\[ E_{m^2} \cap F_{m^2} = \emptyset. \]

Let \( \lambda - m^2 \leq 0 \). Then we can write
\[ E_{m^2} \cap F_{m^2} = \left\{ (m^1, l^1) : 0 \leq m^1 \leq m^2, 0 \leq l^1 \leq m^1 \right\}. \]

Hence
\[ G \leq c \sum_{m^2=[\lambda/3]}^{[\lambda/2]} \sum_{m^1=[\lambda m^2]}^{[\lambda m^2-1]} \sum_{l^1=0}^{m^1} \frac{1}{2m^1} \frac{1}{2m^2} \]
\[ \leq c \sum_{m^2=[\lambda/3]}^{[\lambda/2]} \sum_{m^1=[\lambda m^2]}^{[\lambda m^2-1]} \frac{1}{2m^2} \frac{1}{2m^1} \]
\[ \leq \frac{c}{2(2/3)\lambda}. \] (30)

Let \( \lambda - m^2 > 0 \) and \( \frac{\lambda - m^2}{2} \leq m^2 < \lambda - m^2 \). Then
\[ E_{m^2} \cap F_{m^2} = \left\{ (m^1, l^1) : \frac{\lambda - m^2}{2} \leq m^1 \leq m^2, \lambda - m^2 - m^1 \leq l^1 \leq m^1 \right\}. \]

Consequently,
\[ G \leq c \sum_{m^2=[\lambda/3]}^{[\lambda/2]} \sum_{m^1=[\lambda m^2]}^{[\lambda m^2-1]} \sum_{l^1=[\lambda m^1-2m^2]}^{[\lambda m^1-1]} \frac{1}{2m^2} \frac{1}{2m^1} \]
\[ \leq c \sum_{m^2=[\lambda/3]}^{[\lambda/2]} \sum_{m^1=[\lambda m^2]}^{[\lambda m^2-1]} \frac{1}{2m^2} \frac{m^1 - \frac{\lambda - m^2}{2}}{2m^1} \leq \frac{c}{2(2/3)\lambda}. \] (31)

Let \( \lambda - m^2 > 0 \) and \( \lambda - m^2 \leq m^2 < N \). Then
\[ E_{m^2} \cap F_{m^2} = \left\{ (m^1, l^1) : \frac{\lambda - m^2}{2} \leq m^1 \leq \lambda - m^2, \lambda - m^2 - m^1 \leq l^1 \leq m^1 \right\} \]
\[ \cup \left\{ (m^1, l^1) : \frac{\lambda - m^2}{2} < m^1 \leq m^2, 0 \leq l^1 \leq m^1 \right\}. \]
Consequently,
\[ G \leq c \sum_{m^2 = [\lambda]/2}^{[\lambda]-m^2} \sum_{l^1 = [\lambda]-m^2} \sum_{l^1} \frac{1}{2m^2 2m^1} + c \sum_{m^2 = [\lambda]/2}^{[\lambda]-m^2} \sum_{l^1 = [\lambda]-m^2} \sum_{l^1=0}^{m^1} \frac{1}{2m^2 2m^1} \]
\[ \leq c \sum_{m^2 = [\lambda]/2}^{[\lambda]-m^2} \frac{1}{2m^2} \sum_{l^1 = [\lambda]-m^2} \sum_{l^1} \frac{m^1 - \lambda - m^2}{2m^1} \]
\[ + c \sum_{m^2 = [\lambda]/2}^{[\lambda]-m^2} \frac{1}{2m^2} \sum_{l^1 = [\lambda]-m^2} \sum_{l^1=0}^{m^1} \frac{m^1}{2m^1} \]
\[ \leq \frac{c}{2^{(2/3)\lambda}}. \quad (32) \]

Combining (30)–(32) we obtain that
\[ \sum_{l^1=0}^{N-1} \sum_{m^2=0}^{N-1} \mu \left( \left\{ \left( x^1, x^2 \right) \in (I_{l^1} \setminus I_{l^1+1}) \times (I_{m^2} \setminus I_{m^2+1}) : \sigma^* a \left( x^1, x^2 \right) > c 2^{\lambda} \right\} \right) \]
\[ \leq \frac{c}{2^{(2/3)\lambda}}. \quad (33) \]

Analogously, we can prove that
\[ \sum_{l^1=0}^{N-1} \sum_{m^2=0}^{N-1} \mu \left( \left\{ \left( x^1, x^2 \right) \in (I_{l^1} \setminus I_{l^1+1}) \times (I_{m^2} \setminus I_{m^2+1}) : \sigma^* a \left( x^1, x^2 \right) > c 2^{\lambda} \right\} \right) \]
\[ \leq \frac{c}{2^{(2/3)\lambda}}. \quad (34) \]

From (33) and (34) we obtain
\[ \mu \left( \left\{ \left( x^1, x^2 \right) \in T_N \times T_N : \sigma^* a \left( x^1, x^2 \right) > c 2^{\lambda} \right\} \right) \leq \frac{c}{2^{(2/3)\lambda}}. \quad (35) \]

Let \( (x^1, x^2) \in I_N \times (I_{m^2} \setminus I_{m^2+1}) \). Then using Lemma 9 we obtain that
\[ \sigma^* a \left( x^1, x^2 \right) \leq c \|a\|_\infty \sup_{n \geq 2^N} \int_{I_N \times I_N} \left| K_n \left( x^1 + t^1, x^2 + t^2 \right) \right| d\mu \left( t^1, t^2 \right) \]
\[ \leq c 2^{m^2+N} \sum_{s=m^2}^{N-1} D_{2^s} \left( x^2 + e_{m^2} \right) \]
\[ \leq c 2^{m^2+N} + c 2^{m^2+N} \sum_{s=m^2+2}^{N-1} D_{2^s} \left( x^2 + e_{m^2} \right) \]
\[ = \sigma^*, 3, 1 \left( x^1, x^2 \right) + \sigma^*, 3, 2 \left( x^1, x^2 \right). \]

Let \( \lambda \geq 3N \). Then it is easy to show that
\[ \sigma^*, 3, 1 \left( x^1, x^2 \right) \leq c 2^{3N} \leq c 2^\lambda, \]
consequently,
\[ \mu \left( \left\{ \sigma^*, 3, 1 > c 2^{\lambda} \right\} \right) = 0. \]
Hence we can suppose that
\[ \lambda < 3N. \] (36)

It is simple to show that (see (36))
\[
\sum_{m^2=0}^{N-1} \mu \left( \left\{ (x^1, x^2) \in I_N \times (I_{m^2} \setminus I_{m^2+1}) : \sigma^{*,3,1} (x^1, x^2) > c2^\lambda \right\} \right) \\
\leq \frac{1}{2^N} \sum_{m^2=\frac{\lambda-N}{2}}^{N-1} \frac{1}{2m^2} \leq \frac{c}{2^{(\lambda+N)/2}} \leq \frac{c}{2^{(2/3)\lambda}}.
\] (37)

Using (1) we conclude that
\[
\sigma^{*,3,2} (x^1, x^2) \neq 0, \quad (x^1, x^2) \in I_N \times (I_{m^2} \setminus I_{m^2+1})
\]
implies
\[
x^2 \in I_N \left( 0, \ldots, 0, x^2_m = 1, 0, \ldots, 0, x^2_{q^2} = 1, x^2_{q^2+1}, \ldots, x^2_{N-1} \right)
\]
for some \( q^2, m^2 < q^2 < N \).
Hence \( \sigma^{*,3,2} \leq c2^{m^2+N+q^2} \).
We can suppose that
\[
m^2 + N + q^2 > \lambda.
\]
Denote
\[
T := \left\{ (m^2, q^2) : m^2 \leq q^2 < N \right\}
\]
and
\[
R := \left\{ (m^2, q^2) : \lambda - N - m^2 \leq q^2 < N \right\}.
\]
Let \( \lambda - N > 0 \). Then
\[
T \cap R = \left\{ (m^2, q^2) : 0 \leq m^2 \leq \frac{\lambda-N}{2}, \lambda - N - m^2 \leq q^2 < N \right\} \\
\cup \left\{ (m^2, q^2) : \frac{\lambda-N}{2} < m^2 < N, m^2 \leq q^2 < N \right\}.
\]
Consequently, from (36) we obtain
\[
\sum_{m^2=0}^{N-1} \mu \left( \left\{ (x^1, x^2) \in I_N \times (I_{m^2} \setminus I_{m^2+1}) : \sigma^{*,3,2} (x^1, x^2) > c2^\lambda \right\} \right) \\
\leq \frac{c}{2^N} \left\{ \sum_{m^2=0}^{[\frac{\lambda-N}{2}]} \sum_{q^2 = [\frac{\lambda-N}{2}] - m^2}^{N-1} \frac{1}{2q^2} + \sum_{m^2=[\frac{\lambda-N}{2}]+1}^{N-1} \sum_{q^2 = m^2}^{N-1} \frac{1}{2q^2} \right\} \\
\leq \frac{c}{2^{(\lambda+N)/2}} \leq \frac{c}{2^{(2/3)\lambda}}.
\] (38)
Let $\lambda - N \leq 0$. Then

$$T \cap R = \left\{ (m^2, q^2) : 0 \leq m^2 < N, m^2 \leq q^2 < N \right\}.$$  

Consequently,

$$\sum_{m^2=0}^{N-1} \mu \left( \left\{ (x^1, x^2) \in I_N \times (I_{m^2} \setminus I_{m^2+1}) : \sigma^* (x^1, x^2) > 2^{\lambda/2} \right\} \right)$$

$$\leq \frac{c}{2^N} \sum_{m^2=0}^{N-1} \sum_{q^2=m^2}^{N-1} \frac{1}{2^{q^2}} \leq \frac{c}{2^N} \leq \frac{c}{2^{(2/3)\lambda}}. \quad (39)$$

Combining (37)–(39) we obtain

$$\mu \left( \left\{ (x^1, x^2) \in I_N \times \tilde{I}_N : \sigma^* (x^1, x^2) > c^{2^{\lambda/2}} \right\} \right) \leq \frac{c}{2^{(2/3)\lambda}}. \quad (40)$$

Analogously, we can prove that

$$\mu \left( \left\{ (x^1, x^2) \in \tilde{I}_N \times I_N : \sigma^* (x^1, x^2) > 2^{\lambda/2} \right\} \right) \leq \frac{c}{2^{(2/3)\lambda}}. \quad (41)$$

Combining (25), (35), (40) and (41) we obtain that

$$2^{2\lambda/3} \mu \left\{ (x^1, x^2) \in I_N \times \tilde{I}_N : \left| \sigma^* a (x^1, x^2) \right| > c^{2^{\lambda/2}} \right\} \leq c < \infty.$$

Theorem 1 is proved. \(\square\)

Since $\sigma^*$ is bounded from the Hardy space $L_\infty (G \times G)$ to the space $L_\infty (G \times G)$ the validity of Corollary 1 follows by interpolation (see Weisz [15]) and Theorem 1.

6. $d > 2$ dimensional case

For the $d$-dimensional Walsh–Fourier series the author [6] proved that the maximal operator

$$\sigma^* f = \sup_{n \in \mathbb{D}} \frac{1}{n} \left| \sum_{j=0}^{n-1} S_{j,...,j}(f) \right|$$

is bounded from the $d$-dimensional dyadic martingale Hardy space $H_p (G \times \cdots \times G)$ to the space $L_p (G \times \cdots \times G)$ for $p > d/(d+1)$. In [7] it is proved that in the above-mentioned theorem the assumption $p > d/(d+1)$ is essential, in particular we showed that the maximal operator $\sigma^*$ is not bounded from the Hardy space $H_{d/(d+1)} (G \times \cdots \times G)$ to the space $L_{d/(d+1)} (G \times \cdots \times G)$. By interpolation it follows that $\sigma^* f$ is not bounded from the Hardy space $H_p (G \times \cdots \times G)$ to the space weak-$L_p (G \times \cdots \times G)$ for $0 < p < d/(d+1)$.

**Conjecture 1.** Let $d > 2$. Then the maximal operator of the Marcinkiewicz–Fejér means of the $d$-dimensional Walsh–Fourier series is bounded from the dyadic Hardy space $H_{d/(d+1)} (G \times \cdots \times G)$ to the space weak-$L_{d/(d+1)} (G \times \cdots \times G)$.  

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