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Bott periodicity and calculus of Euler classes on spheres

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ABSTRACT

A variety of computations regarding the Euler class group $E(A_n, A_n)$ and the Grothendieck group $K_0(A_n)$ of the algebraic sphere $Spec(A_n)$ is done. The Euler class of the algebraic tangent bundle on $Spec(A_n)$ is computed. It is also investigated whether every element in the Euler class group $E(A_n, A_n)$ is the Euler class of a projective A_n module of rank n.

1. Introduction

Work on obstruction theory for projective modules started with the work of N. Mohan Kumar and M.P. Murthy [Mk,MkM,Mu1]. It is a result of Murthy [Mu1] that *for a reduced* (*smooth*) *affine algebra A with* $\dim A = n$, *over an algebraically closed field k, the top Chern class map $C_0 : K_0(A) \to CH_0(A)$ is surjective*. This result is a consequence of the result [Mu1] that *given any local complete intersection ideal I of height n, there is a projective A-module P with rank$(P) = n$ that maps surjectively onto I*.

For real smooth affine varieties such propositions will fail. Most common examples are that of real spheres. We denote the real sphere of dimension n by $\mathbb{S}^n$ and A_n denotes the ring of algebraic functions on $\mathbb{S}^n$. We have, the Chow group of zero cycles $CH_0(A_n) = \mathbb{Z}/2$ (see 3.1) and by the theorem of Swan [Sw2], $K_0(A_n) = KO(\mathbb{S}^n)$. By the periodicity theorem of Bott (see 5.10), for nonnegative integers $n = 8r + 3, 8r + 5, 8r + 6, 8r + 7$ $(r \geqslant 0)$ we have $K_0(A_n) = KO(\mathbb{S}^n) = \mathbb{Z}$. In these cases, the top Chern class map $C_0 = 0$ and it fails to be surjective.

On the other hand, by Bott periodicity (see 5.10), $\widetilde{K_0}(A_{8r}) = \widetilde{KO}(\mathbb{S}^{8r}) = \mathbb{Z}$, $\widetilde{K_0}(A_{8r+1}) = \widetilde{KO}(\mathbb{S}^{8r+1}) = \mathbb{Z}/(2)$, $\widetilde{K_0}(A_{8r+2}) = \widetilde{KO}(\mathbb{S}^{8r+2}) = \mathbb{Z}/(2)$, $\widetilde{K_0}(A_{8r+4}) = \widetilde{KO}(\mathbb{S}^{8r+4}) = \mathbb{Z}$. Therefore, in these cases, the question of surjectivity of the top Chern class map $C_0 : K_0(A_n) \to CH_0(A_n)$ fully depends on the top Chern class of the generator τ_n of $\widetilde{K_0}(A_n)$. In analogy to the obstruction theory in topology, it makes

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more sense to consider the Euler class group $E(A_n)$ of A_n as the obstruction group, instead of the Chow group $CH_0(A_n)$.

For (smooth) affine rings A with $\dim A = n \geqslant 2$, over a field k, the original definition of Euler class groups $E(A)$ was given by Nori [MS,BRS2]. For a projective A-module P, with $\det P = A$ and an orientation $\chi : A \xrightarrow{\sim} \det P$, an Euler class $e(P, \chi) \in E(A)$ was defined. We mainly refer to [BRS2], for basics on Euler class groups and Euler classes. For such a ring A, $\mathcal{P}O_n(A)$ will denote the set of all isomorphism classes of pairs (P, χ), where P is a projective A-module rank n, with trivial determinant, and $\chi : A \xrightarrow{\sim} \det P$ is an isomorphism, to be called an orientation.

So, our main question is *whether the Euler class map* $e : \mathcal{P}O_n(A_n) \to E(A_n)$ *is surjective*. In fact, $E(A_n) = \mathbb{Z}$. For reasons given above, the Euler class map fails to be surjective for $n = 8r + 3, 8r + 5, 8r + 6, 8r + 7$ $(r \geqslant 0)$. *In fact, we also prove that for* $n = 8r + 1$ *this map fails to be surjective. For any even integer* $n \geqslant 2$, *we prove that any even class* $N \in E(A_n) = \mathbb{Z}$, *is in the image of* e. *For* $n = 2, 4, 8$ *we prove that* e *is surjective. For* $n = 8r, 8r + 2, 8r + 4 \geqslant 2$, *we prove that* e *is surjective if and only if the top Stiefel–Whitney class* $w_n(\tau_n) = 1$ *where* τ_n *is the generator of* $\widetilde{K_0}(A_n)$. It remains an open question whether $w_n(\tau_n) = 1$. It follows (see 6.4) that $w_n(\tau_n) = 1$ if an only if $C_0(\tau_n) = 1$.

Among other results in this paper, we compute (see 3.3) the Euler class of the algebraic tangent bundle T over $Spec(A_n)$. As in topology (see [MiS]), $e(T, \chi) = -2$, when n is even and zero when n is odd. This provides a fully algebraic proof that the algebraic tangent bundles T over even dimensional spheres $Spec(A_n)$ do not have a free direct summand.

Given any real maximal ideal m of A_n, we attach (see 4.2) a local orientation ω on m in an algorithmic way and compute the class $(m, \omega) = 1$ or $(m, \omega) = -1$ in $E(A_n)$.

2. Preliminaries

Following are some of the notations we will be using in this paper.

Notations 2.1. First, the fields of real numbers and complex numbers will, respectively, be denoted by $\mathbb{R}$ and $\mathbb{C}$. The quaternion algebra will be denoted by $\mathbb{H}$.

1. The real sphere of dimension n will be denoted by $\mathbb{S}^n$. Let
$$A_n = \frac{\mathbb{R}[X_0, X_1, \ldots, X_n]}{(\sum_{i=0}^n X_i^2 - 1)} = \mathbb{R}[x_0, x_1, \ldots, x_n]$$
denote the ring of algebraic functions on $\mathbb{S}^n$.
2. For any real affine variety $X = Spec(A)$, let $\mathbb{R}(X) = S^{-1}A$, where S is the multiplicative set of all $f \in A$ that do not vanish at any real point of $Spec(A)$. Also, $X(\mathbb{R})$ denote the set of all real points of X.
3. For any noetherian commutative ring A and line bundles L on $Spec(A)$, the Euler class group will be denoted by $E(A, L)$ and the weak Euler class group will be denoted by $E_0(A, L)$. Usually, $E(A, A)$ will be denoted by $E(A)$ and similarly $E_0(A)$ will denote $E_0(A, A)$. We refer to [BRS2] for the definitions and the basic properties of these groups.

The following theorem would be obvious to the experts (see [BRS1]).

Theorem 2.2. *Let* $X = spec(A)$ *be a smooth affine variety of dimension* $n \geqslant 2$ *over* $\mathbb{R}$. *Then, the natural map*
$$E_0\big(\mathbb{R}(X)\big) \to CH_0\big(\mathbb{R}(X)\big)$$
is an isomorphism and $CH_0(\mathbb{R}(X)) \approx \mathbb{Z}/(2)^r$ *where* r *is the number of compact connected components of* $X(\mathbb{R})$.

Proof. It follows directly from [BRS1, Theorem 5.5] and Theorem 2.3 below that $E_0(\mathbb{R}(X)) \xrightarrow{\sim} CH_0(\mathbb{R}(X))$. Also, by [BRS1, Theorem 4.10], $CH_0(\mathbb{R}(X)) \approx \mathbb{Z}/(2)^r$. □

Theorem 2.3. *(See [BDM].) Let $X = spec(A)$ be smooth affine variety of dimension $n \geqslant 2$ over $\mathbb{R}$. The following diagram of exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^{\mathbb{C}}(L) & \longrightarrow & E(A, L) & \longrightarrow & E(\mathbb{R}(X), L) & \longrightarrow & 0 \\ & & \wr\downarrow\varphi & & \downarrow\Theta & & \downarrow & & \\ 0 & \longrightarrow & CH(\mathbb{C}) & \longrightarrow & CH_0(A) & \longrightarrow & CH_0(\mathbb{R}(X)) & \longrightarrow & 0 \end{array}$$

commute and the first vertical map φ is an isomorphism.

Proof. We only need to prove that φ is injective. The proof is given in the proof of [BDM, Proposition 4.29]. □

We also include the following easy lemma.

Lemma 2.4. *Let A be any smooth affine ring over $\mathbb{R}$ with $\dim A = n \geqslant 2$ and L be a line bundle on Spec(A). Let P be a projective A-module of rank n and $\det P = L$. Let $\chi, \eta : L \xrightarrow{\sim} \bigwedge^n P$ be two orientations. Suppose $e(P, \chi) = (I, \omega)$ where I is an ideal of height n and ω is a local orientation on I and $\eta = u\chi$ where u is a unit in A. Then $e(P, \eta) = (I, u\omega)$.*

Proof. Write $F = L \oplus A^{n-1}$. By theorem in [BRS2], there is a surjective map $f : F \to I$ that induces (I, ω) as in the commutative diagram:

$$\begin{array}{ccccc} P & \longrightarrow & P/IP & \xleftarrow[\sim]{\gamma\sim\chi} & F/IF \\ f\downarrow & & \downarrow & \swarrow\omega & \uparrow\delta \\ I & \longrightarrow & I/I^2 & \xleftarrow[\omega\delta]{} & F/IF. \end{array}$$

Here γ is an isomorphism with determinant χ, and δ is any isomorphism with $\det(\delta) = u$. So $\gamma\delta \sim u\chi = \eta$ and $e(P, \eta) = (I, u\omega)$. □

3. The tangent bundle

It is well known that the tangent bundle T_n, over the real sphere $\mathbb{S}^n$, of even dimension $n \geqslant 1$, does not have a nowhere vanishing section. The purpose of this section is to compute the Euler class of the algebraic tangent bundle explicitly.

First, note that all line bundles over $\mathbb{S}^n$, with $n \geqslant 2$ are trivial, we have only one Euler class group $E(A_n, A_n)$ to be denoted by $E(A_n)$. Similarly, we have only one weak Euler class group $E_0(A_n)$. The following proposition entails some of the basic facts about Euler class groups of the spheres.

Proposition 3.1. *The Euler class group of the sphere is given by $E(A_n) = \mathbb{Z}$, generated by (m, ω) where m is any real maximal ideal and ω is any local orientation of m. Similarly, the weak Euler class group is given by*

$$E_0(A_n) \approx CH_0(A_n) = \frac{\mathbb{Z}}{(2)}.$$

Proof. From Theorem 2.3, we have the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^{\mathbb{C}}(A_n) & \longrightarrow & E(A_n) & \longrightarrow & E(\mathbb{R}(\mathbb{S}^n)) & \longrightarrow & 0 \\ & & \wr\downarrow\varphi & & \downarrow\Theta & & \downarrow & & \\ 0 & \longrightarrow & CH(\mathbb{C}) & \longrightarrow & CH_0(A_n) & \longrightarrow & CH_0(\mathbb{R}(\mathbb{S}^n)) & \longrightarrow & 0. \end{array}$$

Since complex points in A_n are complete intersection [MS, Lemma 4.2], we have $CH(\mathbb{C}) = E^{\mathbb{C}}(A_n) = 0$ and the above diagram reduces to

$$\begin{array}{ccccccc} 0 & \longrightarrow & E(A_n) & \xrightarrow{\sim} & E(\mathbb{R}(\mathbb{S}^n)) & \longrightarrow & 0 \\ & & \downarrow\Theta & & \downarrow & & \\ 0 & \longrightarrow & CH_0(A_n) & \xrightarrow{\sim} & CH_0(\mathbb{R}(\mathbb{S}^n)) & \longrightarrow & 0. \end{array}$$

We have by Theorem 2.2, $E_0(A_n) \xrightarrow{\sim} CH_0(A_n)$. Therefore, by [BRS1, Theorems 4.13, 4.10]

$$E\big(\mathbb{R}\big(\mathbb{S}^n\big)\big) = \mathbb{Z} \quad \text{and} \quad CH_0(A_n) \approx E_0\big(\mathbb{R}\big(\mathbb{S}^n\big)\big) = \mathbb{Z}/(2).$$

The proof is complete. □

The following definition will be convenient for subsequent discussions.

Definition 3.2. Let $m_0 = (x_0 - 1, x_1, \dots, x_n)$ be the maximal ideal in A_n that corresponds to the real point $(1, 0, \dots, 0) \in \mathbb{S}^n$. Write $F = A_n^n$ and let $e_1, \dots, e_n$ be the standard basis of F. Define local orientation

$$\omega_0 : F/m_0F \to m_0/m_0^2 \quad \text{where for } i = 1, \dots, n, \quad \omega(e_i) = image(x_i).$$

By Proposition 3.1, (m_0, ω_0) will generate the Euler class group $E(A_n) = \mathbb{Z}$. This generator $(m_0, \omega_0) = 1$ will be called the **standard generator** of $E(A_n)$. Similarly, the class of $m_0 = 1$ will be called the **standard generator** of $E_0(A_n) = \mathbb{Z}/(2)$.

Unless stated otherwise, we use these standard generators in our subsequent discussions.

We compute the Euler class of the algebraic tangent bundle over $Spec(A_n)$ as follows.

Theorem 3.3. *Let T_n be the projective A_n-module corresponding to the tangent bundle over $\mathbb{S}^n$. There is an orientation $\chi : A_n \xrightarrow{\sim} \bigwedge^n T_n$ such that, if $n \geqslant 2$ is even, then the Euler class $e(T_n, \chi) = -2 \in E(A_n)$ and if $n \geqslant 3$ is odd, then the Euler class $e(T_n, \chi) = 0 \in E(A_n)$.*

Proof. Write $m_0 = (x_0 - 1, x_1, \dots, x_n)$, $m_1 = (x_0 + 1, x_1, \dots, x_n) \in Spec(A_n)$. Then m_0, m_1 correspond, respectively, to the points $(1, 0, \dots, 0)$, $(-1, 0, \dots, 0)$ in $\mathbb{S}^n$. We have

$$m_0 = (x_1, \dots, x_n) + m_0^2, \qquad m_1 = (x_1, \dots, x_n) + m_1^2,$$

and $m_0 \cap m_1 = (x_1, \dots, x_n)$. Write $F = A_n^n$ and let $e_1, \dots, e_n$ be the standard basis. For $j = 0, 1$ we define local orientations

$$\omega_j : F/m_jF \to m_j/m_j^2 \quad \text{where for } i = 1, \dots, n, \quad \omega_j(e_i) = image(x_i).$$

Therefore, $(m_0, \omega_0) = 1$ is the standard generator of $E(A_n) = \mathbb{Z}$. We write $J = m_0 \cap m_1 = (x_1, \ldots, x_n)$ and define the surjective map

$$\alpha : F \twoheadrightarrow J \quad \text{where for } i \geqslant 1, \quad \alpha(e_i) = x_i.$$

Then, α induces the local orientation

$$\omega : F/JF \to J/J^2 \quad \text{where } \omega(e_i) = image(x_i).$$

Since (J, ω) is global, it follows

$$(m_0, \omega_0) + (m_1, \omega_1) = (J, \omega) = 0.$$

Hence

$$(m_1, \omega_1) = -(m_0, \omega_0) = -1.$$

Since $E(A_n) = E(\mathbb{R}(\mathbb{S}^n))$, we can apply [BDM, Lemma 4.2] and we have

$$(m_1, \omega_1) + (m_1, -\omega_1) = 0.$$

Therefore

$$(m_0, \omega_0) + (m_1, -\omega_1) = 2(m_0, \omega_0) = 2.$$

Let $D = diagonal(-x_0, 1, \ldots, 1) : F/JF \to F/JF$, then D is an automorphism and $\det(D) = image(-x_0)$. Now, let $\eta = \omega D : F/JF \to J/J^2$. In fact,

$$\eta(e_1) = image(-x_0 x_1) \quad \text{and} \quad \eta(e_i) = image(x_i) \quad \forall i > 1.$$

Note that

$$D = diagonal(-1, 1, \ldots, 1) \mod m_0, \qquad D = Id \mod m_1.$$

Since, ω_i are the reductions of ω modulo m_i we have

$$(J, \eta) = (m_0, -\omega_0) + (m_1, \omega_1) = -\big[(m_0, \omega_0) + (m_1, -\omega_1)\big] = -2.$$

Now we apply [BRS2, Lemma 5.1], to $\alpha : F \twoheadrightarrow J$, with $a = b = image(-x_0)$. We have the following:

1. Define T by the exact sequence

$$0 \to T \to A_n \oplus F = A^{n+1} \xrightarrow{\Phi} A_n \to 0$$

where

$$\Phi = -(x_0, x_1, \ldots, x_n) = (b, -\alpha).$$

2. We have (J, ω) is obtained from $(\alpha, \chi_0 = Id_{A_n})$.
3. By [BRS2, Lemma 5.1], T has an orientation $\chi : A_n \to \bigwedge^n T$ such that

$$e(T, \chi) = \big(J, image(-x_0)^{n-1}\omega\big) = \big(m_0, (-1)^{n-1}\omega_0\big) + (m_1, \omega_1).$$

4. If n is **EVEN**, we have

$$e(T,\chi) = (m_0, -\omega_0) + (m_1, \omega_1) = -2.$$

And if, n is **ODD**, we have

$$e(T,\chi) = (m_0, \omega_0) + (m_1, \omega_1) = 0.$$

5. Note that $T = \ker(\Phi) \approx \ker(-\Phi) = T_n$ is the tangent bundle.

So, the proof is complete. □

4. An algorithmic computation in $E(A_n)$

Lemma 4.1. *Let A_n be as above and let $m_1, M_1, m_2, M_2, \ldots, m_N, M_N \in spec(A_n)$ be a set of distinct maximal ideals that correspond to distinct real points in $\mathbb{S}^n$. We will assume that these points are in $\mathbb{S}^1 = \{x_j = 0: \forall j \geqslant 2\} \subseteq \mathbb{S}^n$. For $i = 1, \ldots, N$, let $L_i = 0$, $x_2 = 0, \ldots, x_n = 0$ be the line passing through the pair of points corresponding to m_i and M_i. Then*

$$\bigcap_{i=1}^{N} (m_i \cap M_i) = \left(\prod_{i=1}^{N} L_i, x_2, \ldots, x_n \right).$$

Proof. Let J denote the right-hand side. Claim that

$$J \subseteq m \in Spec(A_n) \quad \Rightarrow \quad m = m_i \quad \text{or} \quad m = M_i \quad \text{for some } i.$$

To see this, note for such an m, we have $L_i \in m$ for some i. Therefore,

$$m_i \cap M_i = (L_i, x_2, \ldots, x_n) \subseteq m.$$

Hence $m = m_i$ or M_i. Let m be such a maximal ideal and assume $m = m_i$. We have, $L_j \notin m_i$ $\forall j \neq i$ and $J_{m_i} = (L_i, x_2, \ldots, x_n)_{m_i} = (m_i)_{m_i}$. The proof is complete. □

Given various points m in $\mathbb{S}^n$, the following is an algorithm to compute class $(m, \omega) \in E(A_n)$.

Theorem 4.2. *As in Definition 3.2, let $(m_0, \omega_0) = 1 \in E(A_n) = \mathbb{Z}$ be the standard generator. Let $p = (a, b, 0, \ldots, 0)$ be a point in $\mathbb{S}^n$ and let $M = (x_0 - a, x_1 - b, x_2, \ldots, x_n) \in Spec(A_n)$ be the maximal ideal corresponding to p. Assume $m_0 \neq M$ and so $a \neq 1$. Let*

$$L = (1-a)x_1 + b(x_0 - 1), \quad \text{so} \quad (L, x_2, x_3, \ldots, x_n) = m_0 \cap M.$$

As in (3.2), $F = A_n^n$ and $e_1, \ldots, e_n$ is the standard basis of F. Define

$$\omega_M : F/MF \to M/M^2 \quad \text{by } \omega_M(e_1) = x_1 - b, \quad \omega_M(e_i) = x_i \quad \forall i \geqslant 2.$$

If $a \neq 0$ (i.e. p is not the north or the south pole), *then ω_M is a surjective map and*

$$(m_0, \omega_0) + \big(M, -sign(a)\omega_M\big) = 0.$$

So, if $a > 0$ then $(M, \omega_M) = 1$ and $a < 0$ then $(M, \omega_M) = -1$.

Proof. Define the surjective map

$$f : F \to m_0 \cap M \quad \text{by } f(e_1) = L, \quad f(e_i) = x_i \quad \forall i \geqslant 2.$$

We will see that f reduces to ω_0 modulo m_0. With $s = -1/2, t = 1/2$ we have, $1 = s(x_0 - 1) + t(x_0 + 1)$. So

$$(x_0 - 1) = s(x_0 - 1)^2 + t\bigl(x_0^2 - 1\bigr) = s(x_0 - 1)^2 + t\sum_{i=1}^{n} -x_i^2 \in m_0^2.$$

Therefore $L - (1 - a)x_1 \in m_0^2$. Also since $M \neq m_0$ we have $a \neq 1$. In fact $a < 1$. Hence f reduces to

$$\omega_0 : F/m_0F \to m_0/m_0^2.$$

Now define

$$\gamma_M : F/MF \to M/M^2 \quad \text{by } \gamma_M(e_1) = L, \quad \gamma_M(e_i) = x_i \quad \forall i \geqslant 2.$$

Since γ_M is the reduction of f, we have

$$(m_0, \omega_0) + (M, \gamma_M) = 0.$$

Let $\omega_M : F/MF \to M/M^2$ be as in the statement of the theorem. We will assume $a \neq 0$ or equivalently, $-1 < b < 1$. In this case, we prove that ω_M is a surjective map. (*Note below that for ω_M to be surjective, we need $a \neq 0$.*) We have

$$L = (1 - a)x_1 + b(x_0 - 1) = (1 - a)(x_1 - b) + b(x_0 - a).$$

We also have $a^2 + b^2 - 1 = 0$. Now again, with $s = -1/2a, t = 1/2a$ we have $1 = s(x_0 - a) + t(x_0 + a)$ and

$$(x_0 - a) = s(x_0 - a)^2 + t\bigl(x_0^2 - a^2\bigr) = s(x_0 - a)^2 + t\bigl(b^2 - x_1^2\bigr) - t\sum_{i=2}^{n} x_i^2.$$

Therefore, ω_M is surjective. Further,

$$\bigl(b^2 - x_1^2\bigr) = (b - x_1)(b + x_1) = (b - x_1)\bigl[2b - (b - x_1)\bigr] = 2b(b - x_1) - (b - x_1)^2.$$

So,

$$(x_0 - a) = s(x_0 - a)^2 + t\bigl[2b(b - x_1) - (b - x_1)^2\bigr] - t\sum_{i=2}^{n} x_i^2 = b/a(b - x_1) + w$$

for some $w \in M^2$. So,

$$(x_0 - a) - b/a(b - x_1) \in M^2.$$

Therefore, *modulo* M, we have

$$L = (1 - a)(x_1 - b) + b(x_0 - a) \equiv (1 - a)(x_1 - b) + b(b/a)(b - x_1)$$

or

$$L \equiv (x_1 - b)\big[1 - a - b^2/a\big] = (x_1 - b)\big[(a-1)/a\big].$$

So, γ_M and ω_M differ by an isomorphism of determinant $(a-1)/a$. Since $a - 1 < 0$, we have $\gamma_M = -sign(a)\omega_M$. Therefore,

$$(m_0, \omega_0) + \big(M, -sign(a)\omega_M\big) = 0.$$

Hence, if

$$a > 0 \quad \Rightarrow \quad (M, \omega_M) = -(M, -\omega_M) = (m_0, \omega_0) = 1$$

and

$$a < 0 \quad \Rightarrow \quad (M, \omega_M) = -(m_0, \omega_0) = -1.$$

So, the proof is complete. □

Remark 4.3. We will continue to use the notations of (3.2, 4.2). As we remarked in the proof of Theorem 4.2, if p is the north pole or the south pole and M is the corresponding maximal ideal, then ω_M, as defined in 4.2, will fail to define a local orientation. If $p = (0, \pm 1, 0, \dots, 0)$ is the north or the south pole, then $M = (x_0, x_1 \mp 1, x_2, \dots, x_n)$. For $p = N$ the north pole or $p = S$ the south pole, a natural local orientation is defined by:

$$\omega_p : F/MF \to M/M^2 \quad \text{where } \omega_p(e_1) = x_0, \quad \omega_p(e_i) = x_i \quad \forall i \geqslant 2.$$

Then $(M, \omega_p) = -1$ if p is the north pole and $(M, \omega_p) = 1$ if p is the south pole.

Proof. Let $p = N = (0, 1, 0, \dots, 0)$ be the north pole. Then $M = (x_0, x_1 - 1, x_2, \dots, x_n)$. Write $L = x_0 + x_1 - 1$. Then $m_0 \cap M = (L, x_2, \dots, x_n)$. Consider the surjective map $f : F \twoheadrightarrow m_0 \cap M$ given by these generators. Note that $L - x_0 = x_1 - 1 \in M^2$. This follows because $1 = -(x_1 - 1)/2 + (x_1 + 1)/2$. So, it follows that f reduces to ω_p. Similarly, f reduces to ω_0 on m_0. Therefore, $(M, \omega_p) = -1$.

If $p = S = (0, -1, 0, \dots, 0)$ is the south pole, then $M = (x_0, x_1 + 1, x_2, \dots, x_n)$. We replace the equation of L by $L = x_0 - x_1 - 1$. Then $L - x_0 = -(x_1 + 1) \in M^2$. It follows, that f reduced to ω_p. Similarly, $L + x_1 = x_0 - 1 \in m_0^2$. This shows that f reduces to $-\omega_0$ on m_0. Therefore, $(M, \omega_p) - (m_0, \omega_0) = 0$. So, $(M, \omega_p) = 1$. The proof is complete. □

Remark 4.4. In the statement of (4.2), we assumed that $p = (a, b, 0, \dots, 0) \in \mathbb{S}^1 \subseteq \mathbb{S}^n$. Now suppose $p \notin \mathbb{S}^1$ is any point in $\mathbb{S}^n$. Let $e_0, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. So, m_0 is the ideal of e_0. There is an orthonormal transformation $(E_0, \dots, E_n)^t = A(e_0, \dots, e_n)^t$ of $\mathbb{R}^{n+1}$ such that $E_0 = e_0$ and $p = aE_0 + bE_1$. Write $A = (a_{ij}\colon i, j = 0, \dots, n)$. It follows, $a_{00} = 1$ and $a_{0j} = a_{j0} = 0$ for all $j = 1, \dots, n$. We can assume $\det A = 1$.

Write $(Y_0, \dots, Y_n)^t = A(X_0, \dots, X_n)^t$. Then $Y_0 = X_0$, and for $i = 1, \dots, n$ we have $Y_i = \sum_{j=1}^n a_{ij} X_j$. It follows that in the Y-coordinates, $e_0 = (1, 0, \dots 0)$ and $p = (a, b, 0, \dots, 0)$. Let ω' be the local orientation on m_0 defined by $(Y_1, \dots, Y_n)$. Since $\det A = 1$, it follows that (m_0, ω') is the standard generator of $E(A_n)$ (see 3.2). Now, we can write down local orientation on M in Y-coordinates, as in (4.2) and the rest of (4.2) remains valid.

5. Bott periodicity

In this section, we will give some background on Bott periodicity, mostly from [ABS,F,Sw1]. We will recall the definition of the Clifford algebras of a quadratic forms.

Definition 5.1. (See [ABS].) Let k be a commutative ring and (V,q) be a quadratic k-module. Then a k-algebra $C(q)$ with an injective map $i: V \to C(q)$ is said to be the **Clifford algebra** of q, if $i(x)^2 = q(x)$ and if it is universal with respect to this property. *Following are some of the properties of* $C(q)$:

1. Note $C(q) = \frac{T(V)}{I(q)}$ where $T(V)$ is the tensor algebra of V and $I(q)$ is the two-sided ideal of $T(V)$ generated by $\{x^2 - q(x){:}\ x \in V\}$.
2. The $\mathbb{Z}_2$-grading on $T(V)$ induces a $\mathbb{Z}_2$-grading on $C(q)$ as $C(q) = C_0(q) \oplus C_1(q)$ where $C_0(q)$ denotes the even part and $C_1(q)$ denotes the odd part.
3. Also, if (V,q') is another quadratic k-module, then
$$C(q \perp q') \approx C(q) \widehat{\otimes} C(q') \quad \text{as graded rings.}$$
This means, the multiplication structure is given by $(u \otimes x_i)(y_j \otimes v) = (-1)^{ij} u y_j \otimes x_i v$ for $x_i \in C_i(q')$, $y_j \in C_j(q)$.
4. If $V = \bigoplus_{i=1}^n k e_i$ is free with basis e_i then
$$C(q) = \bigoplus_{0 \leqslant i_1 < \cdots < i_r \leqslant n;\ r \geqslant 0} k e_{i_1 i_2 \ldots i_r}.$$
We will mostly be concerned with this case where V is free. Further, if $q = \sum_{i=1}^n a_i X_i^2$ is a diagonal form, then
$$\forall i, j = 1, \ldots, n; \quad \text{with } i \neq j, \quad e_i^2 = a_i \quad \text{and} \quad e_i e_j = -e_j e_i.$$

Notations 5.2. We will introduce some notations for our convenience.

1. Let k be a commutative ring and (V,q) be a quadratic k-module and $V = \bigoplus_{i=1}^n k e_i$ is free and $q = q(X_1, \ldots, X_n)$. As in [Sw1], we denote $R_k(q) = R(q) = \frac{k[X_1,\ldots,X_n]}{(q-1)}$. We usually drop the subscript k and use the notation $R(q)$.
2. Suppose C is a ring. Then:
   (a) The category of finitely generated (left) C-modules will be denoted by $\mathcal{M}(C)$.
   (b) If C has a $\mathbb{Z}_2$-grading, the category of finitely generated (left) $\mathbb{Z}_2$-graded C-modules will be denoted by $\mathcal{G}(C)$.
   (c) The category of finitely generated (left) projective C-modules will be denoted by $\mathcal{P}(C)$.
3. Given a category $\mathcal{C}$, with exact sequences, the Grothendieck group of $\mathcal{C}$ will be denoted $K(\mathcal{C})$.
4. Given a ring R, we will denote $K_0(R) = K(\mathcal{P}(R))$. If the rank map $rank: K_0(R) \to \mathbb{Z}$ is defined, we denote $\widetilde{K_0}(R) = rank^{-1}(0)$.
5. Given a connected smooth real manifold X, the Grothendieck group of the category of real vector bundles over X will be denoted by $KO(X)$. As above, $\widetilde{KO}(X)$ will denote the kernel of the rank map.
6. For a commutative noetherian ring R of dimension n and $X = Spec(R)$, the Chow group of zero cycles will be denoted by $CH_0(R)$ or $CH_0(X)$. When the top Chern class is defined, $C_0 = C^n: K_0(R) \to CH_0(R)$ will denote the homomorphism defined by the top Chern class.

5.1. *Generators of* $\widetilde{K_0}(A_n)$

In this subsection, we describe the generators of $\widetilde{K_0}(A_n)$.

Proposition 5.3. *Let k be ring with $1/2 \in k$ and let $q = a_1X_1^2 + \cdots + a_nX_n^2$ be a diagonal form. Let $e_1, \ldots, e_n$ denote the canonical generators of $C(-q)$. Let $M = M^0 \oplus M^1 \in \mathcal{G}(C(-q))$ be a $\mathbb{Z}_2$-graded $C(-q)$-module and*

$$N = R(q \perp 1) \otimes_k M = N^0 \oplus N^1$$

where

$$N^0 = R(q \perp 1) \otimes_k M^0, \qquad N^1 = R(q \perp 1) \otimes_k M^1.$$

Let x_i denote the image of X_i in $R(q \perp 1)$. Define

$$\varphi(x) = \sum_{i=1}^{n} x_i(1 \otimes e_i) : N^1 \to N^0, \qquad \psi(x) = \sum_{i=1}^{n} x_i(1 \otimes e_i) : N^0 \to N^1.$$

Write $q \perp 1 = q + X_0^2$ and let $y = x_0 = image(X_0) \in R(q \perp 1)$. Define

$$\rho_M = \rho = \frac{1}{2}\begin{pmatrix} 1-y & \varphi(x) \\ -\psi(x) & 1+y \end{pmatrix} : N \to N.$$

That means, for $n_0 \in N^0$, $n_1 \in N^1$ we have

$$\rho(n_0, n_1) = \bigl((1-y)n_0 + \varphi(x)n_1, -\psi(x)n_0 + (1+y)n_1\bigr)/2.$$

Then,

$$\varphi\psi(x) = -q(x) : N^0 \to N^0, \qquad \psi\varphi(x) = -q(x) : N^1 \to N^1$$

and ρ is an idempotent homomorphism.

Proof. By direct multiplication, it follows $\varphi\psi = -q(x)$, $\psi\varphi = -q(x)$. Again, we have

$$\rho^2 = \frac{1}{4}\begin{pmatrix} (1-y)^2 - \varphi(x)\psi(x) & 2\varphi(x) \\ -2\psi(x) & -\psi(x)\varphi(x) + (1+y)^2 \end{pmatrix}$$

which is

$$\frac{1}{4}\begin{pmatrix} (1-y)^2 + q(x) & 2\varphi(x) \\ -2\psi(x) & q(x) + (1+y)^2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2(1-y) & 2\varphi(x) \\ -2\psi(x) & 2(1+y) \end{pmatrix} = \rho.$$

This completes the proof. □

Definition 5.4. We use the notation as in Proposition 5.3. Define a functor

$$\alpha : \mathcal{G}\bigl(C(-q)\bigr) \to \mathcal{P}\bigl(R(q \oplus 1)\bigr) \quad \text{by } \alpha(M) = kernel(\rho_M).$$

Since, $k \to R(q \perp 1)$ is flat, it follows easily that α is an exact functor. Therefore, α induces a homomorphism

$$\Theta_q : K\bigl(\mathcal{G}\bigl(C(-q)\bigr)\bigr) \to \widetilde{K_0}\bigl(R(q \perp 1)\bigr)$$

where $\forall M \in \mathcal{G}(C(-q))$

$$\Theta_q\bigl([M]\bigr) = \bigl[\alpha(M)\bigr] - rank\bigl(\alpha(M)\bigr).$$

Before we proceed, we will describe $\alpha(M)$ in (5.4) by patching two trivial bundles on the two (algebraic) hemispheres along the (algebraic) equator, as follows.

Proposition 5.5. *We will use all the notations of* (5.3, 5.4)*. We have* $q = q(X_1, \ldots, X_n)$, $q \perp 1 = q + X_0^2$ *and* $y = x_0 = image(X_0)$*. Let* $M = M^0 \oplus M^1 \in \mathcal{G}(C(-q))$, $N = N^0 \oplus N^1$, φ, ψ *be as in* (5.3)*. Write*

$$F^0 = N^0_{1+y} = R\big(q + X_0^2\big)_{1+y} \otimes M^0, \qquad F^1 = N^1_{1-y} = R\big(q + X_0^2\big)_{1-y} \otimes M^1.$$

Then $\alpha(M)$ *is obtained by patching* F^0 *and* F^1 *via* ψ_{1-y^2}*. In particular, if k is a field,* $rank(\alpha(M)) = \dim_k M/2 = \dim_k M_0$.

Proof. Define

$$\sigma : F^0_{1-y} \to F^1_{1+y} \quad \text{by } \sigma(n_0) = \frac{-\psi(n_0)}{1+y}$$

and

$$\eta : F^1_{1+y} \to F^0_{1-y} \quad \text{by } \sigma(n_1) = \frac{\varphi(n_1)}{1-y}.$$

Then for $n_0 \in F^0_{1-y}$, we have

$$\eta\sigma(n_0) = \frac{-\varphi\psi(n_0)}{1-y^2} = \frac{q(x_1, \ldots, x_n)(n_0)}{1-y^2} = n_0.$$

So, $\eta\sigma = 1$ and similarly, $\sigma\eta = 1$. Consider fiber product

$$\begin{array}{ccc} R(q \perp 1) & \longrightarrow & R(q \perp 1)_{1-y} \\ \downarrow & & \downarrow \\ R(q \perp 1)_{1+y} & \longrightarrow & R(q \perp 1)_{1-y^2} \end{array}$$

and define $P(\sigma)$ by the patching diagram

$$\begin{array}{ccccc} P(\sigma) & \longrightarrow & & \longrightarrow & F^1 \\ \downarrow & & & & \downarrow \\ F^0 & \longrightarrow & F^0_{1-y} & \xrightarrow{\sigma} & F^1_{1+y}. \end{array}$$

Define

$$f_0 : F^0 = N^0_{1+y} \to \alpha(M)_{1+y} \quad \text{by } f_0(n_0) = \left(n_0, \frac{\psi(n_0)}{1+y}\right)$$

and

$$f_1 : F^1 = N^1_{1-y} \to \alpha(M)_{1-y} \quad \text{by } f_1(n_1) = -\left(\frac{-\varphi(n_1)}{1-y}, n_1\right).$$

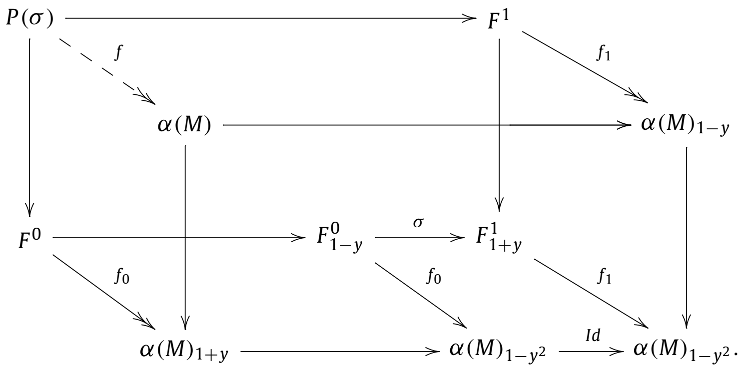
We check that f_0, f_1 are well-defined isomorphisms. Recall that

$$\rho_M = \rho = \frac{1}{2}\begin{pmatrix} 1-y & \varphi(x) \\ -\psi(x) & 1+y \end{pmatrix}.$$

Using the identities $q(x) + y^2 = 1$, $\varphi\psi = -q$, $\psi\varphi = -q$, direct computation shows

$$f_0(n_0) = \left(n_0, \frac{\psi(n_0)}{1+y}\right), \qquad f_1(n_1) = -\left(\frac{-\varphi(n_1)}{1-y}, n_1\right) \in kernel(\rho) = \alpha(M).$$

So, f_0, f_1 are well defined. Clearly, f_0, f_1 are injective and their surjectivity can also be checked directly. Now consider the patching diagram:



We check $f_1\sigma = f_0$. For $n_0 \in F^0_{1-y}$, we have

$$\begin{aligned} f_1\sigma(n_0) &= -\left(\frac{-\varphi(\frac{-\psi(n_0)}{1+y})}{1-y}, \frac{-\psi(n_0)}{1+y}\right) \\ &= -\left(\frac{-q(x)}{1-y^2}(n_0), \frac{-\psi(n_0)}{1+y}\right) = \left(n_0, \frac{\psi(n_0)}{1+y}\right) = f_0(n_0), \end{aligned}$$

since $q(x) = 1 - y^2$. In this patching diagram above, f is obtained by properties of fiber product diagrams. Now, since f_0, f_1 are isomorphisms, $f : P(\sigma) \to \alpha(M)$ is also an isomorphism. Let $P(\psi_{1-y^2})$ denote the projective module obtained by patching F^0 and F^1 via ψ_{1-y^2}. Since $P(\sigma) \approx P(\psi_{1-y^2})$, the proposition is established. □

5.2. Further background on Bott periodicity

For the benefit of the readership, in this subsection, we give some further background on Bott periodicity from [ABS,Sw1]. We establish that Θ_q defined in (5.4) is a surjective homomorphism, when $q = \sum_{i=1}^n X_i^2 \in \mathbb{R}[X_1, \dots, X_n]$. In this case, $R(q \perp 1) = A_n$. We have the following proposition.

Proposition 5.6. *(See [ABS].) We continue to use notations as in* (5.3, 5.4)*. The composition*

$$K(\mathcal{G}(C(-q \perp -1))) \longrightarrow K(\mathcal{G}(C(-q))) \xrightarrow{\Theta_q} \widetilde{K_0}(R(q \perp 1))$$

is zero. Further, as in [ABS,Sw1], *define* $ABS(q)$ *by the exact sequence*

$$K(\mathcal{G}(C(-q \perp -1))) \longrightarrow K(\mathcal{G}(C(-q))) \longrightarrow ABS(q) \longrightarrow 0.$$

So, there is a homomorphism $\alpha_q : ABS(q) \to \widetilde{K_0}(R(q \perp 1))$ *such that the diagram*

$$\begin{array}{ccccccc} K(\mathcal{G}(C(-q \perp -1))) & \longrightarrow & K(\mathcal{G}(C(-q))) & \longrightarrow & ABS(q) & \longrightarrow & 0 \\ & & & \searrow^{\Theta_q} & \downarrow{\scriptstyle \alpha_q} & & \\ & & & & \widetilde{K_0}(R(q \perp 1)) & & \end{array}$$

commute.

Proof. We reinterpret the proof of Swan [Sw1, 7.7], and sketch a direct proof. Let $e_1, \ldots, e_n$ denote the canonical generators of $C(-q)$ and f be the other generator of $C(-q \perp -1)$. Let $M = M^0 \oplus M^1 \in \mathcal{G}(C(-q - Z^2))$. Write $N = M \otimes R(q \perp X_0^2)$ and define $f^* : M \to M$ such that $f^*_{|M^0} = f_{|M^0}, f^*_{|M^1} = -f_{|M^1}$ and similarly define e_i^*.

With the notations as in (5.3, 5.4), we have $\rho = \frac{1-\gamma}{2}$, where

$$\gamma = \begin{pmatrix} y & -\varphi \\ \psi & -y \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix} + \sum_{i=1}^{n} x_i e_i^*.$$

So, $\alpha(M) = \{w \in N\colon\ w = \gamma(w)\}$. Also note $\gamma^2 = 1$. Define

$$L^0 = \ker(f^* - 1) = \{(m_0, fm_0)\colon\ m_0 \in M^0\} = \{(-fm_1, m_1)\colon\ m_1 \in M^1\}$$

and

$$L^1 = \ker(f^* + 1) = \{(fm_1, m_1)\colon\ m_1 \in M^1\} = \{(m_0, -fm_0)\colon\ m_0 \in M^0\}.$$

So, $M = L^0 \oplus L^1$ and $N = Q^0 \oplus Q^1$ with $Q^0 = L^0 \otimes R(q \perp X_0^2)$, $Q^1 = L^1 \otimes R(q \perp X_0^2)$. We check that *diagonal*$(1,-1)L^0 \subseteq L^1$ and $e_i^* L^0 \subseteq L^1$. So, $\gamma(Q^0) \subseteq Q^1$ and similarly, $\gamma(Q^1) \subseteq Q^0$.

We have $Q^0 \cap \alpha(M) \subseteq Q^0 \cap Q^1 = 0$, and for $n = n^0 + n^1$ with $n^i \in Q^i$, we have $n = (n^0 - \gamma(n^1)) + (n^1 + \gamma(n^1)) \in Q^0 + \alpha(M)$. So, $N = Q^0 \oplus \alpha(M)$ and $\alpha(M) \approx N/Q^0 = Q^1$ is free. The proof is complete. □

The following theorem relates topological and algebraic K-groups.

Theorem 5.7. *(See [ABS,F,Sw1].) Let* $q = X_1^2 + \cdots + X_n^2 \in \mathbb{R}[X]$. *Then the following is a commutative diagram of isomorphisms*:

$$\begin{array}{ccc} ABS(q) & & \\ \downarrow{\scriptstyle \alpha_q}\ \wr & \searrow^{\sim} & \\ \widetilde{K_0}(A_n) & \xrightarrow{\sim} & \widetilde{KO}(\mathbb{S}^n). \end{array}$$

In particular, the homomorphism

$$\Theta_q : K\big(\mathcal{G}\big(C(-q)\big)\big) \to \widetilde{K_0}\big(R(q \perp 1)\big) \text{ is surjective.}$$

Proof. Note that $R(q \perp 1) = \frac{\mathbb{R}[X_0, X_1, \ldots, X_n]}{(X_0^2 + q - 1)} = A_n$. The diagonal isomorphism is established in [ABS]. The horizontal (equivalently the vertical) homomorphism is an isomorphism due to the theorem of Swan [Sw2, Theorem 3]. So, the proof is complete. □

5.3. Patching matrices

Proposition 5.5 exhibits the importance of a suitable description of the homomorphism ψ_{1-y^2}, as a matrix. We are interested in the cases of real spheres $Spec(A_n)$ with $K_0(A_n) \approx KO(\mathbb{S}^n)$ nontrivial. So, $q = \sum_{i=1}^n X_i^2 \in \mathbb{R}[X_1, \ldots, X_n]$ and $n = 8r, 8r+1, 8r+2, 8r+4$. We only need to consider the irreducible $\mathbb{Z}_2$-graded modules M over $C_n = C(-q)$.

We will include the following from [ABS] regarding Bott periodicity.

Theorem 5.8. *Let $q_n = q = \sum_{i=1}^n X_i^2 \in \mathbb{R}[X_1, \ldots, X_n]$ and $C_n = C(-q)$. Write $a_n = (\dim_{\mathbb{R}} \mathcal{I}_n)/2 = \dim_{\mathbb{R}} \mathcal{I}_n^0$ where $\mathcal{I}_n = \mathcal{I}_n^0 \oplus \mathcal{I}_n^1$ is an irreducible $\mathbb{Z}_2$-graded C_n-module. The following chart summarizes some information regarding $C_n = C(-q)$:*

n	C_n	$K(\mathcal{G}(C_n))$	$ABS(q_n)$	a_n
1	$\mathbb{C}$	$\mathbb{Z}$	$\mathbb{Z}_2$	1
2	$\mathbb{H}$	$\mathbb{Z}$	$\mathbb{Z}_2$	2
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{Z}$	0	4
4	$\mathbb{M}_2(\mathbb{H})$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	4
5	$\mathbb{M}_4(\mathbb{C})$	$\mathbb{Z}$	0	8
6	$\mathbb{M}_8(\mathbb{R})$	$\mathbb{Z}$	0	8
7	$\mathbb{M}_8(\mathbb{R}) \oplus \mathbb{M}_8(\mathbb{R})$	$\mathbb{Z}$	0	8
8	$\mathbb{M}_{16}(\mathbb{R})$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	8

Further

$$C_{n+8} \approx C_8 \otimes C_n \approx \mathbb{M}_{16}(\mathbb{R}) \otimes C_n$$

and

$$K\big(\mathcal{G}(C_{n+8})\big) \approx K\big(\mathcal{G}(C_n)\big), \qquad ABS(q_{n+8}) \approx ABS(q_8), \qquad a_{n+8} = 16a_n.$$

The following corollary will be of some interest to us.

Corollary 5.9. *With notations as above* (5.8)*, for nonnegative integers r, we have*

$$C_{8r} \approx \mathbb{M}_{16^r}(\mathbb{R}), \qquad C_{8r+1} \approx \mathbb{M}_{16^r}(\mathbb{C}), \qquad C_{8r+2} \approx \mathbb{M}_{16^r}(\mathbb{H}), \qquad C_{8r+4} \approx \mathbb{M}_{2*16^r}(\mathbb{H})$$

and

$$a_{8r} = 16^r/2, \qquad a_{8r+1} = 16^r, \qquad a_{8r+2} = 2 * 16^r, \qquad a_{8r+4} = 4 * 16^r.$$

Proof. First part follows from (5.8). For the later part, let I_n be an irreducible C_n-module. Note that for $n = 8r, 8r+1, 8r+2, 8r+4$, Clifford algebras C_n are matrix algebras. Form general theory, I_n is isomorphic to the module of column vectors. So, $\dim_{\mathbb{R}} I_n$ is easily computable. One can also establish, by induction, that there are $\mathbb{Z}_2$-graded C_n-modules $\mathcal{I}_n$ with $\dim \mathcal{I}_n = \dim I_n$. So, $\mathcal{I}_n$ is irreducible and $a_n = (\dim \mathcal{I}_n)/2 = (\dim I)/2$. This completes the proof. □

Theorem 5.10. *Following chart describes the* $\widetilde{KO}(\mathbb{S}^n)$ *groups.*

n	$8r$	$8r+1$	$8r+2$	$8r+3$	$8r+4$	$8r+5$	$8r+6$	$8r+7$
$\widetilde{KO}(\mathbb{S}^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

Proof. It follows from Theorem 5.7 and Corollary 5.9. For a complete proof the reader is referred to the book [H] or [ABS]. □

Now we state our result on the matrix representation of ψ. In fact, we will do it more formally at the polynomial ring level.

Proposition 5.11. *Let* $n = 8r, 8r+1, 8r+2, 8r+4$ *be a nonnegative integer and* $q(X) = \sum_{i=1}^{n} X_i^2 \in \mathbb{R}[X_1,\dots,X_n]$. *As before,* $C_n = C(-q)$ *and* $e_1,\dots,e_n$ *are the canonical generators of* C_n.

Let $M = M^0 \oplus M^1$ *be a* $\mathbb{Z}_2$*-graded irreducible* C_n*-module. Write* $m = a_n = \dim_{\mathbb{R}} M^0$. *Define*

$$\Psi = \Psi_n = \sum_{i=1}^{n} X_i(1 \otimes e_i) : \mathbb{R}[X_1,\dots,X_n] \otimes M^0 \to \mathbb{R}[X_1,\dots,X_n] \otimes M^1$$

and

$$\Phi = \Phi_n = \sum_{i=1}^{n} X_i(1 \otimes e_i) : \mathbb{R}[X_1,\dots,X_n] \otimes M^1 \to \mathbb{R}[X_1,\dots,X_n] \otimes M^0.$$

Then, there are choices of bases $u_1,\dots,u_m$ *of* M^0 *and* $v_1,\dots,v_m$ *of* M^1 *such that the matrix* Γ *of* Ψ *and the matrix* Δ *of* Φ *have the following properties*:

1. *Each row and column of* Γ, Δ *has exactly n nonzero entries and for* $i = 1,\dots,n$ *exactly one entry in each row and column is* $\pm X_i$.
2. *As a consequence,* $\Delta = -\Gamma^t$ *and they are orthogonal matrices.*

Proof of (1) ⇒ (2). Suppose we have bases of M^0, M^1 as above that satisfy (1). Write $u = (u_1,\dots,u_m)^t$, $v = (v_1,\dots,v_m)^t$. Then we have $-q(x)(u) = \Phi(u) = \Gamma\Delta(v)$. So, $\Gamma\Delta = -q$. Let Γ_i^r denote the ith-row of Γ and Δ_i^c denote the ith-column of Δ. So, $\Gamma_i^r \Delta_i^c = -\sum_{i=1}^{n} X_i^2$. Comparing two sides, we have $\Gamma_i^r = -(\Delta_i^c)^t$. So, $\Delta = -\Gamma^t$. Since $\Gamma\Delta = -q$, we have Γ, Δ are orthogonal matrices. Proof of (1) comes later. □

Before we get into the proof of 5.11, we wish to deal with the initial cases of $n = 2, 4, 8$ with some extra details.

Lemma 5.12. *Let* $q = X_1^2 + X_2^2 \in \mathbb{R}$ *and* $C_2 = C(-q)$. *Then* $C_2 = \mathbb{H}$, *where the canonical basis of* $\mathbb{R}^2 \subseteq C_2$ *is* $e_1 = i, e_2 = j$ *and the matrices of* Ψ_2 *and* Φ_2 *have the property* (1) *of Proposition* 5.11.

Proof. A matrix representation of Ψ is given by

$$\begin{pmatrix} \Psi(1) \\ \Psi(k) \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix}.$$

Similarly, we can get a matrix representation of Φ. The proof is complete. □

Now we consider the case $n = 4$. We will include additional information that will be useful later.

Lemma 5.13. *Let* $q = \sum_{i=1}^{4} X_i^2 \in \mathbb{R}[X_1, X_2, X_3, X_4]$ *and* $C_4 = C(-q)$. *Then*:

1. *We have* $C_4 = \mathbb{M}_2(\mathbb{H})$ *where the canonical basis of* $\mathbb{R}^4 \subseteq C_4$ *is given as follows*:

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad e_2 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix},$$

and

$$e_3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \qquad e_4 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

2. *Following* [F], *write* $w_4 = 1 + e_1e_2e_3e_4$. *We have the following identities,*

$$e_1e_2e_3e_4w_4 = w_4, \qquad -e_1e_2w_4 = e_3e_4w_4, \qquad e_1e_3w_4 = e_2e_4w_4, \qquad -e_1e_4w_4 = e_2e_3w_4.$$

3. *Let* $M = C_4w_4$. *Then, M is irreducible.*
4. *Then* Ψ_4, Φ_4 *have the desired property* (1) *of* (5.11).

Proof. Proof of (1) follows by direct checking. Identities in (2) are obvious. The statement (3) is a theorem of Fossum [F]. To see a proof, let $M = Cw_4 = M^0 \oplus M^1$ be the $\mathbb{Z}_2$-graded decomposition of M. We have

$$M^0 = \mathbb{R}w_4 + \sum_{i<j} \mathbb{R}e_ie_jw_4 + \mathbb{R}e_1e_2e_3e_4w_4; \qquad M^1 = \sum \mathbb{R}e_iw_4 + \sum_{i<j<k} \mathbb{R}e_ie_je_kw_4.$$

Using the identities above, it is easy to check that a basis of M^0 is given by

$$u_1 = w_4, \qquad u_2 = -e_1e_2w_4, \qquad u_3 = -e_1e_3w_4, \qquad u_4 = -e_1e_4w_4$$

and a basis of M^1 is given by

$$v_1 = e_1w_4 = e_1u_1, \qquad v_2 = e_2w_4, \qquad v_3 = e_3w_4, \qquad v_4 = e_4w_4 = e_1e_2e_3w_4.$$

Since dimension of an irreducible module over $C_4 = \mathbb{M}_2(\mathbb{H})$ is eight, M is irreducible. This establishes (3).

Now, we write down the matrix of Ψ_4, Φ_4 with respect to the above bases:

$$\begin{pmatrix} \Psi(u_1) \\ \Psi(u_2) \\ \Psi(u_3) \\ \Psi(u_4) \end{pmatrix} = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ -X_2 & X_1 & X_4 & -X_3 \\ -X_3 & -X_4 & X_1 & X_2 \\ -X_4 & X_3 & -X_2 & X_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

Also

$$\begin{pmatrix} \Phi(v_1) \\ \Phi(v_2) \\ \Phi(v_3) \\ \Phi(v_4) \end{pmatrix} = \begin{pmatrix} -X_1 & X_2 & X_3 & X_4 \\ -X_2 & -X_1 & X_4 & -X_3 \\ -X_3 & -X_4 & -X_1 & X_2 \\ -X_4 & X_3 & -X_2 & -X_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

The proof is complete. □

Now we will consider the case of $n = 8$.

Lemma 5.14. *Let $q = \sum_{i=1}^{8} X_i^2 \in \mathbb{R}[X_1, \ldots, X_8]$ and $C_8 = C(-q)$. Let $E_1, E_2, \ldots, E_8$ be the canonical generators of C_8. Following Fossum* [F], *let*

$$w_8 = (1 + E_1E_2E_5E_6 + E_1E_3E_5E_7 + E_1E_4E_5E_8)(1 + E_1E_2E_3E_4)(1 + E_5E_6E_7E_8).$$

Then $M = C_8w_8$ is irreducible and Ψ_8, Φ_8 have the desired property (1) *of* (5.11)*.*

Proof. It is a theorem of Fossum [F], that $M = C_8w_8$ is irreducible. A basis of M is also given in [F]. We will describe this basis of M and provide a proof of irreducibility. By (5.1), there is an isomorphism $C_8 \approx C_4 \widehat{\otimes} C_4$. As in (5.13), we denote the canonical generators of C_4 by e_1, e_2, e_3, e_4 and $w_4 = 1 + e_1e_2e_3e_4$.

We will identify $C_8 = C_4 \widehat{\otimes} C_4$. Under this identification, the canonical generators of C_8 are given by $E_i = e_i \otimes 1$ for $i = 1, 2, 3, 4$ and $E_i = 1 \otimes e_{i-4}$ for $i = 5, 6, 7, 8$. Also, w_8 is identified as

$$w_8 = (w_4 \otimes w_4 + e_1e_2w_4 \otimes e_1e_2w_4 + e_1e_3w_4 \otimes e_1e_3w_4 + e_1e_4w_4 \otimes e_1e_4w_4).$$

We denote $w = w_8$. Let $M = C_8w_8 = M^0 \oplus M^1$ be the $\mathbb{Z}_2$-graded decomposition of M. We denote the basis [F] of M^0 by u_i as in the table:

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
$=$	w	E_1E_2w	E_1E_3w	E_2E_3w	E_1E_5w	E_2E_5w	E_3E_5w	$E_1E_2E_3E_5w$

and similarly, v_i will denote the basis [F] of M^1 as follows:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$=$	E_1w	E_2w	E_3w	$E_1E_2E_3w$	E_5w	$E_1E_2E_5w$	$E_1E_3E_5w$	$E_2E_3E_5w$

The following multiplication table will be useful for our purpose:

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
E_1	v_1	$-v_2$	$-v_3$	v_4	$-v_5$	v_6	v_7	$-v_8$
E_2	v_2	v_1	$-v_4$	$-v_3$	$-v_6$	$-v_5$	v_8	v_7
E_3	v_3	v_4	v_1	v_2	$-v_7$	$-v_8$	$-v_5$	$-v_6$
E_4	v_4	$-v_3$	v_2	$-v_1$	v_8	$-v_7$	v_6	$-v_5$
E_5	v_5	v_6	v_7	v_8	v_1	v_2	v_3	v_4
E_6	v_6	$-v_5$	$-v_8$	v_7	v_2	$-v_1$	$-v_4$	v_3
E_7	v_7	v_8	$-v_5$	$-v_6$	v_3	v_4	$-v_1$	$-v_2$
E_8	$-v_8$	v_7	$-v_6$	v_5	v_4	$-v_3$	v_2	$-v_1$

This table is constructed by using the identities in (5.13). For the benefit of the reader, we give proof of one of them. We prove $E_6u_1 = E_6w = v_6$. First, we have $E_6w = -E_5(E_5E_6w) = -E_5(1 \otimes e_1e_2)w$. We compute

$$\begin{aligned} E_5E_6w &= (1 \otimes e_1e_2)w \\ &= (w_4 \otimes e_1e_2w_4 + e_1e_2w_4 \otimes e_1e_2e_1e_2w_4 + e_1e_3w_4 \otimes e_1e_2e_1e_3w_4 + e_1e_4w_4 \otimes e_1e_2e_1e_4w_4) \\ &= (-e_1e_2 \otimes 1)[e_1e_2w_4 \otimes e_1e_2w_4 + w_4 \otimes w_4 + e_2e_3w_4 \otimes e_2e_3w_4 + e_2e_4w_4 \otimes e_2e_4w_4] \\ &= -E_1E_2\big[e_1e_2w_4 \otimes e_1e_2w_4 + w_4 \otimes w_4 + (-e_1e_4w_w \otimes -e_1e_4w_4) + e_1e_3w_4 \otimes e_1e_3w_4\big]. \end{aligned}$$

Therefore, $E_5E_6w = (1 \otimes e_1e_2)w = -E_1E_2w$. So,

$$E_6w = -E_5(E_5E_6w) = -E_5(-E_1E_2w) = E_1E_2E_5w = v_6.$$

This establishes $E_6u_1 = v_6$.

A similar multiplication table (E_iv_j) can be constructed using the fact $E_iv_j = u_k \Leftrightarrow -v_j = E_iu_k$. This shows that the vector space V generated by $\{u_i, v_j\colon\ i, j = 1, \dots, 8\}$ is a C_8-(left)module. So $V = M$. Since, an irreducible C_8-module has dimension sixteen, M is irreducible.

Now we compute the matrix Γ of Ψ with respect to these bases. We have, $\Psi(u_i) = \sum_{j=1}^{8} X_jE_ju_i$. The (i, j)th entry of the matrix Γ of Ψ is $\pm X_k$ if and only if $E_ku_i = \pm v_j$. For a fixed i there is exactly one j such that $E_ku_i = \pm v_j$ and similarly for a fixed j there is exactly one i such that $E_ku_i = \pm v_j$. So, for $k = 1, \dots, 8$; $\pm X_k$ appears exactly once in each row and column. In fact, The matrix of Ψ is

$$\Gamma = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & -X_8 \\ X_2 & -X_1 & -X_4 & X_3 & -X_6 & X_5 & X_8 & X_7 \\ X_3 & X_4 & -X_1 & -X_2 & -X_7 & -X_8 & X_5 & -X_6 \\ -X_4 & X_3 & -X_2 & X_1 & X_8 & -X_7 & X_6 & X_5 \\ X_5 & X_6 & X_7 & X_8 & -X_1 & -X_2 & -X_3 & X_4 \\ -X_6 & X_5 & -X_8 & X_7 & -X_2 & X_1 & -X_4 & -X_3 \\ -X_7 & X_8 & X_5 & -X_6 & -X_3 & X_4 & X_1 & X_2 \\ -X_8 & -X_7 & X_6 & X_5 & -X_4 & -X_3 & X_2 & -X_1 \end{pmatrix}.$$

Similar argument can be given for Φ. So, Ψ, Φ have the property (1) of (5.11).

We remark that the property (2) of (5.11) was established as a consequence of property (1). Alternately, we can use the fact $E_iv_j = u_k \Leftrightarrow -v_j = E_iu_k$ to establish property (2). This completes the proof of (5.14). □

Now we are ready to give a complete proof of Proposition 5.11.

Proof of 5.11. We already proved (1) ⇒ (2). So, we only need to prove (1). The case $n = 1$, is obvious and Lemmas 5.12, 5.13, 5.14, respectively, establish the proposition in the cases $n = 2, 4, 8$.

We will use induction, i.e. we assume that (1) of the proposition is valid for some $m = 8r$, $8r + 1, 8r + 2, 8r + 4$ and prove that the same is valid for $n = m + 8$.

First, we set up some notations. For a matrix A, the ith-row will be denoted by ${}_rA_i$ and the ith-column will be denoted by ${}_cA_i$. We have $m + 8$ variables $X_1, \dots, X_m, X_{m+1}, \dots, X_{m+8}$. For $i = 1, \dots, 8$, we will write $Y_i = X_{m+i}$. In our cases, there is only one irreducible $\mathbb{Z}_2$-graded C_m-module, which will be denoted by $M(m) = M(m)^0 \oplus M(m)^1$. We have $C_{m+8} = C_m \widehat{\otimes} C_8$. Comparing dimensions (see 5.9), we have $M(m + 8) = M(m) \widehat{\otimes} M(8)$.

Write $N = \dim_{\mathbb{R}} M(m)^0$. We assume that there are bases $u_1, \dots, u_N$ of $M(m)^0$ and $v_1, \dots, v_N$ of $M(m)^1$ and bases $\mu_1, \dots, \mu_8$ of $M(8)^0$ and $\nu_1, \dots, \nu_8$ of $M(8)^1$ such that

$$\begin{pmatrix} \Psi_m(u_1) \\ \cdots \\ \cdots \\ \Psi_m(u_N) \end{pmatrix} = A(X) \begin{pmatrix} v_1 \\ \cdots \\ \cdots \\ v_N \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Psi_8(\mu_1) \\ \cdots \\ \cdots \\ \Psi_8(\mu_8) \end{pmatrix} = B(Y) \begin{pmatrix} \nu_1 \\ \cdots \\ \cdots \\ \nu_8 \end{pmatrix}$$

where $A(X) = (a_{ij}(X_1, \dots, X_m))$ and $B(Y) = (b_{ij}(Y_1, \dots, Y_8))$ have the properties Γ of the proposition. Also

$$M(m + 8)^0 = M(m)^0 \otimes M(8)^0 \oplus M(m)^1 \otimes M(8)^1 \quad \text{with basis } u_i \otimes \mu_j,\ v_i \otimes \nu_j$$

and

$$M(m + 8)^1 = M(m)^1 \otimes M(8)^0 \oplus M(m)^0 \otimes M(8)^1 \quad \text{with basis } v_i \otimes \mu_j,\ u_i \otimes \nu_j.$$

The canonical generators of C_m will be denoted by $e_1, \dots, e_m$ and the canonical generators of C_8 will be denoted by $e'_1, \dots, e'_8$. With $E_1 = e_1 \otimes 1, \dots, E_m = e_m \otimes 1$; $E'_1 = E_{m+1} = 1 \otimes e'_1, \dots, E'_8 = E_{m+8} = 1 \otimes e'_8$, we have

$$\Psi_{m+8} = \sum_{i=1}^{N} X_i E_i + \sum_{i=1}^{8} Y_i E'_i.$$

We have

$$\begin{aligned}\Psi_{m+8}(u_1 \otimes \mu_1) &= \sum_{i=1}^{N} X_i E_i(u_1 \otimes \mu_1) + \sum_{i=1}^{8} Y_i E'_i(u_1 \otimes \mu_1)\\ &= \sum_{i=1}^{N} a_{1i}(X) v_i \otimes \mu_1 + \sum_{i=1}^{8} b_{1i}(Y) u_1 \otimes \nu_i.\end{aligned}$$

We will use the notations $u = (u_1, \dots, u_N)^t$, $v = (v_1, \dots, v_N)^t$, $\mu = (\mu_1, \dots, \mu_8)^t$, $\nu = (\nu_1, \dots, \nu_8)^t$. With these notation,

$$\Psi_{m+8}(u_1 \otimes \mu_1) = {}_rA(X)_1 v \otimes \mu_1 + {}_rB(Y)_1 u_1 \otimes \nu.$$

For $i = 1, \dots, N$, likewise, we get

$$\Psi_{m+8}(u_i \otimes \mu_1) = {}_rA(X)_i v \otimes \mu_1 + {}_rB(Y)_1 u_i \otimes \nu.$$

Therefore,

$$\Psi_{m+8}(u \otimes \mu_1) = \left(\begin{array}{cccc|cccc} {}_rA_1(X) & 0 & \dots & 0 & {}_rB_1(Y) & 0 & 0 & 0 \\ {}_rA_2(X) & 0 & \dots & 0 & 0 & {}_rB_1(Y) & 0 & 0 \\ \dots & 0 & \dots & 0 & \dots & \dots & \dots & \\ {}_rA_N(X) & 0 & \dots & 0 & 0 & 0 & 0 & {}_rB_1(Y) \end{array}\right) \begin{pmatrix} v \otimes \mu_1 \\ \dots \\ v \otimes \mu_8 \\ u_1 \otimes \nu \\ \dots \\ u_N \otimes \nu \end{pmatrix}.$$

Given a row vector $a(Y)$ of length 8, let $\mathcal{R}(a(Y)) \in \mathbb{M}_{N \times 8N}$ denotes the matrix as on the right-hand side of the above matrix. With such notations,

$$\Psi_{m+8}(u \otimes \mu_1) = \left(\begin{array}{cccc|c} A(X) & 0 & \dots & 0 & \mathcal{R}\big({}_rB_1(Y)\big) \end{array}\right) \begin{pmatrix} v \otimes \mu_1 \\ \dots \\ v \otimes \mu_8 \\ u_1 \otimes \nu \\ \dots \\ u_N \otimes \nu \end{pmatrix}.$$

Using similar calculations for $\Psi_{m+8}(u \otimes \mu_i)$ we have

$$\begin{pmatrix} \Psi_{m+8}(u \otimes \mu_1) \\ \Psi_{m+8}(u \otimes \mu_2) \\ \dots \\ \Psi_{m+8}(u \otimes \mu_8) \end{pmatrix} = \left(\begin{array}{cccc|c} A(X) & 0 & \dots & 0 & \mathcal{R}({}_rB_1(Y)) \\ 0 & A(X) & \dots & 0 & \mathcal{R}({}_rB_2(Y)) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A(X) & \mathcal{R}({}_rB_8(Y)) \end{array}\right) \begin{pmatrix} v \otimes \mu_1 \\ \dots \\ v \otimes \mu_8 \\ u_1 \otimes \nu \\ \dots \\ u_N \otimes \nu \end{pmatrix}.$$

This gives the upper $8N$ rows of the matrix of Ψ_{m+8} which form a $8N \times (8N + 8N)$ matrix. Now we proceed to compute

$$\begin{pmatrix} \Psi_{m+8}(\nu_1 \otimes \nu) \\ \Psi_{m+8}(\nu_2 \otimes \nu) \\ \cdots \\ \Psi_{m+8}(\nu_N \otimes \nu) \end{pmatrix}.$$

Again, the matrices of Φ_m, Φ_8 are respectively $-A(X)^t, -B(Y)^t$. We have,

$$\begin{aligned} \Psi_{m+8}(\nu_1 \otimes \nu_1) &= \sum_{i=1}^{m} X_i(e_i \otimes 1)(\nu_1 \otimes \nu_1) + \sum_{i=1}^{8} Y_i\big(1 \otimes e_i'\big)(\nu_1 \otimes \nu_1) \\ &= \Phi_m(\nu_1) \otimes \nu_1 - \nu_1 \otimes \Phi_8(\nu_1) = -\sum_{j=1}^{N} a_{j1}(X)u_j \otimes \nu_1 + \sum_{j=1}^{8} b_{j1}(Y)\nu_1 \otimes \mu_j \\ &= -{}_rA_1(X)^t u \otimes \nu_1 + {}_rB_1(Y)^t \nu_1 \otimes \mu. \end{aligned}$$

Similarly, for $i = 1, \ldots, 8$, we have

$$\Psi_{m+8}(\nu_1 \otimes \nu_i) = \Phi_m(\nu_1) \otimes \nu_i - \nu_1 \otimes \Phi_8(\nu_i) = -{}_rA_1(X)^t u \otimes \nu_i + {}_rB_i(Y)^t \nu_1 \otimes \mu;$$

and for $k = 1, \ldots, N$, we have

$$\Psi_{m+8}(\nu_k \otimes \nu_i) = \Phi_m(\nu_k) \otimes \nu_i - \nu_k \otimes \Phi_8(\nu_i) = -{}_rA_k(X)^t u \otimes \nu_i + {}_rB_i(Y)^t \nu_k \otimes \mu.$$

So, the left half of the matrix of $\Psi_{m+8}(\nu_1 \otimes \nu)$ is given by

$$\left({}_cB_1^t \quad 0 \quad \cdots \quad 0 \;\middle|\; {}_cB_2^t \quad 0 \quad \cdots \quad 0 \;\middle|\; \cdots \;\middle|\; {}_cB_8^t \quad 0 \quad \cdots \quad 0 \right) \in \mathbb{M}_{8 \times 8N}.$$

Similarly, the left half of the matrix of $\Psi_{m+8}(\nu_2 \otimes \nu)$ is given by

$$\left(0 \quad {}_cB_1^t \quad \cdots \quad 0 \;\middle|\; 0 \quad {}_cB_2^t \quad \cdots \quad 0 \;\middle|\; \cdots \;\middle|\; 0 \quad {}_cB_8^t \quad \cdots \quad 0 \right) \in \mathbb{M}_{8 \times 8N}.$$

So, the left-lower block of the matrix Ψ_{m+8} is given by

$$\left(\begin{array}{cccc|c|cccc} {}_cB_1(Y)^t & 0 & \cdots & 0 & \cdots & {}_cB_8(Y)^t & 0 & \cdots & 0 \\ 0 & {}_cB_1(Y)^t & \cdots & 0 & \cdots & 0 & {}_cB_8(Y)^t & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & {}_cB_1(Y)^t & \cdots & 0 & 0 & \cdots & {}_cB_8(Y)^t \end{array}\right)$$
$$= \begin{pmatrix} \mathcal{R}({}_rB_1(Y)) \\ \mathcal{R}({}_rB_2(Y)) \\ \cdots \\ \mathcal{R}({}_rB_8(Y)) \end{pmatrix}^t = (\textit{Upper-Right Block})^t \in \mathbb{M}_{8N \times 8N}.$$

Now we compute the right half of the same matrix of

$$\begin{pmatrix} \Psi_{m+8}(\nu_1 \otimes \nu) \\ \Psi_{m+8}(\nu_2 \otimes \nu) \\ \cdots \\ \Psi_{m+8}(\nu_N \otimes \nu) \end{pmatrix}.$$

Let $\mathbb{I}_8$ denote the identity matrix of order 8. Then, the right half of this matrix is given by

$$-\begin{pmatrix} a_{11}(X)\mathbb{I}_8 & a_{21}(X)\mathbb{I}_8 & \cdots & a_{N1}(X)\mathbb{I}_8 \\ a_{12}(X)\mathbb{I}_8 & a_{22}(X)\mathbb{I}_8 & \cdots & a_{N2}(X)\mathbb{I}_8 \\ \cdots & \cdots & \cdots & \cdots \\ a_{1N}(X)\mathbb{I}_8 & a_{2N}(X)\mathbb{I}_8 & \cdots & a_{NN}(X)\mathbb{I}_8 \end{pmatrix}.$$

Now, the upper left and lower right blocks of the matrix of Ψ_{m+8} involve only the variables $X_1, \ldots, X_m$ and the upper right and lower left blocks involve only the variables $Y_1, \ldots, Y_8$. Recall that $A(X)$ and $B(Y)$ have the properties of Γ of the proposition. Examining all four blocks of the matrix of Ψ_{m+8}, it follows that the matrix of Ψ_{m+8} also has the property of Γ of the proposition. By symmetry, the matrix of Φ_{m+8} also has the property of Δ of the proposition. This completes the proof of Proposition 5.11. □

6. Complete intersections

In this final section, we consider the question whether a local complete intersection ideal I of A_n, with $height(I) = n$, is the image of a projective A_n-module P of rank n. For an affirmative answer to this question for all such ideals I it is necessary that the top Chern class map $C_0 : K_0(A_n) \to CH_0(A_n)$ is surjective. Since $CH_0(A_n) = \mathbb{Z}_2$ and for $n = 8r+3, 8r+5, 8r+6, 8r+7$, by (5.10), $K_0(A_n) = \mathbb{Z}$, the top Chern class map C_0 fails to be surjective. So, in these cases, the question has a negative answer.

We consider the stronger question whether each element of the Euler class group $E(A_n) = \mathbb{Z}$ is Euler class of a projective A_n-module P of rank n. We start with the following two theorems about even classes and odd classes.

Theorem 6.1. *Let A_n be the ring of algebraic functions on $\mathbb{S}^n$, as in nation (2.1), and $n \geqslant 2$ be* ***even****. Let $N = 2r$ be an even integer. Then there is a stably free A_n-module P of rank n and an orientation $\chi : A_n \xrightarrow{\sim} \det(P)$ such that $e(p, \chi) = N$.*

Proof. By Lemma 2.4, we can assume $N \geqslant 0$. Let $m_1, \ldots, m_N$ be even number of distinct real maximal ideals and assume that the corresponding real points are on $\mathbb{S}^1 = (x_2 = 0, \ldots, x_n = 0)$. As in Lemma 4.1,

$$\bigcap_{i=1}^{N} m_i = (L, x_2, \ldots, x_n) \quad \text{where } L = \prod_{i=1}^{N/2} L_i \quad \text{with } L_i \text{ linear}.$$

Write $F = A_n^n$ and $J = \bigcap_{i=1}^{N} m_i = (L, x_2, \ldots, x_n)$. The standard basis of F will be denoted by $e_1, \ldots, e_n$. Define

$$f : F \twoheadrightarrow J \quad \text{by } f(e_1) = L, \quad f(e_i) = x_i \quad \forall i \geqslant 2.$$

Let

$$\omega : F/JF \to J/J^2 \quad \text{and} \quad \text{for } i = 0, 1 \quad \omega_i : F/m_iF \to m_i/m_i^2,$$

be induced by f. Therefore

$$(J, \omega) = \sum_{i=1}^{N} (m_i, \omega_i) = 0 \in E(A_n).$$

We can assume

$$(m_i,\omega_i)=1 \quad \forall i=1,\dots,s, \qquad (m_i,\omega_i)=-1 \quad \forall i=s+1,\dots,N.$$

Let $u\in A$ be such that $u-1\in m_i$ for $i=1,\dots,s$ and $u+1\in m_i$ for $i=s+1,\dots,N$. Note $u^2-1\in J$. Define P by the exact sequence

$$0\to P\to A_n\oplus F\xrightarrow{(u,-f)}A_n\to 0.$$

By [BRS2, Lemma 5.1], P has an orientation χ such that

$$e(P,\chi)=u^{-(n-1)}(J,\omega)=\sum_{i=1}^{s}(m_i,\omega_i)+\sum_{i=s+1}^{N}-(m_i,\omega_i)=N.$$

The proof is complete. □

Theorem 6.2. *Let A_n be the ring of algebraic functions on $\mathbb{S}^n$. Assume $n=\dim A_n$ is **even**. Then, there exists a projective A_n-module P with rank$(P)=n$ and an orientation $\chi: A_n \xrightarrow{\sim} \det P$ with $e(P,\chi)=N$ for some odd integer N if and only if the same is possible for all odd integers N.*

Proof. Suppose N odd and $e(P,\chi)=N$. By Lemma 2.4, we can assume $N>0$. Now assume M be any other odd integer. Again, we can assume $M>0$. Let $m_1,\dots,m_M$ be distinct real maximal ideals and $(m_i,\omega_i)=1\in EL(A_n)=\mathbb{Z}$. Write $F=A_n^n$ and $I=\bigcap_{i=1}^{M}m_i$. Let $\omega_I: F/IF\to I/I^2$ be obtained from $\omega_1,\dots,\omega_M$. Then $M=(I,\omega_I)$. Note that the weak Euler class group

$$E_0(A_n)\approx CH_0(A_n)=\mathbb{Z}/(2).$$

Therefore, the weak Euler class $e_0(P)=image(N)=1=image(M)=(I)$. So, by proposition [BRS2, Proposition 6.4], there is a projective A_n-module Q of rank n and a surjective map $f: Q\twoheadrightarrow I$, and also $[P]=[Q]\in K_0(A_n)$. Fix an orientation $\chi_0: A_n\xrightarrow{\sim}\det Q$. Using an isomorphism $\gamma: F/IF\xrightarrow{\sim}Q/IQ$, with $\det\gamma\equiv\chi_0$, f induces orientations $\eta: F/IF\to I/I^2$ and $\eta_i: F/m_iF\to m_i/m_i^2$, for $i=1,\dots,M$. Then, by definition,

$$e(Q,\chi_0)=(I,\eta)=\sum_{i=1}^{M}(m_i,\eta_i).$$

We can assume that

$$(m_i,\eta_i)=1 \quad \forall i\leqslant s, \quad \text{and} \quad (m_i,\eta_i)=-1 \quad \forall i>s.$$

Pick $u\in A_n$ such that $u-1\in m_i$ for $i\leqslant s$ and $u+1\in m_i$ for $i>s$. Let Q' be defined by

$$0\to Q'\to A_n\oplus Q\xrightarrow{(u,-f)}A_n\to 0.$$

Then, by [BRS2, Lemma 5.1], Q' has an orientation χ' such that

$$e(Q',\chi')=\left(I,\bar u^{-(n-1)}\eta\right)=\sum_{i=1}^{s}\left(m_i,\bar u^{-(n-1)}\eta_i\right)+\sum_{i=s+1}^{N}\left(m_i,\bar u^{-(n-1)}\eta_i\right).$$

Here n is even. So, $e(Q',\chi')=\sum_{i=1}^{s}(m_i,\eta_i)+\sum_{i=s+1}^{N}(m_i,-\eta_i)=M$. This completes the proof. □

Before we proceed, we need the following proposition that relates top Chern classes with Stiefel–Whitney classes.

Proposition 6.3. *Let A_n be the ring of algebraic functions on the real sphere $\mathbb{S}^n$ and $X = Spec(A_n)$. Then, the following diagram*

$$\begin{array}{ccc} K_0(X) & \longrightarrow & KO(\mathbb{S}^n) \\ {\scriptstyle C_0}\downarrow & & \downarrow{\scriptstyle w_n} \\ CH_0(X) & \longrightarrow & H^n(\mathbb{S}^n, \mathbb{Z}/(2)) \end{array}$$

commutes, where C_0 denotes the top Chern class map and w_n denotes the top Stiefel–Whitney class.

Proof. Note, $CH_0(\mathbb{R}(X)) = \mathbb{Z}/(2)$ and $H^n(\mathbb{S}^n, \mathbb{Z}/(2)) = \mathbb{Z}/(2)$. Any element in $\widetilde{K_0}(X)$ can be written as $[P] - [A_n^n]$, where P is a projective A_n-module of rank n. By Bertini's theorem (see [BRS1, 2.11]), we can find a surjective map $f : P \to I$ where $I = m_1 \cap m_2 \cap \cdots \cap m_N$ is intersection of N distinct maximal ideals. Assume $m_1, \ldots, m_r$ are real maximal ideals and $m_{r+1}, \ldots, m_N$ are the complex maximal ideals. For $i = 1, \ldots, r$, let $y_i \in \mathbb{S}^n$ be the point corresponding to m_i. So, $C_0(P) = \bar{r} \in \mathbb{Z}/(2)$, where $\bar{r}$ is the image of r in $\mathbb{Z}/(2)$.

Let $\widetilde{P}$ denote the bundle on $\mathbb{S}^n$ induced by P. Then f induces a section s on the bundle $\widetilde{P}$, transversally intersecting the zerosection, exactly on the points $y_1, \ldots, y_r$. So, $w_n(\widetilde{P}) = \bar{r}$. The proof is complete. □

Remark. In a subsequent paper [MaSh], a more general version of Proposition 6.3 was proved later.

Now, we have the following corollary to Theorem 6.2.

Corollary 6.4. *Let A_n be the ring of algebraic functions on $\mathbb{S}^n$. Assume $n = \dim A_n \geqslant 2$ is* ***even****. Then, the following are equivalent*:

1. *$e(P, \chi) = 1$ for some projective A_n-module P with rank$(P) = n$ and orientation $\chi : A_n \xrightarrow{\sim} \det P$.*
2. *For some odd integer N, $e(P, \chi) = N$ for some projective A_n-module P with rank$(P) = n$ and orientation $\chi : A_n \xrightarrow{\sim} \det P$.*
3. *For any odd integers N, $e(P, \chi) = N$ for some projective A_n-module P with rank$(P) = n$ and orientation $\chi : A_n \xrightarrow{\sim} \det P$.*
4. *The top Chern class $C_0(P) = 1$ for some projective A_n-module P with rank$(P) = n$.*
5. *The Stiefel–Whitney class $w_n(V) = 1$ for some vector bundle V with rank$(V) = n$.*

Let $n = 8r, 8r + 2, 8r + 4$ and let P_n be a projective A_n-module of rank n such that $[P_n] - n = \tau_n$ is the generator of $\widetilde{K_0}(A_n)$. Then above conditions are equivalent to $w_n(P_n) = 1$ (which is equivalent to *$C_0(P_n) = 1$*).

Proof. By (6.2), (1) ⇔ (2) ⇔ (3). It is obvious that (2) ⇔ (4). Also by (6.3), we have (4) ⇔ (5), because we can assume [Sw2] that V is algebraic.

For the later part, we only need to prove that (5) $\Rightarrow w_n(P_n) = 1$. To prove this assume that $w_n(P_n) = 0$ and let V be any vector bundle of rank n over $\mathbb{S}^n$. We have $[V] - rank(V) = k\tau_n$. So, the total Stiefel–Whitney class $w(V) = w(k\tau_n) = w(\tau_n)^k = 1$. So, $w_n(V) = 0$. The proof is complete. □

Remark 6.5. We have the following summary. Let P denote a projective A_n-module of rank $n \geqslant 2$ and $\chi : A_n \xrightarrow{\sim} \det P$ be an orientation. Then

1. For $n = 8r+3, 8r+5, 8r+7$ we have $\widetilde{K_0}(A_n) = 0$. So, the top Chern class $C_0(P) = 0$. By [BDM], $P \approx Q \oplus A_n$. Therefore $e(P,\chi) = 0$.
2. For $n = 8r+6$, we have $\widetilde{K_0}(A_n) = 0$. So, $C_0(P) = 0$, and hence $e(P,\chi)$ is always even. Further, by (6.1), for any even integer N there is a projective A_n-module Q with $rank(Q) = n$ and an orientation $\eta : A_n \xrightarrow{\sim} \det Q$, such that $e(Q,\eta) = N$.
3. For $n = 8r+1$, we have $\widetilde{K_0}(A_n) = \mathbb{Z}/2$. If $e(P,\chi)$ is even then $C_0(P) = 0$. So, $P \approx Q \oplus A_n$ and $e(P,\chi) = 0$ for all orientations χ. So, only even value $e(P,\chi)$ can assume is zero.
4. Now consider the remaining cases, $n = 8r, 8r+2, 8r+4$. We have $\widetilde{K_0}(A_{8r}) = \mathbb{Z}$, $\widetilde{K_0}(A_{8r+2}) = \mathbb{Z}/2$, $\widetilde{K_0}(A_{8r+4}) = \mathbb{Z}$. As in the case of $n = 8r+6$, for any even integer N, for some (Q,η) the Euler class $e(Q,\eta) = N$. The case of odd integers N was discussed in (6.4).

This leads us to the following question.

Question 6.6. *Suppose $n = 8r, 8r+1, 8r+2, 8r+4$ and τ_n is the generator of $\widetilde{K_0}(A_n)$. Then, whether $w_n(\tau_n) = 1$ (which is equivalent to $C_0(\tau_n) = 1$)?*

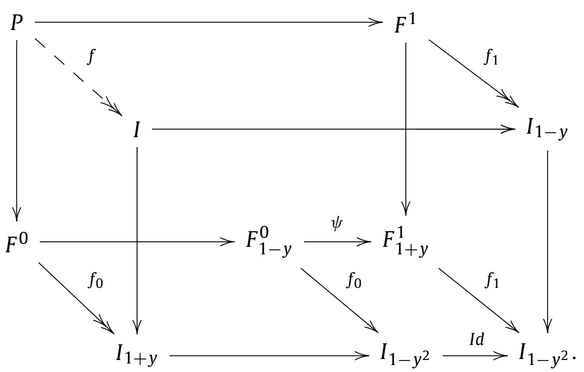
Apparently, answer to this question is not known. For $n = 1$, the question has affirmative answer. We will be able to answer this question for $n = 2, 4, 8$ using the description (5.11) of the patching matrix ψ_n.

Theorem 6.7. *Let $n = 2, 4, 8$ and A_n denote the ring of algebraic functions on $\mathbb{S}^n$. Let τ_n be the generator of $\widetilde{K_0}(A_n)$. Then, the top Chern class $C_0(\tau_n) = 1$.*

Proof. Here $n = 2, 4, 8$ and $q = q_n = \sum_{i=1}^n X_i^2$. If $M = M^0 \oplus M^1$ is an irreducible $\mathbb{Z}_2$-graded C_n-module, then $\tau_n = [P] - rank(P)$, where $P = \alpha(M)$ as defined in Proposition 5.5. We have $R(q + X_0^2) = A_n$ and P is obtained by patching together $F^0 = (A_n)_{1+x_0} \otimes M^0$ and $F^1 = (A_n)_{1-x_0} \otimes M^1$ via $\psi = \psi_n$. In the cases of $n = 2, 4, 8$, by (5.9), $rank(P) = \dim M_0 = n$. By (5.11), with respect some bases of M^0, M^1, the matrix of ψ has the first column $(x_1, \ldots, x_n)^t$.

We write $y = x_0$ and $F = A_n^n$. Let $e_1, \ldots, e_n$ denote the standard basis of F. We identify $F^0 = F_{1+y}$, $F^1 = F_{1-y}$ and consider ψ as a matrix with first column $(x_1, \ldots, x_n)^t$.

Let $I = (y-1, x_1, \ldots, x_n)$ be the ideal of the north pole of $\mathbb{S}^n$. Then, $I_{1+y} = (x_1, \ldots, x_n)$. Define surjective maps $f_0 : F^0 \twoheadrightarrow I_{1+y}$ where $f_0(e_i) = x_i$ for $i = 1, \ldots, n$; and $f_1 : F^1 \twoheadrightarrow I_{1-y}$ where $f_1(e_1) = 1$ and $f_1(e_i) = x_i$ for $i = 2, \ldots, n$. We have the following patching diagram:



The map f is induced by the properties of fiber product diagrams. Since f_0, f_1 are surjective, so is f. Therefore, the top Chern class $C_0(P) = cycle(A_n/I) = 1$. Since $\tau_n = [P] - n$, we have $C_0(\tau_n) = 1$. This completes the proof. $\square$

Corollary 6.8. *Let $n = 2, 4, 8$. Then, given any integer N there is a projective A_n-module Q of rank n and orientation $\chi : A_n \xrightarrow{\sim} \det Q$, such that the Euler class $e(Q, \chi) = N$.*

Also, suppose I is a locally complete intersection ideal of height n and $\omega : (A_n/I)^n \twoheadrightarrow I/I^2$ is a surjective homomorphism. Then, there is a projective A_n-module P of rank n, and orientation $\chi : A_n \xrightarrow{\sim} \det P$ and a surjective homomorphism $f : P \twoheadrightarrow I$ such that (I, ω) is induced by (P, χ).

Proof. First part follows immediately from (6.1, 6.4, 6.7). The later part follows from [BRS2, Corollary 4.3]. □

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