ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Topology 44 (2005) 827–843
TOPOLOGY
www.elsevier.com/locate/top

Fundamental groups of asymptotic cones[☆]

A. Erschler[a,*], D. Osin[b]

[a] CNRS, University Lille 1, UFR de Mathématiques, 59655, Villeneuve d'Ascq Cedex, France
[b] Department of Mathematics, 1326 Stevenson Center, Vanderbilt University, Nashville TN 37240-0001, USA

Received 20 January 2004; received in revised form 26 July 2004; accepted 9 February 2005

Abstract

We show that for any metric space M satisfying certain natural conditions, there is a finitely generated group G, an ultrafilter ω, and an isometric embedding $\imath$ of M to the asymptotic cone $Cone_\omega(G)$ such that the induced homomorphism $\imath^* : \pi_1(M) \to \pi_1(Cone_\omega(G))$ is injective. In particular, we prove that any countable group can be embedded into a fundamental group of an asymptotic cone of a finitely generated group.

MSC: 20F69; 14F35; 20F06

Keywords: Asymptotic cone; Isometric embedding at infinity

1. Introduction

To any metric space X with a distance function $dist$, one can associate a new metric space $Cone_\omega(X)$, the so-called *asymptotic cone* of X, by taking the ultralimit of scaled spaces $(X, (1/n)dist)$ with respect to an ultrafilter ω. Informally speaking $Cone_\omega(X)$ shows what X looks like if the observer is placed at 'infinity' (see the next section for a precise definition). This notion appears in the proof of Gromov's theorem about groups of polynomial growth (it is defined in [3], though in the polynomial growth case,

☆ This work was partially supported by the Swiss National Science foundation. The second author was also supported by the RFBR grant # 99-01-00894.
* Corresponding author. Tel.: +33 2043 4208; fax: +33 20434302.
E-mail addresses: erschler@math.univ-lille1.fr, erschler@pdmi.ras.ru (A. Erschler), denis.ossine@vanderbilt.edu (D. Osin).

0040-9383/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.top.2005.02.003

where for the convergence to the limit one does not need ultralimits, the corresponding limit space is considered already in [6]).

It is known that a group is hyperbolic if and only if any of its asymptotic cones is an *R*-tree [8] (Misha Kapovich pointed out to the first author that if one of the cones of a finitely presented group is an *R*-tree, then this group is hyperbolic).

The application of asymptotic cones to the study of algebraic and geometric properties of hyperbolic groups appears in a different language in [12] and in [18]. For recent progress see [20,19,21] and references therein. Asymptotic cones can be used for proving rigidity theorems for symmetric spaces [9]. For other results about asymptotic cones we refer to [1,2,4,5,15–17,22,10].

The case when X is a finitely generated group G endowed with a word metric is of particular interest. In [8], Gromov pointed out a connection between homotopical properties of $Cone_\omega(G)$ and asymptotic invariants of G. Namely, if $Cone_\omega(G)$ is simply connected with respect to all ultrafilters, then G is finitely presented and the Dehn function is polynomial. A partial converse result was obtained in [15]. However almost nothing was known about algebraic structure of the fundamental group $\pi_1(Cone_\omega(G))$ in case $\pi_1(Cone_\omega(G))$ is nontrivial. In particular, no examples of non-free finitely generated subgroups of such fundamental groups were known until now. (It was observed in [2] that the asymptotic cones of Baumslag–Solitar groups

$$BS_{p,q} = \langle a, b \mid b^{-1}a^p b = a^q \rangle,$$

where $|p| \neq |q|$, contain non-free infinitely generated groups.)

The following question is stated in [8].

Problem 1.1. *Which groups can appear as* (*finitely generated*) *subgroups in fundamental groups of asymptotic cones of finitely generated groups?*

In the present paper we answer this question (for finitely generated groups) by proving the following theorem.

Theorem 1.2. *Let M be a metric space such that*

(M1) *M is geodesic, that is for any two points $x, y \in M$ there is a path joining x and y of length $dist_M(x, y)$;*

(M2) *there is a sequence of compact subsets $M_1 \subseteq M_2 \subseteq \cdots$ of M such that*

$$M = \bigcup_{i=1}^{\infty} M_i.$$

Then there exists a group G generated by 2 elements, an ultrafilter ω, and an isometric embedding $\imath$ of M into the asymptotic cone $Cone_\omega(G)$. If, in addition, M is uniformly locally simply-connected, i.e.,

(M3) *there exists $\varepsilon > 0$ such that any loop in any ball of radius ε is contractible, then G and the embedding of M into the asymptotic cone of G can be chosen in such a way that the induced homomorphism of the fundamental groups*

$$\imath^* : \pi_1(M) \to \pi_1(Cone_\omega(G))$$

is injective.

Obviously any combinatorial complex M with countable number of cells in each dimension admits a (natural) metric which induces the standard topology on M and with respect to which M satisfies (M1)–(M3). Since any countable group is a fundamental group of a countable combinatorial 2-complex, we obtain

Corollary 1.3. *For any countable group H, there exists a finitely generated group G and an ultrafilter* ω *such that* $\pi_1(Cone_\omega(G))$ *contains a subgroup isomorphic to H.*

The construction in the proof of Theorem 1.2 is similar to that in [14] and can be intuitively understood as follows. Given a metric space M with metric $dist_M$ satisfying (M1)–(M3), we first approximate M by finite ε_i-nets Net_i, where $\varepsilon_i \to 0$ as $i \to \infty$. Then we choose a rapidly growing sequence of natural numbers $\{n_i\}$ and use a construction similar to that from [14] to produce embeddings α_i of Net_i into a finitely generated group G endowed with a word metric $dist_G$ such that α_i, being considered as a map from a metric space $(Net_i, dist_M)$ to $(G, (1/n_i)dist_G)$, is a (λ_i, c_i)-quasi-isometry, where $\lambda_i \to 1$ and $c_i \to 0$ as $i \to \infty$. This gives an isometric embedding $\imath : M \to Cone_\omega(G)$ for any ultrafilter ω satisfying $\omega\{n_i\} = 1$. Then condition (M3) allows one to show that $\imath$ induces an injective map on the fundamental group $\pi_1(M)$.

The paper is organized as follows. In the next section we collect main definitions and results which are used in what follows. Some auxiliary results about words with small cancellation properties are proved in Section 3. In Section 4 we construct the group G and the injective map $\imath : M \to Cone_\omega(G)$ mentioned in Theorem 1 and show that $\imath$ is an isometry. In Section 5 we conclude the proof of the main theorem by proving injectivity of $\imath^* : \pi_1(M) \to \pi_1(Cone_\omega(G))$.

2. Preliminaries

2.1. Asymptotic cones

Recall that a *non-principal ultrafilter* ω is a finitely additive non-zero measure on the set of all subsets of $\mathbb{N}$ such that each subset has measure either 0 or 1 and all finite subsets have measure 0. For any bounded function $h : \mathbb{N} \to \mathbb{R}$ its limit $h(\omega)$ with respect to a non-principal ultrafilter ω is uniquely defined by the following condition: for every $\delta > 0$

$$\omega(\{i \in \mathbb{N} : |h(i) - h(\omega)| < \delta\}) = 1.$$

Definition 2.1. Let X be a metric space with a distance function $dist$. We fix a basepoint $O \in X$ and consider the set of all sequences $g : \mathbb{N} \to X$ such that

$$dist(O, g(i)) \leqslant \mathrm{const}_g \cdot i$$

(here the constant $const_g$ depends on g). To any pair of such sequences g_1, g_2 one may assign a function

$$h_{g_1,g_2}(i) = \frac{dist(g_1(i), g_2(i))}{i}.$$

We say that the sequences g_1, g_2 are equivalent if the limit $h_{g_1,g_2}(\omega) = 0$. The set $Cone_\omega(G)$ of all equivalence classes of sequences with the distance

$$dist_{\mathrm{Cone}}(g_1, g_2) = h_{g_1,g_2}(\omega)$$

is called an *asymptotic cone* of X with respect to the non-principal ultrafilter ω. Clearly this space does not depend of the basepoint chosen.

If G is a group generated by a finite set S, we can regard G as a metric space assuming the distance between two elements $a, b \in G$ to be equal to the length of the shortest word in the alphabet $S^{\pm 1}$ representing $a^{-1}b$. Such a distance function is called the *word metric associated to S*. Given an ultrafilter ω, this leads to the asymptotic cone $Cone_\omega(G)$. It is worth noting that in some examples $Cone_\omega(G)$ strongly depends on the choice of the ultrafilter [22].

2.2. Cayley graphs and van Kampen diagrams

Recall that a *Cayley graph* $Cay(G)$ of a group G generated by a set S is an oriented labelled 1-complex with the vertex set $V(Cay(G)) = G$ and the edge set $E(Cay(G)) = G \times S$. An edge $e = (g, s) \in E(Cay(G))$ goes from the vertex g to the vertex gs and has the label $Lab(e) = s$. As usual, we denote the origin and the terminus of the edge e, i.e., the vertices g and gs, by e_- and e_+, respectively. The word metric on G associated to S can be extended to $Cay(G)$ by assuming the length of every edge to be equal to one. Also, it is easy to see that a word W in $S^{\pm 1}$ represents 1 in G if and only if some (or, equivalently, any) path p in $Cay(G)$ labelled W is a cycle.

A planar *map* Δ over a group presentation

$$G = \langle S \mid \mathcal{P} \rangle \tag{1}$$

is a finite oriented connected simply-connected 2-complex endowed with a labelling function $Lab : E(\Delta) \to S^{\pm 1}$ (we use the same notation as for Cayley graphs) such that $Lab(\mathrm{e}^{-1}) = (Lab(e))^{-1}$.

Given a combinatorial path $p = e_1 e_2 \dots e_k$ in Δ (respectively in $Cay(G)$), where $e_1, e_2, \dots, e_k \in E(\Delta)$ (respectively $e_1, e_2, \dots, e_k \in E(Cay(G))$), we denote by $Lab(p)$ its label. By definition, $Lab(p) = Lab(e_1)Lab(e_2)\dots Lab(e_k)$. We also denote by $p_- = (e_1)_-$ and $p_+ = (e_k)_+$ the origin and the terminus of p, respectively. A path p is called *irreducible* if it contains no subpaths of type ee^{-1} for $e \in E(\Delta)$ (respectively $e \in E(Cay(G))$). The length $|p|$ of p is, by definition, the number k of edges of p.

Given a cell Π of Δ, we denote by $\partial\Pi$ the boundary of Π; similarly, $\partial\Delta$ denotes the boundary of Δ. The label of $\partial\Pi$ or $\partial\Delta$ is defined up to a cyclic permutation. A map Δ over a presentation (1) is called a *van Kampen diagram* over (1) if the following holds. For any cell Π of Δ, the boundary label $Lab(\partial\Pi)$ is equal to a cyclic permutation of a word $P^{\pm 1}$, where $P \in \mathcal{P}$. Sometimes it is convenient to use the notion of 0-refinement in order to assume diagrams to be homeomorphic to a disc. We do not explain this notion here and refer the interested reader to [13].

The van Kampen lemma states that a word W over the alphabet $S^{\pm 1}$ represents the identity in the group given by (1) if and only if there exists a simply-connected planar diagram Δ over (1) such that $Lab(\partial\Delta)$ coincides with W [11,13]. A van Kampen diagram is called *minimal*, if it contains the minimal number of cells among all diagrams with the same boundary labels.

2.3. Approximations of metric spaces by graphs

We recall that a subset Z is called a δ-net in a metric space Y, if for all $y \in Y$ there exists $z \in Z$ such that the distance between y and z is less than δ. We say that a subset Z of a metric space Y is a (δ_1, δ_2)-*net* if Z is a δ_1-net and the distance between any two points of Z is greater than δ_2. The following lemma will be used in Section 4.

Lemma 2.2. *Suppose that M is a metric space satisfying conditions* (M1) *and* (M2). *There exists a sequence of finite subsets $Net_1 \subseteq Net_2 \subseteq \cdots$ of M such that for all $i \in \mathbb{N}$, Net_i is a $(2/i, 1/i)$-net in M_i.*

Proof. We proceed by induction on i. Suppose that Net_{i-1} is a $(2/(i-1), 1/(i-1))$-net in M_{i-1} if $i > 1$, and $Net_{i-1} = \emptyset$ if $i = 1$. We consider an arbitrary finite $1/i$-net N in M_i that contains Net_{i-1} as a subset. Let $\mathscr{N}$ denote the set of all subsets L such that $Net_{i-1} \subseteq L \subseteq N$ and for any two different elements $x, y \in L$ we have $dist_M(x, y) > 1/i$. Note that the set $\mathscr{N}$ is non-empty as it contains Net_{i-1}.

Consider a partial order on $\mathscr{N}$ which corresponds to inclusion, i.e., for any $A, B \in \mathscr{N}$, $A \preceq B$ if and only if $A \subseteq B$. Since $\mathscr{N}$ is finite, we can take a maximal subset B with respect to this order. Note that for any $t \in N$, we have $dist_M(t, B) \leqslant 1/i$. Indeed, otherwise $B \cup \{t\} \in \mathscr{N}$ and thus B is not maximal. Therefore, for any $x \in M_i$, we have $dist_M(x, B) \leqslant 2/i$. Thus B is a $(2/i, 1/i)$-net in M_i. □

3. Words with small cancellations

To prove the main result of our paper we will need an infinite set of words satisfying certain small cancellation conditions. We begin with definitions.

Let X be an alphabet and F a free group with the basis X. Throughout the following discussion we write $U \equiv V$ to express the letter-by-letter equality of the words U and V. Given a word W over the alphabet X, by a cyclic word W we mean the set of all cyclic shifts of W. Two cyclic words W_1 and W_2 are equal if and only there exist cyclic shifts U_1, U_2 of W_1 and W_2 respectively such that $U_1 \equiv U_2$. A subword of a cyclic word W is a subword of a cyclic shift of W. By $\|W\|$ we denote the length of a (cyclic) word W. Finally, for a real number r, $[r]$ means the greatest integer which is less than or equal to r.

Definition 3.1. A set $\mathscr{T}$ of cyclic words in X satisfies the condition $C^*(\lambda)$ if for all common subwords A of any two different cyclic words $B, C \in \mathscr{T}^{\pm 1}$, we have $\|A\| < \lambda \min\{\|B\|, \|C\|\}$ and for all cyclic words $B \in \mathscr{T}^{\pm 1}$, all subwords A of B of length $\|A\| \geqslant \lambda \|B\|$ occur in B only once.

Definition 3.2. Given a set $\mathscr{T}$ of words in X, we define a *growth function* of T by the formula

$$\sigma_{\mathscr{T}}(n) = \sharp \mathscr{T}(n),$$

where $\mathscr{T}(n)$ is the set of all words from $\mathscr{T}$ having length exactly n, i.e.,

$$\mathscr{T}(n) = \{W \in \mathscr{T} : \|W\| = n\}.$$

The main result of this section is the following.

Proposition 3.3. *There exists a set $\mathscr{T}$ of words in the alphabet $X = \{a, b\}$ and a non-increasing function $\lambda : \mathbb{N} \to (0, 1)$ satisfying the following conditions.*

(i) *The function $\sigma_{\mathscr{T}}$ is non-decreasing and* $\lim_{n\to\infty} \sigma_{\mathscr{T}}(n) = \infty$.

(ii) $\lim_{n\to\infty}\lambda(n)=0$.
(iii) $\mathscr{T}$ *satisfies* $C^*(1/50)$ *condition and for all* $n\in\mathbb{N}$, *the set* $\bigcup_{k=n}^{\infty}\mathscr{T}(k)$ *satisfies* $C^*(\lambda(n))$.

The proof of Proposition 3.3 is based on four auxiliary lemmas. Recall that for any $l\geqslant 2$, a word W is called l-aperiodic if it has no non-empty subwords of the form V^l. The following lemma can be found in the book [13, Theorem 4.6]

Lemma 3.4. *Denote by* $f(n)$ *the number of all* 6-*aperiodic words of length* $n>0$ *over the alphabet* $X=\{a,b\}$. *Then we have*

$$f(n)>(3/2)^n.$$

Let $\mathscr{X}(k)=\{X_{k,1},\dots,X_{k,f(k)}\}$ be the set of all different 6-aperiodic words of length k in the alphabet $\{a,b\}$. For every $k>8$ and every $i=0,1\dots,([f(k)/k]-1)$, consider the (cyclic) word

$$W_{k,i}=(a^6bX_{k-8,ik+1}b)(a^6bX_{k-8,ik+2}b)\dots(a^6bX_{k-8,ik+k}b). \tag{2}$$

Set

$$\mathscr{A}_k=\left\{W_{k,i}: i=0,1,\dots,\left(\left[\frac{f(k)}{k}\right]-1\right)\right\}.$$

The next lemma is an immediate consequence of (2) and Lemma 3.4.

Lemma 3.5. *For any* $k>8$ *and any* $W\in\mathscr{A}_k$, *we have*

(a) $\|W\|=k^2$;
(b) $\#\mathscr{A}_k\geqslant(3/2)^k/k-1$.

Lemma 3.6. *For any* $k>8$, *the set* $\bigcup_{j=k}^{\infty}\mathscr{A}_j$ *satisfies* $C^*(3/k)$.

Proof. Suppose that $U\in\mathscr{A}_j$, $j\geqslant k$, is a cyclic word and V is a subword of U such that $\|V\|\geqslant(3/k)\|U\|$. Then we have $\|V\|\geqslant(3/j)\|U\|=3j>2j+8$. Note that any subword of U of length greater than $2j+8$ contains a subword of type

$$a^6bX_{j-8,i}ba^6, \tag{3}$$

where $X_{j-8,i}\in\mathscr{X}(j-8)$. Since all words from $\mathscr{X}(j-8)$ are aperiodic and different, such a subword occurs in U only once. Therefore, V occurs in U once.

Further, let U_1,U_2 be two cyclic words from $\bigcup_{j=k}^{\infty}\mathscr{A}_j$ and V a common subword of U_1,U_2 such that $\|V\|>(3/k)\min\{\|U_1\|,\|U_2\|\}$. Arguing as above, we can show that V contains a subword of type (3) for $j=\min\{\|U_1\|,\|U_2\|\}$. It remains to observe that such a subword appears in a unique word from $\mathscr{A}$. □

For each $n \in \mathbb{N}$, $n \geqslant 81$, we construct a set $\mathscr{B}_n$ of words over $\{a, b\}$ as follows. First we divide each set $\mathscr{A}_k$ into $(2k+1)$ disjoint parts such that

$$\mathscr{A}_k = \bigsqcup_{i=1}^{2k+1} \mathscr{A}_{k,i} \tag{4}$$

and

$$\sharp\mathscr{A}_{k,i} \geqslant \left[\frac{\sharp\mathscr{A}_k}{2k+1}\right] \tag{5}$$

for any $i = 1, \ldots, 2k+1$. We set

$$\mathscr{B}_n = \mathscr{A}_{k,l}, \tag{6}$$

where $k = [\sqrt{n}]$ and $l = n - [\sqrt{n}]^2$. Note that $[\sqrt{n}] \geqslant [\sqrt{81}] > 8$ and $l \leqslant n - (\sqrt{n}-1)^2 = 2\sqrt{n} - 1 \leqslant 2k+1$. Thus $\mathscr{B}_n$ is well-defined for $n \geqslant 81$. Furthermore, for any $W \in \mathscr{B}_n$, we have $W \in \mathscr{A}_{[\sqrt{n}]}$. Hence

$$n \geqslant |W| \geqslant (\sqrt{n}-1)^2 > n - 2\sqrt{n} \tag{7}$$

by Lemma 3.5. Finally, given an arbitrary word $W \in \mathscr{B}_n$, we form a new word

$$\overline{W} = Wb^m, \tag{8}$$

where $m = n - |W|$. We call W a *core* of the word $\overline{W}$. Inequality (7) yields

$$0 \leqslant m < 2\sqrt{n}. \tag{9}$$

We set

$$\mathscr{T}(n) = \{\overline{W} : W \in \mathscr{B}_n\}$$

for all $n \geqslant 81$ and $\mathscr{T}(n) = \emptyset$ for $n < 81$.

The proof of the next lemma is straightforward. We leave it to the reader.

Lemma 3.7. *Let A, B, C, D be arbitrary words in the alphabet X. Suppose that* $\max\{\|C\|, \|D\|\} \leqslant y$ *and any common subword of cyclic words A and B has length at most x. Then the length of any common subword of the cyclic words AC and BD is at most* $3x + 2y$.

Proof of Proposition 2.1. Let us take $\mathscr{T}(n)$ as defined above and set $\mathscr{T} = \bigcup_{k=1}^{\infty} \mathscr{T}(k)$. Combining (5), (6), and Lemma 3.5, we obtain

$$\sigma(n) = \sharp\mathscr{T}(n) = \sharp\mathscr{B}_n \geqslant \left[\frac{\sharp A_{[\sqrt{n}]}}{2[\sqrt{n}]+1}\right] \geqslant \left[\frac{(3/2)^{[\sqrt{n}]} - [\sqrt{n}]}{[\sqrt{n}](2[\sqrt{n}]+1)}\right].$$

Evidently we have $\lim_{n\to\infty}\sigma(n) = \infty$. Moreover, passing to a subset of $\mathscr{T}$ if necessary we can always assume that $\sigma(n)$ is non-decreasing.

Let us show that the union $\bigcup_{k=n}^{\infty} \mathscr{T}(k)$ satisfies $C^*(\lambda(n))$ for

$$\lambda(n) = \frac{9}{[\sqrt{n}]} + \frac{2[\sqrt{n}]}{n - 2\sqrt{n}}.$$

Suppose that $\overline{U}_1, \overline{U}_2$ are two different words from $\bigcup_{k=n}^{\infty} \mathscr{T}(k)$, $n \geqslant 81$, and V is a common subword of $\overline{U}_1, \overline{U}_2$. Let U_1 and U_2 be the cores of $\overline{U}_1$ and $\overline{U}_2$ respectively, $l = \min\{\|U_1\|, \|U_2\|\}$. Note that the length of any common subword of U_1 and U_2 is at most $3l/[\sqrt{n}]$ by Lemma 3.6. According to Lemma 3.7 and inequality (9) this yields

$$\frac{\|V\|}{\min\{\|\overline{U}_1\|, \|\overline{U}_2\|\}} \leqslant \frac{(9/[\sqrt{n}])l + 2[\sqrt{n}]}{l} \leqslant \lambda(n).$$

In case $\overline{U} \in \bigcup_{k=n}^{\infty} \mathscr{T}(k)$ and V is a common subword of two different cyclic shifts of $\overline{U}$, we obtain the inequality $\|V\|/\|U\| \leqslant \lambda(n)$ in an analogous way. Finally, let N be an integer such that $\lambda(N) \leqslant \frac{1}{50}$. Then we set $\mathscr{T}(n) = \emptyset$ for all $n \leqslant N$ and redefine $\lambda(n)$ to be equal to $1/50$ for all $n \leqslant N$. $\square$

4. Main construction

Throughout the rest of the paper we fix a metric space M satisfying conditions (M1) and (M2). Let $\mathscr{T}$ be the set of words provided by Proposition 3.3 and $\sigma = \sigma_{\mathscr{T}}$ its growth function. Also, let us fix a sequence

$$Net_1 \subseteq Net_2 \subseteq \cdots$$

constructed in Lemma 2.2 such that Net_i is a $(2/i, 1/i)$–net in M_i, $i \in \mathbb{N}$. By Γ_i we denote the complete (abstract) graph with the vertex set Net_i. Further, we endow Γ_i with a metric in which the length of an edge e with endpoints x and y is $dist_M(x, y)$. Thus there is a map $\Gamma_i \to M$ that maps each vertex of Γ_i to the corresponding point of Net_i and maps edges of Γ_i to geodesics in M. It is clear that the restriction of this map to the set of vertices of Γ_i is an embedding.

We consider a sequence n_i, $i \in \mathbb{N}$, satisfying the following three conditions:

(I) $\{n_i/i\}$ is an increasing sequence of natural numbers.
(II) For any $i \in \mathbb{N}$, $\sigma(n_i/i) \geqslant N_i(N_i - 1)/2$, where $N_i = \sharp Net_i$.
(III) $n_i/i > n_{i-1} diam M_{i-1}$ for all $i \in \mathbb{N}$, $i \geqslant 2$.

Obviously we can always ensure (I)–(III), choosing n_i after n_{i-1}. (Recall that $\sigma(n) \to \infty$ as $n \to \infty$.)

For every $i \in \mathbb{N}$, we take an arbitrary orientation on edges of Γ_i. Let $E(\Gamma_i)$ denote the set of all oriented edges of Γ_i. In the next lemma $\lceil x \rceil$ means the smallest integer y such that $y \geqslant x$.

Lemma 4.1. *There exists an injective labelling function $\phi : \bigcup_{i=1}^{\infty} E(\Gamma_i) \to \mathscr{T}^{\pm 1}$ such that for any edge $e \in E(\Gamma_i)$ with endpoints x, y, we have*

$$\|\phi(e)\| = \lceil n_i dist_M(x, y) \rceil \tag{10}$$

and $\phi(e^{-1}) = \phi(e)^{-1}$. In particular, the set $\{\phi(e) \mid e \in E(\Gamma_i)\}$ of all edge labels of Γ_i satisfies $C^*(\lambda(i))$.

Proof. Suppose that $x, y \in Net_i$ and $u, v \in Net_j$ for some $j > i$. Then combining conditions (I)–(III) and the fact that Net_j is a $(2/j, 1/j)$-net, we obtain

$$\lceil n_i dist_M(x, y) \rceil \leqslant \lceil n_i diam M_i \rceil < \lceil n_{i+1}/(i+1) \rceil \leqslant \lceil n_j/j \rceil \leqslant \lceil n_j dist_M(u, v) \rceil.$$

Thus it suffices to show that the number l_{ik} of unordered pairs $x, y \in Net_i$ such that $[n_i dist_M(x, y)] = k$ is less than the number of words of length k in $\mathscr{T}$ for every possible k. Obviously, we have

$$l_{ik} \leqslant \frac{N_i(N_i - 1)}{2}$$

and

$$\sigma(k) \geqslant \sigma(n_i/i) \geqslant \frac{N_i(N_i - 1)}{2}$$

since σ is non-decreasing and $dist_M(x, y) > 1/i$ for any $x, y \in Net_i$.

The assertion "in particular" can be derived as follows. Note that $n_i/i > i$ by the property (I). Since for any $e \in E(\Gamma_i)$, we have

$$\|\phi(i)\| \geqslant n_i dist_M(e_-, e_+) \geqslant n_i/i > i,$$

$\phi(e)$ belongs to the union $\bigcup_{j=i}^{\infty} \mathscr{T}(j)$. It remains to apply Proposition 3.3. □

If $p = e_1 e_2 \ldots e_n$ is a combinatorial path in Γ_i, where $e_1, e_2, \ldots, e_n \in E(\Gamma_i)$, we define the label $\phi(p)$ to be the word $\phi(e_1)\phi(e_2)\ldots\phi(e_n)$. Let

$$\mathscr{R}_i = \{\phi(p) \mid p \text{ is an irreducible cycle in } \Gamma_i\}$$

and

$$\mathscr{R} = \bigcup_{i=1}^{\infty} \mathscr{R}_i.$$

Finally, we define the group G by the presentation

$$\langle a, b \mid \mathscr{R} \rangle. \tag{11}$$

Let Δ be a van Kampen diagram over (11), Π a cell of Δ. We say that Π has *rank i* if $Lab(\partial\Pi)$ is a word from $\mathscr{R}_i$. Further, we call a word W in the alphabet $\{a^{\pm 1}, b^{\pm 1}\}$ a *Γ_i-word* if W is a label of some irreducible combinatorial path p in Γ_i. (Evidently such a path p is unique as $\mathscr{T}$ satisfies $C^*(\frac{1}{50})$ and ϕ is injective.)

Suppose that p is a path in a van Kampen diagram Δ over (11). If $Lab(p)$ is a Γ_i-word corresponding to the path $e_1 \ldots e_t$ in Γ_i, where $e_1, \ldots, e_t$ are edges of Γ_i, then p can be represented as a product

$$p = p_1 \ldots p_t \tag{12}$$

of its segments $p_1, \ldots, p_t$ with labels $Lab(p_1) = \phi(e_1), \ldots, Lab(p_t) = \phi(e_t)$. In this case we call the decomposition (12) a *canonical* decomposition of p.

Note that for the boundary p of a cell Π in Δ, two edges (say, e and f) adjacent to the vertex $(p_i)_+ = (p_{i+1})_- = e_+ = f_-$ for some i can have mutually inverse labels. This allows one to identify e with f^{-1}; then we can pass to the next pair of edges adjacent to $e_- = f_+$ and so on. Since $\mathscr{T}$ satisfies $C^*(\frac{1}{50})$, not more than $1/50$ of each segment $p_1, \ldots, p_t$ can be cancelled by such reductions. The irreducible path $p'_1 \ldots p'_t$, where p'_i is a subpath of p_i, is called a *reduced boundary* of Π and is denoted by $\partial_{\text{red}}\Pi$.

Thus we have $|p_i'| \geqslant \frac{48}{50}|p_i|$. Also, to each path q in Δ, we assign a path q_{red} which is obtained from q by eliminating edges that do not appear in reduced boundaries of cells in Δ.

It seems more natural to consider the reduced boundary. However, in the sequel, working with the notion of the well-attached cells defined below, it is convenient, for technical reasons, to distinguish between the notion of the boundary and that of the reduced boundary.

Given two cells Π_1, Π_2 of the same rank i in a van Kampen diagram Δ over (11), we say that Π_1 and Π_2 are *well-attached to each other*, if the following is true. Up to a cyclic shift, $\partial\Pi_1$ (respectively $(\partial\Pi_2)^{-1}$) admits a canonical decomposition $\partial\Pi_1 = p_1 \dots p_t$ (respectively $(\partial\Pi_2)^{-1} = q_1 \dots q_s$) associated to a path $e_1 \dots e_t$ (respectively $f_1 \dots f_s$) in Γ_i, where $e_1 = f_1$ and $p_1 = q_1$. Let d be the reduced cycle in Γ_i obtained from $e_2 \dots e_t f_s^{-1} \dots f_2^{-1}$. Then the label of the cycle $c = p_2 \dots p_t q_s^{-1} \dots q_2^{-1}$ is freely equal to the Γ_i-word corresponding to d. Thus, by the definition of $\mathcal{R}_i$, $Lab(c)$ is freely equal to a relator and hence we can replace cells Π_1 and Π_2 with one cell (see [13] for details).

Now suppose that $\partial\Delta = uw$, where $Lab(w)$ is a Γ_i-word. We say that a cell Π of rank i is *well-attached to a segment w of boundary of Δ* if, up to a cyclic shift, $\partial\Pi$ (respectively w^{-1}) admits a canonical decomposition $\partial\Pi = p_1 \dots p_t$ (respectively $w^{-1} = q_1 \dots q_s$) associated to a path $e_1 \dots e_t$ (respectively $f_1 \dots f_s$) in Γ_i, where $e_1 = f_1$ and $p_1 = q_1$. In this case we denote by d the reduced cycle in Γ_i obtained from $f_s^{-1} \dots f_2^{-1} e_2 \dots e_t$. Then the label of the path $v = q_s^{-1} \dots q_2^{-1} p_2 \dots p_t$ is freely equal to the Γ_i-word corresponding to d. Thus, by cutting the cell Π, we obtain a subdiagram Σ of Δ such that $\partial\Sigma = uv$, where v is also a Γ_i-word.

We can summarize these observations as follows.

Lemma 4.2. *Let Δ be a van Kampen diagram over* (11).

(1) *Suppose that Δ is minimal, i.e., it has a minimal number of cells among all diagrams over* (11) *with the same boundary label. Then no two cells of Δ are well attached to each other.*
(2) *Suppose that $Lab(\partial\Delta) = VW$, where W is a Γ_i-word. Assume that a cell Π is well-attached to the subpath of $\partial\Delta$ labelled W. Then there exists a subdiagram Σ of Δ* (*which can be obtained from Δ by cutting the cell Π*) *such that $Lab(\partial\Sigma) = VU$, where U is a Γ_i-word.*

The next lemma provides certain sufficient conditions for two cells (or a cell and a part of boundary of a diagram) to be well-attached.

Lemma 4.3. *Let Δ be a van Kampen diagram over* (11).

(1) *Suppose that Π_1, Π_2 are cells in Δ such that there exists a common subpath p of $\partial_{\mathrm{red}}\Pi_1$ and $(\partial_{\mathrm{red}}\Pi_2)^{-1}$ such that*

$$|p| \geqslant \frac{1}{10}\, \min\{|\partial_{\mathrm{red}}\Pi_1|,\ \ |\partial_{\mathrm{red}}\Pi_2|\}.$$

Then Π_1 and Π_2 are well-attached to each other.
(2) *Suppose that $\partial\Delta = vw$, where $Lab(w)$ is a Γ_i-word. Assume that for a cell Π, there is a common subpath q of w_{red} and $(\partial_{\mathrm{red}}\Pi)^{-1}$ of length*

$$|q| \geqslant \frac{1}{10}\, |\partial_{\mathrm{red}}\Pi|.$$

Then Π is well-attached to the subpath w of $\partial\Delta$.

Proof. Let us prove the first assertion of the lemma. Up to a cyclic shift, $\partial\Pi_1$ admits canonical decomposition $p_1 \dots p_t$. Let $p'_1 \dots p'_t$ be the corresponding decomposition of $\partial_{\text{red}}\Pi_1$, where p'_i is a subpath of p_i. Then p and a certain p'_i have a common subpath q of length at least $(\frac{1}{20})|p'_i| \geqslant (\frac{48}{1000})|p_i|$. Let $q_1 \dots q_s$ be the canonical decomposition of $\partial\Pi_2$, $q'_1 \dots q'_s$ the corresponding decomposition of $\partial_{\text{red}}\Pi_2$. Let Z denote the set of endpoints of paths $q'_1, \dots, q'_s$. If q is cut by vertices from Z into at most two parts, then one of these parts has length at least $\frac{1}{2}|q| \geqslant (\frac{24}{1000})|p_i| > (\frac{1}{50})|p_i|$.If q contains more than one vertex from Z, then q'_j is a subpath of p_i for some j (note that $|q'_j| \geqslant (\frac{48}{50})|q_j|$). In both cases we found a common subpath of p_i and q_j of length at least $(\frac{1}{50})\min\{|p_i|,\ |q_j|\}$. Therefore the labels of edges e_i and f_j of Γ_k and Γ_l respectively corresponding to p_i and q_j contain a common subword of length at least $(\frac{1}{50})\min\{\|\phi(e_i)\|,\ \|\phi(f_j)\|\}$. Since $\mathcal{T}$ satisfies $C^*(\frac{1}{50})$ and ϕ is injective, we have $p_i = q_j$, $k = l$, and $e_i = f_j$. The proof of the second assertion is similar and we leave it to the reader. □

From Lemmas 4.2 and 4.3, we immediately obtain

Corollary 4.4. *Let Δ be a minimal van Kampen diagram over* (11). *Then for any common subpath p of the reduced boundaries any two cells Π_1 and Π_2 of Δ, we have* $|p| < \frac{1}{10}\min\{|\partial_{\text{red}}\Pi_1|,\ |\partial_{\text{red}}\Pi_2|\}$.

Up to notation, the proof of the next lemma coincides with the proof of Lemma 8 in [14]. We provide it for convenience of the reader.

Lemma 4.5. *Suppose that W is a Γ_i-word. Then there exists a word V such that $W = V$ in G, V is of the minimal length among all of the words* (*not necessarily Γ_i-words*) *representing the same element as W in G, and V is freely equal to a Γ_i-word.*

Proof. Let V be a shortest word representing the same element as W in G. We consider a van Kampen diagram Δ over (11) corresponding to this equality. Without loss of generality we may assume that the word W and Δ are chosen in such a way that Δ has the minimal number of cells among all diagrams corresponding to equalities of V to Γ_i-words. We are going to show that Δ contains no cells at all, and thus V is freely equal to a Γ_i-word.

Assume that there is at least one cell in Δ. Denote by Δ' the map obtained from Δ by eliminating all edges that do not appear in reduced boundaries of cells of Δ. Then Δ', as a map, satisfies $C'(\frac{1}{10})$ small cancellation condition (see [11, Chapter 5]) by Corollary 4.4. By Greendlinger's Lemma, this means that Δ contains a cell Π such that there is a common subpath of $\partial_{\text{red}}\Pi$ and $(\partial\Delta)_{\text{red}}$ of length $|p| > 0.7|\partial_{\text{red}}\Pi|$. (We substitute $\lambda = 0.1$ in the Greendlinger's constant $1 - 3\lambda$ from [11]).

The boundary of Δ consists of two parts v and w corresponding to words V and W. If the path p has a common subpath with $w_{\text{red}}^{\pm 1}$ of length at least $0.1|\partial\Pi|$, then Π is well-attached to the subpath w of $\partial\Delta$ by Lemma 4.3. However, by the second assertion of Lemma 4.2 this contradicts the choice of W and Δ. Hence there is a common subpath q of $\partial_{\text{red}}\Pi$ and $v_{\text{red}}^{\pm 1}$ such that $|q| > (0.7 - 0.2)|\partial_{\text{red}}\Pi| = 0.5|\partial_{\text{red}}\Pi|$. Thus $\partial_{\text{red}}\Pi = qq_1$, $|q| > |q_1|$, and the words $Lab(q)$, $Lab(q_1)$ represent the same element in the group G. But $Lab(q)$ is a subword of V and we arrive at a contradiction to our choice of V as a shortest word representing the same element as W in G. □

Definition 4.6. For each $i \in \mathbb{N}$, we construct an embedding

$$\alpha_i : Net_i \to G$$

as follows. Let us fix a point O in Net_1 (and thus $O \in Net_i$ for all i). Then for any $x \in Net_i$ there is a combinatorial path p in Γ_i such that $p_- = O$, $p_+ = x$. We define $\alpha_i(x)$ to be equal to the element of G represented by $\phi(p)$. Note that $\alpha_i(x)$ is independent of the choice of p. Indeed, if q is another path in Γ_i with the origin O and terminus x, then pq^{-1} is a cycle and thus $\phi(p)\phi(q^{-1})$ is a relator from $\mathscr{R}_i$, i.e., $\phi(p)$ and $\phi(q)$ represent the same element of G.

Lemma 4.7. *Let $dist_G$ denote the word metric on G corresponding to the generating set $\{a, b\}$. Then for any $i \in \mathbb{N}$ and any $x, y \in Net_i$, we have*

$$(1 - 2\lambda(i))dist_M(x, y) \leqslant \frac{1}{n_i}\, dist_G(\alpha_i(x), \alpha_i(y)) \leqslant dist_M(x, y) + \frac{1}{n_i}. \tag{13}$$

Proof. If e is an edge in Γ_i such that $e_- = x$, $e_+ = y$, and a, b are edges in Γ_i such that $a_- = b_+ = O$, $a_+ = x$, $b_- = y$, then aeb is a cycle in Γ_i. Therefore $\phi(a)\phi(e)\phi(b)$ labels a cycle c in $Cay(G)$ beginning at 1. Let $c = psq$, where $Lab(p) \equiv \phi(a)$, $Lab(s) \equiv \phi(e)$, $Lab(q) \equiv \phi(b)$. Since by definition $p_+ = \alpha_i(x)$ and $q_- = \alpha_i(y)$, the elements $\alpha_i(x)$ and $\alpha_i(y)$ are connected by the path s in $Cay(G)$. Therefore,

$$dist_G(\alpha_i(x), \alpha_i(y)) \leqslant |s| = \|\phi(e)\| = \lceil n_i dist_M(x, y)\rceil \leqslant n_i dist_M(x, y) + 1.$$

This gives the right-hand side inequality in (13).

Further, by Lemma 4.5, there exists a word V representing the element $(\alpha_i(x))^{-1}\alpha_i(y))$ and a Γ_i-word U freely equal to V such that

$$\|V\| = dist_G(1, \alpha_i(x))^{-1}\alpha_i(y)) = dist_G(\alpha_i(x), \alpha_i(y)). \tag{14}$$

Obviously we have

$$\|V\| \geqslant (1 - 2\lambda(i))\|U\| \tag{15}$$

since the set of edge labels of Γ_i satisfies $C^*(\lambda(i))$ by Lemma 4.1. Let $r = e_1 \dots e_t$ be the path in Γ_i corresponding to U. Then, arguing as in the first case, we can show that $r_- = x$, $r_+ = y$ and thus

$$\|U\| = \sum_{j=1}^{t} \|\phi(e_j)\| = \sum_{j=1}^{t} n_i dist_M((e_j)_-, (e_j)_+) \geqslant n_i dist_M(x, y). \tag{16}$$

Combining (14), (15), and (16) we obtain the left-hand inequality in (13). □

Definition 4.8. We take a non-principal ultrafilter ω such that $\omega(\{n_i\}) = 1$ and consider the asymptotic cone $Cone_\omega(G)$ of G with respect to this ultrafilter. Our next goal is to define an embedding $\imath$ of M to $Cone_\omega(G)$.

Let x be a point of M. Then there is a sequence of points $x_i \in Net_i$ such that $x_i \to x$ as $i \to \infty$. We define $\imath(x)$ to be the point of $Cone_\omega(G)$ represented by an arbitrary sequence $\{g_i\}$, where $g_{n_i} = \alpha_i(x_i)$ for any $i \in \mathbb{N}$. Obviously $\imath$ is well-defined as the point of $Cone_\omega(G)$ representing the sequence $\{g_i\}$ depends on the subsequence $\{g_{n_i}\}$ only.

Proposition 4.9. *Suppose that M is a metric space satisfying* (M1) *and* (M2). *Then the map $\imath$ is an isometry.*

Proof. Let x, y be points of M, $\{x_i\}$, $\{y_i\}$ the sequences of elements of nets Net_i such that $x_i \to x$ as $i \to \infty$ and $y_i \to y$ as $i \to \infty$. Let $\{g_i\}$ and $\{h_i\}$ be the corresponding sequences of elements of G representing $\iota(x)$ and $\iota(y)$. Then applying Lemma 4.7, we have

$$\begin{aligned} dist_M(x, y) &= \lim_{i\to\infty} dist_M(x_i, y_i) = \lim_{i\to\infty} \frac{1}{n_i} dist_G(\alpha_i(x_i), \alpha_i(y_i)) \\ &= \lim_{\omega} \frac{1}{i} dist_G(g_i, h_i) = dist_{\mathrm{Cone}}(\iota(x), \iota(y)). \quad \square \end{aligned}$$

5. Embedding of the fundamental group

All assumptions and notation from the previous section remain in force here. In particular, ι denotes the isometry $M \to Cone_\omega(G)$ constructed in the previous section. In addition we suppose that M satisfies (M3). Also, let ι^* denote the homomorphism $\pi_1(M) \to \pi_1(Cone_\omega(G))$ induced by ι. We conclude the proof of Theorem 1.2 by proving the following.

Proposition 5.1. *Suppose that M satisfies* (M1)–(M3). *Then the map* $\iota^* : \pi_1(M) \to \pi_1(Cone_\omega(G))$ *is injective.*

Proof. Let $S = [0, 1] \times [0, 1]$ be a unit square and $\gamma : \partial S \to M$ a loop in M such that $\iota\gamma$ is contractible in $Cone_\omega(G)$. We want to show that γ is contractible in M.

Since $\iota\gamma$ is contractible in $Cone_\omega(G)$, there exists a continuous map $r : S \to Cone_\omega(G)$ such that the restriction of r to ∂S coincides with $\iota\gamma$. The unit square S is compact, and therefore r is uniformly continuous. Hence there exists δ such that for any $y_1, y_2 \in B$ which lie at a distance at most δ in B, we have

$$dist_{\mathrm{Cone}}(r(x), r(y)) < \varepsilon/20. \tag{17}$$

We can also assume that $1/\delta \in \mathbb{N}$. By $Grid_\delta$ we denote the standard δ-net in S that is the set

$$Grid_\delta = \{(a\delta, b\delta) \mid a, b \in \mathbb{Z},\ 0 \leqslant a, b \leqslant 1/\delta\}.$$

By $r(Grid_\delta)$ we denote the image of $Grid_\delta$ in $Cone_\omega(G)$.

For every point $x \in r(Grid_\delta) \cup (\bigcup_{i=1}^\infty \iota(Net_i))$ we fix an arbitrary sequence $\{x_i\}$ of elements of G that represents x in $Cone_\omega(G)$ (such a sequence will be called a *standard representative* of x). Let ε be the constant from (M3). We take $L \in \mathbb{N}$ such that the following conditions hold:

(L0) $r(Grid_\delta)$ is contained in $\iota(M_L)$;
(L1) $1/L < \varepsilon/20$; in particular, $1/n_L < \varepsilon/20$;
(L2) for any two points $x, y \in r(Grid_\delta) \cup \iota(Net_L)$, we have

$$\left| \frac{1}{n_L} dist_G(x_{n_L}, y_{n_L}) - dist_{\mathrm{Cone}}(x, y) \right| \leqslant \varepsilon/20,$$

where $\{x_i\}$ and $\{y_i\}$ are standard representatives of x and y respectively.

(Note that for any l there exist $L > l$ such that (L0)–(L2) hold.)

We say that two points x, y in $Grid_\delta$ are neighbors if they have the form $x=(a\delta, b\delta)$, $y=((a+1)\delta, b\delta)$ or $x=(a\delta, b\delta)$, $y=(a\delta, (b+1)\delta)$. If $x, y \in Grid_\delta$ are neighbors and $\{x_i\}$, $\{y_i\}$ are standard representatives of $r(x), r(y)$, we fix an arbitrary geodesic in the Cayley graph G going from the element x_{n_L} to y_{n_L} and denote this geodesic by $g(x_{n_L}, y_{n_L})$. Further for every point $x \in Grid_\delta$ which lies on ∂S, we take a point $t^x \in \iota(Net_L)$ which is closest to $r(x)$; in particular, we have

$$dist_{\text{Cone}}(t^x, r(x)) \leqslant 2/L \leqslant 0.1\varepsilon \tag{18}$$

as $r(x) \in \iota(M_L)$ by (L0) and $\iota(Net_L)$ is a $2/L$-net in $\iota(M_L)$ (recall that ι is an isometry). Suppose that $\{t_i^x\}$ is the standard representative of t^x. Then we join elements x_{n_L} and $t^x_{n_L}$ by a geodesic $h(x_{n_L}, t^x_{n_L})$ in $Cay(G)$. Finally, if $x, y \in \partial S$ are neighbors and t^x, t^y are the corresponding points of $\iota(Net_L)$, then we denote by $k(t^x_{n_L}, t^y_{n_L})$ a path in $Cay(G)$ joining $t^x_{n_L}$ to $t^y_{n_L}$ such that the label of k is equal to $\phi(e)$, where e is the edge of Γ_L satisfying the conditions $\iota(e_-)=t^x$, $\iota(e_+)=t^y$. In particular, we have

$$|k(t^x_{n_L}, t^y_{n_L})| = \|\phi(e)\| = \lceil n_L dist_{\text{Cone}}(t^x, t^y)\rceil. \tag{19}$$

Let $x^1, \dots, x^m$, where $m=4/\delta$, be subsequent points of $Grid_\delta \cap \partial S$ (i.e., x^i and x^{i+1} are neighbors, where indices are modulo m). Then the label of the cycle

$$p = k(t^{x^1}_{n_L}, t^{x^2}_{n_L})k(t^{x^2}_{n_L}, t^{x^3}_{n_L}) \dots k(t^{x^m}_{n_L}, t^{x^1}_{n_L}) \tag{20}$$

is a Γ_i-word. We construct a van Kampen diagram Ξ with boundary label $Lab(\partial\Xi) \equiv \phi(p)$ as follows. The net $Grid_\delta$ allows one to regard S as a union of $1/\delta^2$ small squares with sides of length δ. For any such square with vertices x, y, z, t in $Grid_\delta$, we consider a minimal van Kampen diagram (homeomorphic to a disk) with boundary label

$$Lab(g(x_{n_L}, y_{n_L})g(y_{n_L}, z_{n_L})g(z_{n_L}, t_{n_L})g(t_{n_L}, x_{n_L})). \tag{21}$$

Also, if $x, y \in \partial S \cap Grid_\delta$ are neighbors, we consider a minimal van Kampen diagram (homeomorphic to a disk) with boundary label

$$Lab(g(x_{n_L}, y_{n_L})h(y_{n_L}, t^y_{n_L})k(t^y_{n_L}, t^x_{n_L})(h(x_{n_L}, t^x_{n_L}))^{-1}). \tag{22}$$

We call the constructed diagrams with boundary labels (21), (22) *elementary*. Gluing these elementary diagrams together in the obvious way we obtain a diagram Ξ over (11) such that $Lab(\partial\Xi)$ is the Γ_i-word defined by (20).

We are going to show that the perimeter of each elementary diagram is less than $0.7n_L\varepsilon$. Indeed, inequality (17) and condition (L2) together yield

$$|g(x_{n_L}, y_{n_L})| = dist_G(x_{n_L}, y_{n_L}) \leqslant n_L\,(dist_{\text{Cone}}(r(x), r(y)) + 0.05\varepsilon) \leqslant 0.1 n_L\varepsilon \tag{23}$$

for any two neighbors $x, y \in Grid_\delta$. If $x \in Grid_\delta \cap \partial S$, then (L1), (L2) and (18) imply

$$\begin{aligned}|h(x_{n_L}, t^x_{n_L})| &= dist_G(x_{n_L}, t^x_{n_L}) \leqslant n_L(dist_{\text{Cone}}(r(x), t^x) + 0.05\varepsilon)\\ &\leqslant n_L(2/L + 0.05\varepsilon) \leqslant 0.15 n_L\varepsilon.\end{aligned} \tag{24}$$

Finally, if $x, y \in \partial S$ are neighbors, then combining (17), (18), and (19) we obtain

$$
\begin{aligned}
|k(t^x_{n_L}, t^y_{n_L})| &= \lceil n_L dist_{\mathrm{Cone}}(t^x, t^y)\rceil \\
&\leqslant \lceil n_L(dist_{\mathrm{Cone}}(t^x, r(x)) + dist_{\mathrm{Cone}}(r(x), r(y)) + dist_{\mathrm{Cone}}(r(y), t^y))\rceil \\
&\leqslant \lceil 0.25 n_L \varepsilon\rceil \leqslant 0.25 n_L\varepsilon + 1 \leqslant 0.3 n_L \varepsilon. \qquad (25)
\end{aligned}
$$

Therefore, any word of type (21) or (22) has length at most $0.7 n_L\varepsilon$.

Lemma 5.2. *Let Π be a cell of rank L in Ξ, l the loop in Γ_L corresponding to the Γ_L*-word $Lab(\partial\Pi)$. *Then l is contractible in M.*

Proof. Note that Π lies in some elementary diagram Θ. Since any elementary diagram is minimal, it satisfies $C'(1/10)$ small cancellation condition as a map by Corollary 4.4. Hence the length of the reduced boundary of any cell in Θ is not greater than $|\partial\Theta| \leqslant 0.7 n_L\varepsilon$. This means that

$$|\partial\Pi| \leqslant \frac{50}{48}\,|\partial_{\mathrm{red}}\Pi| \leqslant \frac{35}{48}\,n_L\varepsilon < n_L\varepsilon.$$

Let $l = e_1 \dots e_t$, where $e_1, \dots, e_t$ are edges of Γ_L. The length of l satisfies

$$|l| = \sum_{i=1}^{t} |e_i| \leqslant \sum_{i=1}^{t} \frac{1}{n_L}\,\|\phi(e_i)\| = \frac{1}{n_L}\,|\partial\Pi| < \varepsilon.$$

Therefore, l is contractible in M by (M3). □

Lemma 5.3. *Consider a van Kampen diagram Δ with boundary labelled by a G_L-word. Suppose that the boundary label of each cell of rank L in this diagram corresponds to a contractible loop in M. Then the boundary label of the diagram also corresponds to a contractible loop in M.*

Proof. We prove the statement of the lemma by induction on the number s of cells in the diagram. If $s = 0$ the statement is obvious, so we assume that $s \geqslant 1$.

By Grindlinger's lemma at least one of the following two statements holds:

(1) There exist two cells Π_1 and Π_2 and a common subpath p of $\partial_{\mathrm{red}}\Pi_1$ and $\partial_{\mathrm{red}}\Pi_2$ such that $|p| \geqslant \frac{1}{10}\min\{|\partial_{\mathrm{red}}\Pi_1|,\ |\partial_{\mathrm{red}}\Pi_2|\}$.
(2) There exist a cell Π and a common subpath p of $\partial_{\mathrm{red}}\Pi$ and $\partial\Delta$ such that $|p| \geqslant \frac{7}{10}\,|\partial_{\mathrm{red}}\Pi|$.

In the first case Π_1 and Π_2 have the same rank and are well-attached to each other by Lemma 4.3. Arguing as in the proof of Lemma 4.2, we can replace Π_1 and Π_2 by one cell Υ. If $rank\ \Pi_1 = rank\ \Pi_2 \neq L$, the statement is true by the inductive hypothesis. To use the inductive hypothesis in case $rank\ \Pi_1 = rank\ \Pi_2 = L$, we have to check that the cycle s corresponding to the new cell Υ is contractible in M. Indeed, if p, q are cycles corresponding to Π_1 and Π_2 (we may assume that $p_- = q_-$), then s is homotopic to the product of p and q^{-1}. Since p and q are contractible in M by the condition of the lemma, s is contractible in M.

In the second case Π has rank L by Lemma 4.2 and is well-attached to the boundary of Δ. We pass to the subdiagram Σ of Δ obtained by cutting the cell Π. Applying Lemma 4.2 again, we conclude that $Lab(\partial\Sigma)$ is a Γ_L-word. By the inductive assumption the cycle c corresponding to $Lab(\partial\Sigma)$ is contractible in M. Let d be the cycle in Γ_L corresponding to $Lab(\partial\Pi)$, f the cycle corresponding to $Lab(\partial\Delta)$. As in the previous case, f is homotopic to the product of c and d^{-1} and hence is contractible in M. □

Now we return to the proof of the proposition. The two previous lemmas imply that the loop q in Γ_L, corresponding to the boundary label of the diagram Ξ under consideration, is contractible.

As above, let $x_1, \dots, x_m$ be subsequent neighbors in $Grid_\delta \cap \partial S$. For every two neighbors x_i, x_{i+1} (indices are modulo m), we denote by c_i, d_i, e_i, f_i the segment $[r(x_i), r(x_{i+1})]$ of $\iota\gamma$, the geodesic path from $r(x_{i+1})$ to $t^{x_{i+1}}$, the edge e of Γ_i such that $\iota(e_-) = t^{x_{i+1}}$, $\iota(e_+) = t^{x_i}$, and the geodesic path from t^{x_i} to $r(x_i)$, respectively. Note that for any $i = 1, \dots, m$, the cycle $a_i = c_i d_i e_i f_i$ is contained in the ball $B_i = B_i(0.35\varepsilon, x_i)$ of radius 0.35ε around $r(x_i)$ in $Cone_\omega(G)$. Indeed any point of c_i is contained in B_i by (17). Further since d_i and f_i are geodesic, d_i and f_i are contained in 0.1ε-neighborhoods of $r(x_{i+1})$ and $r(x_i)$ respectively according to (18); together with (17) this implies that d_i and c_i lie in B_i. Finally, each point of e_i belongs to B_i as e_i is geodesic, the distance between $r(x_i)$ and the end of e_i is at most 0.1ε, and the length of e_i is at most 0.25ε by the triangle inequality. Thus a_i is contained in B_i. Since ι is an isometry, this means that the preimage of a_i under $\iota : M \to Cone_\omega(G)$ is contractible in M by (M3). Hence $\iota(q)$ is homotopic to $\iota\gamma$ via a homotopy in M. Hence γ is contractible in M according to Lemma 5.3. □

6. Concluding remarks and questions

We have shown that any countable group can be embedded into a fundamental group of an asymptotic cone of some finitely generated group. Note that our proof also shows that any recursively presentable group can be embedded into a fundamental group of some finitely presentable group.

The construction of our group depends on a space M and a scaling sequence n_k. Similarly we can start with a countable set of spaces N_j (satisfying (M1)–(M3)), take a countable set of non-intersecting scaling sequences $n_k^j(N_j)$ and construct a group G, such that for each j there is a scale on which N_j is embedded into the asymptotic cone of G. A natural task is to check that starting with the spaces with very different fundamental groups (e.g. $\mathbb{Z}/p\mathbb{Z}$ for different p) one gets asymptotic cones (on different scales) with infinitely many different fundamental groups. Then under certain conditions on the spaces the group G is recursively presentable and we can embed it into a finitely presentable group. Again, a natural task is to check that one can choose this embedding in such a way that this finitely presented group has different fundamental groups on different scales.

Another natural question is: does there exist a finitely presented group such that the simple connectivity of the asymptotic cone depends on the choice of the ultrafilter?

Finally let us mention that recently Kramer et al. [10] have shown that if continuum hypothesis fails, then there exist finitely presented groups (which are uniform lattices in certain semisimple Lie groups) that have infinitely many different asymptotic cones. However, if continuum hypothesis holds, then the examples from [10] have unique asymptotic cones.

Acknowledgements

The results of this paper were obtained in Spring 2001, as both authors were visiting the University of Geneva. We would like to thank Pierre de la Harpe for his invitation to Geneva and the Swiss National Science Foundation for the support of our work. We are grateful to the referee for useful remarks and suggestions. We are also grateful to Mark Sapir and Cornelia Drutu for the interest they have shown in our results and Alexander Ol'shanskii for helpful comments and remarks.

References

[1] M.R. Bridson, Asymptotic cones and polynomial isoperimetric inequalities, Topology 38 (3) (1999) 543–554.
[2] J. Burillo, Dimension and fundamental groups of asymptotic cones, J. London Math. Soc. (2) 59 (2) (1999) 557–572.
[3] L. van den Dries, A.J. Wilkie, Gromov's theorem on groups of polynomial growth and elementary logic, J. Algebra 89 (2) (1984) 349–374.
[4] C. Drutu, Quasi-isometry invariants and asymptotic cones, International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory, Lincoln, NE, 2000, Inter. J. Algebra Comput. 12 (1–2) (2002) 99–135.
[5] A. Dyubina, I. Polterovich, Explicit constructions of universal $\mathbb{R}$-trees and asymptotic geometry of hyperbolic spaces, Bull. London Math. Soc. 33 (6) (2001) 727–734.
[6] M. Gromov, Groups of polynomial growth and expanding maps, IHES. Publ. Math. 53 (1981) 53–73.
[8] M. Gromov, Asymptotic invariants of infinite groups, Geometric group theory, London Math. Soc. Lecture Note Ser. 182 (1993) 1–295.
[9] B. Kleiner, B. Leeb, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings, IHES Publ. Math. 86 (1998) 115–197.
[10] L. Kramer, S. Shelah, K. Tent, S. Thomas, Asymptotic cones of finitely presented groups, preprint, http://xxx.lanl.gov/abs/math.GT/0306420.
[11] R.C. Lyndon, P.E. Shupp, Combinatorial Group Theory, Springer, Berlin, 1977.
[12] J.W. Morgan, P.B. Shalen, Valuations, trees, and degenerations of hyperbolic structures, I. Ann. Math. 120 (2) (1984) 401–476.
[13] A.Yu. Ol'shanskii, Geometry of defining relations in groups. Mathematics and its Applications (Soviet Series), 70. Kluwer Academic Publishers Group, Dordrecht, 1991.
[14] A.Yu. Ol'shanskii, Distortion functions for subgroups, in: J. Cossey, C.F. Miller, W.D. Neumann, M. Shapiro (Eds.), Group Theory Down Under, Gruyter, Berlin, 1999, 281–291.
[15] P. Papasoglu, On the asymptotic cone of groups satisfying a quadratic isoperimetric inequality, J. Differential Geom. 44 (4) (1996) 789–806.
[16] F. Paulin, Topologie de Gromov equivariante, structures hyperboliques et arbres reels, Inv. Math. 94 (1988) 53–80.
[17] T.R. Riley, Higher connectedness of asymptotic cones, Topology 42 (2003) 1289–1352.
[18] E. Rips, Cyclic splittings of finitely presented groups and the canonical JSJ-decomposition, in: S.D. Chatterji (Ed.), Proceedings of the International Congress of Mathematicians, ICM '94, August 3–11, 1994, Zurich, Switzerland. vol. I, Birkhauser, Basel, 1995, pp. 595–600.
[19] Z. Sela, The isomorphism problem for hyperbolic groups, J. Ann. of Math. 141 (2) (1995) 217–283.
[20] Z. Sela, Endomorphisms of hyperbolic groups, J. The Hopf property, Topology 38 (2) (1999) 301–321.
[21] Z. Sela, Diophantine geometry over groups VIII: The elementary theory of a hyperbolic group, available at http://www.ma.huji.ac.il/∼zlil/
[22] S. Thomas, B. Velickovic, Asymptotic cones of finitely generated groups, Bull. London Math. Soc. 32 (2) (2000) 203–208.

Further reading

[7] M. Gromov, Hyperbolic groups, Essays in group theory 75–263 Math. Sci. Res. Inst. Publ., 8, Springer, Berlin, NY, 1987.