Product topology in the hyperspace of subcontinua

Alejandro Illanes *, Verónica Martínez-de-la-Vega ¹

Instituto de Matemáticas, UNAM, Circuito Exterior, Cd. Universitaria, México 04510, D.F. Mexico

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Abstract

Let $X$ be a metric continuum. Let $C(X)$ be the hyperspace of subcontinua of $X$. Given two finite subsets $P$ and $Q$ of $X$, let $U(P, Q) = \{ A \in C(X): P \subset A \text{ and } A \cap Q = \emptyset \}$. In this paper we consider $C(X)$ with the topology $\tau_P$ which have the sets $U(P, Q)$ as a basis. In this paper we show that, for a dendroid $X$, some topological properties of $X$ are very closely related to the topological structure of $(C(X), \tau_P)$. © 2000 Elsevier Science B.V. All rights reserved.

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Introduction

A continuum is a compact connected metric space. The letter $X$ will always denote a nondegenerate continuum, with metric $d$, let

$$ C(X) = \{ A \subseteq X: A \text{ is a subcontinuum of } X \}. $$

Given two (possibly empty) finite subsets $P$ and $Q$ of $X$, let

$$ U(P, Q) = \{ A \in C(X): P \subseteq A \text{ and } Q \cap A = \emptyset \}. $$

It is easy to show (Proposition 1) that $\{ U(P, Q): P \text{ and } Q \text{ are finite subsets of } X \}$ is a basis for a topology $\tau_P$ of $C(X)$, which will be called the product topology for $C(X)$. This name is justified by Proposition 2 below. We also consider the topology $\tau_H$ in $C(X)$, generated by the Hausdorff metric $H$ in $C(X)$ (see [3, 4.1]). If $p \in X$ and $\varepsilon > 0$, let

$$ B(\varepsilon, p) = \{ q \in X: d(p, q) < \varepsilon \}. $$

The set of rational numbers is denoted by $\mathbb{Q}$. The set of positive integers is denoted by $\mathbb{N}$.
A dendroid is an arcwise connected, hereditarily unicoherent continuum and a dendrite is a locally connected dendroid. An end point of a dendroid \(X\) is a point \(p\) such that \(p\) is an end point of every arc containing \(p\). Given points \(p\) and \(q\) in a dendroid \(X\), let \(pq\) denote the unique arc in \(X\), joining \(p\) and \(q\) if \(p \neq q\) and let \(pq = \{p\}\) if \(p = q\). Given a nonempty subset \(P\) of a dendroid \(X\), let \(I(P)\) be the intersection of all subcontinua of \(X\) containing \(P\). Then \(I(P)\) is the minimum (with respect to inclusion) subcontinuum of \(X\) containing \(P\).

In this paper we show that some topological properties of a dendroid \(X\) are very closely related to the topological structure of \((C(X), \tau_P)\). Among other results we prove the following theorems.

**Theorem 9.** For a dendroid \(X\), there is a countable collection of nonempty open subsets, \(B\), of \((C(X), \tau_P)\) such that every nonempty element in \(\tau_P\) contains an element of \(B\) if and only if the set of end points of \(X\) is countable.

**Theorem 10.** For a dendroid \(X\), \((C(X), \tau_P)\) is separable if and only if there is not an uncountable collection of pairwise disjoint arcs in \(X\).

**Theorem 18.** For a dendroid \(X\), \(\tau_H \subset \tau_P\) if and only if \(X\) is a dendrite.

### 1. General properties

For a continuum \(X\), let

\[ F_1(X) = \{\{p\} \in C(X): p \in X\}. \]

The following proposition is easy to prove.

**Proposition 1.**

(a) \(U(\emptyset, \emptyset) = C(X)\),
(b) \(U(P, Q) \cap U(R, S) = U(P \cup R, Q \cup S)\),
(c) \(\{U(P, Q)\}: P \text{ and } Q \text{ are finite subsets of } X\) is a basis for a topology of \(C(X)\),
(d) \(U(P, Q)\) is open and closed in \((C(X), \tau_P)\),
(e) \(F_1(X)\) is a closed and discrete subspace of \((C(X), \tau_P)\),
(f) if \(Y\) is a subcontinuum of \(X\), then \(C(Y)\) is a closed subset of \((C(X), \tau_P)\) and the restriction of the topology \(\tau_P\) to \(C(Y)\) is the same as the product topology defined on \(C(Y)\), and
(g) \(X\) is an isolated element in \((C(X), \tau_P)\) if and only if there exists a finite subset \(F\) of \(X\) such that \(X\) is irreducible around \(F\) (this means that the only subcontinuum of \(X\) which contains \(F\) is \(X\)).

Now, let \(P(X) = \prod_{p \in X} [0, 1]_p\). We consider \(P(X)\) with the product topology, where, for each \(p \in X\), the space \([0, 1]_p\) is endowed with the discrete topology. There is a natural
way to consider $C(X)$ as a subspace of $P(X)$ by defining the function $g : C(X) \rightarrow P(X)$ given by $g(A) = \{x_p \}_{p \in X}$, where

$$x_p = \begin{cases} 1, & \text{if } p \in A, \\ 0, & \text{if } p \notin A. \end{cases}$$

The following result follows directly from the definition of $\tau_F$ and justifies the title of the paper.

**Proposition 2.** The function $g$ is an embedding from $(C(X), \tau_F)$ into $P(X)$.

**Corollary 3.** $(C(X), \tau_F)$ is a Tychonoff space.

With respect to the normality of $(C(X), \tau_F)$, we have the following question.

**Question 4.** Is there a continuum $X$ such that $(C(X), \tau_F)$ is normal?

Corollary 7 gives a partial answer to Question 4. First, we need the following lemma.

**Lemma 5.** If $(C(X), \tau_F)$ is separable, then $(C(X), \tau_F)$ is not normal.

**Proof.** We use Jones’ Lemma [1, Ex. 3, Section 3, Chapter VII] that says: if $Y$ is a topological space and $Y$ contains a dense subset $D$ and a discrete closed subset $S$ such that $|S| \geq 2^{|D|}$, then $Y$ is not normal.

By hypothesis, $(C(X), \tau_F)$ contains a countable dense subset $D$. We take $S = f_1(X)$. By Proposition 1(e), $S$ is an uncountable closed and discrete subspace of $(C(X), \tau_F)$. Thus, by Jones’ Lemma, $(C(X), \tau_F)$ is not normal.

There are many continua $X$ for which $(C(X), \tau_F)$ is not separable, see, for example, Theorem 10.

**Theorem 6.** If $X$ is the unit interval $[0, 1]$, then $(C(X), \tau_F)$ is not normal.

**Proof.** By Lemma 5, it is enough to check that $(C(X), \tau_F)$ is separable. Let $\mathcal{D} = \{[r, s] \in C(X) : r, s \in \mathbb{Q}\}$. It is easy to show that $\mathcal{D}$ is dense in $(C(X), \tau_F)$.

**Corollary 7.** If $X$ contains an arc, then $(C(X), \tau_F)$ is not normal.

**Proof.** It follows from Proposition 1(f) and Theorem 6.

2. **Semi-basis for $(C(X), \tau_F)$ when $X$ is a dendroid**

A *semi-basis* for a topological space $Y$, is a family $\mathcal{B}$ of nonempty open subsets of $Y$ such that every nonempty open subset of $Y$ contains an element of $\mathcal{B}$. For a continuum $X$, a *Whitney map* for $C(X)$ is a continuous function $\mu : (C(X), \tau_H) \rightarrow [0, 1]$ such that
\[ \mu(X) = 1, \mu([p]) = 0 \] for each \( p \in X \), and if \( A \subseteq B \) and \( A, B \in C(X) \), then \( \mu(A) < \mu(B) \).

It is known that, for very continuum \( X \), \( C(X) \) admits Whitney maps [3, Exercise 4.33].

**Lemma 8.** Let \( X \) be a dendroid and let \( \mu \) be a Whitney map for \( C(X) \). Given \( x \in X \), define \( f_x : X \to [0, 1] \) by \( f_x(p) = \mu(xp) \). Then:

(a) for each locally connected subcontinuum \( Y \) of \( X \), \( f_x|Y \) is continuous, and

(b) for each \( r \in [0, 1] \), \( \{ p \in X : f_x(p) \leq r \} \) is a subcontinuum of \( X \).

**Proof.**

(a) Let \( p \in Y \) and let \( \varepsilon > 0 \). Let \( \delta > 0 \) be such that if \( A, B \in C(X) \) and \( H(A, B) < 2\delta \), then \( |\mu(A) - \mu(B)| < \varepsilon \). Since \( Y \) is locally connected, there exists an open connected subset \( V \) of \( Y \) such that \( p \in V \subseteq cl_Y(V) \subseteq B(\delta, p) \). If \( q \in V \), then \( xp \cup cl_Y(V) \) is a subcontinuum of \( X \) containing \( x \) and \( q \). Thus \( xq \subseteq xp \cup cl_Y(V) \). Similarly, \( xp \subseteq xq \cup cl_Y(V) \). This implies that \( H(xp, xq) < 2\delta \). Hence \( |f_x(p) - f_x(q)| < \varepsilon \). Therefore, \( f_x|Y \) is continuous.

(b) Let \( A = \{ p \in X : f_x(p) \leq r \} \). If \( p \in A \), then \( \mu(xp) \leq r \) and \( xp \subseteq A \). Hence \( A \) is connected. Now, suppose that \( p = \lim p_n \) where each \( p_n \in A \). Since \( (C(X), \tau_p) \) is compact [3, Theorem 4.17], there exists a subsequence \( \{xp_n\}_{n=1}^{\infty} \) of \( \{xp_n\}_{n=1}^{\infty} \) such that \( \{xp_n\}_{n=1}^{\infty} \) converges to an element \( B \in C(X) \), with the Hausdorff metric. Then \( \mu(B) \leq r \) and \( x, p \in B \). This implies that \( xp \subseteq B \). Thus \( \mu(xp) \leq r \). Therefore, \( A \) is closed. \( \square \)

**Theorem 9.** Let \( X \) be a dendroid, then \( (C(X), \tau_p) \) has a countable semi-basis if and only if the set of end points of \( X \) is countable.

**Proof.** Let \( E \) be the set of end points of \( X \).

(\( \Rightarrow \)) Suppose that \( (C(X), \tau_p) \) has a countable semi-basis \( B_0 \). For each \( B \in B_0 \), choose finite sets \( P_B \) and \( Q_B \) such that \( \emptyset \neq U(P_B, Q_B) \subseteq B \). Then the set \( B = \{U(P_B, Q_B) : B \in B_0 \} \) is a semi-basis for \( (C(X), \tau_p) \). Let

\[ G = \left( \bigcup \{P_B : B \in B_0\} \right) \cup \left( \bigcup \{Q_B : B \in B_0\} \right). \]

Then \( G \) is countable.

We need to show that \( E \) is countable. Suppose to the contrary that \( E \) is not countable. Then there exists a point \( p_0 \in E \). Since \( B \) is a semi-basis, there exists \( B \in B_0 \) such that \( U(P_B, Q_B) \subset \bigcup\{p_0}\). Fix an element \( A \in U(P_B, Q_B) \). Then \( P_B \subset A \) and \( A \cap Q_B = \emptyset \).

If \( P_B \) is the empty set, then any point \( q \in X - (Q_B \cup \{p_0\}) \) satisfies that \( \{q\} \in U(P_B, Q_B) \) and \( \{q\} \notin U(q_0) \), which is a contradiction. Hence \( P_B \) is nonempty.

Clearly \( I(P_B) = \{xy : x, y \in P_B\} \). Since \( P_B \subset I(P_B) \subset A \), we obtain that \( I(P_B) \subset U(P_B, Q_B) \subset \bigcup\{p_0\}, \emptyset \). This implies that \( p_0 \in xy \) for some \( x, y \in P_B \). Since \( p_0 \in E \), \( p_0 = x \) or \( p_0 = y \). This is a contradiction since \( x, y \in G \) and \( p_0 \notin G \). The proof of the necessity is complete.

(\( \Leftarrow \)) Suppose that \( E \) is countable. Fix a point \( p_0 \in X \) and fix a Whitney map \( \mu : (C(X), \tau_H) \to [0, 1] \).

Let \( D = E \cup \{x \in X : \mu(xp_0) \notin \mathbb{Q}\} \).

We will show that \( D \) is countable. For each \( e \in E \), let \( D_e = \{x \in ep_0 : \mu(xp_0) \in \mathbb{Q}\} \). Define \( f : ep_0 \to [0, 1] \) be given by \( f(x) = \mu(xp_0) \). Then \( f \) is one-to-one and
$D_e = f^{-1}(Q)$. Thus $D_e$ is countable. Since $X = \bigcup \{e p_0: e \in E\}$, we conclude that $D = E \cup (\bigcup \{D_e: e \in E\})$. Therefore, $D$ is countable.

Now, we will prove that every subarc $pq$ of $X$ contains points of $D$. Let $a$ be the unique point in $pq$ such that $a p_0 \cap pq = \{a\}$. We may assume that $a \neq p$. Then $a p_0$ is a proper subarc of $p p_0$. Since the map $g : p a \to [0, 1]$ given by $g(x) = \mu(x p_0)$ is continuous (Lemma 8) and one-to-one, we have that $g(p a)$ is a nondegenerate subinterval of $[0, 1]$. Then there exists a point $x \in p a \subset pq$ such that $g(x) = \mu(x p_0) \in Q$. Hence, $x \in D \cap pq$.

Let

$$B = \{U(P, Q): P \text{ and } Q \text{ are finite subsets of } D \text{ and } U(P, Q) \neq \emptyset\}.$$ 

We will show that $B$ is a semi-basis of $(C(X), \tau_P)$. Let $P_0$, $Q_0$ be finite subsets of $X$ such that $U(P_0, Q_0)$ is nonempty. If $P_0$ is the empty set, choose an arc $ab$ contained in $X - Q_0$. Thus $U([a, b], Q_0)$ is nonempty and it is contained in $U(P_0, Q_0)$. Hence, it is enough to show that $U([a, b], Q_0)$ contains an element of $B$. Therefore, we may assume that $P_0$ is nondegenerate.

Notice that $I(P_0)$ is a nondegenerate subtree of $X$. Then we may choose a point $x_0 \in I(P_0) \cap D$. Let $P'$ be the set of end points of the tree $I(P_0)$. Clearly, $I(P') = I(P_0)$ and $I(P) \in U(P_0, Q_0)$. For each $p \in P'$, define a point $x_p \in D$ in the following way:

1. if $p \in E$, then $x_p = p$, and
2. if $p \notin E$, then there is a nondegenerate subarc $yp$ in $X$ such that $yp \cap I(P') = \{p\}$ and $yp \cap Q_0 = \emptyset$. Choose a point $x_p \in D \cap yp$.

Let $G = I(P') \cup (\bigcup \{px_p: p \in P'\})$. Then $G \in U(P_0, Q_0)$.

For each $q \in Q_0$, let $z$ be the unique point in $G$ such that $qz \cap G = \{z\}$. Since $q \notin G$, $q \neq z$. Then we may choose a point $y_q \in D \cap qz - \{q, z\}$.

Let $P = \{x_0\} \cup \{x_p: p \in P'\}$ and $Q = \{y_q: q \in Q_0\}$. Since $G \in U(P, Q)$, $U(P, Q) \neq \emptyset$.

Finally, we need to show that $U(P, Q) \subset U(P_0, Q_0)$. Let $A \in U(P, Q)$. For each $p \in P'$, $p \in x_0 x_p \subset A$. Then $P' \subset A$. Hence, $P_0 \subset I(P') \subset A$. If there exists a point $q \in Q_0 \cap A$, then $y_q \in x_0 q \subset A$, which is a contradiction. Hence $A \cap Q_0 = \emptyset$. Therefore, $A \in U(P_0, Q_0)$. This completes the proof that $B$ is a semi-basis of $(C(X), \tau_P)$ and the proof of the theorem.

3. Separability of $(C(X), \tau_P)$ when $X$ is a dendroid

A continuum $X$ is said to be Suslinean if there is no an uncountable family of pairwise disjoint, nondegenerate subcontinua of $X$. It is easy to show that if $X$ is a dendroid, then $X$ is Suslinean if and only if there is no an uncountable family of pairwise disjoint, nondegenerate subarcs of $X$.

**Theorem 10.** If $X$ is a dendroid, then $(C(X), \tau_P)$ is separable if and only if $X$ is Suslinean.

**Proof.** ($\Rightarrow$) Suppose that there exists an uncountable family $\mathcal{A}$ such that the elements of $\mathcal{A}$ are pairwise disjoint subarcs of $X$. We will prove that $(C(X), \tau_P)$ is not separable.

For each $\alpha \in \mathcal{A}$, choose three points $a_\alpha, q_\alpha$ and $b_\alpha$ in $\alpha$ such that $q_\alpha \in a_\alpha b_\alpha - \{a_\alpha, b_\alpha\}$.
Let $\mathcal{U}_a = \mathcal{U}([a_a], \{q_a\})$ and $\mathcal{V}_a = \mathcal{U}([b_a], \{q_a\})$. Notice that $\mathcal{U}_a$ and $\mathcal{V}_a$ are nonempty disjoint basic open of $(C(X), \tau_P)$.

We need the following property of the sets $\mathcal{U}_a, \mathcal{V}_a$.

**Claim 1.** If $\alpha, \beta \in A$, $\alpha \neq \beta$ and $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, then $\mathcal{V}_\alpha \cap \mathcal{V}_\beta = \emptyset$.

**Proof.** In order to prove Claim 1, suppose to the contrary that there exists an element $B \in \mathcal{V}_\alpha \cap \mathcal{V}_\beta$. Let $A \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Then $a_\alpha, a_\beta \in A$, $b_\alpha, b_\beta \in B$, $q_\alpha, q_\beta \notin A$ and $q_\alpha, q_\beta \notin B$. Thus the set $A \cup a_\alpha b_\beta \cup B$ is a subcontinuum of $X$ which contains $a_\alpha$ and $b_\beta$. Hence $q_\alpha \in A \cup a_\beta b_\beta \cup B$. This is a contradiction since $\alpha \cap \beta = \emptyset$. Therefore, Claim 1 is proved.

Let $D$ be a dense set in $(C(X), \tau_P)$. We will prove that $D$ is uncountable.

For each $A \in C(X)$, let $C_A = \{x : A \in \mathcal{U}_x\}$. We analyze two cases:

**Case 1.** There exists $A \in C(X)$ such that $C_A$ is uncountable. Given $\alpha, \beta \in C_A$, $A \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. By (1), $\mathcal{V}_\alpha \cap \mathcal{V}_\beta = \emptyset$. Then $\{\mathcal{V}_\alpha : \alpha \in C_A\}$ is an uncountable family of nonempty pairwise disjoint open sets in $(C(X), \tau_P)$. This implies that $D$ is uncountable.

**Case 2.** For each $A \in C(X)$, $C_A$ is countable. If $D$ is countable, then $\bigcup\{C_D : D \in D\}$ is a countable subset of $A$. Since $A$ is uncountable, there exists $\gamma \in A - \bigcup\{C_D : D \in D\}$. Then $\gamma \notin C_D$ for each $D \in D$. This implies that $D \notin \mathcal{U}_\gamma$ for each $D \in D$. This contradicts the density of $D$. Hence, $D$ is uncountable.

This completes the proof that $(C(X), \tau_P)$ is not separable.

$(\Leftarrow)$ Now, we suppose that $X$ does not contain a family of pairwise disjoint arcs. We will show that $(C(X), \tau_P)$ is separable.

Fix a Whitney map $\mu : (C(X), \tau_H) \rightarrow [0, 1]$.

Given $w \in X$, define $f_w : X \rightarrow [0, 1]$ as in Lemma 8. That is:

$$f_w(x) = \mu(wx).$$

For each $r > 0$ let

$$D(w, r) = \{x \in X : x \text{ is not an end point of } X \text{ and } f_w(x) = r\}.$$ 

We claim that $D(w, r)$ is countable for every $w \in X$ and every $r > 0$.

Suppose to the contrary that $D(w, r)$ is uncountable for some $w \in X$ and some $r > 0$.

For each $x \in D(w, r)$, since $x$ is not an end point, it follows that there exists $b_x \in X$ such that $x \in wb_x$ and $x \neq b_x$.

Let $A = \{xb_x : x \in D(w, r)\}$. We will show that the elements of $A$ are pairwise disjoint. Let $x \neq y$ be elements of $A$. Suppose that $xb_x \cap yb_y \neq \emptyset$. Fix a point $z \in xb_x \cap yb_y$. Then $x, y \in wz$. Thus $wx \subseteq wy$ or $wy \subseteq wx$ and $\mu(wx) = r = \mu(wy)$. This implies that $wx = wy$. Thus $x = y$. This contradiction proves that $A$ is a family of pairwise disjoint subarcs of $X$. Since this contradicts our assumption, we conclude that each $D(w, r)$ is countable.

Now we are ready to construct a dense countable subset of $(C(X), \tau_P)$.

Fix a point $w_0 \in X$. Define $D = \bigcup\{D(w_0, r) : r \in (0, 1) \cap \mathbb{Q}\}$. Notice that $D$ is countable. For each $x \in X$ and each $r \in (0, 1) \cap \mathbb{Q}$, define $P(x, r) = \{y \in X : f_x(y) \leq r\}$. Finally, define
\[ D = \left\{ I([d_1, \ldots, d_n]) \cup \left( \bigcup_{i=1}^{n} P(d_i, r) \right) : n \in \mathbb{N}, d_1, \ldots, d_n \in D \text{ and } r \in \mathbb{Q} \cap (0, 1) \right\}. \]

Clearly, \( D \) is countable. By Lemma 8(b), each element in \( D \) is in \( C(X) \).

Before proving that \( D \) is dense in \( (C(X), \tau_P) \), we will prove that every subarc \( pq \) of \( X \) contains points of \( D \). Let \( a \) be the unique point in \( pq \) such that \( aw_0 \cap pq = \{a\} \). We may assume that \( a \neq p \). Then \( aw_0 \) is a proper subarc of \( pw_0 \). Since the map \( f_{w_0} \) is continuous (Lemma 8) and one-to-one, we have that \( f_{w_0}(pa) \) is a nondegenerate subinterval of \([0, 1]\). Then there exists a point \( x \in pa \subset pq \) such that \( x \neq p, x \neq q \) and \( f_{w_0}(x) = \mathbb{Q} \cap (0, 1) \). Hence, \( x \in D \cap pq \).

Now we are ready to check the density of \( D \).

Let \( P \) and \( Q \) be finite subsets of \( X \) such that \( U(P, Q) \) is nonempty. If \( P = \emptyset \), then choose a point \( p \in X - Q \). Thus \( U([p], Q) \) is a nonempty subset of \( U(P, Q) \). Hence, we may consider \( U(\{p\}, Q) \) instead of \( U(P, Q) \). Therefore, we may assume that \( P \neq \emptyset \).

Let \( r \in \mathbb{Q} \cap (0, 1) \) be such that \( r < \min(\mu(pq): p \in P \text{ and } q \in Q \cup \{1\}) \).

For each \( p \in P \), fix a point \( ap \in X \) such that \( \mu(pa) < r, ap \in D \), and \( ap \neq p \). For each \( q \in Q \), since \( f_q \) is continuous (Lemma 8) and \( f_q(p) > r \), there exists \( dp \in pa - \{p\} \) such that \( f_q(dp) > r \) and \( dp \in D \). Since \( Q \) is finite, we may choose one point \( dp \) which works for each \( q \in Q \). That is, \( dp \) has the property that \( f_q(dp) > r \) for every \( q \in Q \). In the case that \( Q = \emptyset \), we simply put \( dp = a_p \). Hence \( \mu(qd_p) > r \) for every \( q \in Q \) and every \( p \in P \).

Let

\[ A = I([d_p: p \in P]) \cup \left( \bigcup \{P(dp, r): p \in P\} \right). \]

Then \( A \in D \). If \( p \in P \), since \( \mu(pd_p) \leq \mu(pa) < r \), we have that \( p \in P(dp, r) \). Thus \( P \subset A \). Given \( p \in P \), by the choice of \( dp \), \( q \notin P(dp, r) \) and \( q \notin pd_p \) for every \( q \in Q \).

Since \( I([dp: p \in P]) \subset I(P) \cup \left( \bigcup \{pd_p: p \in P\} \right) \), we conclude that \( Q \cap A = \emptyset \).

Hence \( A \in U(P, Q) \). Thus \( D \) is dense in \( (C(X), \tau_P) \).

Therefore, \( (C(X), \tau_P) \) is separable. \( \square \)

4. A characterization of dendrites

**Definition 11.**

(1) A subcontinuum \( Y \) of a dendroid \( X \) is said to be a semi-comb if there exists:

(a) an arc \( A \subset Y \),

(b) two points \( p \neq q \) in \( A \),

(c) a sequence of points \( \{p_n\}_{n=1}^{\infty} \) in \( Y - A \), and

(d) a sequence of points \( \{q_n\}_{n=1}^{\infty} \) in \( A \) such that:

(i) \( Y = A \cup \text{cl}_X(\bigcup \{p_nq_n: n \in \mathbb{N}\}) \),

(ii) \( p_n \to p, q_n \to q \).
By [3, Theorem 5.6], we may choose a point $q_n \in X \setminus \overline{A}$ for every $n \in \mathbb{N}$.

(2) A subcontinuum $Y$ of a dendroid $X$ is said to be a semi-broom if there exists:

(a) an arc $A \subset Y$,
(b) two points $p \neq q$ in $A$, and
(c) a sequence of points $\{p_n\}_{n=1}^{\infty}$ in $Y - A$ such that:
   (i) $Y = A \cup \overline{clX(\bigcup\{p_n q: n \in \mathbb{N}\})}$,
   (ii) $p_n \to p$,
   (iii) $p_n q \cap p_n q = \{q\}$ if $m \neq n$, and
   (iv) $p_n q \cap A = \{q\}$ for every $n \in \mathbb{N}$.

**Lemma 12.** Let $X$ be a nonlocally connected dendroid. Then there exist:

(a) two open subsets $U$ and $V$ of $X$,
(b) two points $p \neq q$ in $\overline{clX(V)}$,
(c) sequences of points $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ in $\overline{clX(V)}$, and
(d) a sequence of pairwise different components $C_0, C_1, C_2, \ldots$ of $U$ such that:
   (i) $\overline{clX(V)} \subset U$,
   (ii) $p q \subset \overline{clX(V)} \cap C_0$,
   (iii) $p_n \to p$, $q_n \to q$, and
   (iv) $p_n q_n \subset \overline{clX(V)} \cap C_n$ for every $n \in \mathbb{N}$.

**Proof.** By [3, Exercise 5.22], there exists a point $p \in X$ such that $X$ is not connected im kleinen at $p$. Then there exists an open subset $U$ of $X$ such that $p \in U$ and $p$ is not in the interior of the component $C_0$ of $U$ which contains $p$. Let $V$ be an open subset of $X$ such that $p \in V \subset \overline{clX(V)} \subset U$.

Let $\varepsilon_1 > 0$ be such that $B(\varepsilon_1, p) \subset V$ and $\varepsilon_1 < 1$. Then we may choose a point $p_1 \in B(\varepsilon_1, p) - C_0$. Let $C_1$ be the component of $U$ which contains $p_1$.

Since $C_1 \neq C_0$ and $C_1$ is closed in $U$, there exists $\varepsilon_2 > 0$ such that $\varepsilon_2 < \min\{\frac{1}{2}, \varepsilon_1\}$ and $B(\varepsilon_2, p) \cap C_1 = \emptyset$. Choose a point $p_2 \in B(\varepsilon_2, p) - C_0$. Let $C_2$ be the component of $U$ which contains $p_2$. Then $C_2 \neq C_1$.

Following with this procedure it is possible to construct a sequence $\{p_n\}_{n=1}^{\infty}$ in $\overline{clX(V)}$ and a sequence pairwise different components $C_0, C_1, C_2, \ldots$ of $U$ such that $p_n \to p$ and $p_n \in C_n$ for every $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let $D_n$ be the component of $\overline{clX(V)}$ which contains $p_n$. Then $D_n \subset C_n$.

By [3, Theorem 5.6], we may choose a point $q_n \in D_n \cap \partial X(V)$. Since $(C(X), \tau_H)$ is compact [3, Theorem 4.17], there exists a subsequence $\{D_{n_k}\}_{k=1}^{\infty}$ of $\{D_n\}_{n=1}^{\infty}$ such that $\{D_{n_k}\}_{k=1}^{\infty}$ converges in $(C(X), \tau_H)$ to an element $D_0 \in C(X)$. We may also assume that $q_{n_k} \to q$ for some $q \in X$.

Notice that $D_0 \subset \overline{clX(V)}$ and $p \in D_0$. Thus $D_0 \subset C_0$. Since $p, q \in D_0$, $pq \subset D_0 \subset \overline{clX(V)} \cap C_0$.

For each $n \in \mathbb{N}$,

$$p_n q_n \subset D_n \subset \overline{clX(V)} \cap C_n.$$
Therefore, the sequences \( \{p_n\}_{k=1}^{\infty}, \{q_n\}_{k=1}^{\infty} \) and \( C_0, C_1, C_2, \ldots \) satisfy the required properties. \( \square \)

**Theorem 13.** Let \( X \) be a dendroid. Then \( X \) is a dendrite if and only if \( X \) contains neither a semi-comb nor a semi-broom.

**Proof.** (\( \Rightarrow \)) Suppose that \( X \) is locally connected and it contains a semi-comb \( Y \) (the case in which \( X \) contains a semi-broom is similar).

Let \( A \subseteq Y, p, q \in A, \{p_n\}_{n=1}^{\infty} \subseteq Y - A, \) and \( \{q_n\}_{n=1}^{\infty} \subseteq A \) as in the definition of a semi-comb. Let \( U \) be an open connected subset of \( X \) such that \( p \in U \) and \( q \notin cl_X(U) \). Let \( N \in \mathbb{N} \) be such that \( p_N \in U \) and \( q_N \notin cl_X(U) \). Since \( A \cup cl_X(U) \) is a subcontinuum of \( X \) that contains \( q_N \) and \( p_N \), we have that \( p_N q_N \subseteq A \cup cl_X(U) \). Thus

\[
p_N q_N = p_N q_N \cap (A \cup cl_X(U)) = (p_N q_N \cap A) \cup (p_N q_N \cap cl_X(U))
\]

This contradicts the connectedness of \( p_N q_N \) and completes the proof that \( X \) does not contain a semi-comb.

(\( \Leftarrow \)) Suppose that \( X \) is not locally connected and suppose that \( X \) contains neither a semi-comb nor a semi-broom.

Let \( U, V \subseteq X, p, q \in cl_X(V), \{p_n\}_{n=1}^{\infty}, \{q_n\}_{n=1}^{\infty} \subseteq cl_X(V), \) and \( C_0, C_1, C_2, \ldots \) components of \( U \) as in Lemma 12.

For each arc \( L \) in \( X \) and each point \( y \in X \), let \( P(y, L) \) be the unique point \( x \) in \( L \) such that \( y_x \cap L = \{x\} \).

The following property is easy to prove.

**Claim 1.** If \( L \) and \( L' \) are arcs in \( X \) such that \( L \cap L' \neq \emptyset \) and \( y \in L' \), then \( P(y, L) \in L' \).

**Claim 2.** If \( L \) is an arc in \( X \), then the sets \( \{n \in \mathbb{N} : p_n \in L\} \) and \( \{n \in \mathbb{N} : q_n \in L\} \) are finite.

**Proof.** In order to prove Claim 2, suppose to the contrary that the set \( \{n \in \mathbb{N} : p_n \in L\} \) is infinite (the analysis for the sequence \( \{q_n\}_{n=1}^{\infty} \) is similar). Since \( p_n \to p, p \in L \), let \( W \) be an open connected set in \( L \) such that \( p \in W \subseteq U \cap L \). Since \( \{n \in \mathbb{N} : p_n \in L\} \) is infinite, there exists \( m \in \mathbb{N} \) such that \( p_m \in W \). Since \( W \) is a connected subset of \( U \), \( W \subseteq C_0 \cap C_m \). This is a contradiction since \( C_0 \neq C_m \). The proof of Claim 2 is complete. \( \square \)

**Claim 3.** If \( L \) is an arc in \( X \), then the set \( J = \{n \in \mathbb{N} : P(p_n, L) \neq P(q_n, L)\} \) is finite.

**Proof.** In order to prove Claim 3, suppose to the contrary that \( J \) is infinite.

By Claim 2, there exists \( N \in \mathbb{N} \) such that, for every \( n \geq N \), \( p_n \notin L \) and \( q_n \notin L \).

Given \( n \in J \), with \( n \geq N \), let \( r_n = P(p_n, L) \) and \( z_n = P(q_n, L) \). Then \( r_n \neq z_n \). If \( L \cap p_n q_n = \emptyset \), then there is only one arc in \( X \) connecting \( L \) and \( p_n q_n \). This implies that \( r_n = z_n \), which is a contradiction. Hence \( L \cap p_n q_n \neq \emptyset \). By Claim 1, \( r_n \in p_n q_n \subseteq cl_X(V) \cap C_n \).

Thus there exists a sequence of positive integers \( N \leq n_1 < n_2 < \cdots \) such that \( r_{n_k} \to r \) for some \( r \in X \), and \( r_{n_k} \in J \) for every \( k \in \mathbb{N} \). Notice that \( r \in cl_X(V) \). Let \( W \) be an open
connected set in $L$ such that $r \in W \subset U \cap L$. Thus there exists $K \in \mathbb{N}$ such that $r_{n_k} \in W$ for every $k \geq K$. Let $C'$ be the component of $U$ which contains $r$. Since $W$ is a connected subset of $U$, $r_{n_k} \in C'$ for every $k \geq K$. Thus $C' = C_{n_k} = C_{n_{k+1}} = \cdots$. This contradicts the choice of $C_1, C_2, \ldots$ and completes the proof of Claim 3.  

By Claims 2 and 3, given an arc $L$ in $X$, there exists $M_L \in \mathbb{N}$ such that, for every $n \geq M_L$, $p_n \notin L$, $q_n \notin L$ and $P(p_n, L) = P(q_n, L)$. 

**Claim 4.** There is not an arc $L$ in $X$ such that $p \in L$ and the set $R = \{P(p_n, L) : n \in \mathbb{N}\}$ is an infinite subset of $L$ and it has an accumulation point $r \in L - \{p\}$. 

**Proof.** In order to prove Claim 4, assume to the contrary that there is an arc $L$ with the described properties. For each $n \in \mathbb{N}$, let $r_n = P(p_n, L)$. Then there exists a sequence of positive integers $M_L < n_1 < n_2 < \cdots$ such that $r_n \to r$ and $r_{n_k} \neq r_n$ if $k \neq j$.

Define $Y = L \cup c_L(\bigcup \{p_n: n \in \mathbb{N}\})$. It is easy to check that $Y$ is a semi-comb in $X$. This contradicts our assumption and proves Claim 4.  

The following claim is similar to Claim 4. 

**Claim 5.** There is no an arc $L$ in $X$ such that $q \in L$ and the set $R = \{P(q_n, L) : n \in \mathbb{N}\}$ is an infinite subset of $L$ and it has an accumulation point $r \in L - \{q\}$. 

**Claim 6.** For every arc $L$ in $X$, the sets $\{P(p_n, L) : n \in \mathbb{N}\}$ and $\{P(q_n, L) : n \in \mathbb{N}\}$ are finite. 

**Proof.** We prove Claim 6. Suppose that there is an arc $L$ in $X$ such that $\{P(p_n, L) : n \in \mathbb{N}\}$ is infinite (by Claim 3, this is equivalent to assuming that $\{P(q_n, L) : n \in \mathbb{N}\}$ is infinite). Let $r$ be an accumulation point of $\{P(p_n, L) : n \in \mathbb{N}\}$. Suppose that $L = ab$. If $p \notin L$, let $L_1 = aP(p, L) \cup pP(p, L)$ and $L_2 = bP(p, L) \cup pP(p, L)$. Then $L_1$ and $L_2$ are arcs in $X$, $p \in L_1 \cap L_2$ and $r$ is an accumulation point of $\{P(p_n, L_1) : n \in \mathbb{N}\}$ or $\{P(p_n, L_2) : n \in \mathbb{N}\}$. Since $r \neq p$, we obtain a contradiction with Claim 4. Thus, $p \in L$. Similarly, $q \in L$. Since $r \neq p$ or $r \neq q$, we obtain a contradiction with Claim 4 or with Claim 5. This ends the proof of Claim 6.  

**Claim 7.** It is possible to construct:

(a) a sequence of infinite subsets $J_1, J_2, \ldots$ of $\mathbb{N}$, and 
(b) a sequence of points $z_0, z_1, z_2, \ldots$ in $X$ such that:

(i) $z_0 \in \{p, q\}$, $z_1 \notin \{p, q\}$, 
(ii) if $n_k = \min J_k$, then $M_{z_0, z_k} \leq n_k < n_{k+1}$ for every $k \in \mathbb{N}$, 
(iii) if $z_0 = q$, then for each $k \geq 2$ and each $n \in J_k$, $z_k = P(q_n, z_0, q_{n_{k-1}})$, 
(iv) if $z_0 = p$, then for each $k \geq 2$ and each $n \in J_k$, $z_k = P(p_n, z_0, p_{n_{k-1}})$, 
(v) $z_0 z_{k-1} \subset z_0 z_k$ for every $k \in \mathbb{N}$, and 
(vi) $J_1 \supset J_2 \supset J_3 \supset \cdots$. 

**Proof.** We will inductively construct $J_1$, $J_2$, ..., and $z_0, z_1, \ldots$.  

By Claim 6, $\{P(q_n, pq): n \in \mathbb{N}\}$ is finite. Then there exists $z_1 \in pq$ such that the set $J_1 = \{n \in \mathbb{N}: n \geq M_{pq} \text{ and } P(q_n, pq) = z_1\}$ is infinite. Choose $z_0 \in \{p, q\} - \{z_1\}$ and let $n_1 = \min J_1$.

Now, suppose that $J_1, \ldots, J_k$ and $z_0, z_1, \ldots, z_k$ have been constructed and they satisfy properties (i) through (vi) and $z_0 = q$ (the construction for the case $z_0 = p$ is similar).

Let $n_k = \min J_k$. Since $n_k \geq M_{z_0,z_k}$, we have that $q_{n_k} \notin z_0z_k$ and $z_k = P(q_{n_k}, z_0q_{n_k-1})$. This implies that $z_k \in z_0q_{n_k}$. Let $L' = z_0q_{n_k}$.

By Claim 6, the set $\{P(q_n, L'): n \in J_k\}$ is finite. Since $J_k$ is infinite, there exists $z_{k+1} \in L'$ such that the set $J = \{n \in J_k: n > \max\{n_k, M_{L'}\}\}$ and $z_{k+1} = P(q_n, L')$ is infinite. Let $J_{k+1} = \{n \in J: n > M_{z_0z_{k+1}}\}$.

Fix $n \in J_{k+1}$. Then $z_k = P(q_n, z_0q_{n_k-1})$. This implies that $z_k \in z_0q_n \cap L'$. By Claim 1, $z_{k+1} \in z_kq_n$. Hence $z_0z_k \subset z_kz_{k+1}$.

This completes the inductive construction and the proof of Claim 7. $\square$

Thus the sequence $\{z_0z_k: k \in \mathbb{N}\}$ is an increasing sequence of arcs in the dendroid $X$. This implies that there exists a point $z \in X$ such that $z_0z_k \subset z_0z$ for every $k \in \mathbb{N}$ and $z_k \to z$. Let $L = z_0z$. Since $n_k \to \infty$ as $k \to \infty$, there exists $K \in \mathbb{N}$ such that $n_k > M_k$ for every $k \geq K$.

**Claim 8.** For each $k \geq K$, $p_{n_k} \notin L$, $q_{n_k} \notin L$ and if $y_k = P(q_{n_k}, L) = P(p_{n_k}, L)$, then $y_k \in z_{k+1}z$.

**Proof.** By definition of $M_L$, in Claim 8, we only need to prove that $y_k \in z_{k+1}z$ for every $k \geq K$. Let $k \geq K$. Suppose, for example, that $z_0 = q$. Since $z_{k+1} = P(q_{n_k+1}, z_0q_{n_k})$, $z_{k+1} \in z_0q_{n_k}$. By Claim 1, $y_k \in z_{k+1}q_{n_k}$. Hence $y_k \notin z_0z_{k+1} - \{z_{k+1}\}$. But $y_k \in L$. Therefore, $y_k \in z_{k+1}z$. $\square$

**Claim 9.** Let $y_k$ be defined as in Claim 8. Then the set $\{y_k \in L: k \geq K\}$ is finite.

Claim 9 is an immediate consequence of Claim 6.

**Claim 10.** If $z_0 = p$ and if $k, m \geq K$ are such that $k \neq m$ and $y_k = y_m$, then $p_{n_k}y_k \cap p_{n_m}y_m = \{y_k\}$.

**Proof.** In order to prove Claim 10, suppose that $k < m$. Since $n_m \in J_m \subset J_{k+1}$, $z_{k+1} = P(p_{n_m}, z_0p_{n_m})$. This implies that $z_{k+1} \in p_{n_m}p_{n_k}$. Since $z_{k+1} \in L$, Claim 1 implies that $y_m \in p_{n_m}z_{k+1}$ and $y_k \in z_{k+1}p_{n_k}$. Since $y_k = y_m$, we obtain that $y_k = z_{k+1} = y_m$. Hence $p_{n_k}y_k \cap p_{n_m}y_m = p_{n_k}z_{k+1} \cap p_{n_m}z_{k+1} = \{z_{k+1}\} = \{y_k\}$. $\square$

The proof of Claim 11 is similar to the proof of Claim 10.

**Claim 11.** If $z_0 = q$ and if $k, m \geq K$ are such that $k \neq m$ and $y_k = y_m$, then $q_{n_k}y_k \cap q_{n_m}y_m = \{y_k\}$. 

By Claim 6, there exists a point \( y \in L \) and there is a sequence of positive integers \( K \leq k_1 < k_2 < \cdots \) such that \( y_{k_j} = y \) for every \( j \in \mathbb{N} \).

We will obtain a final contradiction by showing that \( X \) contains a semi-broom. We assume that \( z_0 = q \), the case \( z_0 = p \) is similar.

We consider the arc \( L \), the points \( q = z_0 \) and \( y \in z_1 z \subset L - \{z_0\} \) (Claim 8). We also consider the sequence \( \{q_{n_j}\}_{j=1}^\infty \) in \( X - L \) and the continuum \( Y = L \cup Cl_X(\bigcup \{q_{n_j}, y: j \in \mathbb{N}\}) \). Clearly \( Y \) is a semi-broom in \( X \).

This contradicts the assumption that \( X \) does not contain a semi-broom and completes the proof of the sufficiency. \( \square \)

5. Comparison of the topologies \( \tau_H \) and \( \tau_P \)

**Lemma 14.** If \( X \) is a continuum \( \tau_P \) is not contained in \( \tau_H \).

**Proof.** Suppose, to the contrary, that \( \tau_P \subset \tau_H \). Let \( p, q \) be two different points in \( X \). Then \( \{p\} \in \mathcal{U}((p), \{q\}) \in \tau_H \). Then there exists \( \varepsilon > 0 \) such that \( \{A \in C(X): H(A, \{p\}) < \varepsilon\} \subset \mathcal{U}((p), \{q\}) \). Fix a point \( x \in B(\varepsilon, p) - \{p\} \). Then \( \{x\} \) must be in \( \mathcal{U}((p), \{q\}) \). This implies that \( \{p\} \subset \{x\} \) and contradicts the choice of \( x \). The proof of the lemma is complete. \( \square \)

**Lemma 15.** If a dendroid \( X \) contains a semi-comb or a semi-broom, then \( \tau_H \) is not contained in \( \tau_P \).

**Proof.** Suppose that \( X \) contains a semi-comb \( Y \) and \( \tau_H \subset \tau_P \). We will obtain a contradiction. A similar contradiction can be obtained if we assume that \( X \) contains a semi-broom and \( \tau_H \subset \tau_P \).

Let \( A \subset Y \), \( p \neq q \in A \), \( \{p_n\}_{n=1}^\infty \subset Y - A \) and \( \{q_n\}_{n=1}^\infty \subset A \) be as in the definition of semi-combs.

Let \( \varepsilon = d(p, q)/2 \). Let \( B = \{B \in C(X): H(B, \{q\}) < \varepsilon\} \). Then \( \{p\} \in B \in \tau_P \). Thus there exist two finite sets \( P \) and \( Q \) of \( X \) such that \( \{q\} \subset \mathcal{U}(P, Q) \subset B \). Since \( P \subset \{q\} \), then \( P = \emptyset \) or \( P = \{q\} \) and \( q \neq Q \). Since \( p_1q_1, p_2q_2, \ldots \) are pairwise disjoint and \( Q \) is a finite set, there exists \( N \in \mathbb{N} \) such that \( p_nq_n \cap Q = \emptyset \) for each \( n \geq N \). Since \( p_n \rightarrow p \), \( q_n \rightarrow q \) and \( A \) is an arc, there exists \( m \geq N \) such that \( d(p_m, p) < \varepsilon \) and \( q_m \subset B(\varepsilon, q) - Q \).

Let \( B = q_m \cup q_mp_m \). Then \( B \in C(X) \), \( P \subset B \) and \( B \cap Q = \emptyset \). This implies that \( H(B, \{q\}) < \varepsilon \). Hence \( p_m \in B(\varepsilon, q) \cap B(\varepsilon, p) \). This contradicts the choice of \( \varepsilon \) and completes the proof of the lemma. \( \square \)

**Definition 16.** A continuum \( X \) is said to be regular if there exists a basis of open sets \( B \) of \( X \) such that every element of \( B \) has finite boundary.

**Theorem 17.** Let \( X \) be a locally connected continuum. Then \( X \) is regular if and only if \( \tau_H \subset \tau_P \).
Proof. \((\Rightarrow)\) Suppose that \(X\) is regular. Let \(\varepsilon > 0\) and \(A \in C(X)\). Let \(\mathcal{C} = \{B \in C(X): H(A, B) < \varepsilon\}\). We need to find finite subsets \(P\) and \(Q\) of \(X\) such that \(A \in \mathcal{U}(P, Q) \subset \mathcal{C}\).

For each point \(a \in A\), let \(U_a\) be an open subset of \(X\) such that \(a \in U_a \subset B(\mathcal{C}, a)\) and \(\text{Bd}(U_a)\) is finite. Since \(A\) is compact, there exists \(n \in \mathbb{N}\) and there exist points \(a_1, \ldots, a_n \in A\) such that \(A \subset U_{a_1} \cup \cdots \cup U_{a_n}\). Let \(U = U_{a_1} \cup \cdots \cup U_{a_n}\). Since \(\text{Bd}(U) \subset \text{Bd}(U_{a_1}) \cup \cdots \cup \text{Bd}(U_{a_n})\) the set \(Q = \text{Bd}(U)\) is finite. Let \(P = \{a_1, \ldots, a_n\}\). Clearly, \(A \in \mathcal{U}(P, Q)\).

Take \(B \in \mathcal{U}(P, Q)\). Then \(B\) is connected, \(P \subset B\) and \(B \cap \text{Bd}(U) = \emptyset\). Thus \(B \subset U\) and \(B \cap U_{a_i} \neq \emptyset\) for each \(i \in \{1, \ldots, n\}\). This implies that \(B \in \mathcal{C}\) and ends the proof of the necessity.

\((\Leftarrow)\) Let \(p \in X\) and let \(\varepsilon > 0\). Let \(D = \{A \in C(X): H(A, \{p\}) < \varepsilon\}\). Then \(D \in \tau_H \subset \tau_P\) and \(\{p\} \in D\). Thus there exist finite subsets \(P\) and \(Q\) of \(X\) such that \(\{p\} \in \mathcal{U}(P, Q) \subset D\). Hence \(P \subset \{p\}\) and \(p \notin Q\). Let \(U\) be component of \(X - Q\) such that \(p \in U\). Since \(X\) is locally connected, \(\text{Bd}(U) \subset Q\). Thus \(\text{Bd}(U)\) is finite. Since \(X\) is locally connected, \(U\) is arcwise connected [3, Theorem 8.26]. Thus, given \(x \in U\), there exists an arc \(\alpha\) in \(U\) such that \(\alpha\) joins \(p\) and \(x\). Notice that \(\alpha \in \mathcal{U}(P, Q)\). Hence, \(H(\alpha, \{p\}) < \varepsilon\). This implies that \(d(p, x) < \varepsilon\). We have proved that \(U \subset B(\varepsilon, p)\). Therefore, \(X\) is regular. \(\square\)

**Theorem 18.** Let \(X\) be a dendroid, then \(X\) is a dendrite if and only if \(\tau_H \subset \tau_P\).

Proof. \((\Leftarrow)\) Suppose that \(X\) is not a dendrite. By Theorem 13, \(X\) contains either a semi-comb or a semi-broom. By Lemma 15, \(\tau_H\) is not contained in \(\tau_P\). This proves the sufficiency.

\((\Rightarrow)\) Suppose that \(X\) is a dendrite. By [3, Theorem 10.20], \(X\) is regular. By Theorem 17, \(\tau_H \subset \tau_P\). \(\square\)

**References**