Nonlinear Functions of Weak Processes. I*

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It is well-known that powers of a (Schwartz) distribution generally fail to exist in nontrivial cases; the same is true of stochastic and operational distributions (i.e. linear mappings into random variables or operators in Hilbert space). However, in these latter cases a notion of "renormalization" is applicable to the powers, which in a number of interesting cases leads again to a distribution.

Section 1 of this paper gives a general theory of renormalization; an intrinsic characterization of renormalized products; their existence in finite-dimensional situations; and a specialization to a certain quantum process underlying all "free Bose–Einstein quantum fields." In the latter case the present renormalized product may be identified with the "Wick product" heuristically treated in connection with quantum field computations by means of a common recursion relation, and the "theorem of Wick" given a simple abstract formulation and treatment.

Section 2 treats the renormalized powers in the case of a process constituting a mathematical formulation of the heuristic notion of "free neutral scalar field in a two-dimensional space-time." An intrinsic characterization for the renormalized powers, as a self-adjoint operator-valued distribution in space, at a fixed time, is developed in terms of simple and natural transformation properties of the distributions, under certain unitary transformations. The existence of the distributions thus characterized is shown by an explicit limiting procedure which yields the commutativity as well as the self-adjointness of all the renormalized powers of the "field" at a fixed time, and provides their simultaneous spectral resolution. These results subsume Theorem 1 of [16].

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† The present work concerned with processes associated with a 2-dimensional space-time constitutes a detailed presentation of material given in a course on the mathematical theory of quantum fields at M.I.T. in 1966–67. It was originally planned to combine this material with more recent work presenting analogous, suitably modified, developments applicable to arbitrary-dimensional space-times. Due to the space which an adequately detailed and rigorous treatment has required, it is being published separately. The higher-dimensional cases will be treated in a forthcoming sequel.
The problem of the meaning and treatment of nonlinear functions of weak functions (i.e. functions defined only through the application of linear functionals) is one that arises in many contexts but has attained as yet comparatively little mathematical development. In the treatment of nonlinear differential equations, for example, the utilization of generalized functions such as the distributions developed by L. Schwartz is severely limited by the lack of any effective nonlinear calculus applicable to such weak functions. The same might appear to be true a fortiori in the case of the nonlinear partial differential equations of relativistic quantum field theory; no nontrivial instance of the complete resolution of any such equation has yet been given. Another case where similar difficulties appear is in the treatment of nonlinear stochastic partial differential equations. In the case of quantum fields, strong (as opposed to weak) functions are ruled out by desiderata of group invariance. Similarly, in the case of stochastic partial differential equations, the desideratum of temporal invariance for the probability measures in question is not clearly consistent with the existence of strong solutions.

This paper originated in the observation that in certain cases of quantized and stochastic weak functions, it was possible to form powers in a certain sense which could be correlated with general mathematical ideas, which were again functions of the same type. Thus the additional structure which is present in these cases permits a treatment which is unavailable in the far simpler case of conventional distributions. The powers in question differ from the limits of corresponding powers of approximating strong functions; in an intuitive sense they differ from these (generally non-existent) limits by certain infinite terms, and so are called "renormalized" powers. Despite this apparent gross mutilation of the formal idea of a power, they have the essential attributes of such functions, from the viewpoints of the theories of partial differential equations and of groups of transformations. In particular, they are local, in the sense that the application of the power commutes with the operation of restriction to an open set; and transform under mappings of the underlying space in the same way as conventional powers. An instance of the utilization of this locality property for the extension of known results concerning non-linear hyperbolic equations from the classical to certain quantized cases is given in [16].

A general theory of renormalized algebraic operations on weak processes is given in Part I of the present paper. These renormalized
operations depend not only on the algebra, but also, in an essential way, on a given linear functional on the algebra. The treatment is designed to be applicable to algebras generated by quantum field operators, the linear functional in question being taken as the "physical vacuum." In anticipation of contemplated applications in this direction, and in order to clarify the theory from a general mathematical viewpoint, the treatment is relatively broad.

In Part II, the particular cases of certain "normal" processes, known in the theoretical physical literature as "free fields," are treated within a context applicable to a general locally compact abelian group (playing the role of the physical "space") and relatively general covariance operator (which is formally equivalent to the object known in the heuristic literature as the "two-point function"). As a quite special case, the space average, relative to an arbitrary weight function in $L_1 \wedge L_2$, of the renormalized $n$th power of a "neutral scalar relativistic field in two space-time dimensions" ($n = 2, 3, \ldots$) is shown to exist as a self-adjoint operator; the simultaneous spectral resolution of all such operators, and various properties of them, follow. In this case, the renormalized products are formally identifiable with objects well-known in heuristic quantum field theory, which have been found effective in standardizing and facilitating perturbative computations; these were studied by G. C. Wick and are known as "Wick products." A mathematical treatment from a fresh viewpoint of the space-time average of formally similar objects was given in [10].

In more than two space-time dimensions the main existential result applies only to processes which are not fully relativistic. In a sequel to this paper it will be shown that there are then analogous results for the relativistic case, in which the space-averaged renormalized powers are suitably generalized operators. An ultimate aim of the present direction of work is the precise formulation and treatment of the fundamental nonlinear relativistic partial differential equations of relativistic field theory, as indicated more fully in [12]. Some of the present and forthcoming work appear there in summary form, as well as in [16], and in the Proceedings of the Conference on Constructive Quantum Field Theory, held at the Massachusetts Institute of Technology in April, 1966.

1. Renormalized Products of Generalized Processes

1.1. Generalized functions whose values are random variables or operators in Hilbert space, and especially the problem of the multi-
plication of such functions, are treated in this paper. The following
terminology will be employed.

**Definition 1.** Let \( M \) be a given set, and \( L \) a real linear vector
space consisting of functions on \( M \) whose values lie in a given real
linear vector space (which in the present work will generally be the
space of real or complex numbers); alternatively, \( L \) may consist of
equivalence classes of functions, two functions being equivalent if
they coincide except on a null set, where it is assumed that a given
ring of subsets of \( M \) has been designated as the ring of null sets.
A (weak) stochastic process in \( M \), with probe space \( L \), is an equivalence
class of real-linear mappings \( \Phi \) from \( L \) into the set of all numerical
random variables on a probability measure space \( \mathcal{P} \); where for two
such mappings \( \Phi \) and \( \Phi' \), \( \Phi \) is said to be equivalent to \( \Phi' \) (symbol-
ically \( \Phi \sim \Phi' \)) if

\[
\Pr[(\Phi(x_1), \ldots, \Phi(x_n)) \in B] = \Pr[(\Phi'(x_1), \ldots, \Phi'(x_n)) \in B]
\]

for every finite ordered set \( x_1, \ldots, x_n \) of vectors in \( L \) and Borel set \( B \) in
numerical \( n \)-space. Here "\( \Pr \)" denotes the probability measure of the
set in question. It is not difficult to establish (and well-known) that
the indicated equivalence condition is equivalent to the condition
that \( E(e^{i\Phi(x)} = E(e^{i\Phi'(x)}) \) for all \( x \in L \), in the case in which the \( \Phi(x) \)
are real-valued random variables, for all \( x \).

Any member \( \Phi \) of the indicated equivalence class will be called a
"concrete" stochastic process; the equivalence class itself may in
distinction be called an "abstract" stochastic process; simply the term
"stochastic process" will be employed when the context makes clear,
or it is immaterial, which type is involved.

An operational process in \( M \), with probe space \( L \), is an equivalence
class of linear mappings \( \Phi \) from \( L \) into the set of all closed densely
defined linear operators in a Hilbert space \( \mathcal{H} \), where strong operations
are employed with the partial algebra of closed densely defined
operators; the strong sum of two such operators is said to exist if the
usual sum has a closure which is a densely defined operator, and is
defined as this operator; and strong multiplication by (the number)
zero carries any closed operator into the (everywhere defined) zero
operator. Here two such maps \( \Phi \) and \( \Phi' \) from \( L \) into the closed
operators in the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}' \) respectively are equivalent
provided there exists a unitary transformation \( U \) of \( \mathcal{H} \) onto \( \mathcal{H}' \) such that

\[
U\Phi(x) U^{-1} = \Phi'(x), \quad x \in L.
\]
Again, any member of the equivalence class will be called a concrete operational process, etc.

A quantum process¹ in \( M \) with probe space \( L \) is an equivalence class of structures \((\Phi, H, v)\) where \((\Phi, H)\) is a concrete operational process, and \(v\) is a unit vector in \( H\); it is cyclic if \(v\) is cyclic for the ring of operators \( R \) generated by the bounded operators determined by the \( \Phi(x)\), by which are meant the partially isometric operators and the spectral projections of the positive self-adjoint operators in (either of the two) canonical polar decompositions of \( \Phi(x) \); the equivalence relation in question is given by the definition: \((\Phi, H, v)\) is equivalent to \((\Phi', H', v')\) provided there exists a unitary transformation \( U \) from \( H \) onto \( H' \) such that

\[
U\Phi(x)U^{-1} = \Phi'(x) \quad Uv = v'.
\]

The linear functional \( E \) on \( R \) given by the equation \( E(T) = \langle Tv, v \rangle \) is called the vacuum¹, or vacuum state, of the process.

Quantum processes constitute essentially a noncommutative generalization of stochastic processes, as indicated by

**Scholium 1.** Let \( \Phi \) denote a concrete stochastic process with probe space \( L \) and probability space \( P \); let \( P' \) denote the subspace whose sigma-ring of measurable sets² is the minimal one with respect to which all the \( \Phi(x) \) are measurable; let \( H = L_2(P') \), and let \( v \) denote the function which is identically 1 on \( P \). Then if \( \Phi'(x) \) denotes for any \( x \in L \) the operation of multiplication by \( \Phi(x) \), the structure \((\Phi', H, v)\) is a concrete cyclic quantum process. Furthermore, equivalent stochastic processes give rise to equivalent quantum processes, and every cyclic quantum process for which \( R \) is commutative arises in this fashion.

**Proof.** The only part of this scholium which is not essentially straight-forward is the association of a given quantum process having

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¹ Operational and quantum processes as here defined are sometimes called "quantum fields," however, since the latter term is often used in an essentially heuristic connotation, and since the terminology in the literature is quite variable, it has seemed desirable to use the more neutral and mathematical term of "process." The term "fundamental state" might for similar reasons be preferable to the term "vacuum" employed below, but it is felt that the relative brevity of the latter supersedes this consideration. In the older heuristic literature, the vector \( v \) described below is itself called the "vacuum," while \( E(T) \) is called the "vacuum expectation value of \( T.\)"

² Strictly speaking, of course, it is the ring of measurable sets modulo null sets which is relevant here, but such considerations are much too well known for it to be worth the circumlocutions required to be explicitly grammatical on this point, here or later in this paper.
a commutative ring $R$ (for brevity, a commutative process) with
a suitable stochastic process; in this connection, cf. [1] or [2].

Remark 1. For clarification of the foregoing and correlation with
alternative terminologies, the following remarks are made. In the
terminology of [3], a stochastic process with probe space $L$ is a weak
(probability) distribution in the dual $L'$ of $L$. In practice it is sometimes
more convenient to use in place of dual spaces, paired linear spaces $L$
and $K$, by which is meant two real linear vector spaces together with
a bilinear nondegenerate form $\langle f, y \rangle (f \in L, y \in K)$ on $L \times K$; $K$
may be identified with $L'$ in the corresponding weak topology. Associated
with the weak distribution $\Phi$ in $K$—which distribution is represented
by definition by a linear mapping from the dual of $K$ to random
variables—there is a probability measure, which is not in general
countably additive, on the ring of subsets of $K$ of the form

$$S = \{y \in K: (y(x_1), ..., y(x_n)) \in B\}$$

for some finite $n$, Borel set $B$ in $R^n$, and vectors $x_1, ..., x_n$ in $L$, which
measure is given by the equation

$$\Pr[S] = \Pr[(\Phi(x_1), ..., \Phi(x_n)) \in B].$$

When this measure is countably additive, the distribution is of the
conventional, so-called "strong" type; but this is rarely the case for
the spaces of interest later in this paper.

Equivalent to the weak distribution, for which the indicated measure
need not be countably additive, is the notion of "generalized random
process" introduced by Gelfand (cf. [4]); the terms "random process"
and "random field" are also used for the concept here called
"stochastic process." The term "random distribution" has been
employed in the case that $M$ is a $C^\infty$ manifold, $L$ consists of all $C^\infty$
functions of compact support, and $\Phi$ enjoys suitable continuity prop-
erties. In order to minimize confusion between the two distinct
meanings of the term "distribution" in these connections, a "weak
distribution" will henceforth be referred to as a "weak probability
measure," and the term "distribution" avoided.

The conventional notion of stochastic process may here be
designated as a strict stochastic process; this is one such that

$$\Phi(x) = \int_M \phi(t) x(t) \, dm(t),$$

where $\phi$ is a measurable mapping from $M$ into random variables, and
$m$ is an abstract Lebesgue measure on $M$; it is assumed here that all elements of $L$ are suitably measurable, and that the integral exists in an absolutely convergent sense. These notions will not be further specified here, since the concept of strict process is introduced only for explanatory purposes; the essential point is that the function-class $\phi(\cdot)$ conventionally representing the stochastic process is uniquely determined by the integrated form of the process, $\Phi(\cdot)$, employed here and applicable more generally to weak processes. Like countably additive probability measures in function space, strict processes occur rather rarely in connection with the present work.

**Definition 2.** An operational process $\Phi$ with probe space $L$ is called "canonical" if $\Phi(x)$ is self-adjoint for all $x \in L$, and if $L$ is a direct sum $L = L_a \oplus L_s$ of subspaces $L_a$ and $L_s$ (the subscripts here are abbreviations for "anti-symmetric" and "symmetric") such that

\begin{align*}
(1.1) \quad & \text{if } x, y \in L_a, \quad \text{then} \quad e^{i\Phi(x)}e^{i\Phi(y)} = e^{iA(x, y)}e^{i\Phi(x)}e^{i\Phi(y)}, \\
(1.2) \quad & \text{if } x, y \in L_s, \quad \text{then} \quad \Phi(x)\Phi(y) + \Phi(y)\Phi(x) = 2S(x, y)I,
\end{align*}

where $A(\cdot, \cdot)$ is a (necessarily anti-symmetric) nondegenerate bilinear form on $L_a$;

\begin{align*}
(1.3) \quad & \text{if } x \in L_a \text{ and } y \in L_s, \text{ then } \Phi(x) \text{ and } \Phi(y) \text{ commute (strongly, i.e. their spectral projections do so)}.
\end{align*}

A quantum process is *canonical* if the associated operational process is such.

**Scholium 2.** If a given operational process $\Phi$ is such that $\Phi(x)$ is always self-adjoint and there exist subspaces $L_a$ and $L_s$ of the indicated types, then $L_s$ and the form $S$ on it, as well as, under the additional hypothesis that $L_a$ is not of finite odd dimension, $L_a$ and the form $A$ on it, are unique.

**Proof.** If $x \in L_s$, then $2\Phi(x)^2 = 2S(x, x)I$, from which it follows that $\Phi(x)$ is bounded. On the other hand, if $x \in L_a$, then by the von Neumann uniqueness theorem for the Schrödinger operators (cf. e.g. [2]), $\Phi(x)$ is bounded only when $x = 0$. It follows that if $x \in L_a$,
and \( y \in L_a \) with \( y \neq 0 \), then \( \Phi(x + y) \) is unbounded. Thus \( L_a \) consists precisely of the set of all vectors \( x \in L \) such that \( \Phi(x) \) is bounded. It is clear that \( S(\cdot, \cdot) \) is uniquely determined and necessarily symmetric.

Thus \( L_a \) is unique. If \( L_a' \) is another space with the same properties as \( L_a \), then any element \( z \in L_a' \) has the form \( z = x + y \) with \( x \in L_a \) and \( y \in L_a \). Since \( x = z - y \), \( \Phi(x) \) is the closure of \( \Phi(z) - \Phi(y) \); since each of \( \Phi(z) \) and \( \Phi(y) \) commute strongly with all \( \Phi(x'), x' \in L_a \), the same is true of \( \Phi(x) \). [In greater detail, if \( A \) and \( B \) are closed densely defined operators in a Hilbert space strongly commuting with the closed densely defined operator \( C \), then the closure of \( A - B \), if it exists, also strongly commutes with \( C \). For the hypothesized commutativity means that the ring \( R_A \) commutes with the ring \( R_C \), as does the ring \( R_B \) with the ring \( R_C \), where the ring in question is the ring of all bounded functions of \( A \), i.e. the ring generated by the partially isometric constituent of \( A \) and the bounded spectral projections of its self-adjoint constituent in its canonical polar decomposition, these rings being the same whether left or right decompositions are employed. Now \( A - B \) is affiliated with the ring \( R_A \vee R_B \), i.e. the ring it determines is contained in \( R_A \vee R_B \); to show this it suffices, as follows from the von Neumann double commutator theorem to show that \( A - B \) commutes with all unitary operators \( V \) which commute with both \( R_A \) and \( R_B \); any such operator \( V \) evidently commutes with \( A \) and \( B \), hence with \( A - B \), and hence with \( A - B \). But \( R_A \vee R_B \) commutes with \( R_C \), since each ring in the union does so; this means that every operator affiliated with \( R_A \vee R_B \) strongly commutes with \( C \), in particular \( A - B \) does so.]

Now the restriction of \( \Phi \) to \( L_a \) extends to a representation of the Clifford algebra over \((L_a, S)\); the restriction of this extension to any subspace of \( L_a \) of finite even dimension is therefore an isomorphism; the image algebra is isomorphic to a finite-dimensional complete matrix algebra, and so has trivial center. Since \( \Phi(x) \) is central, it must be a scalar operator, but this is inconsistent with the isomorphic character of the indicated restriction of the extension of \( \Phi \). Thus \( L_a \) is unique. On taking adjoints in the ("Weyl") relation (1.1) it follows that \(-A(x, y) = A(-y, -x) = A(y, x)\), and it is evident that \( A(x, y) \) is unique.

**Remark 2.** The relations (1.1) are the Weyl form of the so-called "canonical commutation relations," which are associated in theoretical physical applications with so-called "Bose–Einstein" fields. The relations (1.2) are the so-called "anti-commutation relations," similarly associated with Fermi–Dirac fields. The latter will be
treated later; the present work is restricted to the case of the relations (1.1), although there is little doubt that parallel results are valid quite generally.

Remark 3. The main aim of this paper is the study of nonlinear local functions of weak processes. In explication of this aim, it may be helpful to utilize a notation similar to one employed in the theory of distributions of Schwartz, (as well as in heuristic studies of quantum fields) according to which the weak process $\Phi(f)$ is written in the form

$$\Phi(f) = \int \phi(x)f(x) \, dx,$$

where $\phi(.)$ is the function considered earlier when $\phi(.)$ is a strict process, but is otherwise meaningful only as a symbolism in connection with $\Phi(f)$. Since $\phi(.)$ has in general no meaning as a function on $M$, and simply serves as an occasionally convenient symbolism in connection with $\Phi(f)$, the expression $F(\phi(x))$, where $F$ is a given function of a numerical variable, has no a priori meaning, unless $F(c) = cl$ for some constant $c$; for example, $\phi(x)^2$ is without apparent mathematical significance. Nevertheless, such expressions arise naturally in the mathematical treatment of the physical idea of a "local" interaction, as well as in the theory of stochastic nonlinear partial differential equations. It will be shown that an effective meaning can be given to such expressions, in suitable nontrivial instances, but that a certain "infinite renormalization" may be involved.

The obvious approach to the indicated question is by the approximation of weak processes by strict ones, combined with the definition of the square (for example) of the weak process as the suitable limit of the squares of the approximating strict processes. This approach is however ineffective in even the simplest nontrivial cases of present concern, in which there is a significant degree of group-invariance.

Example 1. Consider for example the problem of defining nonlinear functions of fractional derivatives of the Wiener model $x(t)$ for Brownian motion. As shown essentially by Wiener, the derivative of order $l$, $x^{(l)}(t)$ is a strict process for $l < 1/2$, and a weak one for $l \geq 1/2$. The simplest nontrivial case is therefore the fractional derivative $x^{(1/2)}(t)$ of order $1/2$, say $y(t)$. For simplicity, the process may be taken on the circle rather than on the line; only a local question is involved, and familiar probabilistic methods could be used to adapt the following from the circle to the line. On the circle, the
process itself, and not merely its derivative, may be formulated in

group-invariant terms:

\[ x(t) \sim \sum_n \epsilon_n e^{int} n^{-1} \]

where \( n = \pm 1, \pm 2, \ldots \), \( \epsilon_n \) is a complex normally distributed random
variable of mean 0 and unit variance, which is orthogonal to its
complex conjugate, and \( \epsilon_{-n} = \epsilon_n \); apart from the last relation, the \( \epsilon_n \)
are assumed to be mutually independent. Then since there exists
a unitary operator on \( L_2 \) which carries \( e^{int} n^{1/2} \) into \( e^{int} | n |^{1/2} \), it will
suffice to consider the process

\[ y(t) \sim \sum_n \epsilon_n e^{int} n^{-1/2} \]

note that the unitary operator in question transforms the process into
an equivalent process (cf. [6]).

If \( y_N(t) \) is the same sum taken over values \( n \) such that \(| n | \leq N\), then

\[ Y_N = \int_0^\infty (y_N(t))^2 dt = 2\pi \sum_{|n| \leq N} |\epsilon_n|^2 |n|^{-1} \]

by a simple computation; \( \text{Exp}(e^{itN}) \) is readily computed as

\[ \prod_{1 \leq n \leq N} (1 - ci(n))^{-1} = f_N(l) \]

say, where \( c \) is a certain positive constant; and \( f_N(l) \text{Exp}[-2il \sum_{1 \leq n \leq N}] \) is easily seen to have a limit, as \( N \to \infty \), uniformly for \( l \) in any bounded real interval. It follows
that if \( Z_N = Y_N - 2 \sum_{1 \leq n \leq N} n^{-1} \), then \( \{Z_N\} \) is stochastically con-
vergent to a finite random variable \( Z \), which implies that \( \{Y_N\} \) is
is stochastically convergent to \( +\infty \). From the zero-one law, it follows
that \( Y_N \to +\infty \) with probability one.

If \( f(t) \) is any nonnegative, not identically zero, continuous function
on the circle, then there exist, by compactness, a finite set \( s_1, \ldots, s_r \)
on the circle such that \( 1 \leq \sum_{1 \leq i \leq r} f(t + s_i) \). Setting

\[ Y_{N,f} = \int (y_N(t))^2 f(t) dt, \]

it follows from the group-invariance of the Wiener process on the
circle that \( \{\text{Exp}[Y_{N,f}]\} \) is independent of \( i \), where \( f_i(t) = f(t + s_i) \), and hence that \( \text{Exp}[Y_{N,f}] \to \infty \). It may be shown (by simple estimates
of variances; or from the general theory of series of independent
random variables (cf. [5], Ch. 3, Sec. 2); or as a trivial consequence
of theory given later in this paper) that \( Y_{N,f} - \text{Exp}[Y_{N,f}] \) is con-
vergent with probability one to a finite random variable. It follows that $Y_{N,j} \to +\infty$ with probability one.

To summarize, for the Wiener process $x(t)$, and any continuous non-negative function $f(\cdot)$, the average with weight function $f$ of the square of the fractional derivative of order 1.2 exists in essentially a conventional probabilistic sense, and

$$\int (x^{1/2}(t))^2 f(t) \, dt = +\infty$$

with probability one.

This might appear to leave little doubt about the meaninglessness of putative differential equations in which expressions such as $(x^{1/2}(t))^2$ appear. Actually, expressions of a similar but more singular nature appear in the differential equations of heuristic quantum field theory. A somewhat opportunistic solution to the problem in the special case just considered emerges from the incidental result cited, namely the convergence as $N \to \infty$ of $V_{N,j} - \text{Exp}[V_{N,j}]$ to a finite random variable; this suggests the redefinition of the square as the limit of the centered squares of the approximating strict processes (where "centering" refers to the subtraction of the expectation value; cf. [5]). Apart from having no a priori justification other than that it gives a finite nonzero result, this procedure suffers from the difficulty that it fails for cubes and higher powers. Nevertheless, it is the simplest instance of a "renormalized" definition of a nonlinear function of a weak process, which will be shown in this paper to have a simple and natural intrinsic characterization. These intrinsically characterized processes will be shown to exist by more complicated renormalizations than centering, which are quite analogous (and in certain cases, effectively equivalent) to these of "subtraction physics;" and the resulting operations on weak processes will be shown to enjoy many of the mathematical properties that nonlinear operations on strict processes have; most notable among these properties are those of "locality" and group-invariance, as specified below.

1.2. Let $\Phi$ be a weak stochastic process in the given set $M$, relative to the given probe space $L$; assume further that $L$ is endowed with a topology relative to which it is a linear topological space, and that $\Phi$ is continuous, relative to the topology of convergence in measure for the random variables in question. The process is called quasi-invariant in case the associated weak distribution on $L'$ is quasi-invariant in the established sense [6]; this means, specifically, that either one of the following two equivalent conditions is satisfied:

(a) For any element $y \in L'$, there exists a unitary operator $U$ on the
Hilbert space of all square-integrable random variables determined by the \( \Theta(f), f \in L \) such that

\[
UT_fU^{-1} = T_f + y(f)I; \quad T_f = \text{operation of multiplication by } \Theta(f);
I = \text{identity operator};
\]

(b) For any element \( y \in L' \) there exists an automorphism \( A_y \) of the algebra of all bounded random variables determined by the \( \Theta(f), f \in L \), whose induced action carries \( \Theta(f) \) into \( \Theta(f) + y(f) \), for all \( f \in L \), in the sense that if \( b(t_1, \ldots, t_n) \) denotes any bounded Baire function of the numerical variables \( t_1, \ldots, t_n \), then

\[
A[b(\Theta(f_1), \ldots, \Theta(f_n))] = b(\Theta(f_1) + y(f_1), \ldots, \Theta(f_n) + y(f_n)).
\]

Now \( L \) is canonically identifiable with the dual of \( L' \), relative to the weak topology on \( L' \) induced by the elements of \( L \); indeed, this is the basis for the association of a stochastic process in \( M \) with a weak distribution in \( L' \). Such a weak distribution may be called "strong" in case the associated probability measure in \( L' \) is countably additive. When this is the case, quasi-invariance is equivalent to the absolute continuity of the transformations \( x \rightarrow x + y \) in \( L' \), for all \( y \in L' \). In the case of a general (not necessarily strict) distribution in \( L' \), a third alternative formulation for quasi-invariance may be given in terms of a generalized notion of absolute continuity (see [6]).

In the presence of quasi-invariance, there is a natural means of defining the powers of a weak process, which may be motivated in the following fashion. If the element of a distinguished measure on \( M \) is denoted as \( dx \), the notation \( \Theta(f) \sim \int \phi(x)f(x)\,dx \) for the given process is, as earlier indicated somewhat suggestive. Now let \( A_k \) denote the automorphism of the algebra of all numerically-valued random variables induced by translation in the space \( K \) by the vector \( k \in K \); suppose that \( K \) consists of functions on \( M \), and suppose also, for the moment, that the process is strict. Then if \( R = \int \phi(x)^2 f(x)\,dx \), it is easily found that

\[
(*) \quad A_k(R) - R = 2 \int \phi(x)k(x)f(x)\,dx + \int k(x)^2 f(x)\,dx,
\]

on noting that \( A_k \) maps \( \phi(x) \) into \( \phi(x) + k(x) \). This suggests the definition that the square, say \( \phi^{(2)}(\cdot) \) of the process \( \Theta(\cdot) \) exists, and has in its domain the function \( f \) on \( M \), provided there exists a random variable \( R \) (measurable with respect to the ring determined by the \( \Theta(g), g \in S \)) such that equation \((*)\) is satisfied. Evidently, \( R \) is never
unique, for an arbitrary constant may be added to it, but its relative uniqueness is readily dealt with as follows.

Let $R_0$ denote the basic ring of random variables, i.e. the ring of measurable numerical functions modulo null functions, relative to the ring determined by the $\Phi(g), g \in S$, on the probability measure space in question. Now let $R_1$ denote the subring (possibly consisting only of the constants) of all elements $R \in R_0$ such that $A_k(R) = R$ for all $k \in K$. Then $\Phi^{(2)}(f)$ is unique modulo $R_1$, since if both $R$ and $R'$ satisfy equation (*), then

\[(\forall) \quad A_k(R) = R - A_k(R') = R'
\]

for all $k \in K$, from which it follows that $R - R' \in R_1$. The residue class of $\Phi^{(2)}(f)$ modulo $R_2$, say $\Phi^{(2)'}(f)$, is then unique, and $\Phi^{(2)'}(.)$ is a linear mapping from a linear space of functions on $M$ into such residue classes. It is in general not a process, but gives rise to a process in a natural fashion in the case that $\Phi^{(2)}(f)$ contains an integrable element. For this element can be chosen so that its conditional expectation relative to the ring $R_1$ is zero, and remains integrable; it is the only such element, for if $R$ and $R'$ are two such elements, then the defining equality, (\forall), implies that $R - R' \in R_1$, and on taking the conditional expectation relative to $R_1$, it follows that $R - R' = E[R | R_1] - E[R' | R_1] = 0$.

The foregoing process may be extended to higher powers by induction, but on making this extension it becomes apparent that it is convenient to introduce a mapping $\Phi_0$, and to define $\Phi_1$ as slightly different from $\Phi$ itself. Specifically, it is convenient to make the

Assumption A. (1) The probe space $L$ is invariant under multiplication by elements of the sample space: $kf \in L$ if $k \in K$ and $f \in L$.

(2) A linear functional $\Phi_0$ is given on $L$, having the property that $\langle f, k \rangle = \Phi_0(fk)$ ($f \in L, k \in K$).

(3) $K$ is closed under multiplication (i.e. is an algebra).

Scholium 3. Let $\Phi$ be a given quasi-invariant stochastic process satisfying Assumption A. Let $B$ denote the basic ring of random variables (of all random variables measurable with respect to the minimal sigma-ring with respect to which all the $\Phi(f)$ are measurable), and let $R_n$ be defined recursively as follows: $R_0$ is the algebra of all constant random variables; $R_n = [R \in B : A_k(R) = R \in R_{n-1}, k \in K]$ for $n > 0$. Then there exists a unique sequence of functions $\Phi^{(n)}$ ($n = 0, 1,...$), where
\( \Phi^{[n]} \) is defined on a certain domain \( D_n \) contained in \( L \), and \( \Phi^{[n]}(f) \) is a residue class of \( B \) modulo the additive subgroup \( R_{n-1} \), such that

1. \( \Phi^{[0]}(f) = \Phi_0(f)e \), where \( e \) is the unit random variable, and \( D_0 = L \).

2. The domain \( D_n \) of \( \Phi^{[n]} \) is a linear set, and \( \Phi^{[n]} \) is a linear mapping of \( D_n \) into \( B/R_{n-1} \).

3. For \( n > 0 \), \( f \in D_n \) if and only if there exists an element \( R \in B \) such that

\[
A_k(R) - R = \sum_{0 \leq r \leq n-1} \Phi^{[r]}(k^{n-r}f) \binom{n}{r},
\]

where \( \binom{n}{r} \) indicates the binomial coefficient for \( n \) over \( r \); and \( \Phi^{[n]}(f) \) is equal to \( R + R_{n-1} \).

Proof. The existence part of the proof is given by induction. For \( n = 1 \), let \( D_1 = L \), and set \( \Phi^{[1]}(f) = \Phi(f) + R_0 \), \( f \in L \). Now a defining property of the automorphism \( A_k \) is that \( A_k(\Phi(f)) = \Phi(f) + \langle f, k \rangle e \), and since \( A_k \) is an automorphism, \( A_k(e) = e \). Hence if \( R \in \Phi^{[1]}(f) \), say \( R = \Phi(f) + ce \)

\[
A_k(R) - R = (\Phi(f) + \langle f, k \rangle e + ce) - (\Phi(f) + ce) = \langle f, k \rangle e = \Phi_0(fk)e.
\]

Thus, (3) is satisfied for \( n = 1 \).

Now assuming that \( \Phi^{[j]}(.) \) has been defined for \( j = 1, \ldots, n-1 \), so that (2) and (3) hold for these values of \( j \), let \( D_n \) be defined as the set of all \( f \in L \) such that there exist a random variable \( R \) such that for all \( k \in K \),

\[
A_k(R) - R = \sum_{0 \leq r \leq n-1} \Phi^{[r]}(k^{n-r}f) \binom{n}{r};
\]

and let \( \Phi^{[n]}(f) \) be defined for any such \( f \) as \( R + R_{n-1} \). By virtue of the linearity of the \( D_j \) and \( \Phi^{[j]} \) for \( j < n \), the set \( D_n \) is again linear. Furthermore, \( R \) is unique modulo \( R_{n-1} \); for by definition, any two elements of \( \Phi_1(f) \) must differ by a constant; if \( \Phi^{[2]}(f) \) is defined, then any two elements may be seen by the argument indicated earlier to differ by an element of \( R_1 \); it follows recursively by the same argument that for any \( j \), that \( \Phi^{[j]}(f) \), if defined, consists of exactly one residue class modulo \( R_{j-1} \); and finally, by another application of the argument, that \( R \) is unique modulo \( R_{n-1} \). It follows from the definition of \( \Phi^{[n]}(.) \) that \( \Phi^{[n]}(f + g) \in \Phi^{[n]}(f) + \Phi^{[n]}(g) \) if \( f, g \in D_n \); and since,
as just established, $\Phi^{[n]}(h)$ consists of exactly one residue class modulo $R_{n-1}$, it results that

$$\Phi^{[n]}(f + g) = \Phi^{[n]}(f) + \Phi^{[n]}(g).$$

This shows the existence of a sequence $\{\Phi^{[n]}(.)\}$ with the indicated properties. Any other sequence with the same properties is either identical, or differs for some least index $n_0$; but the foregoing argument shows that $\Phi^{[n_0]}(.)$ is uniquely determined by the $\Phi^{[j]}(.)$ for $j < n_0$.

On its maximal domain, the $nth$ power of a stochastic process is not itself precisely a process; but it may be canonically restricted to form a process, as in

**Scholium 4.** *With the same hypothesis as Scholium 1, there exists a unique sequence of stochastic processes $\Phi^{(n)}(.)$, $n = 0, 1, 2,...$, such that $\Phi^{(n)}(.)$ has domain $L_n$ contained in $L$, has values in $B$, and:

1. $\Phi^{(0)}(f) = \Phi_0(f)e$, and $L_0 = L$;
2. For $n > 0$, $f \in L_n$ if and only if there exists an integrable element $R \in B$ such that

$$A_k(R) - R = \sum_{0 \leq r < n-1} \Phi^{(r)}(h^{n-r}f) \binom{n}{r};$$

and $\Phi^{(n)}(f)$ is then defined as the unique such $R$ of vanishing conditional expectation relative to $R_1$.

**Proof.** The existence part is again by induction. Equation (1) serves as a definition for $\Phi^{(0)}(.)$. For $n = 1$, let $L_1$ denote the set of all elements $f \in L$ such that $\Phi(f)$ is integrable, and for such an element, let $\Phi^{(1)}(f) = \Phi(f) - E(\Phi(f) \mid R_1)$, where $E(\mid S)$ denotes the conditional expectation operation relative to the given ring $S$ (of all random variables measurable with respect to a given sigma-ring). Then $\Phi^{(1)}(f)$ has vanishing conditional expectation relative to $R_1$, and satisfies equation (*). It is the unique integrable such random variable, for if $R$ and $R'$ are random variables of vanishing expectation value relative to $R_1$, and each satisfying equation (*), it results that

$$A_k(R - R') = R - R',$$

which implies that $R - R' \in R_1$. This means that $R - R'$ is equal to its conditional expectation relative to $R_1$, but this vanishes by the additivity of the expectation and the vanishing of the conditional expectations of $R$ and of $R'$. 
Now proceeding recursively, suppose that $\Phi^{(j)}(.)$ has been defined for $j < n$ in such a way that (1) and (2) hold. Let $L_n$ denote the set of all elements $f$ such that there exists an integrable random variable $R$ such that

$$A(R) - R = \sum_{0 \leq r \leq n-1} \Phi^{(r)}(k^{n-r}f) \binom{n}{r};$$

setting $R' - R - E(R | R_1)$, it is easily seen that $R'$ satisfies the same relation, and additionally, $E(R' | R_1)$. By the argument of the preceding paragraph, it is the unique integrable random variable satisfying equation (*). Defining $\Phi^{(n)}(f) = R'$, it is evident that (2) holds, and the construction of the $\Phi^{(n)}(.)$ by induction is complete. The argument shows at the same time that the sequence $\Phi^{(n)}(.)$ is unique.

**Definition 3.** The process $\Phi^{(n)}(.)$ is called the *nth renormalized power* of the process $\Phi(.)$. Its symbolic kernel will be denoted as $\phi(x)^n$; thus $\Phi_n(f) \sim \int \phi(x)^n f(x) dx$.

**Corollary 1.** For any given ergodically quasi-invariant stochastic process satisfying Assumption A, the renormalized powers are uniquely determined by the (symbolic) equations

$$(\phi(x) + k(x))^n:$$

$$= \phi(x)^n: + n \phi(x)^{n-1}: k(x) + \frac{n(n-1)}{2} \phi(x)^{n-2}: k(x)^2 + \cdots$$

$$+ n \phi(x) k(x)^{n-1} + k(x)^n e;$$

$$E(\phi(x)^n:) = 0 \quad (n = 1, 2, ...);$$

$$\phi(x)^0: = e.$$

It must be emphasized that the expression $$(\phi(x) + k(x))^n:$$ has no meaning, in general, (i.e. except in the case of a strict process) except as the symbolic kernel of $A_n(\Phi_n(f))$; and that the given equations otherwise acquire meaning only by multiplication by $f(x)$ and integration, i.e. in the form of equation (*) of Scholium 2. It should be recalled that an ergodic process is one such that the only invariant random variables, under the transformations in question, are constants; an ergodically quasi-invariant process is then one such that $R_1 = R_0$.

**Remark 4.** A similar method could be employed to define and treat more generally the product of two not necessarily identical
weak stochastic processes. Certain cases of this are important in practice, but for greater clarity in the presentation of the essentials, the general question will not be treated here.

Example 2. Let \( f(x) \) denote an element of \( L_1(R^1) \) such that \( f(x) > 0 \) for all \( x \), and \( \int f(x) \, dx = 1 \); let \( F \) denote the indefinite integral of \( f \), and let \( P = (R^1, F) \) denote the probability space associated with \( F \). Setting \( L = K = R^1 \), with \( \langle x, y \rangle = xy \), and setting \( \Phi(1) = x \) as a measurable function on \( P \), then \( \Phi \) is an ergodically quasi-invariant process on the space \( M \) consisting of the single point 1, and satisfies Assumption A. The process \( \Phi \) is strict, and there is no difficulty in defining \( \Phi^n; \) specifically, \( \Phi^n(1) = x^n \) as a random variable on \( P \). The renormalized product \( :\Phi(1)^n: \) is represented correspondingly by a polynomial \( p_n(x) \) of degree \( n \), uniquely characterized by the properties: \( (d/dx)p_n(x) = np_{n-1}(x)(n > 0); p_0(x) = 1; E(p_n(x)) = 0 \) for \( n > 0 \). In general these polynomials are not orthogonal in \( L_2(P) \), but in the special case in which \( f(x) = (2\pi)^{-1/2} e^{-x^2/2} \), which is important in connection with quantum processes, they have this additional property, and so coincide with the multiples of the Hermite polynomials which have leading coefficient equal to 1; this can be deduced from the recursion relation \( H_n'(x) = 2nH_{n-1}(x) \) for the Hermite polynomials.

It is easily seen by induction that for an arbitrary, not necessarily quasi-invariant, probability measure \( F \) on \( R^1 \) (countably additive on the Borel subsets of \( R^1 \)), there exists a unique sequence \( \{p_n(x)\} \) of polynomials having the properties just indicated. This serves to illustrate an adaptation of the preceding developments to the case of a process which is not necessarily quasi-invariant. Ultimately, the quasi-invariance will be important, but for the present it may be dispensed with, and an approach developed whose formalism is close to one useful in connection with quantum processes.

1.3. In place of the characterization of powers by their transformation properties under vector translations, one may take the corresponding infinitesimal translation properties. This means that in place of the recursive property of the powers:

\[
(x + a)^n - x^n = \sum_{0 \leq r \leq n} x^r a^{n-r} \binom{n}{r},
\]

one may use the infinitesimal form of this relation as \( a \to 0 \):

\[
(d/dx)x^n = nx^{n-1}.
\]

Although algebraically relatively simple, this infinitesimal approach leads to analytical difficulties, which necessitate
strong restrictions on the process under consideration. In this section, the simplest nontrivial case, in which $M$ is a finite set, will be examined.

**Definition 4.** A "polynomial" (function) on the space $K$ is a function $F(k)$ of the form:

$$F(k) = p(<f_1, k>, ..., <f_r, k>),$$

where $p(t_1, ..., t_r)$ is a polynomial over the complex field in the indeterminates $t_1, ..., t_r$, and the $f_i$ are arbitrary in the dual space $L$. The set of all polynomials (evidently an algebra) is the "polynomial algebra" over $K$.

A process $\Phi$ with probe space $L$ is "nonsingular" if the only polynomial $F(.)$ on the sample space such that the random variable

$$p(\Phi(f_1), ..., \Phi(f_r)) = 0,$$

where $F$ and $p$ are related in the indicated fashion, and the $f_i$ are linearly independent, is identically zero. It is not difficult to show that if $p$ and $p'$ are both polynomials over the complex field in certain numbers of indeterminates which are both related to $F$ in the indicated fashion, then the vanishing of $p(\Phi(f_1), ..., \Phi(f_r))$ implies that of $p'(\Phi(f_1'), ..., \Phi(f_r'))$. It follows that the mapping

$$F(.) \mapsto p(\Phi(f_1), ..., \Phi(f_r))$$

is an algebraic isomorphism of the polynomial algebra $P(K)$ over $K$ into the algebra of random variables (= equivalence classes of measurable functions modulo null functions) on the probability measure space in question; the image algebra is called the "polynomial algebra of the process $\Phi$," or in case it is essential to avoid possible confusion with the algebra $P$, the algebra of "polynomial random variables."

A process $\Phi$ is said to "have moments of all orders" in case the product $\Phi(f_1) \cdots \Phi(f_n)$ is integrable for arbitrary $f_1, ..., f_r \in L$. For any two random polynomials $A$ and $B$, the inner product $\langle A, B \rangle$ is defined as the expectation value of $BA$, where the superscribed bar denotes the complex conjugate; this expectation evidently exists in the case of a process having moments of all orders.

**Scholium 5.** Let $K$ and $L$ be as above, and let $P$ denote the algebra of all polynomials on $K$. For any element $f \in S$, let $\Psi(f)$ denote the
operation on $\mathcal{P}$ which carries $F(k) \rightarrow \langle f, k \rangle F(k)$. For any element $k_0 \in \mathbb{K}$, let $\Pi(k_0)$ denote the operation on $\mathcal{P}$ which carries $F(k)$ into $i(\partial/\partial t) p(k + tk_0)|_{t=0}$. Then the linear mappings $\Psi$ and $\Pi$ from $\mathbb{L}$ and $\mathbb{K}$ (respectively) to the linear operators on $\mathcal{P}$ satisfy the following relations for arbitrary $f, f' \in \mathbb{L}$ and $k, k' \in \mathbb{K}$ (where $I$ indicates the identity operator on $\mathcal{P}$):

$$[\Psi(f), \Pi(k)] = -i\langle f, k \rangle I; \quad [\Psi(f), \Psi(f')] = 0; \quad [\Pi(k), \Pi(k')] = 0$$

Proof. The latter two relations are evident. To establish the former, note that the treatment of the putative equality of $[\Psi(f), \Pi(k)]$, and $-i\langle f, k \rangle q$, where $q$ is a given polynomial on $\mathbb{K}$, is reducible to the case in which $\mathbb{K}$ and $\mathbb{L}$ are finite-dimensional (cf. the treatment of analogous matters in [7], pp. 116–119 and 128–130). In this case the verification of the asserted relation is essentially the same as that of the well-known fact that the Schrödinger operators satisfy the Heisenberg commutation relations.

Scholium 6. Let $\Phi$ be a real stochastic process on the finite set $M$, with each of $\mathbb{L}$ and $\mathbb{K}$ consisting of all functions on $M$ and with $\langle f, k \rangle = \sum_{x \in M} f(x) k(x)$. Suppose that $\Phi$ satisfies Assumption A, has moments of all orders, and is nonsingular. For $k \in \mathbb{K}$, let $\Delta(k)$ denote the operation on the polynomial algebra of the process given by the equation

$$i\Delta(k) p(f_1, \ldots, f_r) = (\partial/\partial t) p(f_1 + t\langle f_1, k \rangle, \ldots, f_r + t\langle f_r, k \rangle)|_{t=0},$$

for an arbitrary complex polynomial $p(t_1, \ldots, t_r)$ in any number of indeterminates and elements $f_1, \ldots, f_r$ in $\mathbb{L}$. Then there exists a unique sequence $\Phi^{(n)}(.)$ of processes with probe space $\mathbb{L}$, such that:

1. $\Phi^{(0)}(f) = \sum_x f(x)$; 2. $E(\Phi^{(n)}(f)) = 0$ for $n > 0$ and all $f$; 3. $\Pi(k) \Phi^{(n)}(f) = -in\Phi^{(n-1)}(fk) \langle f, k \rangle$ for $n > 0$, $f \in \mathbb{L}$, $k \in \mathbb{K}$; $\Phi^{(n)}(f)$ is a random polynomial for all $n$ and $f$.

Proof. That the operation $\Delta(k)$ is well-defined, i.e. independent of the polynomial $p$ representing the random polynomial in question, follows without difficulty, from the nonsingularity of the process.

To establish the stated uniqueness, suppose that $\{\Phi^{(n')}\}$ is a sequence of processes satisfying the same conditions as the sequence $\{\Phi^{(n)}\}$; let $n$ denote the least index for which $\Phi^{(n')} \neq \Phi^{(n)}$ (assuming such exists, as otherwise uniqueness holds). Then $n > 0$, and $\Phi^{(n)}(f) - \Phi^{(n')} (f)$ is a random polynomial which is annihilated by
all the \( \Pi(k) \). To complete the proof of uniqueness, it suffices to show that any such polynomial must be constant; condition (2) then ensures that the constant vanishes, implying that \( \Phi^{(n)}(f) = \Phi^{(n)}(f) \).

By virtue of the isomorphism of the polynomial algebra over \( K \) with the polynomial algebra of the process, in the case of a nonsingular process, and the easily ascertained fact that \( \Pi(k) \) is carried into \( \Delta(k) \) by the induced action on operators of this isomorphism, it suffices to show that the only polynomial on \( K \) which is annihilated by all \( \Pi(k) \) is identically constant; and this reduces to the corresponding well-known fact in a finite number of dimensions.

To establish the existence part of the Scholium, it suffices to establish \( \Phi_n \) as a strict process, whose kernel will be denoted, as earlier, as \( :\phi(x)^n:; \) in terms of this kernel, \( \Phi_n(f) \) is given by the equation

\[
\Phi_n(f) = \sum_{x \in M} :\phi(x)^n: f(x).
\]

Now observe the

**Lemma 1.** Let \( F \) be any probability measure on the (Borel subsets of the) reals, having moments of all orders. Then there exists a sequence \( \{p_n(n)\} \) of polynomials on the reals \( (n = 0, 1, \ldots) \) such that: (i) \( p_0(l) = 1; \) (ii) degree \( (p_n(l) - l^n) < n; \) (iii) \( (d/dl)p_n(l) = np_{n-1}(l) \) and \( \int_{-\infty}^{\infty} p_n(l) dF(l) = 0 \) for \( n > 0 \).

**Proof of Lemma.** Proceeding by induction, \( p_n \) is defined as the indefinite integral of \( np_{n-1} \), with the constant of integration uniquely determined by the requirement that \( \int p_n(l) dF(l) = 0, n > 0 \).

**Resumption of Proof of Scholium.** Given a fixed point \( x \in M \), let \( F_x \) denote the probability distribution of \( \phi(x): F_x(B) = \Pr[\phi(x) \in B] \). Let \( p_n(x) \) denote the polynomial of degree \( n \) given by the lemma when \( F_x \) is substituted for \( F \). Since \( \Pi(k) \) is linear in \( k \), and the proposed \( \Phi_n(f) \) is linear as a function of \( f \), it suffices to establish the relation (3) for the case in which each of \( k \) and \( f \) is the characteristic function of a one-point set. It is not difficult to see that the relation is then equivalent to the equation

\[
i(\partial/\partial l_x') p_n^x(l_x) = np_{n-1}^x(l_x) \delta_{x x'},
\]

where \( \delta_{x x'} \) denotes the Kronecker delta on \( M \times M \). Since \( l_x \) and \( l_{x'} \) are distinct indeterminates for \( x \neq x' \), this equation is a consequence of condition (iii) in the conclusion of the Sublemma.

**Remark 5.** Nonlinear functions other than rational integral ones
are of uncertain importance for applications, and will not be considered here, but the foregoing approach may be adapted to them in the following way. Let $\mathbf{F}$ be a given class of differentiable functions of a real variable, which is closed under differentiation. If $b \in \mathbf{F}$, then heuristically, $\Phi^{(b)}(f) \sim \int b'(x)f(x) \, dx$, and

$$\left[ \Phi^{(b)}(f), \Phi(g) \right] \sim i \int b'(x)f(x)g(x) \, dx.$$ 

One is thereby led to the formulation of $\Phi^{(b)}(f)$ as a suitably continuous map from $\mathbf{F} \times \mathbf{D}(b \in \mathbf{F}, f \in \mathbf{D})$ to operators, having the properties that

$$\left[ \Phi^{(b)}(f), \Phi(g) \right] = i\Phi^{(b')}fg$$  \quad (b' = \text{derivative of } b),$$

and to depend on $f$ in the same fashion as earlier. A quite different but possibly ultimately convergent approach in a special case (certain entire functions of the free scalar field in two-dimensional space-time) has been developed by A. Jaffe (Ann. Phys. 32, 127-156 (1965)).

Having constructed the normal products $\phi(x)^n$, it is natural as well as relevant for later applications to consider the normal products $\phi(x_1) \phi(x_2) \cdots \phi(x_j)$, where the points $x_1, x_2, \ldots, x_j$ are not necessarily distinct. The following extension of Scholium 6 includes this case.

**Theorem 1.** Under the hypotheses of Scholium 5, there exists a unique linear map $\Pi : A \to A$ on the algebra $A$ of all polynomial random variables for the process $\Phi$, having the properties: (1) $\Pi : 1 = 1$; (2) $\Pi(A) = p(A)$ in case $A = p(\Phi(f_1), \ldots, \Phi(f_r))$; (3) $\Pi(k)A = \Pi(k)A$. \quad (k \in \mathbf{K}, A \in \mathbf{A}).$

**Proof.** For the uniqueness, suppose that $N$ and $N'$ are both linear maps of $A$ into $A$ having the properties of the mapping (1), (2) and (3). If $N \neq N'$, there exists a least degree $n$ such that $N(A) \neq N'(A)$ for some element $A \in \mathbf{A}$ of degree $n$ (defined as the degree of the corresponding polynomial). Now

$$\Pi(k)N(A) = N(\Pi(k)A), \quad \Pi(k)N'(A) = N'(\Pi(k)A),$$

and $\Pi(k)A$ has degree less than that of $A$, so that

$$N(\Pi(k)A) = N'(\Pi(k)A);$$

thus,

$$\Pi(k)N(A) = \Pi(k)N'(A), \quad k \in \mathbf{K}.$$
It follows that \( N(A) \) and \( N'(A) \) differ by a constant, but in view of condition (2), the constant must vanish, showing that \( N(A) = N'(A) \), showing finally that necessarily \( N = N' \).

For the existence, the following generalization of Lemma 1 may be used.

**Lemma 2.** Let \( F \) be any probability measure on \((\text{the Borel subsets of)}\) \( \mathbb{R}^r \), having moments of all orders. Then there exist polynomials \( p_{n_1, n_2, \ldots, n_r}(t_1, t_2, \ldots, t_r) \) on \( \mathbb{R}^r \), defined for nonnegative integral indices \( n_1, n_2, \ldots, n_r \), such that:

1. \( p_{0, 0, \ldots, 0}(t_1, t_2, \ldots, t_r) = 1 \);
2. \( \deg(p_{n_1, n_2, \ldots, n_r}(t_1, t_2, \ldots, t_r) - t_1^{n_1}t_2^{n_2} \cdots t_r^{n_r}) < \sum_j n_j \);
3. \( (\partial/\partial t_j) p_{n_1, n_2, \ldots, n_r} = n_j p_{n_1, n_2, \ldots, n-1, \ldots, n_r} \)

(with the convention that \( p \)'s with negative indices are identically zero);
4. \( \int p_{n_1, n_2, \ldots, n_r} dF = 0 \)

except when all \( n_j = 0 \).

**Proof.** The proof is by induction. Assuming that the \( p_{n_1, \ldots, n_r} \) are defined so that the stated conclusions hold when \( \sum n_j < n \), let \( p_{n_1, \ldots, n_j} \) be defined in the case \( \sum n_j = n \) as the unique polynomial satisfying the conditions (iii) and (iv). That (iii) as an equation for \( p_{n_1, \ldots, n_r} \) has a solution follows from the Poincaré lemma, since it is readily verified that

\[
(\partial/\partial t_k) n_j p_{n_1, n_2, \ldots, n_{j-1}, \ldots, n_r} = (\partial/\partial t_j) n_k p_{n_1, n_2, \ldots, n_{k-1}, \ldots, n_r};
\]

the solution is evidently unique within an additive constant, which is fixed by condition (iv).

Resuming the proof of the Scholium, set

\[ :\phi(x_1)^{n_1} \phi(x_2)^{n_2} \cdots \phi(x_r)^{n_r} : = p_{n_1, n_2, \ldots, n_r}(\phi(x_1), \phi(x_2), \ldots, \phi(x_r)), \]

when \( x_i \neq x_j \) for \( i \neq j \), applying the lemma to the joint distribution of \( \phi(x_1), \phi(x_2), \ldots, \phi(x_r) \) and extend the mapping \( : \) to all of \( \mathcal{P} \) by linearity, as is uniquely possible since the \( \phi(x_1)^{n_1} \phi(x_2)^{n_2} \cdots \phi(x_r)^{n_r} \) form a linearly independent set which spans \( \mathcal{A} \). Conditions (2) and (3), being linear in \( \mathcal{A} \), remain satisfied by this extension.
1.4. The mapping $\Pi$ from $K$ to the operators on $P$ has itself many of the attributes of a stochastic process. It is a linear mapping, and the operators in the range are mutually commutative; with additional regularity assumptions it is indeed possible to derive a stochastic process along such lines, in which the probabilities are derived from the expectation values corresponding to the function identically 1 on the probability space in question. This will not be done here, as the construction is largely superseded by later developments; but the parallelism between the mappings $\Phi$ and $\Pi$ which is here indicated is important for quantum processes. In these processes it is relevant to extend the normal mapping $\cdot \cdot$ not only to polynomials in the $\Pi(k)$, but also to (noncommutative) polynomials in the noncommuting operators $\Pi(k)$ and the operations $\Psi(f)$ of multiplication by the $\Phi(f)$. In a somewhat different formalism from earlier, this extension is derived in the present section from a result in [7]; the present formalism is symmetrical between $\Phi$ and $\Psi$; it is independent of the stochastic process viewpoint just developed; but it is readily interpreted from this viewpoint in the light of the results here since and including Scholium 5.

**Theorem 2.** Let $A(\cdot, \cdot)$ be a nondegenerate anti-symmetric form on the linear vector space $M$, over the field $F$ of real or complex numbers, and let $E$ denote the corresponding (infinitesimal) Weyl algebra over $(M, A)$. Let $E$ be any given linear functional on $E$ such that $E(e) = 1$. Then there exists a unique mapping $\cdot \cdot$ from monomials in $E$ to $E$ such that

$$[z_1 z_2 \cdots z_n : z'] = \sum_1^n A(z_i, z') z_1 \cdots \hat{z}_i \cdots z_n : E(z_1 \cdots z_n :) = 0$$

for arbitrary $z_1, \ldots, z_n$ and $z'$ in $M$.

**Proof.** With regard to uniqueness, suppose there exist two mappings $N$ and $N'$ from the set $N$ of all monomials in $E$, into $E$, satisfying the indicated conditions. Let $n$ be the least degree such that $N$ and $N'$ differ on a monomial of degree $n$. Then $n > 1$, for from the relations $[z_1 : z'] = A(z_1, z') = [z_1, z']$ for all $z'$, it results that

$$[z_1 : z] = 0$$

for all $z'$; since $E$ has trivial center, it results that

$$z_1 : = z_1 - E(z_1) e$$

for all $z_1$.

Subtracting the first defining equation for $N$ from that for $N'$, it results that $[N'(u) - N(u), z'] = 0$ for a monomial $u$ of degree $n$ such that $N(u) \neq N(u')$. It follows that $N'(u) - N(u) = ce$ for some
scalar \( c \), which must vanish by virtue of the second condition on \( N \) and \( N' \). Thus the mapping \( \phi \) is unique, if it exists at all.

To prove existence, suppose, as the basis of an induction argument, that a mapping \( \phi \) defined on monomials of degree less than \( n \), satisfying conditions, exists; it will then be shown that the same is true with \( n \) replaced by \( n + 1 \). Let \( z \) be arbitrary in \( \mathcal{M} \), and let \( K(z) \) be the element of \( \mathbb{E} \) given by the equation

\[
K(z) = \sum_{i=1}^{n} (z_1 \cdots \hat{z}_i \cdots z_n : - z_1 \cdots \hat{z}_i \cdots z_n)[z_1, z].
\]

A simple computation gives, for arbitrary \( z' \in \mathcal{M} \),

\[
[K(z), z'] = \sum_{1 \leq i, j \leq n} (z_1 \cdots \hat{z}_i \cdots \hat{z}_j \cdots z_n:
- z_1 \cdots \hat{z}_i \cdots \hat{z}_j \cdots z_n)[z_1, z][z_j, z'].
\]

The interchange of \( z \) and \( z' \) is equivalent to the interchange of \( i \) and \( j \) in the product of two commutators on the right, while the expression preceding this product is symmetric in \( i \) and \( j \). It follows that

\[
[K(z), z'] = [K(z'), z]
\]

for arbitrary \( z \) and \( z' \) in \( \mathcal{M} \). It now results from \([7]\) that there exists an element \( u \in \mathbb{E} \) such that \([u, z] = K(z)\) for all \( z \in \mathcal{M} \). Now defining \( z_1 \cdots z_n : = u + z_1 \cdots z_n - ce \), where \( c \) is determined (uniquely) by the condition that \( \mathbb{E}(z_1 \cdots z_n : ) = 0 \), a straightforward computation shows that the stated conditions on \( z_1 \cdots z_n : \) are satisfied.

**Corollary 2.** \( z_1 \cdots z_n : - z_1 \cdots z_n \) is of degree at most \( n - 1 \).

**Proof.** From the form \( z_1 \cdots z_n : \), this is evident in the case \( n = 1 \). If the conclusion of the Corollary is assumed valid for monomials of degree \( n - 1 \), it then follows that the \( K(z) \) of the preceding proof is at most of degree \( n - 2 \). It follows from \([7]\) that the element \( u \) just obtained is of degree at most \( n - 1 \), and the Corollary is proved.

**Corollary 3.** \( z_1 \cdots z_n : \) is a symmetric function of \( z_1, \ldots, z_n \).

**Proof.** Clear if \( n = 1 \). Use an induction argument; assuming conclusion valid for lesser values of \( n \), it suffices to show that

\[
[:z_1, \ldots, z_n : , z']
\]

\({}^3\) Proved there only for the case \( F = \text{reals} \), but the proof applies also without any essential change to the case \( F = \text{complex number field} \).
is a symmetric function of $z_1, \ldots, z_n$, since these commutators, together with the symmetrical requirement of vanishing expectation value uniquely determine $z_1 \cdots z_n$. The commutator in question is

$$\sum_j : z_1 \cdots : z_j : \cdots : z_n : A(z_j, z'),$$

which under the permutation $p$ goes over into

$$\sum_j : z_{p(1)} \cdots : z_{p(j)} \cdots : z_{p(n)} : A(z_{p(j)}, z').$$

Now employing the induction hypothesis, $z_{p(1)} \cdots z_{p(j)} \cdots z_{p(n)} : = z_1 \cdots z_{p(j)} \cdots z_n :$, so the sum in question is

$$\sum_j : z_1 \cdots : z_{p(j)} \cdots : z_n : A(z_{p(j)}, z'),$$

which on making the transformation $p^{-1}$ on the summation variable gives the required $p$-independent result.

**Corollary 4.** $z_1 \cdots z_n :$ is linear in each $z_i$ separately.

**Proof.** Follows readily by an induction argument similar to that just given.

**Corollary 5.** If $N$ is an abelian subalgebra of $M$ (i.e. $A(N, N) = 0$), then $z$ extends to a linear mapping $N$ of $E(N)$ into itself which is uniquely determined by the conditions: $[N(u), z] = N([u, z])$ ($u \in E(N)$ and $z \in L$), and $E(N(u)) = 0$.

**Remark 6.** It should be noted that $ad(z)$ leaves $E(N)$ invariant for any $z \in L$, as follows from an easy computation. The Corollary asserts essentially that the action of $N$ on $E(N)$ depends only on the restriction of $E$ to $E(N)$, and not on the value of $E$ outside of $E(N)$; it is probable that $E$ is never determined on all of $E(L)$ by its values on an $E(N)$.

**Proof of Corollary.** As earlier noted, the monomials in a set of basis vectors for $N$ span $E(N)$, so that $N$ is uniquely determined by linearity from the values of $z$ on these monomials. To show that $N$ depends only on $E \mid E(N)$, suppose $N'$ is another mapping associated with a linear functional $E'$ on $E(L)$, and that $E$ and $E'$ agree on $E(N)$. Let $u$ be a monomial in $E(N)$ of the least degree such that $N(u) \neq N'(u)$ (evidently, this degree exceeds 0). Then for $z \in L$, by virtue of the defining properties of $N$ and $N'$ and the assumption on the degree of $u$,

$$[N(u), z] = N([u, z]) = N'([u, z]) = [N'(u), z],$$
showing that $N(u)$ and $N'(u)$ can differ only by a constant. Since $E$ and $E'$ agree on $E(N)$, this constant must vanish.

1.5. The mapping $\phi_\phi$ transforms a monomial $u \in E$ into another element of $E$ having $u$ as its term of highest degree, while having distinguished commutation properties. A similar mapping can be defined which has distinguished orthogonality properties, as in the theory of orthogonal polynomials. More specifically, in the important special case in which $A$ has pure imaginary values, an inner product $\langle u, v \rangle_E$ may be defined on $E$ by first defining the mapping $u \mapsto u^*$ of $E$ onto $E$ as the unique anti-linear anti-automorphism (or adjunction operation) which leaves fixed all elements of $L$, and second, setting

$$\langle u, v \rangle = E(v^*u), \quad u \text{ and } v \text{ arbitrary in } E.$$  

Observe next that although the set of all elements of $E$ of a given degree $d$ is not a linear set (where the degree of $u$ is the minimal number such that there exists a linear subspace $S$ of $L$ of dimension $d$ such that $u \in E(S)$, while degree $(0) = -\infty$ and degree $cI = 0$ for $c \neq 0$), the set of all elements whose degree is at most $d$ is a linear set. The projection $P_d$ of any given element $u$ of $E$ onto the submanifold consisting of the elements of $E$ of degree $\leq$ any given number $d$ is consequently well-defined, providing this submanifold is complete relative to the indicated inner product. The mapping $z_1 \cdots z_r \mapsto z_1 \cdots z_r - P_{r-1}(z_1 \cdots z_r)(z_1, \ldots, z_r \in L)$ is then the cited analog to the mapping $\phi_\phi$.

In general, these two mappings are distinct, but in the most important special case for applications, they are the same. This section develops the theory of this special case. As a consequence, the present mapping $\phi_\phi$ in the special case in question, will be identified with the similarly denoted mapping discussed in treatments of quantum fields satisfying symmetric (Bose–Einstein) statistics.

**Definition 5.** An admissible complex structure on a real linear vector space $L$ relative to a given nondegenerate symmetric bilinear form $A$ on $L$, having pure imaginary values, is a real-linear transformation $J$ on $L$ having the properties: (1) $A(Jz, Jz') = A(z, z')$; (2) $J^2 = -1$; (3) $-iA(Jz, z) \geq 0$, for all $z, z' \in L$. For any element $z \in L$, the element $C(z) = (z - ijz)/2^{1/2}$ is called a creator; the set of all creators denoted as $C$. Similarly, the element $C(z)^* = (z + ijz)/2^{1/2}$ is called an annihilator, and the set of all annihilators denoted as $C^*$. The normal vacuum on $E$ relative to $J$ is defined as the linear functional $E$ such that $E(1) = 1$ and
$E(a_1 \cdots a_r b_1 \cdots b_s) = 0$ if $a_1, \ldots, a_r \in \mathbb{C}$ and $b_1, \ldots, b_s \in \mathbb{C}^*$, while $r + s > 0$. (This functional is easily seen to be unique, if it exists; that it does exist is shown below.)

**Scholium 7.**

a. $L$ is a complex pre-Hilbert space relative to the inner product

$$\langle x, x' \rangle = -iA(jz, z') + A(z, z'),$$

and to $J$ as the complex unit.

b. The mapping $C: L \rightarrow \mathbb{C}$ is an isomorphism of this space into $\mathbb{C}$ as a subspace of $E$, as a Hilbert space relative to the inner product

$$\langle u, v \rangle = E(v^*u),$$

$E$ being a normal vacuum, and to the already given complex structure in $E$.

c. The complex extension $L + iL$ of $L$ is the direct sum of $\mathbb{C}$ and $\mathbb{C}^*$; and $E(\mathbb{C})$ and $E(\mathbb{C}^*)$ are orthogonal in $E$.

d. For any unitary (or anti-unitary) operator $U$ on $L$, there is a unique automorphism (or anti-automorphism) $\gamma(U)$ of $E$ which carries $C(z)$ into $C(Uz)$.

e. The following commutation relations are valid:

$$[C(z), C(z')] = 0 = [C(z)^*, C(z')^*]$$

$$[C(z), C(z')^*] \subseteq \langle z, z' \rangle I.$$

f. If $L$ is finite dimensional, there exists a faithful representation of $E$ by anti-holomorphic linear differential operators with polynomial coefficients acting in the space of all (anti-holomorphic) polynomials $p(w)$ on $L$ as a complex linear space, in such a way that $C(z)$ corresponds to the operation of multiplication by $-\langle z, w \rangle \sqrt{2i}$; $C(z)^*$ to that of $(i/2) \times$ differentiation $\partial \partial \langle z, w \rangle$ in the direction $z$; the functional $E(u) = \int_L \phi(u) | \exp(-(1/4)|z|^2 dz$ is a normal vacuum; and a continuous representation $\Gamma$ of the unitary group on $L$, which $\Gamma(U)$ acts as follows: $p(w) \rightarrow p(U^{-1}w)$; leaves the normal vacuum invariant; and implements $\gamma(U)$ in the sense that $C(Uz) = \Gamma(U) C(z) \Gamma(U)^{-1}$.

g. The normal vacuum is unique and invariant under all $\Gamma(U)$, for $L$ of arbitrary dimensionality, and satisfies the equation $E(u^*) = \overline{E(u)}$.

**Proof.** Ad a. Since $A(z, z') = 0$, $\langle z, z \rangle \geqslant 0$. Further it is evident that $\langle z, z' \rangle = \langle z', z \rangle$. Now $\langle Jz, z' \rangle = -iA(-z, z') + A(Jz, z') = -iA(-z, z') + A(jz, z')$.
i\langle z, z' \rangle. In case \langle z, z \rangle = 0, it follows from Schwarz' inequality that 
\langle z, z' \rangle = 0 for all z', which implies z = 0, by virtue of the non-
degeneracy of \( A \).

Ad b. It is straightforward to verify that \( C(Jz) = iC(z) \), and to verify the relations given in c. Applying \( E \) to both sides of the equation
\[ [C(z), C(z')^*] = \langle z, z' \rangle I, \]
and using the defining condition for a vacuum, it follows that
\[ E(C(z')^* C(z)) = \langle z, z' \rangle. \]

Ad c. Given an arbitrary element \( x + iy \in L + iL \), it can be represented in the form \( z + z' \) with \( z \in C \) and \( z' \in C' \) by solving simple equations having an evidently unique solution. If \( u \in E(C) \) and \( v \in E(C^*) \), then \( v^*u \in E(C) \) and it follows that \( E(v^*u) = 0 \).

Ad d. If \( T \) is any symplectic transformation on \( L \), i.e. one preserving the bilinear form \( A \), it follows from general considerations that there exists a unique automorphism \( \theta \) of \( E \) such that \( \theta(z) = Tz \) for \( z \in L \). If \( T \) is in addition unitary, it commutes with \( J \), and the relation \( C(z)^\theta = C(Tz) \) is easily verified. The anti-unitary case follows similarly.

Ad e. This has been indicated in the proof of b.

Ad f. This is established in [8], for a particular normal vacuum.

Ad f. The indicated representation is given in [8]. It is immediate that the indicated functional \( E \) is a normal vacuum, etc.

Ad g. The treatment in [8] covers the infinite-dimensional case as well. The invariance under all \( \Gamma(U) \) is independently evident from the invariance of the polynomial \( I \) under the \( \overline{\Gamma(U)} \). The property \( E(u^*) = \overline{E(u)} \) follows readily.

**Corollary 6.** Relative to the normal vacuum, \( :ab: = ab \) if \( a \) is any product of creators and \( b \) any product of annihilators (where either one of the products may be replaced by \( e \)).

**Proof.** Evidently, the result is correct when \( a = b = e \). To treat the case when \( a \) is a product of \( r \) creators and \( b \) a product of \( s \) annihilators, use induction on \( r + s \). Evidently when \( r + s > 0 \), \( E(ab) = 0 \). Now if \( c \) is either a creator or annihilator, \( [ab, c] \) is a sum of similar products \( a'b' \) with \( r' + s' \leq r + s - 1 \). Applying the induction hypothesis, it follows that \( [ab, c] = [:ab:, c] \), and the Corollary follows.
This new mapping satisfies relations including those defining the old mapping, and so coincides with it on a product of elements of \( L \). In terms of this extension of the \( \mathcal{O} \) mapping, it is useful to note

**Corollary 7. Relative to the normal vacuum \( E \),**

\[
E(z_1 z_2 \cdots z_r; z_{r+1}) = 0
\]

for arbitrary \( z_i \in L + iL(r > 1) \).

**Proof.** By linearity, it suffices to consider the case in which all the \( z_i \) are creators or annihilators. It is no essential loss of generality to assume that \( z_{r+1} \) is a creator, as otherwise the conclusion is obvious, and in view of the symmetry of \( \mathcal{O}_1 \cdots \mathcal{O}_r \) as a function of \( z_1, \ldots, z_r \), it may likewise be assumed that \( \mathcal{O}_1 \cdots \mathcal{O}_r = z_1 \cdots z_r = ab \), where \( a \) is a product of creators and \( b \) a product of annihilators, not both of degree 0. Since \( E(ab) = E(b^a^*) \), it may be assumed that \( a = e \). On making the unitary transformation \( z \rightarrow e^{-it_x} (z \in \mathbb{C}) \), the induced automorphism carries \( b \rightarrow e^{-itr}b \), and \( z_{r+1} \rightarrow e^{it}z_{r+1} \), and thus multiplies the expectation value in question by \( e^{-itr(t-1)/i} \). Since this expectation is invariant, and \( r > 1 \), this requires that the expectation vanish.

The correspondence between the earlier defined map \( \mathcal{O} \) and the indicated generalization of the orthogonal polynomial construction follows from the next result. This result constitutes a statement in objective mathematical terms of an important principle in quantum field theory known as Wick's theorem. (Cf. e.g. \([9]\) for one of the clearest accounts in the quantum-field-theoretic literature; the proof below is somewhat shorter than that indicated there.)

**Theorem 1.3.** Let \( \mathcal{O} \) denote the renormalization map on \( E(L) \) relative to the normal vacuum \( E \) (relative in turn to the nondegenerate anti-symmetric pure-imaginary-valued bilinear form \( A \) on \( L \) and admissible complex structure \( J \) on \( L \), as above). Then for arbitrary \( z_1, \ldots, z_{r+1} \) in \( L + iL \),

(a) \[
\mathcal{O}_1 \cdots \mathcal{O}_r \mathcal{O}_{r+1} = \mathcal{O}_1 \cdots \mathcal{O}_r \mathcal{O}_{r+1} + \sum_{1 \leq j \leq r} \mathcal{O}_1 \cdots \mathcal{O}_j \cdots z_i \cdots \mathcal{O}_r \mathcal{O}_{r+1} E(z_i z_{r+1})
\]

(b) \[
z_1 \cdots z_r = z_1 \cdots z_r + \sum_{i \leq j} z_1 \cdots \hat{z}_i \cdots \hat{z}_j \cdots z_r \mathcal{O}_{z_i z_j} + \sum_{i_1 < i_2 < j_2} \cdots z_1 \cdots z_{i_1} \cdots z_{i_2} \cdots z_{j_2} \cdots z_r \mathcal{O}_{z_{i_1}, z_{i_2}} \mathcal{O}_{z_{j_1}, z_{j_2}} E(z_{i_1} z_{j_1}) + \cdots + \sum_{i_1 < j_1, \ldots, i_s < j_s} \mathcal{O}_1 \cdots \mathcal{O}_s \mathcal{O}_{z_{i_1}, z_{j_1}} \cdots \mathcal{O}_{z_{i_s}, z_{j_s}} E(z_{i_1} z_{j_1}) \cdots E(z_{i_s} z_{j_s}).
\]
where $s$ is the integral part of $r/2$, and $u = e$ when $r$ is even, and otherwise $u = z_k$, where $k \in \{1, \ldots, r\} - \{i_1, j_1, \ldots, i_s, j_o\}$.

**Proof.** To prove (a) by induction, it suffices to show that the expectation values of both sides agree, and that their commutators with an arbitrary element $z \in \mathbf{L} + i\mathbf{L}$, are the same. In case $r = 1$, (a) states that $z_1z_2 = z_1z_2$, which is easily seen to be valid. For $r > 1$, the identity of the expectation values follows from Corollary 7. Now if $z$ is arbitrary in $\mathbf{E}_1$, then denoting the left and right sides of equation (a) as $L$ and $R$, respectively,

$$
[L, z] = z_1 \cdots z_r; [x_{r+1}, z] + \sum_{1 \leq i \leq r} z_1 \cdots \hat{z}_i \cdots z_r; [z_j, z] z_{r+1};
$$

$$
[R, z] = z_1 \cdots z_r; [x_{r+1}, z] + \sum_{1 \leq i \leq r} z_1 \cdots \hat{z}_i \cdots z_r; [x_i, z]
$$

$$
+ \sum_{1 \leq i \leq r} \sum_{1 \leq j \leq r, i \neq j} z_1 \cdots \hat{z}_i \cdots \hat{z}_j \cdots z_r; E(z_j x_{r+1})[x_i, z].
$$

Reversing the order of summation in the double sum and using the induction hypothesis yields the expression

$$
\sum_{1 \leq i \leq r} \{z_1 \cdots \hat{z}_i \cdots z_r; x_{r+1} - z_1 \cdots z_i \cdots z_{r+1};[z_i, z];
$$

combining this expression with the earlier terms in $[R, z]$, and relabeling the summation variable as $j$, the identity with $[L, z]$ follows.4

To prove (b) along similar lines, first note that the identity of $[L, z]$ with $[R, z]$ for arbitrary $z \in \mathbf{E}_1$ follows directly from the induction hypothesis; $[L, z]$ is a sum of $r$ terms over $i$, in the $i$th term $z_i$ being deleted, while the characterization of $u$ in terms of commutators provides a precisely corresponding sum for $[R, z]$.

---

4 Since (a) gives $z_1 \cdots z_{r+1}$ in terms of lower order renormalized products, it provides an alternative characterization for these products. At the same time, it permits specialization to the cases of conventional Wick products by virtue of the occurrence of such recursive equations in the literature of quantum field theory (cf. [9], p. 163, (16.18)).

In the latter reference, the expression "$A_1 \cdots A_n B$" should be replaced by the expression "$A_1 \cdots A_n B$". A different approach to Wick products employing a definition due to Caienello, is given in [10]. A similar recursion formula is derived for the case of a "free scalar field"; see equation (3.35), where however in the expression "$\phi(g_1) \cdots \phi(g_b) \cdots \phi(g_l)\$", $l$ should be replaced by $l - 1$. 
It remains only to show that \( E(z_1 \cdots z_r) = 0 \) if \( r \) is odd, while if \( r = 2s, \) \( s \) integral,
\[
E(z_1 \cdots z_r) = \sum_{k=1}^{s} \prod_{i=1}^{k} E(z_i z_i),
\]
the sum being taken over the set of all exhaustive collections of mutually disjoint 2-element subsets of \( 1, \ldots, r. \) The vanishing of \( E(z_1 \cdots z_r) \) for \( r \) odd follows from the invariance of \( E \) under the induced action of the unitary transformation \( z \to -z, \) which transforms \( E(z_1 \cdots z_r) \) into its negative. When \( r \) is even, the application of the induction hypothesis to the representation of \( z_1 \cdots z_{r-1} \) as a sum of renormalized products, together with the application of the result (a) to the effect of multiplication by \( z_r, \) shows that the only nonvanishing contributions to \( E(z_1 \cdots z_r) \) in its corresponding representation arise from the last term. The last term in \( z_1 \cdots z_{r-1} \) is
\[
\sum_{j=1}^{r-1} \sum' z_j \prod_{k=1}^{s-1} E(z_{i_k} z_{j_k}),
\]
where \( j \) is different from all the \( i_k \) and \( j_k \), and \( \sum' \) is taken over the set of all exhaustive collections of mutually disjoint 2-element subsets of \( 1, 2, \ldots, r - 1 - j. \) On multiplication with \( z_k \) and application of \( E, \) the stated conclusion results.

2. Denseness of the Domains of Renormalized Products

It has now been shown that the renormalized product is well-defined and unique on a certain domain, under fairly general conditions, but the precise extent of this domain has been left open. In this section a readily applicable condition for the density of this domain will be given, and further relevant properties developed, for a class of processes associated with given locally compact abelian groups. Among such processes are certain which may be described heuristically as those describing the neutral scalar antisymmetric quantum field, satisfying an appropriate linear partial differential equation with constant coefficients, at a fixed time; and in particular, the density and other properties will follow for the scalar relativistic field in two space-time dimensions. This section is concerned solely with the “static” situation, which it is necessary to make precise before the dynamical equations can be given mathematical meaning. In order to give the subject an appropriate general setting, one may make the
DEFINITION 2.1. Let $\Omega = (L, A, G, V)$ be a system consisting of a real linear topological vector space $L$, a real continuous non-degenerate anti-symmetric bilinear form $A$ on $L$, a given topological group $G$, and a given continuous linear representation $V$ of $G$ on $L$, by transformations preserving the form $A$; such a system may be called a covariant classical (linear) structure. A covariant symmetric process over $\Omega$ is defined as a system $(\Psi, K, v, \Gamma)$ such that $(\Psi, K, v)$ is a symmetric process over $(L, A)$ in the sense earlier indicated, while $\Gamma$ is a continuous unitary representation of $G$ on $K$ having the properties

$$\Gamma(a) v = v, \quad \Gamma(a) \Psi(x) \Gamma(a)^{-1} = \Psi(V(a) x) \quad (a \in G, x \in L).$$

Now let $M$ be a real locally convex linear topological vector space, and suppose there is given a continuous representation $U$ by invertible continuous linear operators of a given group $G$. If $M^*$ denotes the dual of $M$, if $L$ is the direct sum $M \oplus M^*$, if $A$ is defined as the form on $L$ given by the equation

$$A(x \oplus f, x' \oplus f') = f'(x) - f(x),$$

and if $V(a) = U(a) \oplus U(a)^{-1}$, then $\Omega = (L, A, G, V)$ is a covariant classical structure; and a covariant anti-symmetric process over $\Omega$ will be said to be built on the system $(M, G, U)$; when $G$ and $U$ are trivial, $\Omega$ is said to be built on $M$. For such a process $(\Psi, K, v, \Gamma)$, the process $(\Psi \mid M, K, v)$ may be called the basic process, and the process $(\Psi \mid M^*, K, v)$ called the conjugate process. (Note that each of these processes separately is essentially a somewhat structured classical process, i.e. the process operators commute; the quantum features result from the interrelations between the two processes, and in particular the renormalized powers involve this interrelation, as indicated in the first section, either implicitly or explicitly.)

There are many ways of describing a concrete symmetric process. Within unitary equivalence of processes, which is all that is here relevant, the so-called generating functional $\langle e^{i \Psi(x)} v, v \rangle$ provides an economical description for cyclic processes (cf. [II]), and may be used to describe the processes to be considered here.

DEFINITION 2.2. The covariance form of a given operational process $(\Psi, K, v)$ is the form $C(x, y)$ defined on the domain $D$ of all vectors $x$ such that $v$ is in the domain of $\Psi(x)$, by the equation:

$$C(x, y) = \langle \Psi(x)v, \Psi(y)v \rangle.$$

A self-adjoint process is called normal
(or Gaussian) in case there exists a symmetric form $Q$ on the probe space $L$ such that $\langle e^{i\psi(x)} v, v \rangle = \exp[-Q(x, z)/4].$

It is readily verified that the covariance form of such a process has the form $C(x, y) = Q(x, y),$ showing that $Q$ is necessarily positive semi-definite; however, $Q$ is not at all arbitrary among such forms, when $A$ is given, but must satisfy certain nontrivial conditions in order to be the covariance form of a process over $(L, A).$ It is known [II] that there is a unique normal process over $(L, A)$ having a given covariance form $Q$ (if any exists at all) which is cyclic.

In case $L$ has defined on it the structure of a not necessarily complete complex Hilbert space, with inner product $\langle \cdot, \cdot \rangle,$ in such a way that $\langle x, y \rangle = \text{Im} (\langle x, y \rangle), \quad Q(x, y) = \text{Re} (\langle x, y \rangle),$

and the process is cyclic, it is called the isonormal process over $L$ (as a complex pre-Hilbert space). It is known that in this case, if $U$ is an arbitrary unitary operator on $L,$ there exists a unique unitary operator $\Gamma(U)$ on $K$ such that $\Gamma(U)v = v,$ $\Gamma(U)\Psi(x)\Gamma(U)^{-1} = \Psi(Ux);$ and $\Gamma(.)$ is a continuous representation of the group of all unitary operators on $L.$ This representation $\Gamma$ extends uniquely to the full unitary group on the completion $H$ of $L$ by continuity, and the extension will also be denoted as $\Gamma(.)$, the precise domain being either evident from the context or immaterial.

Other normal processes may be derived from the isonormal process, and the following construction will be particularly relevant here. Let $(M, m)$ denote a regular locally compact measure space ($M$ being the locally compact space, and $m$ denoting the measure); let $G$ denote a topological group, and suppose given a continuous action of $G$ on $M,$ leaving the measure $m$ invariant; the notation $gx$ will denote the transform of $x \in M$ by the element $g$ of $G.$ Let $C$ denote a positive self-adjoint operator on $RL_2(M),$ i.e. the $L_2$-space of real functions on $M,$ and let $M$ denote the Hilbert space consisting of the completion of the domain $D_C$ of $C,$ relative to the inner product $\langle x, y \rangle_M = \langle Cx, Cy \rangle.$ Assuming further that $C$ is invariant under the induced action of $G$ on $L_2(M, R),$ i.e. under the orthogonal transformations $U_0(g): f(a) \rightarrow f(g^{-1}a),$ there is a unique continuous orthogonal representation $U(.)$ of $G$ on $M$ which coincides with $U_0(.)$ where both are defined and comparable. There is then a certain covariant symmetric process $\Omega = (\Psi, K, v, \Gamma)$ built on $(M, G, U),$ which may be characterized as follows; it is the unique normal such
process such that if $\Phi$ denotes the basic process and $\Phi^*$ the conjugate process, while $E$ denotes the vacuum state ($E(T) = \langle Tv, v \rangle$), then

$$
2E(\Phi(x) \Phi(y)) = \langle Cx, Cy \rangle, \quad 2E(\Phi^*(x) \Phi^*(y)) = \langle C^{-1}x, C^{-1}y \rangle,
$$

$$
2E(\Phi(x) \Phi^*(y)) = \langle x, y \rangle i.
$$

(Note that there is a material distinction between the basic process and the conjugate process, in that if $\Phi$ and $\Phi^*$ are interchanged, and if $C$ and $C^{-1}$ are interchanged, the sign of $E(\Phi(x) \Phi^*(y))$ must also be changed, as a result of the commutation relation $[\Phi(x), \Phi^*(y)] \subseteq -i\langle x, y \rangle$.) This process will be called the \textit{standard normal process built from $(M, G, C)$}; and $C^2$ will be called the \textit{variance operator} of the process.

The existence and uniqueness of this process both follow from corresponding properties of the isonormal process. More specifically, setting $\Psi_1(x) = \Phi(C^{-1}x)$ and $\Psi_2(x) = \Phi(Cx)$ (making the obvious convention about the extensions of $C^{-1}$ and $C$ to $M$ and $M^*$), and defining $H_C$ as the complex Hilbert space $M \oplus M^*$ with the complex structure

$$
\langle x \oplus f, x' \oplus f' \rangle = \langle Cx, Cx' \rangle + \langle C^{-1}x, C^{-1}x' \rangle + i(f(x') - f'(x)),
$$

the standard normal process built from $(M, G, C)$ appears as the isonormal process over $H_C$, with the action of $g \in G$ being that derived from the representation $\Gamma$ associated with the isonormal process; and conversely, from the isonormal process on this space, the standard normal process derives by reversal of the foregoing. Concerning uniqueness, cf. loc. cit. If $A$ is a dense $G$-invariant linear set in both $D_C$ and $D_{C^{-1}}$, relative to their natural inner products, the process $(\Psi | A \oplus A, K, v)$ will be said to be built on $(A, G, U(\cdot), C)$.

As a consequence of a uniqueness result for the vacuum of certain quantum processes studied in [8], it may be deduced that any mathematical representation of the conventional heuristic notion of "free neutral scalar quantum field," satisfying the general features indicated in [12], may be represented at a fixed time by a standard normal process built from $(M, G, C)$, where $M$ is euclidean space, $G$ is the euclidean group, and $C = (cI - \Delta)^{1/4}$, $\Delta$ being the usual Laplacian and $c$ being a positive constant. In this representation, the conventional
Wick products for this field, at the fixed time in question, may be formally identified with the processes $\Phi^{(m)}$ defined earlier.

No simple condition on the operator $C$ is known in general which insures the existence of a dense set of vectors $f$ for which $\Phi^{(m)}(f)$ is defined. With however the assumption that $M$ is locally compact abelian group, $m$ is Haar measure, and $G$ is the group of all translations and measure-preserving automorphisms of $M$, a readily applicable condition is given below. This covers the conventional case just indicated, as well as common modifications thereof, notably the case in which space is toroidal (described in the heuristic literature as "the imposition of periodic boundary conditions in space"). Note that with the indicated assumption, $C$ is transformed by the Fourier transformation on $M$ into the operation of multiplication by a function on the dual group $M^*$, acting on $RL_x(M^*)$; this function will be denoted as $C(.)$, and will be called the spectral function for $C$ (or for the process).

Quite conceivably, the assumption that the group is commutative may be superfluous, in the light of the duality theory given in [13] and [14]; in particular, the case in which $G$ is a compact non-abelian group will be treated elsewhere in connection with certain only approximately relativistic particle models. To emphasize the group structure, the notation will be changed so that the underlying space is now $G$.

In the definition of renormalized power given earlier in connection with quasi-invariant stochastic processes, the sample space is required to be an algebra, as is natural in relation to the formation of powers. On the other hand, the domain of the conjugate process of a normal process may be extended to a Hilbert space, which does not form an algebra with respect to the relevant multiplication; nor is there in practice a unique natural subalgebra which is dense in this Hilbert space. An appropriate procedure here is to define the renormalized powers relative to a particular choice of subalgebra as sample space, and then to show that resulting normalized powers are unchanged if an appropriate larger algebra is employed in place of the original algebra. This procedure is entirely in keeping with Section 1, but has the feature that the sample space will be dual to the probe space only in a relatively technical topology, whose precise character is largely irrelevant. In order to minimize these technical problems without essential loss of generality in the results, it will be convenient to broaden formally the treatment of renormalized product, and at the same time present the treatment in purely operator-theoretic terms. Equivalent results in terms of stochastic processes follow directly.
on application of commutative spectral theory, in a fashion previously
developed.

**Scholium 2.1.** Let \((\Psi, K, \nu)\) be an irreducible symmetric process
over the classical system \((L, A)\), where \(L\) has the form \(A \oplus A\), \(A\) being
an algebra of real integrable functions on the measure space \((M, m)\),
and \(A\) has the form:

\[ A(f \oplus g, f' \oplus g') = \int (f(a) g'(a) - f'(a) g(a)) \, dm(a). \]

Let \(\Phi(f) = \Psi(f \oplus 0)\) and \(\Phi(f) = \Psi(0 \oplus f)\), and let \(R\) denote the ring
of operators determined by the \(\Phi(f)\), \(f \in A\). Then there exist unique mappings \(\Phi^{(n)}(.) (n = 0, 1, \ldots)\) defined on domains \(A_n \subset A\), having the
properties:

(a) \(A_n\) is linear, \(\Phi^{(n)}(tf) = t\Phi^{(n)}(f)\) if \(t \in \mathbb{R}, t \neq 0\), and \(\Phi^{(n)}(f + g)\)
is the closure of \(\Phi^{(n)}(f) + \Phi^{(n)}(g)\) for arbitrary \(f\) and \(g\) in \(A_n\):

(b) \(\Phi^{(0)}(f) = (\int f) I, I = \text{identity on } K.\)

(c) For arbitrary \(f \in A_n\) and \(g \in A\), \(\Phi_n(f)\) is affiliated with \(R\), and
\(e^{-i\Phi(f)}\Phi^{(n)}(f)e^{i\Phi(f)}\) is the closure of

\[ \Phi^{(n)}(f) + n\Phi^{(n-1)}(fg) + \cdots + \binom{n}{r} \Phi^{(n-r)}(fg^r) + \cdots + \Phi^{(0)}(fg^n) \]

(when the latter is defined).

(d) If \(f \in A\) and there exists a self-adjoint operator \(T\) affiliated with \(R\)
having \(\nu\) in its domain such that \(e^{-i\Phi(f)} Te^{i\Phi(f)}\) is the closure of the following
operator, when it is defined:

\[ T + n\Phi^{(n-1)}(fg) + \cdots + \binom{n}{r} \Phi^{(n-r)}(fg^r) + \cdots + \Phi^{(0)}(fg^n), \]

and such that \(\langle Tv, \nu \rangle = 0\), then \(f \in A_n\), and \(\Phi^{(n)}(f) = T.\)

**Proof.** This is by induction on \(n\). Let it be assumed that
for \(j < n, j \geq 0\), it has been shown that there exist mappings
\(\Phi^{(j)}(.)\) on domains \(A_j\), having the indicated properties. Now define
\(A_n\) to be the set of all \(f \in A\) such that the relation given in (d) holds;
and observe that the operator \(T\) in question is unique. For if \(T'\) is
another such operator, then on subtracting the two relations in
question, and employing the calculus of operators affiliated with the
abelian ring \(R\), (cf. [15]) it results that

\[ e^{-i\Phi(f)}(T - T') e^{i\Phi(f)} = T - T' \quad (T - T' \text{ is here the strong difference}). \]
By virtue of the irreducibility of the process, this equation can hold for all \( g \) only if \( T - T' = cf \) for some constant \( c \). From the conditions that \( \langle T_v, v \rangle = \langle T'_v, v \rangle = 0 \), it follows that \( T = T' \). Thus \( \Phi(n)(.) \) is well-defined and has the required properties.

**Definition 2.3.** The process \( \Phi(n)(.) \) described in Scholium 2.1 is called the renormalized \( n \)th power of the basic process \( \Phi \), relative to the given process \((\Psi, \mathbf{K}, \nu)\).

As in the case of the "canonical commutation relations," the relations defining the unbounded operators \( \Phi(n)(f) \) may be effectively replaced by relations which deal only with unitary operators, as in

**Scholium 2.2.** With the hypothesis of Scholium 2.1, there exist unique mappings \( W_n(.) \) having domains \( D_n \subset A \), and range in the set of unitary operators on \( \mathbf{K} \), such that

(a) \( D_n \) is linear, \( W_n(tf) \) is a strongly continuous function of \( t \) for any fixed \( f \in D_n \), and \( W_n(f + g) = W_n(f) W_n(g) \), \( g \) and \( f \) being arbitrary in \( D_n \).

(b) \( W_0(f) = \exp\{if\} \).

(c) For arbitrary \( f \in D_n \) and \( g \in A \),

\[
e^{-i\phi(g)} W_n(f) e^{i\phi(g)} = W_n(f) W_{n-1}(fg) \cdots W_{n-r} \left( \binom{n}{r} f \cdots g \right) \cdots W_0(fg^n),
\]

when all the operators in question here are defined.

(d) If \( f \in A \) and there exists a continuous one-parameter unitary group on \( \mathbf{K} \), \( S(.) \), in the ring of operators determined by the \( \Phi(h), h \in A \), such that

\[
e^{-i\phi(g)} S(t) e^{i\phi(g)} = S(t) W_{n-1}(fg) \cdots W_{n-r} \left( \binom{n}{r} f \cdots g \right) \cdots W_0(fg^n), \quad t \in R^1,
\]

whenever the right side of the foregoing equation is defined, and such that \( \langle S(t)v, v \rangle = 1, t \in R^1 \), then \( f \in A_n \) and \( W_n(f) = S(1) \).

Furthermore, the domain \( D_n \) is the same as the domain \( A_n \) of Scholium 2.1, and \( W_n(f) = \exp(i\Phi(n)(f)), f \in A_n \).

**Proof.** Proceeding by induction as before, consider the set \( D_n \) of all elements \( f \in A \) such that a one-parameter group, say \( S(t) = S_f(t), t \in R^1 \), exists as in (d). Then this group is unique, for if \( S'(t) \) is another such group, it is readily verified that \( S(t) S'(t)^{-1} \) commutes with all \( e^{i\phi(g)} \), and being in the ring \( R \) generated by the
bounded functions of the \( \Phi(h) \), must be a scalar by virtue of the irreducibility of the process. Hence \( S'(t) = cS(t) \) for some constant \( c \), which together with the condition that \( \langle S(t)v, v \rangle = 1 = \langle S'(t)v, v \rangle \) implies that \( S(.) = S'(.) \). Now defining \( W_n(f) = S(1) \) for any such element \( f \), it is straightforward to verify the remaining conclusions. It is easily seen from this proof and the calculus of closed operators affiliated with an abelian ring that \( D_n \) and \( W_n(f) \) have the indicated forms.

**Theorem 2.1.** Let \( G \) be a locally compact abelian group, and let \( C \) denote a nonnegative self-adjoint operator in \( RL_2(G) \) which is invariant under the regular representation \( U(.) \) of \( G \): \( U(a)f = f(a^{-1}x) \). Suppose also that \( C \) annihilates no nonzero vector, and that the spectral function of \( C^{-2} \) lies in \( L_p(G^*) \) for all \( p > 1 \). Let \( A \) denote the algebra of all real functions on \( G \) whose Fourier transforms are in all the spaces \( L_p(G^*) \) for \( p \geq 1 \).

Then the renormalized nth power \( \Phi^{(n)}(.) \) of the basic process for the process built on \( (A, G, U(.), C) \) has in its domain the space \( L_2(G) \).

**Proof.** Let \( R \) denote the ring of operators on \( K \)—where the process under consideration is \( (\Psi, K, v) \)—generated by the bounded functions of the \( \Phi(f) \), for all \( f \). Let \( E \) denote the functional on \( R \) given by the equation \( E(T) = \langle Tv, v \rangle \). Now \( v \) is cyclic for \( R \), as follows from the simultaneous diagonalization of \( R \) provided by the real wave representation for the isonormal process; according to this representation, \( K \) is unitarily equivalent to \( L_2(\mathbb{H}_r) \), where \( \mathbb{H}_r \) is any real subspace of the underlying complex Hilbert space \( \mathbb{H} \) such that as a real space, \( \mathbb{H} = \mathbb{H}_r + i\mathbb{H}_r \), in such a way the \( \Psi(x) \) for \( x \in \mathbb{H}_r \) are represented as the multiplication operators \( F(y) \rightarrow c \langle y, x \rangle F(y) \), where \( c \) is a numerical constant and \( \langle y, x \rangle \) refers to the inner product in \( \mathbb{H}_r \); since the functionals \( y \rightarrow \langle y, x \rangle \) separate the probability measure space in question here (cf. [J]), their bounded functions generate the multiplication algebra of the measure space, and in particular the function identically \( 1 \) on the space, which corresponds in the unitary equivalence to \( v \), is cyclic for this algebra.

The idea of the proof is to construct the \( \Phi^{(n)}(f) \) as limits of operators affiliated with \( R \). Because of the isomorphism between \( R \) and \( L_\infty(\mathbb{H}_r) \), all the operators relevant here could be represented as (abstract-Lebesgue) measurable functions on an associated measure space, and the limit taken in appropriate \( L_p \)-spaces with respect to this measure space. It will however be algebraically more straightforward, and more clearly invariant, to work directly in spaces of operators.
affiliated with \( R \), employing the theory given in [15]. According to
this theory, for any "gage" \( E \) on a ring of operators \( N \), there is an
associated space \( L_p(R, E) \) which is quite analogous to a conventional
\( L_p \)-space \( (1 \leq p \leq \infty) \); the elements of \( L_p(R, E) \) consist of closed
densely defined operators on \( K \) which are affiliated with \( R \).

For \( y \in G \), let \( U(y) \) denote the induced action on \( H \) of the translation
\( x \rightarrow x - y \) on \( G \), and let \( \Gamma(U(y)) \) denote the corresponding unitary
operator on \( K \). Then \( \Gamma(U(y)) \Phi(g) \Gamma(U(y))^{-1} = \Phi(g_y) \),
so that \( \Gamma(U(y)) \Phi(g_y)^* \Gamma(U(y))^{-1} = \Phi(g_y)^* \). Since \( \Gamma(U(y))v = v \),
the automorphism \( a_y \) on \( R \) which is given by the equation
\( a_y(X) = \Gamma(U(y)) X \Gamma(U(y))^{-1} \) leaves \( E \) invariant. It follows that the
induced action of \( a_y \) on \( L_p(R, E) \), for \( p < \infty \), is continuous, and hence
that \( \Phi(g_y)^* \) is continuous as a function of \( y \) with values in \( L_p(R, E) \),
for any positive integer \( r \), and for \( p < \infty \). It follows that \( \Phi(g_y)^* \) :is
also a continuous function from \( G \) into \( L_p(R, E) \), and hence that the integral
\( \int \Phi(g_y)^* f(y) \, dy \) exists as a strong integral for an
\( L_p(R, E) \)-valued functions, provided \( f \in L_1(G) \). The idea of the proof
is to obtain \( \int \Phi(g)^* f(y) \, dy \), i.e. \( \Phi^{(n)}(f) \), as the limit in \( L_2(A, E) \)
of \( \int \Phi(g_y)^* f(y) \, dy \), as \( g \) approaches the "delta function." On the
other hand, it will be convenient to work in the representation in which
\( C \) is diagonalized. Let \( \hat{\Phi} \) denote the process whose domain is the set \( \hat{A} \)
of all Fourier transforms \( j \) of elements \( f \in A \), and such that
\( \hat{\Phi}(f) = \Phi(f) \). The symbolic notation

\[ \Phi(F) = \int_{G^*} F(k) \phi(k) \, dk \]

(the element of measure being that of Haar measure on \( G^* \)) is employed in the sense earlier indicated.

Let \( N \) denote the complex Hilbert space \( L_q(R, E) \), and consider
the norm in \( N \) of \( u(g) - u(g') \), where

\[ u(g) = \int \Phi(g_y)^* f(y) \, dy, \]

\( g \) being a given integrable function on \( G \), assumed for simplicity to
have the property that \( g(x) = g(-x) \), as is no essential loss
of generality in the present connection, and \( f \) is a given real function
in \( L_1(G) \cap L_q(G) \). Evidently,

\[ \| u(g) - u(g') \|^2 = \langle u(g), u(g) \rangle + \langle u(g'), u(g') \rangle \]

\[ - \langle u(g), u(g') \rangle - \langle u(g'), u(g) \rangle. \]
Now \( \langle u(g), u(h) \rangle = \int \langle \Phi(g_y)^n, \Phi(h_y)^n \rangle f(y) f(y') dy dy' \) in view of the circumstance that the integral defining \( u(g) \) is convergent in \( L_q(\mathbb{R}, E) \). By Theorem 1.3, \( E(z^n : z'^n) = n! E(zz')^n \), employing the notation of that theorem, so that

\[
\langle \Phi(g_y)^n, \Phi(h_y)^n \rangle = E(\Phi(g_y)^n : \Phi(h_y)^n)
\]

\[
= n! \left( \int \hat{g}(k) e^{ik\cdot y} \hat{h}(k) e^{-ik\cdot y'} b(k) dk \right)^n (b = C(\cdot)^g),
\]

= \( A(y - y')^n n! \), where \( A \) is the Fourier transform of \( \hat{g} \hat{h} b \). The latter function, say \( a \), is in \( L_p(G^*) \) for all \( p > 1 \), in as much as \( \hat{g} \) and \( \hat{h} \) are bounded, being Fourier transforms of integrable functions, and \( b \) is assumed to be in all the \( L_p(G^*) \) for \( p > 1 \).

By the Hausdorff–Young theorem, the \( n \)-fold convolution of \( a \) with itself is again in all \( L_p(G^*) \), \( p > 1 \); by \( L_p \)-Fourier transform theory, the Fourier transform of this convolution is \( A^n \). Thus

\[
\langle u(g), u(h) \rangle = \int A(y - y')^n f(y) f(y') dy dy',
\]

where \( A^n \) is the Fourier transform of a function in all \( L_p(G^*) \). The convolution \( A(y - y')^n f(y) dy \) is by another application of the Hausdorff–Young theorem in all \( L_p(G) \), \( p \geq 2 \), in as much as \( A^n \) is in all \( L_q(G) \) for \( q \geq 2 \) and \( f \in L_1(G) \). Since \( \langle u(g), u(h) \rangle \) is the inner product of this convolution with \( f \), it results from the Plancherel theorem that

\[
\langle u(g), u(h) \rangle = \int (a * \cdots * a)(Y) |\hat{f}(Y)|^2 dY.
\]

Applying this result to the evaluation of \( \| u(g) - u(g') \|^2 \), it follows that

\[
\| u(g) - u(g') \|^2 = \int \left[ (b | \hat{g} |^2)^{(n)} + (b | \hat{g}' |^2)^{(n)} \right.
\]

\[
- 2(b \hat{g} \hat{g}')^{(n)}(Y) |\hat{f}(Y)|^2 dY
\]

where the notation \( p^{(n)} \) indicates the \( n \)-fold convolution of \( p \) with itself. (\( p^{(1)} = p, p^{(2)} = p * p \), etc.)

Now as \( g, g' \to \delta, \hat{g} \) and \( \hat{g}' \to 1 \) uniformly on every compact subset of \( G^* \), and are in addition uniformly bounded by 1. The expression
say $U(Y)$ in square brackets under the integral sign in the foregoing equation, may be expressed as

$$U(Y) = \int_{G^{(n-1)}} b(Y - Y_1) b(Y_1 - Y_2) \cdots$$

$$\times b(Y_{n-2} - Y_{n-1}) b(Y_{n-1}) D(Y, Y_1, \ldots, Y_{n-1}) dY_1 \cdots dY_{n-1},$$

where

$$D = F_{g^2} + F_{g^2} - 2F_{g^2}$$

and

$$F_h(Y, Y_1, \ldots, Y_{n-1}) = h(Y - Y_1) \cdots h(Y_{n-2} - Y_{n-1}) h(Y_{n-1}).$$

Now $|U(Y)| \leq 2b^{(n)}(Y)$, by a direct estimate, so that the integrand in the expression for $||u(g) - u(g')||^2$ is dominated by $b^{(n)}(Y)|f(Y)|^2$, which is a fixed integrable function, in view of the circumstance that $b^{(n)}$ is in all $L_p(G^*)$, $p > 1$, and $f$ is in all $L_q(G^*)$ for $q \geq 2$. As $g \to \delta$, $\delta \to 1$ uniformly on every compact subset of $G^*$, from which it follows that $F_{g^2} \to 1$ uniformly on every compact subset of $G^{*n}$, and is in addition bounded by 1. It follows by dominated convergence that $u(g) - u(g') \to 0$.

Now set $\lim_{g \to \delta} \int \Phi(g_y) f(y) dy = \Omega_n(f)$; it will be shown next that $\Omega_n(f)$ has the characteristic properties of $\Phi^{(n)}(f)$. It is evident from the construction that all the $\Omega_n(f)$ are affiliated with the (maximal abelian) ring generated by the $\exp[i\Phi(f)]$. Furthermore, the $\Omega_n(f)$ are self-adjoint, for they are hermitian, being limits of hermitian operators, and normal, as all elements of $L_1(R, E)$. By virtue of the affiliation of the $\Omega_n(f)$ with $R$,

$$e^{i\Phi(h)} \Omega_n(f) e^{-i\Phi(h)} = \Omega_n(f).$$

Furthermore, $\Omega_n$ is linear, relative to the strong operations on the closed operators affiliated with $R$, by virtue of the linearity of the approximating expressions. To complete the identification of $\Omega_n(f)$ with $\Phi^{(n)}(f)$, it is only necessary to show that

$$e^{i\Phi(h)} \Omega_n(f) e^{-i\Phi(h)} = \Omega_n(f) + n\Omega_{n-1}(fh) + \cdots + \binom{n}{m} \Omega_{n-m}(fh^m) + \cdots$$

for all $h \in A$.

To this end note that

$$e^{-i\Phi(h)} = (\Phi(f) + \langle f, h \rangle e^n, \quad n = 1, 2, \ldots$$
from which it follows, observing that \( \Phi(f)^n \): is a polynomial in \( \Phi(f) \), that
\[
e^{-i\phi(h)} :\Phi(f)^n : e^{i\phi(h)} u = \left( \sum_{r=0}^{n} \binom{n}{r} :\Phi(f)^{n-r} : \langle f, h \rangle^r \right) u
\]
if \( u \) is any vector in a domain invariant under \( \Phi(f) \) and the \( e^{i\phi(h)} \). Such a domain \( D \) is provided for example by the bounded vectors for \( \Phi(f) \), i.e. the vectors in the range of the bounded spectral projections on \( \Phi(f) \). This domain is also strongly dense in the sense of [15].

Now the operator \( e^{-i\phi(h)} :\Phi(f)^n : e^{i\phi(h)} \) is affiliated with \( R \), since \( \Phi(f) \) is so affiliated, and transformation by \( e^{-i\phi(h)} \) leaves this ring invariant (since it maps the operators \( \Phi(h) \) which determine the ring into operators affiliated with \( R \)). The same is true of the operator \( \sum_{r=0}^{n} \binom{n}{r} :\Phi(f)^{n-r} : \), which is a sum of operators affiliated with \( R \). Since they agree on a strongly dense domain, their closures are the same.

Now
\[
e^{-i\phi(h)} \int :\Phi(g_{y})^{n} : f(y) dy e^{i\phi(h)} = \int e^{-i\phi(h)} :\Phi(g_{y})^{n} : e^{i\phi(h)} f(y) dy,
\]
by virtue of the

**Lemma 2.1.** Let \( F \) be a continuous and bounded function from the regular measure space \( G \) to \( L_2(\mathbb{R}, E) \); let \( V \) be a unitary operator such that \( VRV^{-1} \subset R \) and \( VF(.)V^{-1} \) is also a continuous and bounded function from \( G \) to \( L_2(\mathbb{R}, E) \). Then if \( k \) is an integrable scalar function on \( G \),
\[
V \int \Gamma(y) k(y) dy V^{-1} = \int VF(y) V^{-1}k(y) dy.
\]

**Proof.** It is convenient to prove first the

**Sublemma.** The operation \( T \rightarrow VTV^{-1} \) (defined on the domain \( D \) of all operators \( T \in L_2(\mathbb{R}, E) \) such that \( VTV^{-1} \) is again in \( L_2(\mathbb{R}, E) \)) is closed as a linear operator in \( L_2(\mathbb{R}, E) \).

**Proof of Sublemma.** Suppose that \( T_n \in D \) \( (n = 1, 2, ...) \), that \( T_n \rightarrow T \) in \( L_2(\mathbb{R}, E) \), and that \( VTV^{-1} \rightarrow T' \) in \( L_2(\mathbb{R}, E) \). Then there exists a subsequence \( T_{n_k} \), such that \( T_{n_k} \rightarrow T \) nearly everywhere (cf. [15]); this means that, given \( \epsilon > 0 \), there exists a sequence \( P_{n_k} \) of projections in \( R \) such that \( P_{n_k}(\epsilon) \uparrow I \) as \( n \uparrow \infty \), and such that \( \|(T_{n_k} - T) \cdot P_{n_k}(\epsilon)\| \rightarrow 0 \). Note that the algebra of closed operators affiliated with \( R \) is invariant under the transformation \( A \rightarrow VATV^{-1} \),
for the spectral projections of these (necessarily normal) operators are carried into the spectral projections of the transformed operators, which are therefore all in \( \mathbb{R} \), which is equivalent to affiliation with \( \mathbb{R} \) in the case of a normal operator. It follows that

\[
\|(VT_{n_1}V^{-1} - I'TV^{-1}) \cdot VP_n(e) V^{-1}\| \to 0;
\]

but the sequence \( VP_n(e)V^{-1} \) has the property of being in \( I' \), and of being in \( \mathbb{R} \), so that it results that \( VT_{n_1}V^{-1} \to VTV^{-1} \) nearly everywhere. On the other hand, \( VT_{n_1}V^{-1} \to T' \) in \( L_2(\mathbb{R}, E) \), so that some subsequence of the \( VT_{n_1}V^{-1} \) is convergent nearly everywhere to \( T' \). It follows that \( T' = VTV^{-1} \), showing that \( T \in \mathcal{D} \), and that the indicated operation is closed as stated.

**Proof of Lemma.** Consider first the case in which \( k \) is continuous on \( G \) and has compact support. Then \( \int F(y) k(y) \, dy \) exists as a Riemann integral, say as the limit of the finite sums \( \sum F(y_i) k(y_i) m_i \) (\( m_i = \) measure of the small set in question). Now \( \sum VF(y, V^{-1}k(y, m) \) is an approximating Riemann sum for the integral \( \int VF(y) V^{-1}k(y) \, dy \), which likewise exists as a Riemann integral, and so converges to the latter integral as the original Riemann sum converges to the former integral. It follows from the sublemma that \( \int F(y) k(y) \, dy \) is in the domain of the transformation \( T \to VTV^{-1} \) in \( L_2(\mathbb{R}, E) \), and that \( V \int F(y) k(y) \, dy V^{-1} \to \int VF(y) V^{-1}k(y) \, dy \).

If now \( k \) is an arbitrary integrable scalar function on \( G \), there is a sequence \( k_n \) of continuous functions of compact support converging to it in \( L_1(G) \). By a simple estimate, \( \int F(y) k_n(y) \, dy \to \int F(y) k(y) \, dy \) and \( \int VF(y) V^{-1}k_n(y) \, dy \to \int VF(y) V^{-1}k(y) \, dy \) (both limits in \( L_2(\mathbb{R}, E) \)). Another application of the sublemma now leads to the conclusion of the lemma.

Consider now the equality stated just before the statement of the lemma. As already observed, \( :\Phi(g_y)^n : \) is a continuous function of \( y \), as a function into \( L_2(\mathbb{R}, E) \). On the other hand, by the expression for \( V :\Phi(g_y)^n : V^{-1} \), where \( V = e^{-ih} \), derived earlier, this is a finite linear sum of continuous maps of \( G \) into \( L_2(\mathbb{R}, E) \), and hence is itself such. Thus the lemma is applicable, and the specialization of its conclusion is just the stated equality.

It remains to show that

\[
\int :\Phi(g_y)^n \cdot h(y)^r f(y) \, dy - \int :\Phi(g_y)^n \cdot h(y)^r f(y) \, dy \to 0
\]
in $L_2(\mathbb{R}, E)$, as $g \to \delta$. On setting $\langle g_y, h \rangle \delta f(y) = F_\delta(y)$, the difference whose convergence is in question may be written as

$$\int \Phi(g_y)^{n-r} F_\delta(y) \, dy - \int \Phi(g_y)^{n-r} F_\delta(y) \, dy + \int \Phi(g_y)^{n-r} F_\delta(y) \, dy - \int \Phi(y)^{n-r} F_\delta(y) \, dy,$$

where $F_\delta(y) = h(y)\delta f(y)$. The second term of this difference has already been estimated and shown to converge to zero. The first term has the form

$$\int \Phi(g_y)^{n-r} (F_\delta(y) - F_\delta(y)) \, dy,$$

the norm of which in $L_2(\mathbb{R}, E)$ is

$$\int \int E^{(r)}(\Phi(g_y)^{n-r}) (F_\delta(y) - F_\delta(y))(F_\delta(y') - F_\delta(y')) \, dy \, dy'.$$

This is similar to terms treated in the earlier part of the proof; by a similar analysis, it reduces to a finite sum of terms of the form

$$\int (b * \cdots * b)(k) \| F_\delta(k) - \hat{F}_\delta(k) \|^2 \, dk,$$

where the indicated convolution is $(n-r)$-fold. This convolution lies in $L_p(\mathbb{G}^*)$ for all $p > 1$, in particular in $L_2(\mathbb{G}^*)$, and so to complete the proof it suffices to show that $\| \hat{F}_\delta(\cdot) - \hat{F}_\delta(\cdot) \|^2 \to 0$ in $L_2(\mathbb{G}^*)$, or equivalently that $| \hat{F}_\delta(\cdot) - \hat{F}_\delta(\cdot) | \to 0$ in $L_1(\mathbb{G}^*)$.

Now $\langle g_y, h \rangle = (g * h)(y)$, so that

$$\hat{F}_g = f * (\hat{h} * \cdots * \hat{h}),$$

the convolution in question being $r$-fold. It follows that

$$|| \hat{F}_g - \hat{F}_\delta ||_1 \leq || \hat{f} ||_1 || \hat{h} * \cdots * \hat{h} ||_1 - \hat{h} * \cdots * \hat{h} ||_1.$$

Since convolution is a continuous operation on $L_1(\mathbb{G}^*)$, it suffices now to show that $\hat{h} \to \hat{h}$ in $L_1(\mathbb{G}^*)$ as $g \to \delta$, and this follows from the fact that the $\hat{g}$ are uniformly bounded by 1, and converge uniformly on every compact set to 1.

Various properties of the renormalized powers are important in applications. I begin with the locality.
**Definition 2.4.** If $H$ is any open subset of $G$, $R_H$ denotes the ring of operators determined by the $\Phi(h)$, as $h$ ranges over the set of elements of $A$ the closure of whose support is contained in $H$.

**Corollary 2.1.** The process $\Phi^{(n)}$ has the property that if $f$ is an element of $L_1(G) \wedge L_2(G)$ which is supported by the measurable subset $K$ of $G$, and if $N$ is an arbitrary neighborhood of the unit $e$ in $G$, then $\Phi^{(n)}(f)$ is affiliated with the ring of operators determined by the $\Phi(h)$, as $h$ ranges over the elements of $A$ which are supported by $KN$.

**Proof.** The proof showed that $\Phi^{(n)}(f)$ is the limit in $L_2(\mathbb{R}, E)$ of $\int :\Phi(g_y)^n : f(y) \, dy$, as $g \to \delta$. Now if $g$ is chosen to have support in $N$, then $g_y$ has support in $KN$, provided $y \in K$; and other values of $y$ do not contribute to the integral. Now for arbitrary $h \in A$, having support interior to the open set $H$, $\Phi(h)$ $\eta_R_H$, from which it follows that $\Phi(h)^n : \eta_R_N$ $(n = 1, 2, \ldots)$; in particular, excluding values of $y$ which do not contribute to the integral, $\Phi(g_y)^n : \eta_R_{KN}$. It follows directly that $\int :\Phi(g_y)^n : f(y) \, dy \, \eta_R_H$, for as a strong Banach-valued integral, the value is contained in the span of the range of the integrand; and the subset of $L_2(\mathbb{R}, E)$ which is affiliated with any given subring of $R$ is closed by the general theory of these spaces.

By the extended automorphism group of the group $G$ I shall mean the group of transformations on $G$ generated by: (a) translations, $x \to xa$ ($a$ fixed in $G$), (b) measure-preserving automorphisms of $G$. For the given locally compact abelian group $G$, let $G^e$ denote the extended automorphism group, and let $U$ denote the unitary representation of $G^e$ on $L_2(G)$ given by the action, $U(l); f(x) \to f(l^{-1}x)$. Another significant property of the renormalized powers just treated are their invariance under the induced action on the quantum process of the representation $U(.)$ of $G^e$.

**Corollary 2.2.** $\Gamma(U(l)) \Phi^{(n)}(f) \Gamma(U(l))^{-1} = \Phi^{(n)}(U(l)f)$, for arbitrary $l \in G^e, f \in L_1(G) \wedge L_2(G)$, and $n = 1, 2, \ldots$.

**Proof.** Since the $\Gamma(U(l))$ leave invariant the state $E$ (as do the $\Gamma(V)$ for arbitrary unitary operators), the transformation $X \to \Gamma(U(l)) X \Gamma(U(l))^{-1}$ is unitary on $L_2(\mathbb{R}, E)$. Now the problem is to show that

$$\lim_{g \to \delta} \Gamma(U(l)) \int :\Phi(g_y)^n : f(y) \, dy \, \Gamma(U(l))^{-1} = \lim_{g \to \delta} \int :\Phi(g_y)^n : (U(l)f)(y) \, dy.$$ 

There is no difficulty in passing the transformation by $\Gamma(U(l))$ under
the integral sign on the left; the renormalization operation : : is invariant under transformation by an operator of the form \( \Gamma(V) \), for unitary \( V \), since this conserves brackets and the state \( E \), in terms of which renormalized products are uniquely defined. The left side of \( \Gamma \) the foregoing equation is therefore identical to the integral

\[
\int :\Phi(\Gamma(U(l)) \Phi(g) \eta_\Gamma \Gamma(\Gamma(U(l))^{-1} f(y) dy,\] (1)

which in turn equals

\[
\int :\Phi(U(l) g)^n(y) dy,
\]

It suffices to treat the cases in which \( \Gamma \) is either a translation or a measure-preserving automorphism. If \( \Gamma \) is the translation \( x \to x + a \), writing the group additively, then the last integral is

\[
\int :\Phi(g_{y,p})^n f(y) dy = \int :\Phi(g_p)^n f(y - a) dy,
\]

which is identical with the integral on the right in the equation in question. If on the other hand, \( \Gamma \) is a measure-preserving automorphism, then

\[
\int :\Phi(U(l) g)^n f(y) dy = \int :\Phi(g(l^{-1}x - y))^n f(y) dy
\]

where \( x \) is a bound variable. The last expression is the same as

\[
\int :\Phi(U(l) g)^n :U(l) f(y) dy;\] (2)

but as \( g \to \delta \), \( U(l) g \to \delta \) for each \( l \), so that the left side of the equation above is (after the passage to the limit \( g \to \delta \)) \( \Phi^{(n)}(U(l) f) \), in agreement with the right side.

In order to treat differential equations involving the renormalized powers, various estimates of them are required. A simple but useful one is given by

**Corollary 2.3.** With the notation of Theorem 2.1, \( \Phi^{(n)}(f) \in L_p(\mathbb{R}, E) \) for all \( p \in [1, \infty) \).

**Proof.** It evidently suffices to show that \( \text{E}(\{\Phi^{(n)}(f)\})^{2r} < \infty \) for all positive integral \( r \). To this end, note that according to Fatou's lemma, if \( f_n \to f \) in \( L_p \), then \( \int |f|^p \leq \limsup_n \int |f_n|^p; \) as a consequence,

\[
\text{E}(\{\Phi^{(n)}(f)\})^{2r} \leq \limsup_{r \to \infty} E\left(\left(\int :\Phi(g)^n :f(y) dy\right)^{2r}\right),
\]
showing that it suffices to show the uniform boundedness in $g$ of $F((\int \Phi(g_\nu)^n \cdot f(y) \, dy)^{2n})$.

Observe next that in the case of an isonormal process $F$ over a Hilbert space $K$, the map $x \mapsto F(x)^n$ is continuous from $H$ into $L_p(H, F)$, for all positive integral $n$. For $\|F(x)^n\|^2 \leq \text{const} \|x\|^n$ by a simple estimate deriving from the unitary invariance of the isonormal process and the finiteness of the $n$th moment of a one-dimensional normal probability distribution. Now

$$F(x)^n - F(y)^n = (F(x) - F(y)) \sum_{i=0}^{n-1} F(x)^i F(y)^{n-i};$$

by Hölder's inequality,

$$\|F(x)^n - F(y)^n\|_2 \leq \|F(x) - F(y)\|_4 \sum_{i=0}^{n-1} \|F(x)^i F(y)^{n-i}\|_4;$$

on the other hand, $\|F(x) - F(y)\|_4 = \|F(x - y)\|_4 \leq \text{const} \|x - y\|$ from the inequality noted, while

$$\|F(x)^i F(y)^{n-i}\|_4 \leq \|F(x)^i\|_8 \|F(y)^{n-i}\|_8 \leq \text{const} \|x\|^i \|y\|^{n-i}$$

by the same inequality.

It follows as a matter of integration theory that $F(x) \in L_p(H, F)$ for all $p \in [1, \infty)$, and that the mapping $x \mapsto F(x)^m$ is a continuous mapping from $H$ into $L_p(H, F)$, for $p \in [1, \infty)$. Now $F(x)^n$ is a finite linear combination of the $F(x)^m$ for $m \leq n$, with coefficients which are positive integral powers of $\|x\|$ (cf. Theorem 1.3). It follows that the map $x \mapsto F(x)^n$ is also a continuous mapping from $H$ into $L_p(H, F)$, for all $p \in [1, \infty)$. The map

$$(x_1, \ldots, x_t) \mapsto F(x_1)^n : F(x_2)^n : \cdots : F(x_t)^n;$$

similarly carries the $n$-fold direct sum of $H$ with itself continuously into $L_p(H, F)$, by virtue of the fact (readily deduced from Hölder's inequality) that for any measure space $(S, \mathcal{S})$, the common part of the $L_p(S, \mathcal{S})$ for $p \in [1, \infty)$ forms a topological algebra, in the topology in which $f_n \to f$ if this is true in every $L_p$-norm, $p \in [1, \infty)$. (This algebra will be denoted as $L_{[1, \infty)}(S, \mathcal{S})$.)

Now $\Phi$, as the basic process of a normal anti-symmetric process, with covariance operator $C^{-2}$, can be identified with the restriction of the isonormal process in the space $K$, consisting of $RL_2(G)$ with the revised inner product: $\langle x, y \rangle' = \langle C^{-1}x, C^{-1}y \rangle$, to the dense
subspace $A$. The mapping $y \rightarrow g_y$ is continuous from $G$ into $K$ for $g_y(k) = \hat{g}(k)e^{ik\cdot y}$, so that

$$\|g_y - g\|_{D_{C-1}} = \int |e^{ik\cdot y} - 1|^2 |\hat{g}(k)|^2 b(k)^{-1} \, dk \to 0 \quad \text{as} \quad y \to 0,$$

since $|\hat{g}(k)|^2 b(k)^{-1}$ is integrable. Hence, composing the map $y \rightarrow g_y$ with the isonormal process, it follows that the map

$$(y_1, \ldots, y_t) \rightarrow \Phi(g_{y_1})^n : \cdots : \Phi(g_{y_t})^n;$$

is continuous from $G^t$ into $L_1(\mathbb{R}, E)$, in the indicated topology; in particular it is continuous into $L_2$, from which it follows that

$$\int \Phi(g_{y_1})^n : \cdots : \Phi(g_{y_t})^n : f(y_1) \cdots f(y_t) \, dy_1 \cdots dy_t$$

exists as a vector-valued integral of a function from $G \times \cdots \times G$ into $L_2(\mathbb{R}, E)$, and has expectation (a continuous linear functional on $L_2(\mathbb{R}, E)$) equal to

$$\int E\left(\Phi(g_{y_1})^n : \cdots : \Phi(g_{y_t})^n : f(y_1) \cdots f(y_t) \, dy_1 \cdots dy_t\right).$$

At this point, the following observation is relevant:

**Sublemma.** Let $P$ be a given probability measure space; let $(M, m)$ be a given regular locally compact measure space, let $f$ and $g$ be continuous and bounded functions from $M$ into $L_1(P)$ and $L_1(P)$, respectively; and suppose that $\gamma^{-1} = \alpha^{-1} + \beta^{-1}$, $\gamma \in [1, \infty]$. Then the integral

$$\int \int f(a) g(b) h(a) k(b) \, da \, db \quad (h, k \in L_1(M, m))$$

exists as a strong vector-valued integral of a function on $(M, m)^2$ to $L_{\gamma}(P)$, and equals $\int f(a) h(a) \, da \int g(a) k(a) \, da$.

**Proof of Sublemma.** If $f$ and $g$ are replaced by simple functions, i.e. finite linear combinations of characteristic functions of measurable sets in $(M, m)$ of finite measure, whose coefficients are in $L_1(P)$ and $L_1(P)$, respectively, the indicated conclusion is easily verified. Now it is no essential loss of generality to suppose that $M$ is sigma-compact, for $h$ and $k$ are supported by a fixed such set, and $M$ may be replaced by this set without any essential change. Making then this supposition, it is well-known that $f$ and $g$ are limits of uniformly bounded sequences of simple functions. Applying the indicated conclusion to the elements of these sequences, it is straightforward to pass to the limit on both sides of the equality, obtaining the stated conclusion.

It follows that if the hypothesis of the Sublemma is valid for all $\alpha$ and $\beta$ in $[1, \infty)$, then the conclusion is valid for all $\gamma \in [1, \infty)$. By
repeated application of this result, it remains valid for any finite number of factors. It follows in particular that

$$\int \Phi(g_u)^n \cdots \Phi(g_u)^n : f(y_1) \cdots f(y_i) \, dy_1 \cdots dy_i = \left( \int \Phi(g_u)^n : f(y) \, dy \right)^4.$$

Thus, it suffices to show that $\int E(\Phi(g_u)^n) \cdots \Phi(g_u)^n : \Pi f(y_i) \, dy_i$ remains bounded as $g \to \delta$.

Now observe that, by Theorem 1.3, and with the same notation,

$$E(z_1^n \cdots z_t^n) = \sum_{q(i,j) \in Q} \prod_{i,j} E(z_{i,j})^q_{(i,j)},$$

where $Q$ is the set of all functions $q(i,j)$ having nonnegative integral values, defined for $i, j = 1, 2, \ldots, t$ and $i \neq j$, and such that $\Sigma_i q(i,j) = n$ for all $j$, and $q(i,j) = q(j,i)$. It is evident that $Q$ is a finite set, and it therefore suffices to show that any individual term in the corresponding expression for $\int E(\Phi(g_u)^n) \cdots \Phi(g_u)^n : \Pi f(y_i) \, dy_i$ remains bounded.

The generic such term has the form

$$\int \prod_{i < j} E(\Phi(g_u), \Phi(g_v)) \Phi(g_u)^q_{(i,j)} \prod_{k} f(y_k) \, dy_k.$$

Now

$$E(\Phi(g_u), \Phi(g_v)) = \int g(h)^2 e^{ih(u - v)} b(h) \, dh = w_q(y_i - y_j),$$

where $w_q$ is the inverse Fourier transform of $g^2 b$. By $L_p$-Fourier transform theory, $\|w_q\|_{p'} \leq \text{const} \|g^2 b\|_p$ for $1 \leq p \leq 2$; since $g^2 \leq 1$, this means that $\|w_q\|_{p'}$ remains bounded as $g \to \delta$ for all sufficiently large $p' < \infty$. From Hölder's inequality, it follows that the same is true of $\|w_q^q\|_{p''}$, for any positive integer $q$, say $q = q(i,j)$ in the special integral under consideration. This means that the term in question has the form

$$\int \prod_{i < j} w^{q(i,j)}(y_i - y_j) \prod_{k} f(y_k) \, dy_k,$$

where the $w^{q(i,j)}$ are functions which depend on $g$, but either have their
norms in $L_{p'}$, for sufficiently large $p'$, bounded independently of $g$, or are identically one (according as $g(i,j) \neq 0$ or $g(i,j) = 0$). To conclude the proof, it therefore suffices to establish the

**Sublemma.** If $f \in L_{1,\infty}(G)$, and if $w^{ij}$ ranges over a set of functions on $G$ which are in $L_{p'}$ for all sufficiently large $p' > 1$, or alternatively are identically 1, then the integral $\int \prod_{i<j} w^{ij}(y_i - y_j) \prod_k f(y_k) \, dy_k$ remains bounded provided the norms of the $w^{ij}$ in $L_{p'}$ remain bounded, for each sufficiently large $p'$.

**Proof of Sublemma.** This is by induction on $t$. The result is trivial for $t = 1$. Now assuming that the conclusion is valid up to $t$, consider the case of $t + 1$. Taking absolute values, applying the Fubini theorem, and integrating first over $y_t$, the integral takes the form

$$\int \left[ \int | \prod_{1<j} w^{ij}(y_1 - y_j)f(y_1) | \, dy_1 \right] G(y_2, \ldots, y_t) \prod_{k>1} dy_k,$$

where the induction hypothesis assures the boundedness of the integral of $G$. Thus it suffices to show that $\int | \prod_{1<j} w^{ij}(y_1 - y_j)f(y_1) | \, dy_1$ remains uniformly bounded as a function of $y_2, \ldots, y_t$, as the $w^{ij}$ vary in the indicated fashion. To this end, let $m$ be the number of $j$ such that $w^{ij}$ is not identically 1, and for such a $w^{ij}$, let $p_j$ be such that $1 < p_j < \infty$ and $w^{ij} \in L_{p}(G)$ for $p_j \leq p \leq \infty$; finally, set $\alpha$ for the maximum of $m + 1$ and all the $p_j$, and define $\beta$ by the equation $\beta^{-1} = 1 - m\alpha^{-1}$. Then evidently $w^{ij} \in L_{\alpha}(G)$ for all $j$, while $\infty > \beta > 1$, so that $f \in L_{\beta}(G)$. Applying Hölder's inequality, it results that $\int | \prod_{1<j} w^{ij}(y_1 - y_j)f(y_1) | \, dy_1 \leq \prod_{1<j} \| w^{ij} \|_{\omega} \| f \|_{\beta}$, which is the required bound.

**Remark 2.1.** The use of the space $L_{\alpha}(A, E)$, of all square-integrable operators with respect to the gage $E$, could be avoided in the foregoing proof through the use in place of it of the real wave representation. The space $L_{\alpha}(A, E)$ is unitarily equivalent to the Hilbert space $K$, in such a way that the identity operator $I$ corresponds to the vacuum vector $v$; and multiplication operations $X \to AX$ in $L_{\alpha}(A, E)$ (for $A$ given in $A$) correspond to the elements of $A$ itself. Through the use of this unitary equivalence, $\Phi^{(n)}(f)$ could be defined equally well by reduction to a limit in $K$, or in $L_{\alpha}(H_r)$, to which $K$ is equivalent, according to the real wave representation. On balance, neither approach seems significantly simpler than the other, and the approach in terms of $L_{\alpha}(A, E)$ appears to be more compact and susceptible to generalization.
Remark 2.2. The neutral scalar relativistic quantized field in 
(n + 1)-dimensional space-time dimensions corresponds to the 
special case in which \( G \) is the additive group of \( \mathbb{R}^n \), and \( C \) has the form 
\( (m^2I - \Delta)^{1/4} \), \( m \) being the "mass," and here supposed positive. 
It is only when \( n = 1 \) that the associated spectral function 
\( b(k) = (m^2 + k^2)^{-1/2} \) satisfies the condition of being in all the spaces 
\( L_p(G^*) \) for \( p > 1 \). When \( n > 1 \), it is demonstrable that, for example, 
\( \Phi^{(2)}(f) \) exists when \( n > 1 \) only for \( f = 0 \) (cf. [12]). In a certain 
extended sense, in which the values of \( \Phi^{(n)}(f) \) are permitted to be 
suitably generalized operators, the renormalized powers exist also 
for \( n > 1 \); this work will be presented in a separate publication.

The cited result for scalar fields in two-dimensional space-time 
is a special case of the following result applicable to euclidean-
invariant and temporally invariant equations in arbitrary-dimensional 
space-times. Concerning the mathematical formulation of the notion 
of "quantized field associated with a given equation," cf. [12] and [17].

**Corollary 2.4.** Let \( \Delta \) denote the laplacian in \( \mathbb{R}^n \); let \( p \) denote 
a polynomial such that \( p(l) > 0 \) for all \( l > 0 \); let \( \Phi(f, t) \) denote the 
quantum field at time \( t \), averaged over space with respect to the function \( f \), 
associated with the equation 
\[
\frac{\partial^2 \phi}{\partial t^2} + p(-\Delta) \phi = 0.
\]

If \( p(x \cdot x)^{-1/2} \) is in \( L_q(\mathbb{R}^n) \) for all \( q > 1 \), then the renormalized powers 
\( \Phi^{(n)}(f) \) exist for \( f \in L_1 \cap L_2 \), and satisfy the conditions and corollaries 
previously given in connection with Theorem 2.1.

**Proof.** The operator \( C \) relevant to this quantized field is \( p(-\Delta)^{1/4} \). 
The spectral function for \( C^{-2} \) is consequently \( p(x \cdot x)^{-1/2} \).

The analog to this corollary when \( \mathbb{R}^n \) is replaced by toroidal space 
\( T^n \) holds by essentially the same proof.

Remark 2.3. A further generalization of the present results is to 
the case of the dynamical process obtained in the following fashion. 
Let \( M \) denote \( (n + 1) \)-dimensional Minkowskian space-time, and 
let \( \mathcal{H} \) denote the space of all real measurable functions \( f \) on \( M \) whose 
Fourier transforms \( \hat{f} \) exist suitably and vanish outside the dual 
lightcones in the dual space, and are such that the inner product 
\( \langle f_1, f_2 \rangle = \int |\hat{f}(k)|^2 \ w(k) \, dk \) is finite, \( w \) being a given positive weight 
function. Temporal displacement then has a positive generator, 
and for suitable choice of \( w \), the associated unique positive-
energy quantum process (relative to the anti-symmetric form $A(f, g) = \text{Im} \langle f, g \rangle$) will be such that its space averages, for fixed times, form well-defined processes, to which the foregoing theory is applicable. The vacuum vector $\psi$ is no longer necessarily cyclic, as a result of which Remark 1 is no longer applicable, but the process in space which results is a simple direct integral of those considered in Theorem 2.1. The space-time process here described has been called a "generalized free field" in the heuristic literature.

Remark 2.4. Work in a direction partially similar to the present one has been published by A. Jaffe, and further work, especially in the direction of dynamical applications indicated in [16], is presented in preprints by A. Jaffe and J. Glimm, which have kindly been conveyed to me by the authors. For example, "Wick polynomials at fixed times," \textit{J. Math. Phys.} 7 (1966), 1250–1255, treats a case which is formally a particular case ($G = \mathbb{R}^1, C(k) = (m^2 + k^2)^{1/4}$) of the processes to which the present work applies. Comment on the connection between these works and the present one may therefore be in order.

In most of the cited work, essential use is made of definitions and results from the theoretical physical literature. As a result of this and of the foundational direction of the work, its precise mathematical status is not readily apparent. I regret that I have not yet been able to make any comprehensive separation of a mathematically unexceptionable part from the part which rests on theoretical physical considerations.

Some impression of the connection may possibly be afforded by reference to a central result, Theorem 3.1 of "\lambda \phi^4 quantum field theory without cut-offs. I," by Jaffe and Glimm (preprint, 1968). This is to the effect that a certain operator is essentially self-adjoint. The operator in question is formally identifiable with the restriction of a renormalized power as treated in Theorem 2.1 (or in Theorem 1 of [16]) to a certain dense domain. The restriction of a self-adjoint operator to a dense domain is essentially self-adjoint provided the operator and domain are invariant under each of a set of unitary operators whose commutator ring is "finite" (cf. [15]). In the present case the $e^{i\zeta(\rho)}$ provide such a unitary set, so that the cited result of Jaffe and Glimm may be regarded as a corollary to our cited work, modulo the cited identification. On the other hand, this formally valid identification seems necessarily extra-mathematical. For, while referring to the mathematically precise work [10], which treats space-time averaged Wick products using its own definition, Jaffe and Glimm
make implicit use, particularly in the relatively singular case of a fixed time, of the conventional calculus of Wick powers given in the physical literature.

References