Units of Integral Semigroup Rings

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It is proved that both the Bass cyclic and bicyclic units generate a subgroup of finite index in \( \mathbb{Z}S \), assuming \( S \) is a finite semigroup such that \( \mathbb{Q}S \) is semisimple Artinian and does not contain certain types of simple components. © 1996 Academic Press, Inc.

1. INTRODUCTION

It is a fundamental problem to compute the unit group of the integral group ring \( \mathbb{Z}G \) of a finite group \( G \). Recently, Jespers and Leal [3] described finitely many generators for a subgroup of finite index in the unit group of \( \mathbb{Z}G \), for many groups \( G \). To be more precise, generators are given for all finite groups \( G \) which are such that every non-abelian homomorphic image is not fixed point free and, furthermore, the rational group algebra \( \mathbb{Q}G \) has no simple components of the following types:

(i) a \( 2 \times 2 \)-matrix ring over the rationals;
(ii) a \( 2 \times 2 \)-matrix ring over a quadratic imaginary extension of the rationals;
(iii) a \( 2 \times 2 \)-matrix ring over a noncommutative division algebra.

The reason we exclude these types of simple rings is that the congruence subgroup theorems for \( 2 \times 2 \)-matrix rings over maximal orders in the respective division rings fail. For a survey on units of integral group rings and other results on generators of the unit group the reader is referred to [7].

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To prove the above mentioned results, it is essential that $\mathbb{Z}G$ is a $\mathbb{Z}$-order in the semisimple Artinian ring $\mathbb{Q}G$. A natural question therefore is to investigate if the results on group rings can be extended to other $\mathbb{Z}$-orders in semisimple Artinian rings. In this paper, we consider this question for the integral semigroup ring $\mathbb{Z}S$ (with identity) of finite semigroups $S$ such that $\mathbb{Q}S$ is a semisimple Artinian ring. The latter is equivalent with $\mathbb{Q}S$ having a principal series

$$S^0 = S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \{0\},$$

where, for $i = 0, 1, \ldots, n$, each principal factor $S_i / S_{i+1} \cong \mathbb{M}^0(G_i; m_i, m_i; P_i)$ is a completely 0-simple Rees matrix semigroup with regular sandwich matrix $P_i$ (here, $G_i$ is a group and $m_i$ is a positive integer), and moreover, the matrix $P_i$ is invertible in the classical matrix ring $M_{m_i}(\mathbb{Q}G_i)$. In the first part of the paper, we will therefore mainly deal with semigroup rings of a completely 0-simple semigroup. In the second part, we then consider the general case. However, as in the group ring case, we need to exclude the existence of certain simple components of $\mathbb{Q}S$.

As in [3], we consider two types of generators: the Bass-cyclic units and units described via idempotents. The former ones first have to be constructed in the more general context of semigroup rings. The hardest part of the paper is to prove that, as in the group ring case, they “exhaust” the central units. The second type of units are needed to generate subgroups of finite index in certain special linear groups. Since regular completely 0-simple semigroups (with $m > 1$) contain many idempotents, the latter is easy to prove based on the results in [3].

2. CONSTRUCTING UNITS

Throughout the paper, $S$ is a finite semigroup such that its integral semigroup ring $\mathbb{Z}S$ has an identity. Note that it is not assumed that $S$ has an identity element. Its unit group is denoted by $\mathcal{U}(\mathbb{Z}S)$.

We introduce here two types of units. First, we extend the definition of a Bass-cyclic unit of a group ring to the semigroup ring. For this we need some notations. By $GR(S)$, we denote the union of all subgroups of $S$. Further, for $a \in S$, we denote by $\langle a \rangle$ the cyclic subsemigroup generated by $a$. Note that $\langle a \rangle$ is a cyclic group if $a \in GR(S)$. In the latter case, we write $\hat{a} = \sum_{i=1}^{n} a^i$, where $n$ is the order of $a$, i.e. the order of $\langle a \rangle$. Note that $n^{-1}\hat{a}$ is an idempotent in $\mathbb{Q}S$. 


**Definition 2.1.** Let \( \mathbb{Z}S \) be a semigroup ring with identity \( 1 \). A Bass cyclic unit of \( \mathbb{Z}S \) is an element of the form
\[
u = \left(1 + \sum_{j=1}^{i-1} a^j\right)^{\phi(n)} + \frac{1 - i^{\phi(n)}}{n} \hat{a},
\]
where \( a \in GR(S) \), \( n \) is the order of \( \langle a \rangle \), \( \phi \) is the Euler function; \( 1 < i < n \), and \( (i, n) = 1 \).

By \( B_S \), we denote the set of all Bass cyclic units of \( \mathbb{Z}S \). Further, if \( S \) has a zero element, then the natural image of \( B_S \) in the contracted semigroup algebra \( \mathbb{Q}_0S \) (see cf. [2] for the definition) is denoted by \( B^0_S \).

Let us first show that such an element \( \nu \) is indeed a unit of \( \mathbb{Z}S \). This can easily be done using the group ring results and the Weddernburn decomposition of \( \mathbb{Z}(\langle a \rangle) \). However, we will give an elementary proof by describing explicitly the inverse of a Bass-cyclic unit. Since \( i, n \) is relatively prime, there exist two integers, say \( k \) and \( l \), such that \( ik - ln = 1 \). Further, we can assume that \( 1 < k < n \). Indeed, write \( k = n' + k' \) for some integers \( n' \) and \( k' \) with \( 0 \leq k' < n \), then \( k' \neq 0 \) since \( ik + ln = 1 \), and \( ik' + ln + l'n' = 1 \) with \( 1 \leq k' < n \). As \( 1 < i < n \) it follows easily that \( 1 < k' \), hence showing the claim. It is obvious that then \( l < 0 \). Let
\[
u = \left(1 + a^1 + \cdots + a^{(k-1)}\right)^{\phi(n)} + \frac{1 - k^{\phi(n)}}{n} \hat{a}.
\]
Then \( \nu w = X + Y + W + T \), where
\[
X = \left(1 + a + \cdots + a^{i-1}\right) \left(1 + a^1 + \cdots + a^{(k-1)}\right)^{\phi(n)}
= \left(1 + a + a^2 + \cdots + a^{i-1} + a^i + a^{i+1} + \cdots + a^{2i-1} + \cdots + a^{ik-1}\right)^{\phi(n)}
= \left(1 + (a + a^2 + \cdots + a^{i-1} + a^i) + a^i(a + a^2 + \cdots + a^{i-1}) + \cdots + a^{i(k-1)}(a + a^2 + \cdots + a^{i-1})\right)^{\phi(n)}
= \left(1 - k\hat{a}\right)^{\phi(n)}
\]
and
\[
Y = \left(1 + a + a^2 + \cdots + a^{i-1}\right)^{\phi(n)} \left(1 - k^{\phi(n)}\right) \hat{a}
= \left(1 + a + a^2 + \cdots + a^{i-1}\right)^{\phi(n)} \left(1 - k^{\phi(n)}\right) \hat{a}
= \frac{\hat{a}}{n} i^{\phi(n)} (1 - k^{\phi(n)}).
\]
Similarly,
\[ W = (1 + a^i + \cdots + a^{i(k-1)}) \frac{1 - i^{\phi(n)}}{n} \hat{a} = \frac{n}{k} \hat{a} (1 - i^{\phi(n)}) \]

Finally,
\[ T = \left( \frac{1 - i^{\phi(n)}}{n} \right) \hat{a} \left( \frac{1 - k^{\phi(n)}}{n} \right) \hat{a} = (1 - i^{\phi(n)}) (1 - k^{\phi(n)}) \frac{\hat{a}}{n} \]

So,
\[ Y + W + T = \left( 1 - (ki)^{\phi(n)} \right) \frac{\hat{a}}{n}, \]
and
\[ uv = (1 - l\hat{a})^{\phi(n)} + (1 - (ki)^{\phi(n)}) \frac{\hat{a}}{n}. \]

Consequently,
\[ uv \left( 1 - \frac{\hat{a}}{n} \right) = \left( 1 - l\hat{a} - \frac{\hat{a}}{n} + l \left( \frac{\hat{a}}{n} \right)^2 \right)^{\phi(n)} = \left( 1 - \frac{\hat{a}}{n} \right)^{\phi(n)} = 1 - \frac{\hat{a}}{n} \]

and
\[ uv \left( 1 - \frac{\hat{a}}{n} \right) = \left( \frac{\hat{a}}{n} - l \left( \frac{\hat{a}}{n} \right)^2 \right)^{\phi(n)} + (1 - (ki)^{\phi(n)}) \frac{\hat{a}}{n} \]
\[ = \left( \frac{\hat{a}}{n} - l\hat{a} \right)^{\phi(n)} + (1 - (ki)^{\phi(n)}) \frac{\hat{a}}{n} \]
\[ = \frac{\hat{a}}{n} ((1 - nl)^{\phi(n)} + 1 - (ki)^{\phi(n)}) \]
\[ = \frac{\hat{a}}{n} ((ik)^{\phi(n)} + 1 - (ik)^{\phi(n)}) \]
\[ = \frac{\hat{a}}{n}. \]

Therefore, \( uv = 1 \). Since \( u \) and \( v \) commute, \( vu = 1 \). Hence, \( u \) is a unit.

Important in the construction of the Bass cyclic units is that \( a \) generates a finite subgroup and that the ring has an identity. Hence, this way one can
define a Bass cyclic unit in any ring $R$ with identity $1$. It is convenient for further computations to introduce the following notation. Let $x \in R$ be an element for which the subsemigroup $\langle x \rangle$ generated by $x$ is a finite group. Let $n$ be the order of $\langle x \rangle$, $(i, n) = 1$, and $1 < i < n$. Denote

$$u_R(x, i) = \left(1 + \sum_{j=1}^{i-1} x^j\right)^{\phi(n)} + \frac{1 - i^{\phi(n)}}{n} \hat{x}.$$

Further, we will write also $\hat{x}$ for $\sum_{j=1}^{n-1} x^j$ in $R$. So in particular, a Bass cyclic unit in $\mathbb{Z}S$ can be denoted by $u(a, i)$, or $u_S(a, 1)$. It is clear that if $f$ is a ring isomorphism from $R$ to $R'$ then $f(u_R(x, i)) = u_R(f(x), i)$.

Second, we introduce the bicyclic units.

**Definition 2.2.** A bicyclic unit of $\mathbb{Z}S$ is an element of the form $u = 1 + (1 - a)s\hat{a}$, or $u = 1 + \hat{s}(1 - a)$, where $s \in S$, and $a \in GR(S)$.

Note that these elements are indeed units as they are of the form $1 + a$ with $a^2 = 0$. We decided to call both units of the latter type bicyclic units. In the case of group rings, only the elements of the former type were called so in [5, 7]; however, this will not create any confusion.

For the remainder of the paper (except for Section 6), we always assume that $QS$ is semisimple Artinian. The main part of the paper is now devoted to showing that the Bass cyclic units and the bicyclic units generate a subgroup of finite index, assuming $QS$ does not contain simple components of the type mentioned in the introduction.

### 3. BASIC SEMIGROUP FACTS

In this section, we introduce some more notations and recall some well-known facts on semigroups and Munn algebras. For further information, we refer the readers to [2, 4].

If $T$ is a semigroup without an identity, we can adjoin one simply by adding a new element 1 and extend the multiplication by defining $1t = t1 = t$ for all $t \in T \cup \{1\}$. We denote this new semigroup by $T^1$. If $T$ has an identity already, then we agree that $T^1 = T$. In a similar way, we can always adjoin a zero element, denoted $\theta_T$, and we write $T^0 = T \cup \{\theta_T\}$ for this new semigroup.

Recall that a finite semigroup $T$ with zero $\theta_T$ is called completely 0-simple semigroup if $T^2 \neq \theta_T$ and $T$ has no ideals other than $T$ and $\theta_T$.

Now, let $G$ be a group, $n_1$ and $n_2$ positive integers, and let $P$ be a $n_1 \times n_2$ matrix with entries from $G^0$. For $g \in G^0$, $i \in \{1, 2, \ldots, n_1\}$, and $j \in \{1, 2, \ldots, n_2\}$, write $(g)_{ij}$ for the $n_1 \times n_2$ matrix with $(i, j)$-entry $g$ and
all other entries $\theta_G$. Let $\mathcal{M}^0(G; n_1, n_2; P)$ be the set whose elements are all such $(g)_{ij}$, with all $(\theta_G)_{ij}$ identified, and define a multiplication on this set by

$$AB = A \circ P \circ B,$$

where $A, B \in \mathcal{M}^0(G; n_1, n_2; P)$ and $\circ$ denotes ordinary multiplication of matrices. So, $(y)_{ij}(h)_{ij} = (yp_{jk}h)_{ij}$, where $p_{ij}$ is the $(i, j)$-entry of the matrix $P$. The set $\mathcal{M}^0(G; n_1, n_2; P)$ endowed with this operation is a semigroup, called a Rees matrix semigroup. The matrix $P$ is called the sandwich matrix. Also, $P$ is called regular if it has at least one non-zero entry in every row and column. The following results are very basic for a completely 0-simple semigroup (see [2, Theorem 3.5] and [4, p. 7]).

**Theorem 3.1.** A finite semigroup is completely 0-simple if and only if it is isomorphic to a Rees matrix semigroup $\mathcal{M}^0(G; n_1, n_2; P)$ with a regular sandwich matrix $P$ and finite group $G$.

**Lemma 3.1.** If $T = \mathcal{M}^0(G; n_1, n_2; P)$ is a completely 0-simple semigroup, then

(i) $\{(p_{ij}^{-1})_{ij} \in T | p_{ij} \neq \theta_G\}$ is the set of non-zero idempotents of $S$, where $P = (p_{ij})$ as usual notation;

(ii) for any $i, j$, if $p_{ij} \neq \theta_G$, then

$$T^{(i)}_{(j)} \rightarrow G^0: (x)_{ij} \rightarrow xp_{ij}$$

is an isomorphism, where

$$T^{(i)}_{(j)} = \{(x)_{ij} \in T | x \in G^0\};$$

(iii) the maximal subgroups of $T$ are all $G_{ij} = T^{(i)}_{(j)} \setminus \{\theta_T\}$, with $p_{ij} \neq \theta_G$.

The construction of a Munn algebra is rather similar to that of a Rees matrix semigroup. Again $n_1$ and $n_2$ are positive integers, but instead of a group $G$, we now consider an algebra $A$. Let $P$ be an $n_2 \times n_1$ matrix with entries in $A$. The Munn algebra $R = \mathcal{M}(A; n_1, n_2; P)$ is the set of all $n_1 \times n_2$-matrices with entries in $A$. Addition and scalar multiplication by elements of $K$ are defined component-wise. Matrices multiply by insertion of the sandwich matrix $P$, that is, if $X$ and $Y$ are two elements of $R$, then the product of $X$ and $Y$ is

$$XY = X \circ P \circ Y,$$

where $\circ$ denotes the ordinary matrix multiplication. In case $n_1 = n_2$ and $P$ is the identity matrix $I$, then $R = \mathcal{M}(A; n_1, n_2; I)$ is the classical matrix ring over $A$. We denote this ring by $M_n(A)$. The lemma below [2, pp. 162–163] is very important to our coming discussion.
**Lemma 3.2.** Let \( m \) be a natural number. Let \( T = \mathcal{M}^0(G; m, m; P) \) be a finite completely 0-simple semigroup such that \( QT \) is semisimple Artinian. Let \( Q_T = QT / Q \theta_f \) be the contracted algebra of \( T \) over \( Q \), and let \( \mathcal{M}(Q G; m, m; P) \) be the Munn algebra over \( Q G \). Then

(i) \( P \) is invertible in \( M_m(Q G) \);

(ii) \( Q_T \cong \mathcal{M}(Q G; m, m; P) \); and

(iii) the mapping \( f_p: \mathcal{M}(Q G; m, m; P) \to M_m(Q G) \) \( A \to A \circ P \) is a ring isomorphism, in particular, \( P^{-1} \) is the identity of \( \mathcal{M}(Q G; m, m; P) \);

(iv) if \( P^{-1} \in \mathcal{M}(Z G; m, m; P) \), then \( f_p(\mathcal{M}(Z G; m, m; P)) = M_m(Z G) \).

Note that, as in the lemma, we will often make abuse of notations by identifying the sandwich matrix \( P \) (defining the completely 0-simple semigroup) in a natural way with a matrix in \( M_m(Q G) \). Similarly, we often identify the elements of \( T \) with their matrix representation used in (ii). Further, in the remainder of this paper, we will often skip the matrix multiplication symbol \( \cdot \). This will not create confusion as it will be clear from the context which product has to be taken.

We will use the notation \( M \cong_{iso} N \) to indicate that \( M \) is isomorphic to \( N \) under the map iso.

Now let \( S \) be our finite semigroup as in the previous section. Then \( S^0 \) has a principal series

\[
S^0 = S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \{ \theta_S \}
\]

with \( S_i / S_{i+1} \cong \mathcal{M}^0(G_i; m_i, m_i; P_i) \), \( i = 0, 1, \ldots, n \). Since \( QS \) is semisimple Artinian we have the following well-known fact (this can be shown using a very standard method (see, for example, [2])):

**Lemma 3.3.**

\[
QS^0 \cong \bigoplus_{i=0}^n Q(S_i / S_{i+1}) \oplus Q\theta_S \cong \bigoplus_{i=0}^n \mathcal{M}(Q G; m_i, m_i; P_i) \oplus Q\theta_S \\
\cong \bigoplus_{i=0}^n \mathcal{M}_m(Q G_i) \oplus Q\theta_S.
\]

Further, since \( QS \) has an identity, each matrix \( P_i \) is invertible in \( M_m(Q G_i) \).

**Remark.** Denote \( S_i = \mathcal{M}^0(G_i; m_i, m_i; P_i) \), then \( ZS^i \) does not necessarily have an identity (although \( ZS \) has an identity). However, \( QS^i \) has an identity, say \( 1_i \), and \( ZS^i + Z1_i \) is a subring of \( QS^i \), with an identity. Hence,
we still can define the Bass cyclic unit and bicyclic unit in this ring but with respect to 1, and a in a subgroup of S'.

4. SEMIGROUP RINGS OF COMPLETELY 0-SIMPLE SEMIGROUPS

Throughout this section, T denotes a finite completely 0-simple semigroup of the type $T_0^G(G; m, m; P)$, where G is a group with identity e and the sandwich matrix P is regular. Further, we assume that $m > 1$, and that P is invertible in $M_m(QG)$.

Since P is regular, for any $i = 1, 2, \ldots, m$, we fix a $j_i \in \{1, 2, \ldots, m\}$ such that $p_{ij_i} \neq \theta_T$. As before, let $G_{kt} = T_{(k)}(\{ \theta_T \})$. Then $GR(T) = \bigcup_{p_{ij} \neq \theta_T} G_{kji}$, and $\bigcup_{i=1}^m f_P(G_{kji}) = \bigcup_{p_{ij} \neq \theta_T} f_P(G_{kji}) = \bigcup_{p_{ij} \neq \theta_T} G_{kji} P$.

In the first part of this section we investigate what a Bass cyclic unit looks like in a matrix format. To do so, we first need the following Lemmas. Further, for $\alpha \in Z$, $A = (a_{ij}) \in M_m(ZG)$, we denote by $\alpha A = (\alpha a_{ij})$.

**Lemma 4.1.** Let $I$ be the identity of $M_m(ZG)$, that is

$$I = \begin{bmatrix}
  e & 0 & \cdots & 0 \\
  0 & e & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & e
\end{bmatrix}.$$

Let $A = (a_{ij}) \in M_m(ZG)$ be a matrix with all entries zero except possibly those in the $i$th row. Then, for any positive integer $n$, $A^n = a_{ii}^{n-1} A$, and

$$(I + A)^n = I + B,$$

where

$$B = \sum_{h=0}^{n-1} (e + a_{ii})^h A.$$ 

**Proof.** We prove the first part by induction on $n$. The case $n = 1$ being clear. Assume $A^n = a_{ii}^{n-1} A$. Then $A^{n+1} = A^n A = a_{ii}^{n-1} A^2$. But, $A^2 = a_{ii} A$. Hence, $A^{n+1} = a_{ii}^{n-1} a_{ii} A = a_{ii}^n A$. We also prove the second part by induction. It is clear for $n = 1$. Now assume that, for $n$ larger than or equal to 1, $(I + A)^n = I + s A$, where $s = \sum_{h=0}^{n-1} (e + a_{ii})^h$. Then $(I + A)^{n+1} = (I + A)^n (I + A) = (I + s A)(I + A) = I + A + s A + s^2 A = I + (e + s) A + s a_{ii} A = \ldots$
I + (e + s + sa_{ii})A. But \( e + s + sa_{ii} = e + s(e + a_{ii}) = e + \sum_{h=0}^{n-1}(e + a_{ii})^h \). \[
\]

**Corollary 4.1.** Given \( p_{ij} \neq 0 \), then \( G_{ij}A \) is a subgroup of \( M_m(\mathbb{Z}G) \) with identity \((p_{ij}^{-1})_{ij}A\); the order of \( A = (g)_{ij}P \in G_{ij}A \) and the order of \( gp_{ij} \in G \) are the same.

Furthermore, for any positive integer \( n \)

\[(I + A)^n = I + B,
\]

where

\[B = \sum_{h=0}^{n-1} (e + gp_{ij})^h A.\]

**Proof.** That \( G_{ij}A \) is a subgroup with identity \((p_{ij}^{-1})_{ij}A\) of the multiplicative semigroup \( M_m(\mathbb{Z}G) \) follows from Lemma 3.2. By Lemma 4.1, for any natural number \( n \), \( A^n = (gp_{ij})^{n-1}A \). Since \((p_{ij}^{-1})_{ij}A\) is the identity of the group \( G_{ij}A \), it is easily verified that \( A^n = (p_{ij}^{-1})_{ij}A \) if and only if \((gp_{ij})^n = e\), the identity of \( G \). Hence, the first part of the result follows.

The second part follows at once from Lemma 4.1 applied to the matrix \((a_{ij}) \) with \( a_{ij} = gp_{ij} \), for \( c = 1, 2, \ldots, m \), \( a_{ii} = gp_{ij} \), and \( a_{ij} = 0 \) for \( q \neq i \) in Lemma 4.1.

In the following Lemma we describe the matrix representation, under the mapping \( f_P \), of the Bass cyclic units defined via the elements of the group \( G_{ij} \).

**Lemma 4.2.** Let \( (g)_{ij} \in G_{ij} \) be of order \( l \), and let \( d \) be an integer with \( 1 < d < l \), and \( (d, l) = 1 \). Then

\[f_P(u_{Z_2}((g)_{ij}, d)) = f_P((g)_{ij}, d) = f_P((g)_{ij}, P, d) = I + B,
\]

where \( B = (u - e)(p_{ij}^{-1})_{ij}P \) and \( u = u_{Z_2}(gp_{ij}, d) \).

**Proof.** Let \( A = (g)_{ij}P \) and let \( A' = (a'_{ij}) \) be the matrix \( \sum_{r=1}^{d-1}A' \). By Lemma 4.1, \( A' = \sum_{h=0}^{d-2}(gp_{ij})^h A \), and thus \( a'_{ij} = \sum_{h=0}^{d-2}(gp_{ij})^h a_{ij} \), and \( a'_{ij} =\)
0 for $q \neq i$. So, again by Lemma 4.1, $(I + A')^{\phi(l)} = I + B$, where $B = \sum_{h=0}^{\phi(l)-1} (e + a_{ii})^{h} A^h = (b_{q,c})$. Thus

$$b_{ic} = \sum_{h=0}^{\phi(l)-1} (e + a_{ii})^{h} a_{ic}^{'}$$

$$= \left( \sum_{h=0}^{\phi(l)-1} \left( e + \sum_{k=0}^{d-2} (gp_{ji})^{k} a_{ii} \right)^{h} \right) \left( \sum_{k=0}^{d-2} (gp_{ji})^{k} a_{ic} \right)$$

$$= \left( \sum_{h=0}^{\phi(l)-1} \left( e + \sum_{k=0}^{d-2} (gp_{ji})^{k} a_{ii} \right)^{h} \right) \times \left( -e + \left( e + \sum_{k=0}^{d-2} (gp_{ji})^{k} a_{ii} \right) a_{ic}^{''} a_{ic} \right)$$

$$= \sum_{h=0}^{\phi(l)-1} \left( e + \sum_{k=0}^{d-2} (gp_{ji})^{k} a_{ii} \right)^{h+1} a_{ic}^{''} a_{ic}$$

$$- \sum_{h=0}^{\phi(l)-1} \left( e + \sum_{k=0}^{d-2} (gp_{ji})^{k} a_{ii} \right)^{h} a_{ic}^{''} a_{ic}$$

$$= \left( e + \sum_{k=0}^{d-2} (gp_{ji})^{k} a_{ii} \right)^{\phi(l)} - e \right) a_{ic}^{''} a_{ic}$$

$$= \left( \sum_{k=0}^{d-1} (gp_{ji})^{k} \right)^{\phi(l)} - e \right) p_{ij}^{-1} p_{ij,c}$$

and $b_{q,c} = 0$ for $q \neq i$. Therefore,

$$(I + A')^{\phi(l)} = I + \left( \sum_{k=0}^{d-1} (gp_{ji})^{k} \right)^{\phi(l)} - e \right) (p_{ij}^{-1})_{ij} P.$$
\[
\begin{align*}
&= (p_{j,i}^{-1})_{ij}P + \sum_{h=0}^{l-2} (gp_{j,i})^h(g)_{ij}P \\
&= (p_{j,i}^{-1})_{ij}P + \sum_{h=0}^{l-2} (gp_{j,i})^h(gp_{j,i})(p_{j,i}^{-1})_{ij}P \\
&= \sum_{h=0}^{l-1} (gp_{j,i})^h(p_{j,i}^{-1})_{ij}P.
\end{align*}
\]

Hence by the previous
\[
u_{M_n(ZG)}((g)_{ij}, P, d)
= I + \left( \left( \sum_{h=0}^{d-1} (gp_{j,i})^h \right)^{\phi(l)} - e \right) + \frac{1 - d^{\phi(l)}}{l} \sum_{h=0}^{l-1} (gp_{j,i})^h(p_{j,i}^{-1})_{ij}P
= I + (u_{ZG}(gp_{j,i}, d) - e)(p_{j,i}^{-1})_{ij}P. \quad \blacksquare
\]

Remark. Recall that
\[
u_{ZG}((g)_{ij}, d)^{-1} = u_{ZG}(((g)_{ij})^d, q),
\]
where \(1 < q < l, (q, l) = 1\). Hence we obtain that
\[
f_p(u_{ZG}((g)_{ij}, d))^{-1} = u_{M_n(ZG)}(((g)_{ij})^d, q)
= u_{M_n(ZG)}(((g)_{ij})P^d, q)
= u_{M_n(ZG)}(((g)_{ij})^dP, q)
= I + C,
\]
where \(C = (v - e)(p_{j,i}^{-1})_{ij}P\), and \(v = u_{ZG}((gp_{j,i})^d, q) = u^{-1}\), with \(u\) as in the lemma.

We now give a series of three lemmas to show that the projection on a simple component of the group generated by the Bass cyclic units and bicyclic units contains a “nice” set of diagonal matrices.

**Lemma 4.3.** Let \(G\) be a finite group and let \(n\) be a positive integer. There exists a positive number \(r\) such that for any \(u \in \mathcal{U}(ZG)\) and any positive integer \(v\),
\[
\sum_{i=0}^{v-1} u^i = 0,
\]
where \( \overline{u} \) and \( \overline{0} \) denote the images of \( u \) and \( 0 \) under the natural map from \( \mathbb{Z}G \) to \( \mathbb{Z}_G \). We denote the smallest such number \( r \) by \( b(G,n) \).

**Proof.** Let \( u \in \mathcal{U}(\mathbb{Z}G) \) and let \( v \) be a positive integer. Since \( |\mathcal{U}(\mathbb{Z}_G)| = d < \infty \) we obtain that \( \overline{u^d} = \overline{1} \). Thus

\[
\sum_{i=0}^{md-1} \overline{u^i} = \sum_{i=0}^{d-1} \overline{u^i (1 + \overline{u^d} + \cdots + \overline{u^{(m-1)d}})} = \sum_{i=0}^{d-1} \overline{u^i \overline{n}} = \overline{0}.
\]

Taking \( r = nd \), the result follows. \( \square \)

**Lemma 4.4.** Let \( P_T = \{ I + gPsP(I - gP) | s, g \in T, g^2 = g \} \). Then the group \( \langle P_T \rangle \) generated by \( P_T \) contains the following elements:

\[
\{ I + gP\alpha P(I - gP) | \alpha \in \mathfrak{f}_P(\chi(\mathbb{Z}_0T)) \},
\]

where \( \chi(\mathbb{Z}_0T) = \mathfrak{f}(\mathbb{Z}G; m, m; P) \).

**Proof.** Note that \( (gP)^2 = gP \). Hence, for any \( k, l \in \mathbb{Z} \); \( s, s' \in T \), and \( s, s' \in T \),

\[
(I + gPsP(I - gP))^k (I + gPs'P(I - gP))^l
= (I + gPksP(I - gP))(I + gPls'P(I - gP))
= I + gP(k s + l s') P(I - gP).
\]

Hence the result follows. \( \square \)

Throughout we denote by \( E_{ij}, 1 \leq i, j \leq m \), the classical matrix units of \( M_m(\mathbb{Z}G) \).

**Lemma 4.5.** Let \( (g)_{ij} \in G_{ij} \) be an element of order \( l \), and let \( d \) be an integer with \( 1 < d < l \), \( (d,l) = 1 \). Let \( V = f_p(u_{ij}, (g)_{ij}, d) \), and \( v = u_{ij} (gP_{ij}, d) \). Let \( t = (p_{ij}^{-1})_{ij} \). Then, for any positive integer \( r \), \( V^r \) has a decomposition \( V^r = D^rF \), with

\[
D^r = I + (v^r - e)E_{ii},
\]

and

\[
F = I + tP(v^{-r} - e)E_{ii}(I - tP).
\]

Furthermore, if \( r \) is a multiple of \( b(G, m(P)) \), where \( m(P) \) is the smallest positive integer such that \( m(P)P^{-1} \in M_m(\mathbb{Z}G) \) (such an element always exists), then \( F \in \langle P_T \rangle \).
Proof. By Lemma 4.2, \( V = I + B \), where
\[
B = (b_{ij}) = (v - e)(p_{ij}^{-1})_i, P.
\]
Note that \( b_{ii} = (v - e) \). So by Lemma 4.1,
\[
V' = (I + B)' = I + \sum_{h=0}^{r-1} (e + (v - e))^h B
\]
\[
= I + \sum_{h=0}^{r-1} v^h B = I + \sum_{h=0}^{r-1} v^h (v - e)tP = I + (v' - e)tP.
\]
Let
\[
D' = I + (v' - e)E_{ii},
\]
and
\[
F = I + tP(v' - e)E_{ii}(I - tP).
\]
Note that if \( r \) is a multiple of \( b(G, m(P)) \) then each integral coefficient of \((v' - e) = (v^{-1} - e)(v^{-1} - 1 + \cdots + v^{-1} + e)\) is a multiple of \( m(P) \) by Lemma 4.3. Hence \( \beta = (v' - e)E_{ii}P^{-1} \) is a matrix with entries in \( \mathbb{Z}G \), and thus \((v' - e)E_{ii} = \beta P \) with \( \beta \in \chi(\mathbb{Z}G; m, m; P) \). Therefore, \( F \in \langle P_t \rangle \) by Lemma 4.4.

Finally, we check that \( D'F \) is really a decomposition of \( V' \) as claimed. Since
\[
F = I + tP(v' - e)E_{ii}(I - tP)
\]
\[
= I + tPE_{ii}(v' - e)(E_{ii} - tP)
\]
\[
= I + E_{ii}(v' - e)(E_{ii} - tP)
\]
\[
= I + (v' - e)(E_{ii} - tP),
\]
we obtain indeed that
\[
D'F = I + (v' - e)E_{ii} + (v' - e)(E_{ii} - tP)
\]
\[
+ (v' - e)E_{ii}(v' - e)(E_{ii} - tP)
\]
\[
= I + [(v' - e) + (v' - e)(v' - e) + (v' - e)(v' - e)]E_{ii}
\]
\[
- [(v' - e) + (v' - e)(v' - e)]tP
\]
\[
= I + (v' - e)tP = V' \quad \blacksquare
\]
Lemma 4.6. With notations as in Lemma 4.5, let $r$ be a multiple of $b(G, m(P))$, where $m(P)$ is such that $m(P)P^{-1} \in M_m(Z_G)$. Then the group generated by $f_p(B^0_G)$ and $\langle P_T \rangle$ contains the subset

$$D(P) = \left\{ \sum_{i=1}^{m} u_i' E_{ii} | u_i' \in B_G, i = 1, 2, \ldots, m \right\},$$

where $B_G$ is the set of Bass cyclic units of $Z_G$ and $B^0_G$ the set of the natural images in $Z_T$ of the Bass cyclic units of $Z_T$.

Proof. Let $r$ be as in the statement of the Lemma and let $V \in f_p(B^0_G)$. Because of Lemma 4.5, $D' = V'F^{-1} \in \langle f_p(B^0_G), P_T \rangle$. Hence, as $\sum_{i=1}^{m} u_i' E_{ii}$ (with each $u_i' \in B_G$) is a product of $m$ such $D'$s, the result follows.

In the remainder of this section, we show that the group generated by $B^0_G$ and the images in $Z_T$ of bicyclic units of $Z_T$ contains a subgroup of finite index in the unit group of the ring $Z_T + Zf$, where $f$ is the identity of $Q_T$.

First we need to introduce some more notations. Since $Q_G$ is semisimple, write

$$Q_G = \bigoplus_{j=1}^{t} Q_G e_j,$$

where each $e_j$ is a primitive central idempotent of $Q_G$, $j = 1, 2, \ldots, t$. Let $Q_G e_j = M_n(D_j)$, $D_j$ is a division ring, $j = 1, 2, \ldots, t$. Then

$$M_m(Q_G) = \bigoplus_{j=1}^{t} M_{mn}(D_j) = \bigoplus_{j=1}^{t} M_m(Q_G) f_j,$$

where $f_j = e_j I_m$ is a primitive central idempotent of $M_m(Q_G)$, $M_m(Q_G) f_j = M_{mn}(D_j)$, $j = 1, 2, \ldots, t$. Furthermore, let

$$\Lambda = \bigoplus_{j=1}^{t} \Lambda_j,$$

where $\Lambda_j$ is a maximal $Z$-order in $M_m(Q_G) f_j$ containing $M_m(Z_G) f_j$. Let $O_j$ be some maximal $Z$-order in $D_j$, $j = 1, 2, \ldots, t$. Then $M_m(O_j)$ is a second maximal order in $M_m(Q_G) f_j$. We will write $GL_j$ or $GL(mn, O_j)$ for its group of units, and $SL_j$ or $SL(mn, O_j)$ for the subgroup in $GL_j$ that consists of all elements having reduced norm one. We will several times abuse notations by identifying in the natural way $SL_j$ with a subgroup of $\mathcal{U}(O, M_m(O_j))$.

Lemma 4.7. Let $J$ be a subgroup of $\mathcal{U}(M_m(Z_G))$ which contains a subgroup $\prod_{j=1}^{t} J_j$ with $J_j$ a subgroup of finite index in $SL_j$, $j = 1, 2, \ldots, t.$
Then the group \( \langle f_p(B_\mathcal{T}_0^T), J, P_T \rangle \) contains a subgroup of finite index in the center of \( \mathcal{Z}(M_m(\mathbb{Z}G)) \).

Proof. Let \( r \) be a multiple of \( b(G, m(P)) \), and let \( w \) denote the natural map from \( \mathcal{Z}(\mathbb{Z}G) \) to \( K_1(\mathbb{Z}G) \). Further, denote by \( \pi_j \) the projection of \( \mathbb{Q}G \) onto \( \mathbb{Q}G e_j = M_m(D_j) \), \( j = 1, 2, \ldots, t \). It is well known that the group generated by \( w(B_G) \) is of finite index in \( K_1(\mathbb{Z}G) \) [1].

Clearly,

\[
Z = Z(\mathcal{Z}(M_m(\mathbb{Z}G))) = \mathcal{Z}(Z(M_m(\mathbb{Z}G))) = \left\{ \sum_{i=1}^m zE_{ii} | z \in Z(\mathcal{Z}(\mathbb{Z}G)) \right\}.
\]

Hence for \( z = \sum_{i=1}^m zE_{ii} = zI_m \in Z \) it follows as in the proof of Lemma 3.2 of [5] and the proof of Lemma 2.3 in [3] that there exists an integer \( k \) (independent of \( z \)) such that \( w(z^k b_z^{-1}) = 1 \) and \( \pi_j(z^k b_z^{-1}) \in M_m(O_j) \), for some \( b_z \in \langle B_G \rangle \). Thus \( \pi_j(z^k b_z^{-1}) \) and therefore also \( \pi_j(z^k b_z^{-1})I_m \) has reduced norm one. So, by the assumptions, there exists a natural number \( v \) such that \( \pi_j(z^{kr} b_z^{-vr})I_m \in J_j \), \( j = 1, 2, \ldots, t \). Note that since \( K_1(\mathbb{Z}G) \) is abelian, we may assume that \( b_z^{vr} \) is a product of \( r \)-th powers of Bass cyclic units. Because of Lemma 4.6, \( b_z^{vr} = b_z^{sr}I_m \in \langle f_p(B_\mathcal{T}_0^T), P_T \rangle \). Hence

\[
z^{kr} b_z^{-vr} = \prod_{j=1}^t \pi_j(z^{kr} b_z^{-vr})I_m \in \prod_{j=1}^t J_j \subseteq J.
\]

Therefore \( z^{kr} \in \langle f_p(B_\mathcal{T}_0^T), P_T, J \rangle \). Since \( Z \) is finitely generated, the result follows.

Lemma 4.8. Under the assumptions of Lemma 4.7

\[
\langle f_p(B_\mathcal{T}_0^T), P_T, J \rangle
\]

is of finite index in \( \mathcal{Z}(M_m(\mathbb{Z}G)) \).

Proof. Since \( I = \sum f_j \), \( \Lambda = \oplus \Lambda_j \), and \( M_m(\mathbb{Z}G)f_j \subseteq \Lambda_j \),

\[
M_m(\mathbb{Z}G) \subseteq \sum M_m(\mathbb{Z}G)f_j \subseteq \sum \Lambda_j = \Lambda
\]

Then \( [\mathcal{Z}(\mathcal{Z}(\mathbb{Z}G)) : \mathcal{Z}(M_m(\mathbb{Z}G))] < \infty \) [7, Lemma 4.6]. It is easy to check

\[
Z(\mathcal{Z}(\Lambda)) \cap \mathcal{Z}(M_m(\mathbb{Z}G)) = Z(\mathcal{Z}(M_m(\mathbb{Z}G))).
\]

Hence

\[
[Z(\mathcal{Z}(\Lambda)) : Z(\mathcal{Z}(M_m(\mathbb{Z}G)))] < \infty.
\]
Lemma 4.7 yields that \( \langle f_0( B_T^0), P_T, J \rangle \) contains a subgroup \( V \) such that \( [Z(\mathcal{U}(M_m(\mathbb{Z}G))): V] < \infty \). So \( [Z(\mathcal{U}(\Lambda)): V] < \infty \). Furthermore, since \( Z(\mathcal{U}(\Lambda)) = \prod Z(\mathcal{U}(\Lambda_i)) \), we obtain \( [Z(\mathcal{U}(\Lambda_i)): V \cap Z(\mathcal{U}(\Lambda_i))] < \infty \). The proof now continues as in the proof of Lemma 2.4 of [3].

Next, we construct a finite set of generators for a group \( J \) with the properties as in Lemma 4.7.

**Lemma 4.9.** Let \( t \) be a nonzero idempotent in \( T \), then \( tPf \) is a non-central idempotent of \( M_m(\mathbb{Q}G_f) \), for \( j = 1, 2, \ldots, t \).

**Proof.** For the sake of simplicity and convenience, we may assume that \( t = (e_{11}) \) (see, for example, [2, Corollary 3.12, p. 106]). Now \( Z(\mathcal{U}(\mathbb{Q}G_f)) = \{ \sum_{i=1}^m e_i | c \in Z(\mathcal{U}(\mathbb{Q}G_f)) \} \). It follows that \( (e_{11})Pf_j = eE_{11} + \sum_{k=1}^m p_{k1} e_i E_{11} = \sum_{k=1}^m p_{k1} e_i E_{11} = ee_{11} + \sum_{i=2}^m p_{i1} e_i E_{11} \neq 0 \), and \( (e_{11})Pf_j \notin Z(M_m(\mathbb{Q}G)) \) since \( m > 1 \). Hence the result follows.

**Lemma 4.10.** Assume that \( m > 2 \) and let \( t \) be a nonzero idempotent in \( T \). Let \( J_{tP} \subseteq M_m(\mathbb{Z}G) \) be the group generated by the elements:

\[
I + tPsP(I - tP)
\]

and

\[
I + (I - tP)sPtP,
\]

where \( s \in T \). Then \( J_{tP} \) contains a subgroup of finite index in \( SL_j = SL(mn_{j}, O_j), j = 1, 2, \ldots, t \).

**Proof.** By Lemma 4.9, \( tPf_j \) is a non-central idempotent of \( M_m(\mathbb{Q}G_f) \). Therefore the result is proved as in Proposition 3.2 in [3].

**Proposition 4.1.** Assume that \( m > 2 \). Then the group generated by \( B_T^0 \) and

\[
\{ f + ts(f - t), f + (f - t)s, t \in T, and t^2 + t = \theta_T \}
\]

is of finite index in the unit group of the ring \( \mathbb{Z}_oT + \mathbb{Z}f \), where \( f \) is the identity of \( \mathbb{Q}_oT \).

**Proof.** From Lemma 4.8 and Lemma 4.10, it follows that the group generated by \( f_0( B_T^0) \) and the units \( I + tPsP(I - tP) \) and \( I + (I - tP)sPtP, t = t^2 \neq \theta_T, s \in T \), is a subgroup of finite index in the unit group of \( M_m(\mathbb{Z}G) \), and hence in \( \sum_{t \in T} \mathbb{Z}sP + \mathbb{Z}I_m \). So the result follows from Lemma 3.2.
5. THE MAIN THEOREM

We are now in a position to prove our main result. Recall that $S$ is any finite semigroup such that $Q_1$ is a semisimple ring and such that $ZS$ contains an identity $1$.

Before the proof of the main theorem, we need the following fact in group theory.

**Lemma 5.1.** Let $A_i, 1 \leq i \leq n$, be group with identity $e_i$, and $\mathcal{A}_i, 1 \leq i \leq n$ be subgroups of the product $A = \prod_{i=1}^{n} A_i$ such that, for each $i$, $\pi_i(\mathcal{A}_i)$ is a finite index subgroup of $A_i$, and $\pi_i(\mathcal{A}_i) = e_i$ for $j < i$, where $\pi_i$ is the natural projection of $A$ onto $A_i$. Then, the group generated by $\bigcup_{i=1}^{n} \mathcal{A}_i$ is a finite index subgroup of $A$.

**Proof.** We prove the lemma by induction on $n$. It is clear for $n = 1$. Now assume that the lemma is true for $n (1 \leq n)$. Thus if $A = \prod_{i=1}^{n+1} A_i$, then $A = (\prod_{i=1}^{n} A_i) \times A_{n+1} = C \times A_{n+1}$ with $C = \prod_{i=1}^{n} A_i$, and the group generated by $\pi_C(\bigcup_{i=1}^{n} \mathcal{A}_i)$ is a finite index subgroup of $C$, where $\pi_C$ is the projection onto $C$ of the product $A = C \times A_{n+1}$, because $\pi_j(\mathcal{A}_j) = \pi_j(\mathcal{A}_j)$ with $\pi_j, 1 \leq j \leq n$, denoting the projection onto $A_j$ of $C$. Therefore, it is sufficient to prove the lemma in case $n = 2$. Let $A = A_1 \times A_2$, and $A_1 = \bigcup_{i=1}^{q} \pi_1(\mathcal{A}_1) a_{1q}$, $A_2 = \bigcup_{i=1}^{k} \pi_2(\mathcal{B}_2) a_{2k}$. Then we claim that $A = \bigcup_{i=1}^{q+k} \pi_1(\mathcal{A}_1, \mathcal{A}_2) a_{1+k}$, where $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ denotes the group generated by $\mathcal{A}_1$ and $\mathcal{A}_2$. Indeed, for any $(a_1, a_2) \in A$, let $a_1 = \pi_1(u_1) a_{1q}$, $a_2 = \pi_2(u_2) a_{2k}$, for some $u_1 \in \mathcal{A}_1$, $u_2 \in \mathcal{A}_2$. Then $(a_1, a_2) = (\pi_1(u_2 u_a u_{-1}^{a_{1q}}, \pi_2(u_2 u_{-1}^{a_{2k}})) = u_2 u_2 a_{1q}, \pi_2(u_1^{*}) a_{2k})$. But $\pi_2(u_1^{*}) a_{2k} \in A_2$, say it is $\pi_2(u_2 a_{2k})$, with $u_2 \in \mathcal{A}_2$. Thus

$$(a_1, a_2) = u_2 u_2 (\pi_1(u_2) a_{1q}, \pi_2(u_2) a_{2k}) = u_2 u_2 u_2^{*} (a_{1q}, a_{2k}).$$

**Theorem 5.1.** Assume that if $S^0$ has a principal factor

$$S_i/S_{i+1} \cong \mathcal{G}^0(G; m, P)$$

with $m = 1$ then $G$, does not have a non-abelian homomorphic image which is fixed point free. Further suppose $Q_1$ does not have simple components of the following type:

(i) a non-commutative division algebra other than a totally definite quaternion algebra;
(ii) a $2 \times 2$ matrix ring over the rationals;
(iii) a $2 \times 2$ matrix ring over a quadratic imaginary extension of the rationals;
(iv) a $2 \times 2$ matrix ring over a non-commutative division algebra.

Then the group generated by the following elements is of finite index in $\mathcal{U}(ZS)$:

(i) the Bass cyclic units $B$ of $ZS$;

(ii) the bicyclic units $(1 + \hat{g}s(1 - g), 1 + (1 - g)s\hat{g}|s \in S, \ g \in GR(S))$.

Proof. Note that if $S$ does not have a zero element, then $ZS^0 = ZS \oplus Z_\theta S$ (a direct product of rings). Hence to prove the result we may assume that $S$ has a zero element $\theta_S$.

We know that

$$Q S \cong \bigoplus_{i=0}^{n} Q_0(S_i/S_{i+1}) \oplus Q_\theta S.$$ 

Further let $\pi_i$ denote the projection of the latter ring onto the $i$-th summand. Let $f_i$ be the identity of $Q_0(S_i/S_{i+1})$. Then

$$\psi(ZS) \subseteq \left( \bigoplus_{i=0}^{n} Z_0(S_i/S_{i+1}) + Zf_i \right) \oplus Z_\theta S.$$ 

Let $\mathcal{U}_i$ be the subgroup of $\mathcal{U}(ZS)$ generated by $B_S$ and $(1 + \hat{g}s(1 - g), 1 + (1 - g)s\hat{g}|s \in S_i, \ g \in GR(S_i))$. Then, in case $m_i \geq 3$, from Proposition 4.1 it follows that $\pi_i(\psi(\mathcal{U}_i))$ is of finite index in $\mathcal{U}(Z_0(S_i/S_{i+1}) + Zf_i)$. In case $m_i = 1$, the same holds because of the results in [3] (note that $\pi_i(\psi(ZS))$ is a group ring). Finally, note that for each $i$ and $u_i \in \mathcal{U}_i$, $\pi_i(\psi(u_i)) = f_i$ for $j < i$. By Lemma 5.1, the group generated by $\psi(\cup \mathcal{U}_i)$ is of finite index in $\mathcal{U}(\oplus(Z_0(S_i/S_{i+1}) + Zf_i) \oplus Z_\theta S)$. Hence the result follows.

6. FINITE UNIT GROUPS

As a last consideration we will study the conditions under which the unit group $\mathcal{U}(ZS)$ is finite. Here we will only assume that $S$ is a finite semigroup such that $ZS$ contains an identity. Our result is the following

**Theorem 6.1.** Let $S$ be a finite semigroup such that $ZS$ contains an identity. Then the order of unit group $\mathcal{U}(ZS)$ is finite if and only if $S$ is an inverse semigroup which is a union of disjoint groups which each are either abelian of exponent 1, 2, 3, 4, or 6 or are a Hamiltonian 2-group.

Proof. First, assume the order $|\mathcal{U}(ZS)| < \infty$. Then it is well known and easy to prove that $QS$, and thus also $ZS$, does not have non-zero nilpotent
elements. Indeed, let \( x \in \mathbb{Z}S \), and assume \( n \geq 2 \) such that \( y = x^{n-1} \neq 0 \), \( x^n = 0 \). Since \( y^2 = 0 \)

\[
(1 - zy)(1 + zy) = 1,
\]

for any \( z \in \mathbb{Z} \). Hence we have an infinite set \( (1 + zy \mid z \in \mathbb{Z}) \) of units, a contradiction. This proves the claim and hence \( QS \) is a semisimple Artinian ring without nilpotent elements. Thus Lemma 3.3 yields that \( S^0 \) has a principal series

\[
S^0 = S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \{ \theta_S \},
\]

where each principal factor \( S_i/S_{i+1} \cong G_i^0 \), each \( G_i \) a group, \( 1 \leq i \leq n \).

The isomorphisms used in Lemma 3.3 yield that

\[
\mathbb{Z}S^0 \cong \bigoplus_{i=0}^n \mathbb{Z}G_i \oplus \mathbb{Z}\theta_S.
\]

Thus from \( \mathbb{Z}(S^0) \cong \prod_{i=0}^n \mathbb{Z}(G_i) \times \mathbb{Z}(\mathbb{Z}) \) we obtain that each \( |\mathbb{Z}(G_i)| \) is finite. Consequently by [6, p. 57, Theorem 4.1], each \( G_i \) is either abelian of exponent \( 1, 2, 3, 4, \) or \( 6 \) or \( G_i \) is a Hamiltonian 2-group.

Conversely, assume \( S \) is an inverse semigroup which is a union of disjoint groups which each are either abelian of exponent \( 1, 2, 3, 4, \) or \( 6 \) or are a Hamiltonian 2-group. Then \( S \) is an inverse semigroup which is a union of groups. By using the same method as above, we obtain \( \mathbb{Z}S^0 \cong \bigoplus_{i=0}^n \mathbb{Z}G_i \oplus \mathbb{Z}\theta_S \). Hence \( \mathbb{Z}(S^0) \cong \prod_{i=0}^n \mathbb{Z}(G_i) \times \mathbb{Z}(\mathbb{Z}) \). The result then follows again by using [6, p. 57, Theorem 4.1].

### REFERENCES