Discreteness Conditions for the Spectrum of Ordinary Differential Operators*

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1. INTRODUCTION

Consider the differential expression

\[ My = w^{-1} \left[ \sum_{j=0}^{n} (-1)^j (p_j y^{(j)})^{(j)} \right] \tag{1.1} \]

on the interval \((0, \infty)\). Throughout the paper we assume that the weight function \(w\) and the coefficients \(p_j\) are real-valued and satisfy the minimal conditions

\[ w > 0, \quad p_n > 0, \quad w, p_n^{-1}, p_j \in L_{10c}(0, \infty), \quad 0 \leq j \leq n - 1. \tag{1.2} \]

It is well known [6] that the expression \(M\) determines a minimal operator \(T_0\) in the weighted Hilbert space \(L^2_w(0, \infty)\) which is closed, symmetric, densely defined and has self-adjoint extensions. All self-adjoint extensions of \(T_0\) are known [6] to have the same essential spectrum. (Although these results are established in [6] only for the case \(w(t) = 1\) and under stronger hypotheses on the coefficients \(p_j\), they can be extended to the results mentioned above by the same methods.)

Here we are interested in finding conditions on \(w\) and \(p_j\) which ensure that the spectrum of every self-adjoint extension of \(T_0\) is discrete, i.e., the essential spectrum is empty. Such conditions have been found by many authors including Berkowitz [1], Brinck [2], Friedrichs [4, 5], Glazman [6], Ismagilov [11], Hinton [9], Hinton and Lewis [8, 10], Lewis [12], Müller-Pfeiffer [18], Molchanov [17], Read [19], Rollins [20], Tkachenko (see [6]). This list is not intended to be comprehensive—the literature on this problem is voluminous.

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Following Hinton and Lewis [10] we say that $T_0$ or, equivalently, $M$ has property BD if the spectrum of each self-adjoint extension of $T_0$ is bounded below and discrete. Two of the best known conditions for property BD are the following:

$$p_j(t) > 0, \ 1 \leq j \leq n - 1 \quad \text{and} \quad \lim_{t \to \infty} p_0(t)/w(t) = \infty \quad (1.3)$$

or

$$p_n(t) = 1 = w(t), \quad p_j(t) = 0, \ 1 \leq j < n, \quad p_0 \text{ is bounded below} \quad (1.4)$$

and

$$\lim_{t \to \infty} \int_{t}^{t+a} p_0(x) \, dx = \infty \quad \text{for each} \quad a > 0. \quad (1.5)$$

The sufficiency of (1.3) for BD is an immediate consequence of the decomposition method—see [6, p. 35]. Condition (1.5) is due to Molchanov [17] who showed that it is necessary and sufficient for the special case (1.4).

Our purpose in this paper is twofold: (i) To find sufficient conditions for property BD in the case of two term expressions. Our conditions can be viewed as extensions of (1.5) to general weight functions $w$ and general leading coefficients $p_n$; (ii) to find sufficiency criteria in the general case of expression (1.1) with middle terms.

Although our conditions are general and seem to be of a completely new type, it is our approach we wish to emphasize. This is based on some recently discovered norm inequalities in weighted $L^p$ spaces. In the case of two term expressions our method is based on certain norm inequalities of "regular" type in [14]. Then "singular" type norm inequalities from [15] are used to reduce the general case of (1.1) with middle terms to the two term case. Thus our results on two term expressions can then be applied to the general case. Moreover, our method yields the extension of other known results for two term expressions to the general case with middle terms.

2. TWO TERM EXPRESSIONS

Only results are stated here. Proofs will be given in Section 4. To avoid unduly complicated subscripts we change the notation for this case. Consider the expression $M$ given by

$$My = w^{-1} [(-1)^n (py^{(n)})^{(n)} + qy]. \quad (2.1)$$
Here \( w, p, q \) are assumed to be real valued and to satisfy the minimal conditions
\[
w > 0, \quad p > 0, \quad w, p^{-1}, q \in L_{\text{loc}}(0, \infty).
\] (2.2)

Let \( q = q^+ - q^- \), where \( q^+(t) = \max\{0, q(t)\} \).

**Definition.** Given a positive number \( a \) and a positive function \( f(t) \), let
\[
Q(w, f(t)) = \inf \int_{t}^{t + af(t)} q^+(x) \, dx
\]
where the infimum is taken over all intervals \( J \subset [t, t + af(t)] \cap (0, \infty) \) of length \( 3^{1-n}af(t) \).

Our main result on two term expressions is:

**Theorem 1.** The expression \( M \) given by (2.1) has property BD if there exists a positive function \( f(t) \) such that
\[
\lim_{t \to \infty} \left\{ \limsup_{t \to \infty} \left[ \int_{t}^{t + af(t)} (w + q^-)^{2(n-1)} \right] \right\} = 0
\] (2.3)

and for each \( a > 0 \)
\[
\lim_{t \to \infty} \int_{t}^{t + af(t)} \frac{w + q^-}{Q(t, a, f(t))} = 0.
\] (2.4)

The complicated condition (2.4) can be simplified if the function \( w + q^- \) satisfies an additional mild restriction.

**Theorem 2.** The expression \( M \) given by (2.1) has property BD if, in addition to (2.3) we have
\[
\int_{t}^{t + af(t)} (w + q^-) \leq K \int_{s}^{s + 3^{1-n}af(s)} (w + q^-)
\] (2.5)

for some fixed \( K > 0 \), all \( t \geq N \), all sufficiently small \( a > 0 \), and for all \( s \) such that \([s, s + 3^{1-n}af(s)] \subset [t, t + af(t)]\); and for each \( a > 0 \)
\[
\lim_{t \to \infty} \int_{t}^{t + af(t)} (w + q^-) q^+ = 0.
\] (2.6)

Condition (2.6) can be viewed as an extension of the Molchanov criterion. The role of the function \( f(t) \) in conditions (2.3) through (2.6) is to allow the intervals of integration to have varying lengths in contrast with (1.5). The presence of \( f(t) \) in these conditions broadens the class of functions \( p, q, w \) which satisfy the conditions. For some classes of functions condition (2.6) even with \( f(t) = 1 \) is necessary and sufficient for BD to hold.
THEOREM 3. Assume \( q(t) \geq -cw(t) \) for \( t \geq 0 \) and some \( c > 0 \). Suppose \( w(t) \leq t^\alpha u(t) \) and \( p(t) \geq t^\alpha v(t) \) for some \( \alpha > 0 \), where \( u \) and \( v \) are locally integrable functions.

(i) If \( 0 < \delta < u(t) < D < \infty \) and \( v(t) \geq \delta > 0 \), then (2.6) with \( f(t) = 1 \) is sufficient for property BD.  
(2.7)

(ii) If \( 0 < D < u(t) < \infty \) and \( v(t) \geq D < \infty \), then (2.6) with \( f(t) = 1 \) is necessary for \( M \) to have property BD.  
(2.8)

Remark. The special case \( p = w = f = 1 \) of Theorem 3 is the well-known result of Molchanov [17].

In the second order case condition (2.5) of Theorem 2 is not needed. For the convenience of the reader we state this case in full.

THEOREM 4. Let \( p, q \) and \( w \) satisfy (2.2). In the Hilbert space \( L^2_w(0, \infty) \) the spectrum of every self-adjoint extension of the minimal operator of the expression

\[
My = w^{-1} \left[ -(py')' + qy \right]
\]

(2.9)

is bounded below and discrete if there is some positive function \( f \) such that

\[
\lim_{a \to 0} \left\{ \limsup_{t \to \infty} \int_t^{t+af(t)} (w + q^-) p^{-1} \right\} = 0
\]

(2.10)

and for each \( \alpha > 0 \)

\[
\lim_{t \to \infty} \int_t^{t+af(t)} (w + q^-) q^+ = 0.
\]

(2.11)

The special case \( w = 1 \), \( q^- = 0 \) of Theorem 4 is closely related to a result of Müller-Pfeiffer [18]. However conditions (2.10), (2.11) are more explicit than those in [18]. Also our proof is different.

According to Theorem 3, for some classes of functions condition (2.6) is necessary and sufficient for BD to hold. It turns out that nevertheless (2.6) can be weakened if (2.3) is appropriately strengthened.

THEOREM 5. The expression \( M \) has property BD if there exists a positive function such that for some number \( \alpha > 0 \) we have

\[
\lim_{t \to \infty} f(t)^{2(n-1)} \int_t^{t+af(t)} (w + q^-) p^{-1} = 0
\]

(2.12)

and

\[
\lim_{t \to \infty} \int_t^{t+af(t)} \frac{(w + q^-)}{Q(t, \alpha, f(t))} = 0.
\]

(2.13)
Corollary 1. Assume (2.5) holds. Then M has property BD if (2.12) and (2.6) hold for some positive function f and some positive number a.

Corollary 2. Let n = 1. Then M has property BD if
\[ \lim_{t \to \infty} \int_{t}^{t+a} (w + q^-) p^{-1} = 0 \]
and
\[ \lim_{t \to \infty} \int_{t}^{t+a} (w + q^-) q^+ = 0 \]
hold for some positive function f and some positive number a.

Corollary 3. Let n = 1. Suppose
\[ q(t) \geq -c, \quad w(t) \leq B \quad \text{and} \quad \lim_{t \to \infty} p(t) = \infty \]
or
\[ q(t) \geq -c, \quad 0 < b \leq p(t) \quad \text{and} \quad \lim_{t \to \infty} w(t) = 0. \]
Then M has property BD if
\[ \lim_{t \to \infty} \int_{t}^{t+a} q^+ = \infty \]
for some \( a > 0 \).

Note that (2.18) allows \( q \) to be identically constant on intervals \( I_n \to \infty \) of length less than \( a \). Recently Read [19] has shown that when \( n = 1 \) property BD can hold even for potentials \( q(t) \) which have the property that \( q(t) \to -\infty \) as \( t \to \infty \) through a sequence of certain intervals \( I_n \).

Corollary 4. Let \( n = 1 \) or \( n > 1 \) and assume (2.5) holds. The expression M has property BD if
\[ q^- \in L^u(0, \infty), \quad p^{-1} \in L^s(0, \infty), \quad w \in L^v(0, \infty), \]
\[ 1 \leq s, u, v \leq \infty; \]
and
\[ \liminf_{t \to \infty} \int_{t}^{t+a} q \geq c > 0, \quad \text{for some} \quad a > 0. \]
A few simple examples are mentioned here to illustrate some of the above conditions.

**Example 1.** Let \( p(t) = t, \ w(t) = 1, \ q^-(t) = 0, \) i.e., \( q(t) \geq 0. \) Then (2.3) holds with \( f(t) = \sqrt{t}. \) Thus, property BD holds by Theorem 2, if, in addition, \( q \) satisfies

\[
\lim_{t \to \infty} \frac{\int_t^{t + a\sqrt{t}} q}{\sqrt{t}} = \infty \quad \text{for each fixed} \quad a > 0.
\]

**Example 2.** Let \( w(t) = 1, \ q^-(t) = 0, \) i.e., \( q(t) > 0, \ p(t) = t^{-1}, f(t) = t^{-1/2}. \) Then (2.3) holds. Also property BD holds, according to Theorem 2, if in addition for each fixed \( a > 0 \)

\[
\lim_{t \to \infty} \frac{\int_t^{t + a\sqrt{t}} q}{\sqrt{t}} = \infty.
\]

**Example 3.** Let \( n = 1, \ p(t) = t^{-1}, \ w(t) = 1, \ q^-(t) = 0, \ a = 1, \ f(t) = t^{-1/2 - \epsilon}, \) \( \epsilon > 0. \) Then (2.14) holds. Thus (2.15) implies BD.

**Example 4.** Let \( n = 1, \ p(t) = t, w(t) = 1, \ q^-(t) = 0, \ a = 1, f(t) = t^{1/2 + \epsilon}, \) \( \epsilon > 0. \) Then (2.14) holds. Thus (2.15) implies BD.

### 3. Expressions with Middle Terms

Our results are based on the class of admissible weight function \( s \) in the inequality

\[
\int_J |y^{(k)}|^2 s \leq K \left( \int_J |y^{(n)}|^2 s \right)^{\alpha} \left( \int_J |y|^2 s \right)^{\beta}.
\]

(H3.1) Here \( n, k \) are integers with \( 1 \leq k < n, \) \( \alpha = (n - k)/n, \beta = k/n, \) and \( J \) is a half line \( J = (a, \infty), \) \( -\infty < a < \infty. \) For a fixed constant \( K, \) \( 0 < K < \infty, \) let \( W_{n,k}(K) \) be the class of all locally integrable non-negative functions \( s \) such that (3.1) holds for all functions \( y \) such that \( y^{(n+1)} \) is locally absolutely continuous on \( J \) and the two integrals on the right in (3.1) are finite. That (3.1) does not hold for arbitrary weight functions \( s \) can be seen from the simple example: \( n = 2, \ k = 1, \ y(t) = t, \) \( s(t) = \exp(-t). \)

Let

\[
W = \bigcup_{0 < K < \infty} W_{n,k}(K).
\]
Note that $W$ does not depend on $n$ and $k$ since (3.1) holds for all $n > 2$, $k = 1, \ldots, n-1$ if it holds for $n = 2$, $k = 1$. This follows from an induction argument.

The classes $W$ and $W_{n,k}(K)$ are not well defined in the sense that it is not clear which functions they contain. It is known [15] that $W$ contains all (non-negative) non-decreasing functions.

Clearly if (3.1) holds for some positive constant $K$ then there is a smallest such $K$. This smallest constant does not depend on the particular half-line $J = (a, \infty)$, $-\infty < a < \infty$, but does depend, in general, on $n$, $k$ and $s$ and so we denote it by $K(n, k, s)$. In the special case $s(t) = 1$ we also denote this constant by $K(n, k)$. The exact values of $K(n, k)$ are not known except for $n = 2, 3, 4$ [13], however, an algorithm for their computation is known [13, 16].

**Lemma 1.** Let $s \in W$; i.e., assume (3.1) holds for some $K > 0$. Then

(a) for any $\varepsilon > 0$ there is some $K(\varepsilon) > 0$ such that

$$\int_J |y^{(k)}|^2 s \leq \varepsilon \int_J |y^{(n)}|^2 s + K(\varepsilon) \int_J |y|^2 s$$

(3.2)

for all functions $y$ with $y^{(n-1)}$ locally absolutely continuous and for which both integrals on the right are finite;

(b) for any $\varepsilon > 0$ there exists a $K(\varepsilon) > 0$ such that

$$\int_J |y^{(k)}|^2 s \leq \varepsilon \int_J |y|^2 s + K_k(\varepsilon) \int_J |y^{(n)}|^2 s$$

(3.3)

for all $y$ with $y^{(n-1)}$ locally absolutely continuous and such that the two integrals on the right are finite.

Furthermore, if $w$ is non-decreasing (and non-negative) then $K(\varepsilon)$ can be taken as

$$K_k(\varepsilon) = [K(n-k)(Kk/n)^{\varepsilon-1}]/\varepsilon - \varepsilon K(n-k)$$

(3.4)

in (3.2) and as

$$K_k(\varepsilon) = K(k/n)(K(n-k)/n)^{\varepsilon-1} / e^{-(n-k)\varepsilon}$$

(3.5)

in (3.3).
Proof. From (3.1) and the general inequality between weighted arithmetic and geometric means [7] we get for any \( \varepsilon > 0 \)
\[
\int |y^{(k)}|^2 s \leq K \left( \varepsilon \int |y^{(n)}|^2 s \right)^{\alpha} \left( \varepsilon \int |y|^2 s \right)^{\beta}
\]
\[
\leq K \left[ \beta \varepsilon \int |y^{(n)}|^2 s + \alpha \varepsilon \int |y|^2 s \right],
\]
where \( a = \alpha^{-1}, b = -\beta^{-1} \). Now (3.2) follows by setting \( \varepsilon = K\beta \varepsilon_1 \); this yields (3.4). The proof of (3.5) is entirely similar.

**Theorem 6.** Let \( M \) be given by (1.1) with \( w \) and \( p_j \) satisfying (1.2). Choose \( s_j(t) > 0 \) in \( W \) such that
\[
P_j(t) > -s_j(t), \quad j = 1, \ldots, n-1, \quad t \geq 0.
\]
Let \( \varepsilon > 0 \) and choose \( K_k(\varepsilon) \) so that (3.2) holds with \( s = s_k \); let \( K(\varepsilon) = \max K_k(\varepsilon), k = 1, \ldots, n-1 \). Then \( M \) has property BD if
\[
p_n(t) \geq \varepsilon \sum_{j=1}^{n-1} s_j(t)
\]
and
\[
\left[ p_0(t) - K(\varepsilon) \sum_{j=1}^{n-1} s_j(t) \right]/w(t) \to \infty \quad \text{as} \quad t \to \infty
\]
(3.8)
or if
\[
p_n(t) \geq K^*(\varepsilon) \sum_{j=1}^{n-1} s_j(t)
\]
(3.9)
and
\[
\left[ p_0(t) - \varepsilon \sum_{j=1}^{n-1} s_j(t) \right]/w(t) \to \infty \quad \text{as} \quad t \to \infty,
\]
(3.10)
where \( \varepsilon > 0 \) is arbitrary and \( K^*(\varepsilon) = \max K_k(\varepsilon), k = 1, \ldots, n-1 \) and each \( K_k(\varepsilon) \) is given by (3.5).

Given an arbitrary weight function \( s \) for which (3.1) holds we do not have an upper bound on the constant \( K \). However, if \( s \) is non-decreasing then Kwong and Zettl showed that
\[
K = K(n, k, s) \leq K(n, k),
\]
(3.11)
where $K(n, k)$ is the best constant in (3.1) when $s(t) = 1$. As mentioned above, these constants can theoretically be computed by the algorithm of Ljubic [16] or Kupcov [13] but the exact values of $K(n, k)$ are not known except for $n = 2, 3, 4$.

An upper bound for $K(n, k)$ in terms of the known constant [7] $K(2, 1) = 2$ can be obtained by an induction argument. Thus

$$K(n, k) \leq (2)^{n^k(n-k)/2}. \quad (3.12)$$

The bound (3.12) is very poor. Using (3.11) and (3.12) in (3.4) we get an explicit constant for $K(\varepsilon)$ in conditions (3.7) through (3.10).

**Corollary 1.** Let $M$ be given by (1.1) with $w$ and $p_j$ satisfying (1.2). Suppose $w$ is nondecreasing. Then $M$ has property BD if (3.6), (3.7) and (3.8) hold with

$$K(\varepsilon) = \max K_k(\varepsilon), \quad k = 1, \ldots, n - 1,$$

where $K_k(\varepsilon)$ is given by (3.4) with $K = 2^k(n-k)/2$.

**Corollary 2.** Let $M$ be as in Theorem 6 and suppose (3.6) holds with $0 \leq s_j(t) \in W$. Let $S = s_1 + s_2 + \cdots + s_{n-1}$. Then $M$ has property BD if, for some $\varepsilon > 0$ and some $T > 0$, we have

$$p_n(t)/S(t) \geq \varepsilon > 0 \quad \text{for} \quad t \geq T \quad (3.13)$$

and

$$[p_0(t) - KS(t)]/w(t) \to \infty \quad \text{as} \quad t \to \infty \quad \text{for every} \quad K > 0. \quad (3.14)$$

**Corollary 3.** Let $M$ be as in Theorem 6 and suppose (3.6) holds with $0 \leq s_j(t) \in W$. Let $S = s_1 + s_2 + \cdots + s_{n-1}$. Then $M$ has property BD if

$$p_n(t)/S(t) \to \infty \quad \text{as} \quad t \to \infty \quad (3.15)$$

and

$$[p_0(t) - \varepsilon S(t)]/w(t) \to \infty \quad \text{as} \quad t \to \infty \quad (3.16)$$

for every sufficiently small $\varepsilon > 0$.

The result of the next corollary is probably known but we have not seen it in the literature.
COROLLARY 4. Let $M$ be given by (1.1) with $w$ and $p_j$ satisfying (1.2). Suppose

$$p_j(t) \geq -c_j, \quad \text{for some } c_j > 0, j = 1, 2, \ldots, n - 1,$$

(3.17)

$$p_n(t) \geq \varepsilon \sum_{j=1}^{n-1} c_j, \quad \text{for some } \varepsilon > 0,$$

(3.18)

$$\left[ p_0(t) - K \sum_{j=1}^{n-1} c_j \right] w(t) \to \infty \quad \text{as } t \to \infty \text{ for any } K > 0,$$

(3.19)

then $M$ has property BD.

The case $c_j = 0, j = 1, \ldots, n - 1$ reduces to (1.5). Given any coefficients $p_j$ and any weight function $w$ we can find $p_n$ and $p_0$ so that the hypothesis of Theorem 6 hold.

COROLLARY 5. Given any $p_j, j = 1, \ldots, n - 1$ and $w$ satisfying (1.2) there exist $p_n$ and $p_0$ such that $M$ has property BD.

Next we combine the method of Section 3 with our results in Section 2 to get strong results in the general case.

THEOREM 7. Let $M$ be given by (1.1) and assume (1.2) holds. Choose $s_j(t) \geq 0$ in $W$ such that (3.6) holds for $j = 1, 2, \ldots, n - 1$. (Such a choice is always possible since any non-decreasing non-negative function is in $W$.) Let $S = \sum_{j=1}^{n-1} s_j$. Set

$$p = p_n - \varepsilon S, \quad q = p_0 - K(\varepsilon) S = q^+ - q^-,$$

(3.20)

where $K(\varepsilon)$ is a positive constant depending on $\varepsilon$ and is defined as in Theorem 6.

If, for some $\varepsilon$, (2.3) and (2.4) or (2.3), (2.5) and (2.6), hold; then $M$ has property BD.

We remark that our method of "sliding" the middle terms $p_j, j = 1, \ldots, n - 1$ over to the two end terms $p_n$ and $p_0$ can be used to extend any known results for two term expressions (2.1) to the general case (1.1).

We also remark that an inequality more general than (3.1) is established in [15]. The weights in the three integrals may be chosen differently provided they are suitably related. Starting with the more general version of (3.1), one can deduce more general BD criteria. We do not pursue this point further here.
4. PROOFS

All our proofs are based on the characterization of property BD given by Lemma 2, below. This is a consequence of the decomposition method (see [6, p. 35]).

The proofs of the results in Section 2 are based on norm inequalities of Kwong and Zettl in [14] and the results in Section 3 depend on norm inequalities of a different type due to Kwong and Zettl [15]. Our method for establishing Theorems 6 and 7 can be used to extend any known results for two term expressions (2.1), given in terms of the coefficients, to the general case of expression (1.1) with middle terms.

For $N > 0$, let $Q_N$ denote the set of all complex-valued functions $y$ defined on $[0, \infty)$ with compact support in $(N, \infty)$ such that $y^{(j)}$ is locally absolutely continuous for $j = 0, 1, \ldots, n - 1$ and $y^{(n)} \in L^1(0, \infty)$. For $y$ in $Q_N$, let

$$I(y, N) = \sum_{j=0}^{n} \int_{-N}^{\infty} p_j |y^{(j)}|^2. \quad (4.1)$$

**Lemma 2.** Let $M$ be given by (1.1) where the coefficients $p_j$ and weight function $w$ are real valued and satisfy (1.2). Then $M$ has property BD if and only if for each real number $\lambda$ there exists a corresponding number $N > 0$ such that

$$I(y, N) \geq \lambda \int_{-N}^{\infty} w y^2 \quad (4.2)$$

for each real-valued $y$ in $Q_N$.

**Proof.** The proof of Lemma 2 is given in [6, Section 10] for the case $w = 1$ and $p_j$ continuous. The modifications needed in this proof to prove Lemma 2 are straightforward and hence omitted.

The next lemma is a special case of Theorem 3 in [14]. The notation in [14] has been changed to conform with our notation here and the result adapted for a compact interval.

**Lemma 3.** Let $I = [a, b]$ be a compact interval of length $L$. Let $n \geq 1$ be a positive integer. Assume that

$$p^{-1}, w, q \in L(I) \quad \text{and} \quad q \geq 0, \quad \text{and} \quad 0 < \int_I q. \quad (4.3)$$

Then the inequality

$$\int_I y'w \leq C \int_I |y^{(n)}|^2 p + D \int_I y'q \quad (4.4)$$
holds for any function $y$ such that $y^{(n-1)}$ is absolutely continuous on $I$ and the two integrals on the right of (4.4) are finite where

$$C = C(n, p, w, I) = 2^n L^{2(n-1)} \int_I w \int_I y^{-1}, \quad (4.5)$$

$$D = D(n, q, w, I) = 2 \int_I w \left[2^{n-1} L^{2(n-1)} B(n - 1, q, I) + 2^{n-2} L^{2(n-2)} B(n - 2, q, I) + \cdots + 2 L^2 B(1, q, I) + B(0, q, I) \right], \quad (4.6)$$

with

$$B(k, q, I) = \inf_{I} B(k, q, I) \quad (4.7)$$

and

$$B(k, q, I) = \frac{1}{k} \left( \min_{I_i} \int_I \right) \min \left( \int_q \right). \quad (4.8)$$

Here the infimum in (4.7) is taken over all partitions $I = \{I_i = [a_i, b_i], i = 1, 2, \ldots, 2^{k+1} - 1\}$, where $a_i = a$, $b_i = a_{i+1}$, $i = 1, 2, \ldots, 2^{k+1} - 2$ such that

$$\int_I q > 0, \quad \text{for} \ B = I_i, i = 1, 3, 5, \ldots, 2^{k+1} - 1; \quad (4.9)$$

the first minimum in (4.8) is taken over $i = 2, 4, \ldots, 2^{k+1} - 2$ and the second over $J = I_i$, $i = 1, 3, \ldots, 2^{k+1} - 1$.

Furthermore if $q > 0$ a.e. on $I$ then the constant $D$ can be chosen as

$$D = D(n, w, q, I) = 2 \left( \int_I w \right) \left[2^{n-1} E(n - 1) + 2^{n-2} E(n - 2) + \cdots + 2 E(1) + E(0) \right] / M_{n-1} = E \int_I w / M_{n-1}, \quad (4.10)$$

where

$$E(r) = (2^{r+1} - 1)^2, \quad r = 0, 1, \ldots, n - 1 \quad (4.11)$$

$$M_{n-1} = \min \int_I q, \quad (4.12)$$

where the minimum is taken over all intervals $J = I_i$, $i = 1, 3, 5, \ldots, 2^{n+1} - 1 = m$ and where the intervals $I_i$, $i = 1, 2, \ldots, m$ form a partition of $I$ into subintervals of equal length.
Proof of Theorem 1. We show that (4.2) holds. Let $0 < \varepsilon < 1$. By (2.3) we can choose $a > 0$ small enough so that

$$2^n(a f(t))^{2(n-1)} \int_t^{t+a f(t)} (w + q^-) \int_t^{t+a f(t)} p^{-1} < \varepsilon$$

holds. Choose $N > 0$ so large that

$$\int_t^{t+a f(t)} (w + q^-)/Q(t,a,f(t)) < \varepsilon \quad \text{for all} \quad t \geq N.$$

This can be done by (2.4).

Let $y \in Q_N$ with compact support $J = [t_1, b] \subset (N, \infty)$. Define $t_{i+1} = t_i + a f(t_i)$ and let $J_i = (t_i, t_{i+1})$, $i = 1, 2, 3, \ldots$. Then $J \subset \bigcup_{i=1}^{\infty} J_i$.

Let $I = [t_i, t_i + a f(t_i)]$. Then (4.3) holds with $w$ replaced by $w + q^-$ and $q$ replaced by $q^+$—we may assume that $\int_I q^+ > 0$ by (2.4) and the definition of $Q(t,a,f(t))$. Hence, by Lemma 3, we have

$$\int_I y^2(w + q^-) \leq C \int_I |y^{(n)}|^2_p + D \int_I y^2 q^+,$$

with $C$ given by (4.5) and $D$ by (4.10), (4.11), (4.12).

From (4.5) and (4.8) we have $C < \varepsilon$. From (4.10), (4.11), (4.12), (4.14) and the definition of $Q(t,a,f(t))$ we can take $D < \varepsilon$. From this we conclude that

$$\int_I y^2(w + q^-) \leq \varepsilon \int_I |y^{(n)}|^2_p + \varepsilon \int_I y^2 q^+ + \int_I y^2 q^- + \int_{J_i} y^2 q^+.$$

Summing (4.16) over $I$ and rearranging the terms we get

$$\int_I y^2 w \leq \varepsilon \int_I |y^{(n)}|^2_p + \varepsilon \int_I y^2 q^+ - \int_I y^2 q^- \leq \varepsilon \int_I |y^{(n)}|^2_p + \varepsilon \int_I y^2 q.$$

Dividing (4.17) by $\varepsilon$ we obtain (4.2) and the proof is complete.

Proof of Theorem 3. Replacing $w^{-1} M$ by $w^{-1} M + c + 1$ we may assume that $q(t) \geq w(t) > 0$ a.e. By (4.2) it is sufficient to establish the case $w(t) = t^a u(t)$ and $p(t) = t^a v(t)$.

To prove the sufficiency note that from (i) and a calculation we get

$$\int_t^{t+a} w = aO(t^a)$$

(4.18)
and

\[ \int_t^{t+\alpha} p^{-1} = aO(t^{-\alpha}). \quad (4.19) \]

Now (2.3) and (2.5) follow from (4.18) and (4.19). Thus part (i) follows from Theorem 2.

To prove the necessity suppose that (2.6) with \( f(t) = 1 \) does not hold. Then there exist \( a > 0, \varepsilon > 0 \) and a sequence \( t_k \to \infty \)

\[ \int_{t_k}^{t_k+\alpha} q \leq \varepsilon^{-1} \int_{t_k}^{t_k+\alpha} w \leq \varepsilon^{-1} D \int_{t_k}^{t_k+\alpha} t^\alpha dt = \varepsilon^{-1} DO(t_k^\alpha), \quad (4.20) \]

for all sufficiently large \( k \).

Let \( 0 < \delta < a/2 \) and choose a function \( \phi \) in \( C^n[0, a] \) with the following properties: \( \phi(t) = 0 \) when \( t = 0 \) and \( t = a \), \( \phi(t) = 1 \) when \( \delta < t < a - \delta \). Let

\[ y_k(t) = t_k^{-\alpha/2} \phi(t - t_k), \quad t_k \leq t \leq t_k + a \]

\[ = 0, \quad 0 \leq t \leq t_k, t_k + a < t. \quad (4.21) \]

Then \( y_k \) is in \( Q_k \).

Note that

\[ \int_{t_k}^{\infty} w y_k^2 = \int_{t_k}^{t_k+\alpha} w y_k^2 \geq d \int_{t_k}^{t_k+\alpha} \phi^2(t - t_k) dt \geq d(a - 2\delta) > 0. \quad (4.22) \]

Also

\[ \int_{t_k}^{\infty} (p y_k^{(n)} + q y_k^2) = \int_{t_k}^{t_k+\alpha} \{ p y_k^{(n)} + q y_k^2 \} \leq K_n \int_{t_k}^{t_k+\alpha} \phi^{(n)}(t - t_k) dt \]

\[ + K_{n-1} \int_{t_k}^{t_k+\alpha} \phi^{(n-1)}(t - t_k) dt + \cdots + K_0 \int_{t_k}^{t_k+\alpha} \phi^2(t - t_k) dt \]

\[ + \int_{t_k}^{t_k+\alpha} q(t) t_k^{-\alpha} \phi^2(t - t_k) dt, \quad (4.23) \]

where the \( K_j \) are positive constants independent of \( k \). The first \( n + 1 \) integrals on the right are all bounded uniformly in \( k \) since \( \phi^{(j)} \) is bounded, \( j = 0, \ldots, n \). Using (4.20) we get

\[ \int_{t_k}^{t_k+\alpha} q(t) t_k^{-\alpha} \phi^2(t - t_k) dt \leq K \quad (4.24) \]
for all \( k \) sufficiently large. Hence

\[
\int_{t_k}^{\infty} \{ p y_k^{(n)^2} + q y_k^2 \} \leq C
\]  

(4.25)

uniformly in \( k \).

From (4.22) and (4.25) it follows that for \( \lambda \) large enough we have

\[
\lambda \int_{t_k}^{\infty} w y_k^2 > \int_{t_k}^{\infty} \{ p y_k^{(n)^2} + q y_k^2 \}.
\]  

(4.26)

Finally from (4.26) and Lemma 2 we conclude that \( M \) does not have property BD.

Proof of Theorem 4. This proof is based on the following lemma. Although this result is known [14] we state and prove it here for the sake of completeness.

**Lemma 4.** Let \( J = [a, b] \) be a compact interval of the real line. Let the real valued functions \( p, q, w \) satisfy

\[
p(t) > 0 \text{ a.e., } w(t) > 0 \text{ a.e., } q(t) \geq 0 \text{ a.e., \hspace{1cm} (4.27)}
\]

\[
\int J q > 0, \hspace{1cm} p^{-1}, q, w \in L^1(J).
\]

Then, for any absolutely continuous function \( y \) on \( J \) for which \( \int J p y^2 \) exists, we have

\[
\int J y^2 w \leq A \int J p y^2 + B \int J q y^2,
\]  

(4.28)

where

\[
A = 2 \int J p^{-1} J w, B = 2 \int J w \int J q.
\]  

(4.29)

**Proof.** Let \( |y| \) achieve its minimum at \( t_0 \in J \). Then

\[
|y(t)| \leq |y(t_0)| + \left| \int_{t_0}^t y' \right|, \hspace{1cm} t \in J.
\]  

(4.30)

Using the Schwarz inequality, we have

\[
\left| \int_{t_0}^t y' \right| \leq \int_{t_0}^t |y'| \leq \left( \int_{t_0}^t p^{-1} \right)^{1/2} \left( \int_{t_0}^t p y'^2 \right)^{1/2}
\]  

(4.31)
for \( t \geq t_0 \) and a similar inequality holds when \( t < t_0 \). Since \( y^2(t_0) \) is minimum

\[
y^2(t_0) \int_q q = \int_q y^2(t_0) q(t) \, dt \leq \int_q q(t) y^2(t) \, dt.
\]

Hence

\[
y^2(t_0) \leq \int_q q y^2 \int_q q. \tag{4.32}
\]

From (4.30), (4.31), (4.32) and the elementary inequality \((a + b)^2 \leq 2(a^2 + b^2)\) we obtain

\[
\int_q y^2 w \leq 2 \int_q w y^2(t_0) + 2 \int_q w y^2 \leq 2 \left[ \int_q w \int_q q \right] \int_q q y^2 + 2 \left[ \int p^{-1} \int w \right] \int_q p y^2. \tag{4.33}
\]

This is (4.28), (4.29).

To prove Theorem 4 let \( 0 < \varepsilon < 1 \). Let \( q = q^+ - q^- \), where \( q^+(t) = \max(0, q(t)) \). By (2.10) there exist \( a > 0, N_1 > 0 \) such that

\[
2 \int_t^{t + af(t)} (w + q^-) p^{-1} < \varepsilon, \tag{4.34}
\]

for all \( t \geq N_1 \).

By (2.11) there is an \( N_2 > 0 \) such that

\[
\int_t^{t + af(t)} q^+ > 0
\]

and

\[
\int_t^{t + af(t)} (w + q^-) \int_t^{t + af(t)} q^+ < \varepsilon, \tag{4.35}
\]

when \( t \geq N_2 \). Let \( N = \max\{N_1, N_2\} \). Let \( y \) in \( Q_N \) have support in the compact interval \( J = [t_1, b] \subset (N, \infty) \). Let \( t_{i+1} = t_i + af(t_i) \). Set \( J_i = [t_i, t_{i+1}] \). Then \( J \subset \bigcup_{i=1}^\infty J_i \). By Lemma 4, with \( q \) replaced by \( q^+ \) and \( w \) by \( w + q^- \), and (4.34), (4.35), we have

\[
\int_t (w + q^-) y^2 \leq \varepsilon \int_t q^+ y^2 + \varepsilon \int_t p y^2. \tag{4.36}
\]
for $I = J_i$, $i = 1, 2, 3, \ldots$. Summing over $i$ we get
\[ \int_J (w + q^-) y^2 \leq \varepsilon \int_J py'^2 + \varepsilon \int_J q^+ y^2. \]  \hfill (4.37)

Rearranging terms we get
\[ \int_N^\infty wy^2 \leq \varepsilon \int_N^\infty py'^2 + \varepsilon \int_N^\infty q^+ y^2 - \int_N^\infty q^- y^2 \leq \varepsilon \int_N^\infty py'^2 + \varepsilon \int_N^\infty qy^2. \]  \hfill (4.38)

Dividing (4.38) by $\varepsilon$ yields (4.2) and the proof is complete.

Proof of Theorem 5. This is similar to the proof of Theorem 1 and so we only outline it here. We obtain inequality (4.15) as in the proof of Theorem 1. Using (2.12) we can get $C < \varepsilon$ and by (2.13) $D < \varepsilon$, both on intervals of the form $I = (t, t + a\varepsilon(t))$ for $t \geq N$. Then 4.17 follows as before.

Corollaries 1 and 2 follow by noting, as we did in Theorems 2 and 4, that if $n > 1$ and (2.5) holds or if $n = 1$ then (2.4) can be replaced by (2.6).

In the case of Corollary 3 conditions (2.16) or (2.17) imply (2.14) and (2.15) reduces to (2.18) since $q$ can be assumed positive and
\[ \int_{t + a\varepsilon(t)}^{t + a\varepsilon(t)} w \leq B\varepsilon(t), \]
which is bounded when $f(t) = 1$.

Corollary 4 follows from the observation that $\int_t^{t + a} p^{-1}$ is bounded when $s = \infty$ in (2.19),
\[ \int_t^{t + a} p^{-1} \leq a^{(s-1)/s} \left( \int_t^{t + a} p^{-s} \right)^{1/s} \to 0 \quad \text{as } t \to \infty \]
when $1 \leq s < \infty$; and
\[ \int_t^{t + a} w \leq a^{(s-1)/s} \left( \int_t^{t + a} w^s \right)^{1/s} \to 0 \quad \text{as } t \to \infty \]
and a similar inequality holds with $w$ replaced by $q^-$. When $v = \infty$ (or $u = \infty$) then $\int_t^{t + a} w$ (or $\int_t^{t + a} q^-$) is bounded.

References