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A comment on the χ_y genus and supersymmetric quantum mechanics

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Abstract

In this Letter we show that a simple modification of supersymmetric quantum mechanics involving a mass term for half the fermions naturally leads to a derivation of the integral formula for the χ_y genus, a quantity that interpolates between the Euler characteristic and arithmetic genus. We note that this modification naturally arises in the moduli space dynamics of monopoles or instantons in theories with 16 supercharges partially broken to 8 supercharges by mass terms.

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Index theory, as developed by Atiyah and Singer in Ref. [1], has many important applications in theoretical physics. For the physicist, many of the mathematical complications of index theory can be avoided by following Alvarez-Gaumé [3] (see also [4]) and formulating it in the context of a suitable supersymmetric quantum mechanical system. The crucial idea is to find a supersymmetric quantum mechanical system whose Witten index yields the topological index of the elliptic complex in question.

On the other hand, there are situations in which supersymmetric quantum mechanics arises naturally. The one we have in mind here, is in the semi-classical quantization of solitons in field theory. In the classical limit the dynamics can often be described in the terms of motion on the moduli space of the soliton (the space of classical solutions). Semi-classical effects are then described by quantum mechanics on the moduli space. In a supersymmetric theory, soliton solutions generally preserve half the supersymmetries of the parent theory and these are inherited by the quantum mechanical system.

An example of this set up is in five-dimensional gauge theories which have soliton solutions consisting of conventional instanton solutions embedded in the four spatial dimensions. Semi-classical effects are described by quantum mechanics on the moduli space of Yang–Mills instantons of a given charge $\mathfrak{M}$ [5]. If the parent theory has 16 supercharges ($\mathcal{N}=4$ in four dimensions) then the quantum mechanical system is the one associated to the de Rahm complex which usually admits 2 supercharges but since $\mathfrak{M}$ is hyper-Kähler this is enhanced to 8. On the other hand if the parent theory has 8 supercharges ($\mathcal{N}=2$ in four dimensions) then the quantum mechanical system is the one associated to the Dolbeault—or equivalently,

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since $\mathfrak{M}$ is hyper-Kähler, the Dirac—complex which usually admits 1 supercharge but, as above, this is enhanced to 4. A similar application concerns the semi-classical quantization of monopoles in the same theories in four dimensions. In this case the same kind of quantum mechanical systems arise but now associated to the monopole moduli space (see, for example, [6–8]).

It is well known that the gauge theories with 16 supercharges can be broken to one with 8 supercharges by adding suitable mass terms. In four dimensions the mass deformed theory is sometimes called the $\mathcal{N}=2^*$ theory. These mass terms induce terms in the quantum mechanical system describing the soliton dynamics which have the effect of breaking one half of the supersymmetries. This suggests that the system with the supersymmetry-breaking terms will somehow interpolate between the de Rahm complex and Dolbeault complex and, in particular, its Witten index will interpolate between the associated topological indices; that is the *Euler characteristic* and the *arithmetic genus*. In the applications to instanton-solitons in five-dimensional supersymmetric gauge theory the index is directly relevant because it determines the instanton contributions to the prepotential.

Although the problem was inspired by instanton solitons in five-dimensional gauge theory (or monopoles in the associated four-dimensional theory) we shall divorce our discussion from these particular examples because there are additional complications in these cases. In particular, the moduli spaces of instantons or monopoles are non-compact and this leads to subtleties in defining the Witten index. We shall work with a target space $\mathfrak{M}$ which is compact. In addition, in both the instanton and monopole examples, VEVs for scalar fields in the parent theory lead to more complicated quantum mechanical systems involving potentials induced by vector fields on the moduli space, as one can see in the instanton case in [5] and in the monopole case in [8]. This leads to an equivariant generalization of index theory and once again we shall avoid these complications in this Letter.

Following Alvarez-Gaumé [3], we start with the quantum mechanical system associated to the de Rahm complex of a compact manifold $\mathfrak{M}$. Let X^μ, $\mu=1,\dots,n$, be local coordinates for $\mathfrak{M}$ which become one-dimensional fields $X^\mu(t)$ in the quantum mechanical system. Associated to these bosonic quantities, we have 2-component fermions $\psi^\mu_\alpha(t)$, $\alpha=1,2$, which are Grassmann-valued fields. The basic Lagrangian that defines the system is

$$\begin{aligned}\mathcal{L}=\frac{1}{2}g_{\mu\nu}\dot X^\mu\dot X^\nu+\frac{i}{2}g_{\mu\nu}\psi^\mu_\alpha\dot\psi^\nu_\alpha\\ +\frac{i}{2}g_{\mu\nu}\psi^\mu_\alpha\Gamma^\nu_{\sigma\rho}\dot X^\sigma\psi^\rho_\alpha+\frac{1}{4}R_{\mu\nu\sigma\rho}\psi^\mu_1\psi^\nu_1\psi^\sigma_2\psi^\rho_2,\end{aligned}\tag{1}$$

where $g_{\mu\nu}(X)$ is the metric on $\mathfrak{M}$ and $R_{\mu\nu\sigma\rho}(X)$ is the usual Riemann tensor associated to the Levi-Civita connection $\Gamma^\nu_{\sigma\rho}(X)$.

The quantization of the theory follows by imposing the following canonical (anti-)commutation relations:

$$\left[X^\mu,p^\nu\right]=ig^{\mu\nu},\qquad \left\{\psi^\mu_a,\psi^\nu_b\right\}=\delta_{ab}g^{\mu\nu}.\tag{2}$$

It is useful to define the supercovariant momentum

$$\pi_\mu=p_\mu+\Gamma_{\mu\nu\lambda}\psi^\nu_1\psi^\lambda_2.\tag{3}$$

The system is invariant under 2 supersymmetries generated by the supersymmetry charges

$$Q_\alpha=\psi^\mu_\alpha\pi_\mu.\tag{4a}$$

It is important to realize that operator ordering here is significant. The Hamiltonian is obtained by the anti-computation of two supercharges:

$$\{Q_\alpha,Q_\beta\}=2\delta_{\alpha\beta}\mathcal{H},\tag{5}$$

yielding

$$\mathcal{H}=\frac{1}{2\sqrt{g}}\pi_\mu\sqrt{g}\,g^{\mu\nu}\pi_\nu.\tag{6}$$

The Hilbert space of the model is realized in terms of a fermionic Fock space, with creation operators and annihilation operators given by the combinations

$$\begin{aligned}b^{\mu\dagger}&=\frac{1}{\sqrt2}\left(\psi^\mu_1-i\psi^\mu_2\right),\\ b^\mu&=\frac{1}{\sqrt2}\left(\psi^\mu_1+i\psi^\mu_2\right).\end{aligned}\tag{7}$$

The states

$$f_{\mu_1\cdots\mu_p}(X)b^{\mu_1\dagger}\cdots b^{\mu_p\dagger}|0\rangle\tag{8}$$

are in one-to-one correspondence with the de Rahm complex of $\mathfrak{M}$; for instance, the above state corresponds to the p-form

$$f_{\mu_1\cdots\mu_p}(X)\,dX^{\mu_1}\wedge\cdots\wedge dX^{\mu_p}.\tag{9}$$

Under this correspondence the supercharges Q_α are realized in terms of the exterior derivative d and its adjoint $d^\dagger$:

$$Q_1 = -\frac{i}{\sqrt{2}}\big(d - d^\dagger\big), \qquad Q_2 = \frac{1}{\sqrt{2}}\big(d + d^\dagger\big). \tag{10}$$

We now suppose that $\mathfrak{M}$ is a Kähler manifold for which g is the Kähler metric. So $\mathfrak{M}$ has a Kähler form ω that is closed. The Kähler metric g is Hermitian with respect to the complex structure $\boldsymbol{I}$ and furthermore $g(\boldsymbol{I}X, Y) = \omega(X, Y)$, for 2 arbitrary tangent vectors X and Y. Under these circumstances, it is well known that the quantum mechanical system admits 2 additional supersymmetries generated by the supercharges

$$Q'_\alpha = (\boldsymbol{I} \cdot \psi_\alpha)^\mu \pi_\mu. \tag{11}$$

Since $\mathfrak{M}$ is a complex manifold we can choose (anti-) holomorphic coordinates $(z^j, \bar{z}^j)$, $j = 1, \ldots, (1/2)n$, compatible with the complex structure: $(\boldsymbol{I} \cdot z)^j = iz^j$ and $(\boldsymbol{I} \cdot \bar{z})^j = -i\bar{z}^j$. In this coordinate system, the Hilbert space can be built on a fermionic Fock space for which ψ_1^j and $\bar{\psi}_2^j$ are the creation operators while $\bar{\psi}_1^j$ and ψ_2^j are the annihilation operators. States in the Hilbert space are naturally identified with elements of the Dolbeault complex via the correspondence

$$\begin{aligned} &\psi_1^{j_1} \cdots \psi_1^{j_p} \bar{\psi}_2^{k_1} \cdots \bar{\psi}_2^{k_q} |0\rangle \\ &\quad \longleftrightarrow \quad dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_q}. \end{aligned} \tag{12}$$

Since $\mathfrak{M}$ is Kähler, we can add a kind of mass for one of the species of fermions to the Lagrangian:

$$\begin{aligned} \mathcal{L}_m &= -\frac{1}{2} m \omega(\psi_2, \psi_2) + c = -\frac{1}{2} m \psi_2^\mu \omega_{\mu\nu} \psi_2^\nu + c \\ &= -\frac{1}{2} m \psi_{2\mu} (\boldsymbol{I} \cdot \psi_2)^\mu + c, \end{aligned} \tag{13}$$

where m is a parameter. In the application to instantons, such a term was derived in the effective quantum mechanics on the moduli space in [5]. In the monopole application, such a term can be extracted indirectly from [8], by choosing the matter content of the $\mathcal{N} = 2$ theory to transform in the adjoint representation. In both cases m is the mass of the adjoint hypermultiplet in the parent theory.

The term (13) is only invariant under half the original supersymmetries; namely those generated by Q_1 and Q'_1, following from the fact that the Kähler form is covariantly constant on a Kähler manifold. In (13), c is a constant that arises via a normal ordering prescription and its value is fixed as follows. In the canonical formalism, we require that the term (13) leads to a modification of the Hamiltonian operator of the normal-ordered form

$$\mathcal{H}' = \mathcal{H} + \mathcal{H}_m, \qquad \mathcal{H}_m = \frac{1}{2} m {:}\psi_{2\mu} (\boldsymbol{I} \cdot \psi_2)^\mu{:}, \tag{14}$$

in order that it annihilates the vacuum state $|0\rangle$. Notice in the language of the Dolbeault complex, $\mathcal{H}_m$, up to the factor of m, simply counts the anti-holomorphic degree. Ensuring that (13) leads to (14), fixes

$$c = \frac{mn}{2}. \tag{15}$$

With the mass term added, the supersymmetry algebra gains a central charge:

$$\begin{aligned} &Q_1^2 = \mathcal{H}' - \mathcal{Z}, \qquad Q_1'^2 = \mathcal{H}' - \mathcal{Z}, \\ &\{Q_1, Q'_1\} = 0, \end{aligned} \tag{16}$$

where it is immediately apparent—since Q_1 is unchanged—that

$$\mathcal{Z} = \mathcal{H}_m. \tag{17}$$

The question is what does the modification do to the Witten index of the model? Since we have remarked that $\mathcal{H}_m$ has a very simple action on the Dolbeault complex—it simply counts the anti-holomorphic degree of a form multiplied by m—the Witten index of the deformed system will be given by

$$\mathrm{ind}_W = \sum_{i,j} (-1)^{i+j} b_{i,j} e^{-\beta m j}, \tag{18}$$

where $b_{i,j}$ are the Betti numbers of the Dolbeault complex with i and j being the holomorphic and anti-holomorphic degrees, respectively.

So the addition of the term (13) or (14), which breaks half the supersymmetry, is to deform the index. When $m = 0$ we recover the Euler characteristic. In the limit, $m \to \infty$, the index reduces to the arithmetic or Todd genus $\sum_i (-1)^i b_{i,0}$. In fact the interpolating quantity (18) is the χ_y genus of Hirzebruch [2]

$$\chi_y = \sum_{i,j} (-1)^i y^j b_{i,j}, \qquad y = -e^{-\beta m}. \tag{19}$$

Now we see how it can naturally be obtained from a deformed quantum mechanical supersymmetric σ-model. Note that the χ_y genus has also been related to supersymmetric quantum mechanics in [9], via a twisted of the boundary conditions on the fermions. Our approach obtained by adding the deformation (13), motivated by the application to soliton dynamics, is different. In related work, a geometric interpretation of the χ_y genus has been given for hyper-Kähler geometries in [10,11]. In particular, this is relevant for the application of our results to soliton quantization for which the relevant geometry is hyper-Kähler.

It remains to derive the known integral expression for the χ_y genus by computing the partition function of the deformed quantum mechanical system. The steps are a simple generalization of the standard derivation of index densities of supersymmetric quantum mechanics [3]. As usual the Witten index can be calculated by computing the Euclidean functional integral with fields being periodic in t. We define β to be the period. The resulting functional integral expression is then independent of β—apart from via the combination βm—and may be readily evaluated in the limit $\beta \to 0$ (with fixed βm). In this limit, constant configurations of $X^\mu(t)$ and $\psi^\mu_\alpha(t)$ dominate the functional integral and one can integrate out the fluctuations to Gaussian order. To this end we expand around constant configurations:

$$X^\mu \to x^\mu + \delta X^\mu(t), \qquad \psi^\mu_\alpha \to \eta^\mu_\alpha + \delta\psi^\mu_\alpha(t). \tag{20}$$

We can now integrate out the fluctuations separately and this is greatly facilitated by choosing at each x^μ normal co-ordinates for which:

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \mathcal{O}\big(\delta X^2\big). \tag{21}$$

The Euclidean action then splits in two:

$$S = S_c + \int_0^\beta dt\, \mathcal{L}_f. \tag{22}$$

The constant part is

$$S_c = -\frac{1}{4}\beta R_{\mu\nu\sigma\rho}\eta^\mu_1\eta^\nu_1\eta^\sigma_2\eta^\rho_2 + \frac{1}{2}\beta m\eta^\mu_2\omega_{\mu\nu}\eta^\nu_2 + \frac{1}{2}\beta nm, \tag{23}$$

where the final term arises from the normal-ordering constant in (13). This expression implies that the fermions zero-modes η^μ_1 scale like $\beta^{-1/2}$, while η^μ_2 do not scale with β (remember that βm is fixed). The fluctuation part is then

$$\mathcal{L}_f = \frac{1}{2}\delta X^\mu \Delta^B_{\mu\nu}\delta X^\nu + \frac{1}{2}\delta\psi^\mu\Delta^F_{\mu\nu}\delta\psi^\nu + \cdots, \tag{24}$$

where the ellipsis represent non-Gaussian terms which only contribute at a higher order in β and hence can be ignored. Using the fact that in normal coordinates (21)

$$\psi^\mu_1 \Gamma_{\mu\sigma\rho}(X)\dot{X}^\sigma\psi^\rho_1 = \frac{1}{2}\delta X^\mu R_{\mu\nu\sigma\rho}(x)\eta^\sigma_1\eta^\rho_1\delta\dot{X}^\nu + \cdots \tag{25}$$

to leading order and up to total derivatives, the bosonic operator is

$$\Delta^B_{\mu\nu} = -\delta_{\mu\nu}\partial_t^2 - \frac{1}{2}R_{\mu\nu\sigma\rho}\eta^\sigma_1\eta^\rho_1\partial_t + \cdots, \tag{26}$$

to leading order in β. The fermionic operator is matrix-valued:

$$\Delta^F_{\mu\nu} = \begin{pmatrix} \delta_{\mu\nu}\partial_t & 0 \\ 0 & \delta_{\mu\nu}\partial_t + \frac{1}{2}R_{\mu\nu\sigma\rho}\eta^\sigma_1\eta^\rho_1 - \beta m\omega_{\mu\nu} \end{pmatrix} + \cdots, \tag{27}$$

to leading order in β.

We can now integrate out the fluctuations δX^μ and $\delta\psi^\mu$, as well as the constant modes η^μ_2 keeping careful track of the overall normalization of the functional integral. As usual we can write the resulting integral over X^μ and η^μ_1 as an integral over differential forms by identifying $\eta^\mu_1 = dX^\mu$. Finally we have

$$\mathrm{ind}_W = \left(\frac{i}{2\pi}\right)^{n/2} e^{-n\beta m/2} \times \int_{\mathfrak{M}} \det{}^{1/2}\left(\frac{R/2\sinh(R/2-\beta m\omega/2)}{\sinh(R/2)}\right), \tag{28}$$

where R is the matrix-valued curvature 2-form. This can be written as

$$\mathrm{ind}_W = \int_{\mathfrak{M}} \prod_{i=1}^{n/2} \frac{x_i(1+ye^{-x_i})}{1-e^{-x_i}}, \tag{29}$$

where x_i are the skew eigenvalues of $R/4\pi$ and $y = -e^{-\beta m}$. This reproduces the integral form for the χ_y genus [2].

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