Existence of entire explosive positive radial solutions for a class of quasilinear elliptic systems

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Abstract

We show the existence of entire explosive positive radial solutions for quasilinear elliptic systems

\[
\frac{\text{div}(|\nabla u|^{m-2}\nabla u)}{x} = p(|x|)g(v), \quad \frac{\text{div}(|\nabla v|^{n-2}\nabla v)}{x} = q(|x|)f(u)
\]

on \(\mathbb{R}^N\), where \(f\) and \(g\) are positive and non-decreasing functions on \((0, \infty)\) satisfying the Keller–Osserman condition.

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1. Introduction

Existence and non-existence of solutions of the quasilinear elliptic system

\[
\begin{align*}
\text{div}(\nabla u^{m-2}\nabla u) + f(u, v) &= 0, \quad x \in \mathbb{R}^N, \\
\text{div}(\nabla v^{n-2}\nabla v) + g(u, v) &= 0, \quad x \in \mathbb{R}^N,
\end{align*}
\]

(1)

has received much attention recently. See, for example, [2,5,6,10,16,17]. Problem (1) arises in the theory of quasiregular and quasiconformal mappings as well as in the study of non-Newtonian fluids. In the latter case, the pair \((m, n)\) is a characteristic of the medium. Media with \((m, n) > (2, 2)\) are called dilatant fluids and those with \((m, n) < (2, 2)\) are called pseudoplastics. If \((m, n) = (2, 2)\), they are Newtonian fluids.

When \(m = n = 2\), system (1) becomes

\[
\begin{align*}
\Delta u + f(u, v) &= 0, \quad x \in \mathbb{R}^N, \\
\Delta v + g(u, v) &= 0, \quad x \in \mathbb{R}^N,
\end{align*}
\]
for which the existence and the non-existence of positive solutions and explosive positive solution has been investigated extensively. We list here, for example, [1,3,8,11–13,15] and refer to the references therein.

When \( m = n = 2 \), \( f = -p(|x|)v^\alpha \), \( g = -q(|x|)u^\beta \), system (1) becomes

\[
\begin{align*}
\Delta u &= p(|x|)v^\alpha, \quad x \in \mathbb{R}^N, \\
\Delta v &= q(|x|)u^\beta, \quad x \in \mathbb{R}^N,
\end{align*}
\]

for which existence results for entire explosive positive solutions can be found in a recent paper by Lair and Wood [8]. Lair and Wood established that all positive entire radial solutions of (2) are explosive provided that

\[
\int_0^\infty t^{p(t)} dt = \infty, \quad \int_0^\infty t^{q(t)} dt = \infty.
\]

If, on the other hand,

\[
\int_0^\infty t^{p(t)} dt < \infty, \quad \int_0^\infty t^{q(t)} dt < \infty,
\]

then all positive entire radial solutions of (2) are bounded.

Cirstea and Radulescu [1] extended the above results to a larger class of systems

\[
\begin{align*}
\Delta u &= p(|x|)g(v), \quad x \in \mathbb{R}^N, \\
\Delta v &= q(|x|)f(u), \quad x \in \mathbb{R}^N.
\end{align*}
\]

In this paper, we consider the following quasilinear elliptic system:

\[
\begin{align*}
\text{div}(|\nabla u|^{m-2}\nabla u) &= p(|x|)g(v), \quad x \in \mathbb{R}^N, \\
\text{div}(|\nabla v|^{n-2}\nabla v) &= q(|x|)f(u), \quad x \in \mathbb{R}^N,
\end{align*}
\]

where \( N \geq 3 \), \( m > 1 \), \( n > 1 \) and \( p, q \in C(\mathbb{R}^N) \) are positive functions. Throughout this paper we assume that \( f, g \in C[0, \infty) \) are positive and non-decreasing on \( (0, \infty) \).

We are concerned here with the existence of entire explosive positive solutions of (3), that is, positive solutions that satisfy \( u(x) \to \infty \) and \( v(x) \to \infty \) as \( |x| \to \infty \). Such problems arise in the study of the subsonic motion of a gas [14], the electric potential in some bodies [9], and Riemannian geometry [4].

Our purpose is to generalize the results in [1,8]. The main results of the present paper are new and extend the results in [1,8]. Using an argument inspired by Lair and Wood [8] and Cirstea and Radulescu [1], we obtain the following main results:

Theorem 1. Suppose \( \eta(|x|) = \min\{p(|x|), q(|x|)\} \geq C > 0 \), and

\[
\lim_{t \to \infty} \frac{g(cf^{1/(n-1)}(t))}{pm-1} = 0 \quad \text{for all} \quad c > 0.
\]

Then there exists an entire positive radial solution of (3) with any central values

\[
u(0) = a \geq 0, \quad v(0) = b \geq 0.
\]
If, in addition, the functions $p$ and $q$ satisfy

$$
\int_0^\infty \left( \frac{t^{1-N}}{s^{N-1}} p(s) \right)^{1/(m-1)} dt = \infty,
$$

$$
\int_0^\infty \left( \frac{t^{1-N}}{s^{N-1}} q(s) \right)^{1/(n-1)} dt = \infty,
$$

(5)

then all entire positive radial solutions of (3) are explosive solutions. On the other hand, if $p$ and $q$ satisfy

$$
\int_0^\infty \left( \frac{t^{1-N}}{s^{N-1}} p(s) \right)^{1/(m-1)} dt < \infty,
$$

$$
\int_0^\infty \left( \frac{t^{1-N}}{s^{N-1}} q(s) \right)^{1/(n-1)} dt < \infty,
$$

(6)

then all entire positive radial solutions of (3) are bounded.

**Remark 1.** If condition (4) of Theorem 1 is replaced by

$$
\lim_{t \to \infty} \frac{f(c g^{1/(m-1)}(t))}{t^{n-1}} = 0 \quad \text{for all } c > 0,
$$

then the conclusion of Theorem 1 still holds.

If $f$ and $g$ satisfy the stronger regularity $f, g \in C^1[0, \infty)$, then we drop the assumption (4) and require, in turn,

$$(H_1) \quad f(0) = g(0) = 0, \quad \lim_{u \to \infty} \frac{f(u)}{g(u)} = \sigma > 0,
$$

and the Keller–Osserman condition

$$(H_2) \quad \int_1^\infty \frac{dt}{\sqrt{G(t)}} < \infty, \quad \text{where } G(t) = \int_0^t g(s) ds.
$$

**Remark 2.** Observe that assumptions $(H_1)$ and $(H_2)$ imply that $f$ satisfies condition $(H_2)$, too.

We use the notation $\mathbb{R}^+ = [0, +\infty)$, and define the

$$
\mathcal{G} = \{ (a, b) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid u(0) = a, \ v(0) = b, \text{ and } (u, v) \text{ is an entire radial solution of (3)} \}.
$$
Theorem 2. Let \( f, g \in C^1[0, \infty) \) satisfy (H\(_1\)) and (H\(_2\)). Assume (6) holds, \( \eta(|x|) \geq C > 0 \). Then set \( G \neq \emptyset \) and is a closed bounded subset of \( \mathbb{R}^+ \times \mathbb{R}^+ \).

Theorem 3. Let \( f, g \in C^1[0, \infty) \) satisfy (H\(_1\)) and (H\(_2\)). Assume (6) holds, \( \eta(|x|) \geq C > 0 \) and \( v = \max\{m(0), n(0)\} > 0 \). Let \( E(G) \) be the closure of the set \( \{(a,b) \in \partial G \mid a > 0, b > 0\} \). Then any entire positive radial solution \((u,v)\) of (3) with central value \((u(0), v(0)) \in E(G)\) is explosive.

Remark 3. If condition \( \eta(|x|) = \min\{p(|x|), q(|x|)\} \geq C > 0 \) is replaced by \( \eta(x) \) is non-negative on \( \Omega \subseteq \mathbb{R}^N \) and satisfies the following: if \( x_0 \in \Omega \) and \( \eta(x_0) = 0 \), then there exists a domain \( \Omega_0 \) such that \( x_0 \in \Omega_0 \subset \Omega \) and \( \eta(x) > 0 \) for all \( x \in \partial \Omega_0 \), then the conclusions of Theorems 1–3 still hold.

2. Preliminaries

We first consider quasilinear elliptic inequalities of the form

\[
\text{div}(\left|\nabla u\right|^{m-2} \nabla u) \geq p(x)f(u), \quad x \in \mathbb{R}^N (N \geq 2),
\]

where \( m > 1, \nabla u = (\nabla_1 u, \ldots, \nabla_N u) \), \( p(x) : \mathbb{R}^N \to (0, \infty) \) and \( f : (0, \infty) \to (0, \infty) \) are continuous functions. A positive entire solution of the inequality (7) is defined to be a positive function \( u \in C^1(\mathbb{R}^N) \) satisfying (7) at every point of \( \mathbb{R}^N \).

From Ref. [7], we give the following lemma:

**Lemma 1 (Weak comparison principle).** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N (N \geq 2) \) with smooth boundary \( \partial \Omega \) and \( \theta : (0, \infty) \to (0, \infty) \) is continuous and non-decreasing. Let \( u_1, u_2 \in W^{1,m}(\Omega) \) satisfy

\[
\int_{\Omega} \left|\nabla u_1\right|^{m-2} \nabla u_1 \nabla \psi \, dx + \int_{\Omega} \theta(u_1) \psi \, dx \leq \int_{\Omega} \left|\nabla u_2\right|^{m-2} \nabla u_2 \nabla \psi \, dx + \int_{\Omega} \theta(u_2) \psi \, dx
\]

for all non-negative \( \psi \in W^{1,m}_0(\Omega) \). Then the inequality

\[ u_1 \leq u_2 \quad \text{on} \quad \partial \Omega \]

implies that

\[ u_1 \leq u_2 \quad \text{in} \quad \Omega. \]

**Lemma 2.** Let \( f(u) \) satisfy the following condition:

(i) \( f(s) \) is a single-value, real, continuous function defined for \( s \in \mathbb{R} \), and there exists a positive non-decreasing continuous function \( F(s) \) such that \( f(s) \geq F(s) \) and

\[
\int_{0}^{\infty} \left[ \int_{0}^{t} F(z) \, dz \right]^{-1/m} \, dx < \infty.
\]
(ii) There exists a constant $\beta > 0$ such that $p(x) \geq \beta > 0$ for $x \in \mathbb{R}^N$.

(iii) $u$ is a solution of

$$\text{div}(|\nabla u|^{m-2}\nabla u) \geq p(x)f(u), \quad x \in D,$$

in a domain $D \subset \mathbb{R}^N \ (N \geq 2)$ and is continuous on its boundary $S$.

Then there exists a decreasing function $g(R)$ determined by $F(u)$ such that

$$u(P) \leq g(R(P)). \quad (8)$$

Here, $P$ denotes a point in $D$ and $R(P)$ denotes its distance from $S$. The function $g(R)$ has the limits

$$g(R) \to \infty \quad \text{as} \quad R \to 0, \quad (9)$$

$$g(R) \to -\infty \quad \text{as} \quad R \to \infty. \quad (10)$$

**Proof.** Each point $P \in D$ can be the centre of a sphere of radius $R(P)$, which lies in $D$. Therefore it suffices to prove the theorem in $D$ as a sphere of radius $R$, and suppose $u$ is defined continuously on $S$. We define a function $v$ in $D$ and $S$ as the solution of the problem

$$\text{div}(|\nabla v|^{m-2}\nabla v) = F_1(v), \quad x \in D, \quad (11)$$

$$v = \alpha \quad \text{on} \quad S. \quad (12)$$

In Eqs. (11), (12), $F_1(v) = \theta F(v)$, where $F(v)$ is the function occurring in condition (i) and $\theta$ is a constant, $0 < \theta < \beta$. Thus, from conditions (i), (ii) and Eqs. (11), (12) we have

$$\text{div}(|\nabla u|^{m-2}\nabla u) - F_1(u) \geq \text{div}(|\nabla v|^{m-2}\nabla v) - F_1(v).$$

Moreover, $\alpha$ is a positive constant which satisfies

$$u \leq \alpha \quad \text{on} \quad S.$$

The existence and uniqueness of a positive solution $v$ of Eqs. (11), (12) is assured because $F_1$ is a non-decreasing function. In fact, the existence can be obtained by the standard variational method, and the uniqueness can be obtained by an idea similar to that in the proof of Lemma 2.1 (see [7]). From (7), Eqs. (11), (12) and Lemma 2.1, we have

$$u \leq v \quad \text{in} \quad D. \quad (13)$$

We now define a function $g(R)$ by

$$g(R) = \lim_{\alpha \to \infty} v(P).$$

Then, since $v$ is an increasing function of $\alpha$, we have $v(P) \leq g(R)$ for every $\alpha$. Combining this inequality with (13), we obtain

$$u(P) \leq g(R). \quad (14)$$

Inequality (14) is the desired inequality (8) of Lemma 2. It remains to show that $g(R)$ is finite in order that (14) is non-trivial, and that $g(R)$ satisfies (9) and (10) and is a decreasing function of $R$. 

To this end we must examine \( v \), the solution of Eqs. (11), (12). First we show that \( v \) must be a function of \( r \) only, where \( r \) denotes the distance from the centre of the sphere. We can find a positive radial solution \( v(r) \) of Eqs. (11), (12) by the variational method to the equivalent form of Eqs. (11), (12) in \( r \):

\[
\left( \Phi_m(v') \right)' + \frac{N-1}{r} \Phi_m(v') = F_1(v),
\]

\[ v'(0) = 0, \quad v(R) = \alpha, \]

(15)

where \( \Phi_m(s) = |s|^{m-2}s \). The uniqueness of the positive solution \( v \) of Eqs. (11), (12) implies that \( v \) is just the radial solution \( v(r) \) of the problem (15), (16).

Since \( v(0) \) is a monotonic increasing function of \( \alpha \), \( \alpha \) is itself uniquely determined by \( v(0) \). Let \( v(0) = v_0 \). As \( v_0 \) increases, \( \alpha = v(R) \) also increases. We will show that \( \alpha = v(R) \) becomes infinite for some finite value of \( v_0 \).

This value of \( v_0 \) is denoted by \( \lim_{\alpha \to \infty} v_0 \), which can be used to define the function \( g(R) \) in this lemma.

It is convenient to rewrite Eq. (15) in the form

\[
(r^{N-1} \Phi_m(v'))' = r^{N-1} F_1(v).
\]

(17)

Integrating Eq. (17) from 0 to \( r \) yields

\[
\Phi_m(v') = r^{1-N} \int_0^r s^{N-1} F_1(v(s)) \, ds.
\]

(18)

From Eq. (18) we see that \( v' \geq 0 \). Therefore, \( v \) is a non-decreasing function and we can obtain from Eq. (18) that

\[
\Phi_m(v') \leq r^{1-N} \left[ F_1(v(r)) \right]^{\frac{r^N}{N}} = \frac{r}{N} F_1[v(r)].
\]

(19)

Inserting (19) into Eq. (15) gives

\[
\left( \Phi_m(v') \right)' \geq F_1(v). \]

(20)

Since \( v' \geq 0 \), Eq. (15) also yields \( (\Phi_m(v'))' \leq F_1(v) \). Combining this with (20) leads to

\[
F_1(v) \geq (\Phi_m(v'))' \geq \frac{F_1(v)}{N}. \]

(21)

We now multiply (21) by \( v' \) and integrate from 0 to \( r \) to obtain

\[
\int_0^r F_1(v)v'(s) \, ds \geq \left( \frac{m-1}{m} \right) (v')^m \geq \int_0^r \frac{1}{N} F_1(v(s))v'(s) \, ds,
\]

that is,

\[
H(v, v_0) \geq (v')^m \geq \frac{1}{N} H(v, v_0).
\]
where
\[
H(v, v_0) = \frac{m}{m-1} \int_{v_0}^{v} F_1(z) \, dz.
\] (22)

This implies
\[
H_1^{1/m}(v, v_0) \geq v' \geq \left(\frac{1}{N}\right)^{1/m} H(v, v_0).
\]

Then we consider \( r \) as a function of \( v \), and have
\[
\int_{v_0}^{v} H^{-1/m}(z, v_0) \, dz \leq r \leq N^{1/m} \int_{v_0}^{v} H^{-1/m}(z, v_0) \, dz.
\] (23)

By condition (i) of this lemma, the integral in (23) converges as \( v \) becomes infinite when \( v_0 = 0 \). But then the integral also converges for any value of \( v_0 > 0 \). If we denote its limit by \( A(v_0) \), letting \( v \to \infty \), (23) yields
\[
A(v_0) \leq r_\infty \leq N^{1/m} A(v_0),
\] (24)

where
\[
A(v_0) = \int_{v_0}^{\infty} H^{-1/m}(z, v_0) \, dz, \quad r_\infty = \lim_{v \to \infty} r(v).
\] (25)

From Eq. (25) we see that for each \( v_0 \), \( v \) becomes infinite at a finite value of \( r_\infty \) in the range indicated in (24). Therefore, \( r_\infty \) is a function of \( v_0 \) and is denoted by \( r_\infty(v_0) \).

The function \( r_\infty(v_0) \) is continuous and non-increasing. If it were increasing, then two solutions corresponding to different value of \( v_0 \) would have to be equal at some value of \( r \). This is impossible because a solution of Eq. (15) with a prescribed value on the surface of a sphere is unique. Furthermore, the integral \( A(v_0) \) tends to \( +\infty \) as \( v_0 \) tends to \( -\infty \), and to zero as \( v_0 \) tends to \( +\infty \). Therefore, by (24), \( r_\infty(v_0) \) behaves in the same way. We now define \( g(R) := \min\{v_0 \mid r_\infty(v_0) = R\} \). This function is decreasing and satisfies Eqs. (9), (10), so it is the desired \( g(R) \) of Lemma 2. This completes the proof of Lemma 2.

\[\triangleright\]

Remark 4. If condition (i) of Lemma 2 is replaced by

There exists a positive non-decreasing continuous \( F(s) \) such that \( f(s) \geq F(s) \) and
\[
\int_{a}^{\infty} \left[ \int_{0}^{x} F(z) \, dz \right]^{-1/m} \, dx < \infty \quad \forall a > 0,
\]
then conclusion of Lemma 2 still holds.

Lemma 3. If \( f(u) \) is non-decreasing and satisfies conditions (i), (ii) of Lemma 2, then in any bounded domain \( D \) there exists a solution of (7) which becomes infinite on \( S \).
Proof. We note that for any constant $\alpha$ and any domain $D$ there exists in $D$ a solution $u_{\alpha}$ of (7) which is equal to $\alpha$ on $S$, provided that $f(u)$ is non-decreasing (see [7] for the existence proof). Furthermore, at each point of $D$, $u_{\alpha}$ increases with $\alpha$. If $f(u)$ satisfies conditions (i) and (ii) of Lemma 2, then Lemma 2 holds, and at each point $P$ in $D$ all of the $u_{\alpha}$ are bounded above. Thus in every closed sub-domain, $u_{\alpha}$ converges uniformly to a limit $u$. This limit is also a solution of (7). As $P$ approaches $S$, $u(P)$ increases infinitely, since on $S$, $u_{\alpha} = \alpha$ becomes infinite. Thus $u$ is the desired solution and Lemma 3 is proved. $\square$

Lemma 4. Suppose $f$ is non-decreasing and satisfies $(H_2)$. Then

$$\int_{1}^{\infty} \frac{ds}{f^{1/(m-1)}(s)} < \infty.$$  \hspace{1cm} (26)

Proof. If we can prove that there exist positive numbers $\delta$ and $M$ such that

$$\frac{f^{1/(m-1)}(s)}{s} \geq \delta^m \text{ for } s \geq M,$$  \hspace{1cm} (27)

then we will be done since

$$F(s) = \int_{0}^{s} f(t) \, dt \leq sf(s) \leq \frac{f^{m/(m-1)}(s)}{\delta^m} \text{ for } s \geq M,$$

which, in turn, yields

$$\left( F(s) \right)^{-1/m} \geq \frac{\delta}{f^{1/(m-1)}(s)} \text{ for } s \geq M$$

so that $(H_2)$ implies (26). To prove (27), we assume it is false. That is, we assume there exists an increasing sequence $\{s_j\}$ of real numbers such that $\lim_{j \to \infty} s_j = \infty$ and $f^{1/(m-1)}(s_j)/s_j < 1/j$ for all $j$. Since $f$ is increasing, we have $f(s) \leq f(s_j)$ for all $s \in [0, s_j]$, which, in turn, produces $F(s) \leq sf(s) \leq sf(s_j)$ for $s \in [0, s_j]$. Hence,

$$\int_{s_1}^{s_j} \left[ F(s) \right]^{-1/m} \, ds \geq \int_{s_1}^{s_j} \left[ sf(s_j) \right]^{-1/m} \, ds \geq \left( \frac{j}{s_j} \right) \left( s_j \right)^{(m-1)/m} \int_{s_1}^{s_j} s^{-1/m} \, ds$$

$$= \frac{m}{m-1} \left( s_j \right)^{(m-1)/m} \left( s_j^{(m-1)/m} - s_1^{(m-1)/m} \right)$$

$$= \frac{m}{m-1} \left( j \right)^{(m-1)/m} \left( 1 - \left( s_1/s_j \right)^{m-1} \right) \to \infty,$$

as $j \to \infty$, contradicting $(H_2)$. Thus (27) must be true. This completes the proof. $\square$
Lemma 5. Suppose $f$ satisfies (H$_2$), $f \in C^1[0, \infty)$, $f(0) = 0$, $f'(u) \geq 0$ and $p(|x|) \geq C > 0$ for $x \in \mathbb{R}^N$ and the following:

$$\int_0^\infty \left( t^{1-N} \int_0^t s^{N-1} p(s) \, ds \right)^{1/(m-1)} \, dt < \infty.$$ 

Then equation 

$$\text{div}(|\nabla u|^{m-2} \nabla u) = p(|x|) f(u)$$

has an entire explosive positive radial solution.

Proof. From Lemma 3, we have that for each $k \in \mathbb{N}$ the boundary-value problem

$$\text{div}(|\nabla v_k|^{m-2} \nabla v_k) = p(|x|) f(v_k), \quad |x| < k,$$

$$v_k(x) \to \infty \quad \text{as} \quad |x| \to k$$

has a positive solution. Furthermore,

$$v_1 \geq v_2 \geq \cdots \geq v_k \geq v_{k+1} \geq \cdots > 0$$

in $\mathbb{R}^N$. To prove our result, we need only prove

(A) There exists $w \in C(\mathbb{R}^N)$, $w > 0$ such that $v_k \geq w$ in $\mathbb{R}^N$ for all $k$ and

(B) $v \to \infty$ as $|x| \to \infty$, where $v = \lim_{k \to \infty} v_k$.

To prove (A), from theorem condition implies

$$z(r) = C - \int_0^r \left( t^{1-N} \int_0^t s^{N-1} p(s) \, ds \right)^{1/(m-1)} \, dt,$$

where

$$C = \int_0^\infty \left( t^{1-N} \int_0^t s^{N-1} p(s) \, ds \right)^{1/(m-1)} \, dt,$$

is the unique positive solution of the following problem:

$$\text{div}(|\nabla z|^{m-2} \nabla z) = -p(r), \quad z \to 0 \quad \text{as} \quad r = |x| \to \infty, \ x \in \mathbb{R}^N.$$

By Lemma 4, we can define

$$F(s) = \int_s^\infty \frac{dt}{f^{1/(m-1)}(t)}, \quad \text{for all} \ s > 0.$$ 

Note also that

$$F'(s) = \frac{-1}{f^{1/(m-1)}(s)} < 0 \quad \text{and} \quad F''(s) = \frac{f'(s)}{(m-1)(f^{1/(m-1)}(s))^m} > 0.$$
A simple calculation shows that
\[
\text{div}(|\nabla F(v_k)|^{m-2} \nabla F(v_k)) = -|F'(v_k)|^{m-1} \text{div}(|\nabla v_k|^{m-2} \nabla v_k) + (m-1)|F'(v_k)|^{m-2} F''(v_k) |\nabla v_k|^m
\]
\[
\geq -|F'(v_k)|^{m-1} p(r) f(v_k) = -p(r).
\]
Thus
\[
-\text{div}(|\nabla F(v_k)|^{m-2} \nabla F(v_k)) \leq -\text{div}(|\nabla v|^m) \quad \text{in} \ |x| < k,
\]
and from Lemma 1 we obtain \( F(v_k) \leq z \) if \(|x| \leq k \). Let \( w = F^{-1}(z) \) and note that \( v_k \geq w \) in \( \mathbb{R}^N \). Consequently, \( v \geq w \) in \( \mathbb{R}^N \) and (A) is proved since \( w \to \infty \) as \(|x| \to \infty \) (here using the fact that \( \lim_{s \to 0^+} F^{-1}(s) = \infty \)). It is clear that (B) follows easily from (A).

**Lemma 6.** The problem
\[
\text{div}(|\nabla l|^{m-2} \nabla l) = (p(|x|) + q(|x|))(f(l) + g(l)), \quad (28)
\]
\[
\text{div}(|\nabla h|^{m-2} \nabla h) = (p(|x|) + q(|x|))(f(h) + g(h)), \quad (29)
\]
has an entire explosive positive radial solution provided that functions \( \eta(|x|) \geq C > 0 \) satisfy (6) and \( f, g \) satisfy (H1)–(H2).

**Proof.** From Lemma 3, for each natural number \( k \), let \( v_k \) be a positive solution of the boundary-value problem
\[
\text{div}(|\nabla v_k|^{m-2} \nabla v_k) = (p(|x|) + q(|x|))(f(v_k) + g(v_k)), \quad |x| < k, \quad v_k \to \infty, \quad |x| \to k.
\]
Again, from Lemma 1, we can show that
\[
v_1 \geq v_2 \geq \cdots \geq v_k \geq v_{k+1} \geq \cdots > 0
\]
in \( \mathbb{R}^N \). To complete the proof, it is sufficient to show that there exists a function \( w \in C(\mathbb{R}^N) \) such that \( w \to \infty \) as \(|x| \to \infty \) and \( v_k \geq w \) in \( \mathbb{R}^N \) for all \( k \). To do this, we note first that condition (H1), set \( 0 < \lambda < \min\{\sigma, 1\} \) and let \( \delta = \delta(\lambda) \) be large enough so that
\[
f(t) \geq \lambda g(t) \quad \forall t \geq \delta.
\]
We consider the equation
\[
\text{div}(|\nabla u|^{m-2} \nabla u) = 2/\lambda[p(|x|) + q(|x|)] f(u), \quad (30)
\]
By Lemma 5, Eq. (30) has a positive solution \( u \) on \( \mathbb{R}^N \) such that \( u(x) \to \infty \) as \(|x| \to \infty \). We claim that \( w = u - 1 \) is a desired lower boundary for \( v_k \). Indeed, since
\[
\text{div}(|\nabla v_k|^{m-2} \nabla v_k) = \text{div}(|\nabla v_k|^{m-2} \nabla v_k) = (p + q)(f(v_k) + g(v_k)) \leq (p + q)(f(v_k + 1) + g(v_k + 1)) \leq 2/\lambda(p + q)f(v_k + 1) \quad \text{for} \ |x| > k,
\]
and clearly \( v_k + 1 > u \) as \(|x| \to k \). Lemma 1 implies that \( v_k + 1 \geq u \) for \(|x| \leq k \). Hence \( v = \lim_{k \to \infty} v_k \geq u - 1 \) on \( \mathbb{R}^N \). Again, by the standard regularity argument for elliptic
problems, it is a straightforward argument to prove that $v$ is the desired solution of (28). By a similar argument, we can show that (29) has an entire explosive positive radial solution. 

By similar argument with Lemma 6 of [8], it is easy to prove the following lemma:

**Lemma 7.** Let $l, h$ be any entire explosive positive radial solution of (28), (29) given in Lemma 6 and define the sequences $\{u_k\}$ and $\{v_k\}$ by

\[
\begin{align*}
  u_k(r) &= a + \int_0^r \left( r^{N-1} \int_0^t s^{N-1} p(s) g(v_{k-1}(s)) \, ds \right)^{1/(m-1)} \, dt, \quad r \geq 0, \\
  v_k(r) &= b + \int_0^r \left( r^{N-1} \int_0^t s^{N-1} q(s) f(u_{k-1}(s)) \, ds \right)^{1/(n-1)} \, dt, \quad r \geq 0,
\end{align*}
\]

where $u_0 = a$, $0 \leq a \leq \min\{l(0), h(0)\}$ and $v_0(r) = b$, $0 \leq b \leq \min\{l(0), h(0)\}$. Then

(a) $u_k(r) \leq u_{k+1}(r)$ and $v_k(r) \leq v_{k+1}(r)$, $r \in \mathbb{R}^+$, $k \geq 1$, and 
(b) $u_k(r) \leq l(r)$ and $v_k(r) \leq h(r)$, $r \in \mathbb{R}^+$, $k \geq 1$.

Thus $\{u_k\}$ and $\{v_k\}$ converge and the limit functions are entire positive radial solutions of system (3).

3. Proof of main theorems

**Proof of Theorem 1.** Since the radial solutions of (3) are solutions of the ordinary differential equations system

\[
\begin{align*}
  \left( r^{N-1} |u'|^{m-2} u \right)' &= r^{N-1} p(r) g(v(r)), \\
  \left( r^{N-1} |v'|^{n-2} v \right)' &= r^{N-1} q(r) f(u(r))
\end{align*}
\]

for $r > 0$, it follows that the radial solutions of (3) with $u(0) = a > 0$, $v(0) = b > 0$ satisfy

\[
\begin{align*}
  u(r) &= a + \int_0^r \left( r^{N-1} \int_0^t s^{N-1} p(s) g(v(s)) \, ds \right)^{1/(m-1)} \, dt, \quad r \geq 0, \quad (31) \\
  v(r) &= b + \int_0^r \left( r^{N-1} \int_0^t s^{N-1} q(s) f(u(s)) \, ds \right)^{1/(n-1)} \, dt, \quad r \geq 0. \quad (32)
\end{align*}
\]

Define $v_0(r) = b$ for all $r \geq 0$. Let $(u_k)_{k \geq 1}$ and $(v_k)_{k \geq 1}$ be two sequences of functions given by

\[
\begin{align*}
  u_k(r) &= a + \int_0^r \left( r^{N-1} \int_0^t s^{N-1} p(s) g(v_{k-1}(s)) \, ds \right)^{1/(m-1)} \, dt, \quad r \geq 0, \quad (33)
\end{align*}
\]
\[ v_k(r) = b + \int_0^r \left( \frac{1}{1-N} \int_0^t s^{N-1} q(s) f(u_{k-1}(s)) \, ds \right)^{1/(n-1)} \, dt, \quad r \geq 0. \quad (34) \]

Since \( v_1(r) \geq b \), we find \( u_2(r) \geq u_1(r) \) for all \( r \geq 0 \). This implies \( v_2(r) \geq v_1(r) \) which further produces \( u_3(r) \geq u_2(r) \) for all \( r \geq 0 \). Proceeding at the same manner we conclude that \( u_k(r) \leq u_{k+1}(r) \) and \( v_k(r) \leq v_{k+1}(r) \), \( \forall r \geq 0 \) and \( k \geq 1 \).

We now prove that the non-decreasing sequences \((u_k(r))_{k \geq 1}\) and \((v_k(r))_{k \geq 1}\) are bounded from above on bounded sets. Indeed, we have

\[ u_k(r) \leq u_{k+1}(r) \leq a + g^{1/(m-1)}(v_k(r)) A(r), \quad \forall r \geq 0, \quad (35) \]

and

\[ v_k(r) \leq b + f^{1/(n-1)}(u_k(r)) B(r), \quad \forall r \geq 0, \quad (36) \]

where

\[ A(r) = \int_0^r \left( \frac{1}{1-N} \int_0^t s^{N-1} p(s) \, ds \right)^{1/(m-1)} \, dt, \]

\[ B(r) = \int_0^r \left( \frac{1}{1-N} \int_0^t s^{N-1} q(s) \, ds \right)^{1/(n-1)} \, dt. \]

Let \( R > 0 \) be arbitrary. By (35) and (36) we find

\[ u_k(R) \leq a + g^{1/(m-1)}(b + f^{1/(n-1)}(u_k(R)) B(R)) A(R), \quad \forall k \geq 1, \]

or, equivalently

\[ 1 \leq \frac{a}{u_k(R)} + \frac{g^{1/(m-1)}(b + f^{1/(n-1)}(u_k(R)) B(R))}{u_k(R)} A(R) \]

\[ = \frac{a}{u_k(R)} + \left( \frac{g(b + f^{1/(n-1)}(u_k(R)) B(R))}{u_k^{m-1}(R)} \right)^{1/(m-1)} A(R) \quad \forall k \geq 1. \quad (37) \]

By the monotonicity of \((u_k(R))_{k \geq 1}\), there exists \( \lim_{k \to \infty} u_k(R) = L(R) \). We claim that \( L(R) \) is finite. Assume the contrary. Then, by taking \( k \to \infty \) in (37) and using (4) we obtain a contradiction. Since \( u_k'(r), v_k'(r) \geq 0 \) we get that the map \((0, \infty) \ni R \to L(R)\) is non-decreasing on \((0, \infty)\) and

\[ u_k(r) \leq u_k(R) \leq L(R), \quad \forall r \in [0, R], \quad \forall k \geq 1, \quad (38) \]

\[ v_k(r) \leq b + f^{1/(n-1)}(L(R)) B(R), \quad \forall r \in [0, R], \quad \forall k \geq 1. \quad (39) \]

It follows that there exists \( \lim_{R \to \infty} L(R) = \bar{L} \in (0, \infty] \) and the sequences \((u_k(r))_{k \geq 1}\), \((u_k(r))_{k \geq 1}\) are bounded above on bounded sets. Thus, we can define \( u(r) = \lim_{k \to \infty} u_k(r) \) and \( v(r) = \lim_{k \to \infty} v_k(r) \) for all \( r \geq 0 \). By standard elliptic regularity theory we obtain that \((u, v)\) is a positive entire solution of (3) with \( u(0) = a \) and \( v(0) = b \).

We now assume that, in addition, condition (6) is fulfilled. We have that \( \lim_{r \to \infty} A(r) = \overline{A} < \infty \) and \( \lim_{r \to \infty} B(r) = \overline{B} < \infty \). Passing to the limit as \( k \to \infty \) in (37) we find
\[ 1 \leq \frac{\alpha}{L(R)} + \frac{g^{1/(m-1)}(b + f^{1/(n-1)}(L(R))B(R))}{L(R)}A(R) \]
\[ \leq \frac{\alpha}{L(R)} + \frac{g^{1/(m-1)}(b + f^{1/(n-1)}(L(R))B(R))}{L(R)}A(R) \]

Letting \( R \to \infty \) and using (4) we deduce \( L(R) < \infty \). Thus, taking into account (38) and (39) we obtain
\[ u_k(r) \leq \mathcal{L} \quad \text{and} \quad v_k(r) \leq b + f^{1/(n-1)}(\mathcal{L}B), \quad \forall r \geq 0, \quad \forall k \geq 1. \]

So, we have found upper bounds for \((u_k(r))_{k \geq 1}\) and \((v_k(r))_{k \geq 1}\) which are independent of \( r \). Thus, the solution \((u, v)\) is bounded from above. This shows that any solution of (31), (32) will be bounded from above provided (6) holds.

Let us now drop the condition (6) and assume that (5) is fulfilled. In this case, \( \lim_{r \to \infty} A(r) = \infty = \lim_{r \to \infty} B(r) \). Let \((u, v)\) be an entire positive radial solution of (3).

Using (31) and (32) we obtain
\[ u(r) \geq a + g^{1/(m-1)}(bA(r)), \quad v(r) \geq b + f^{1/(n-1)}(a)B(r), \quad \forall r \geq 0. \]

Taking \( r \to \infty \) we get that \((u, v)\) is an entire explosive solution. This concludes the proof of Theorem 1.

\[ \Box \]

**Remark 5.** We now give some examples of non-linearities \( f \) and \( g \) which satisfy the assumptions of Theorem 1.

1. Let \( f = (1 + t^m)^{\gamma/m} \) and \( g(t) = (1 + t^{m(n-1)})^{\theta(m-1)/m} \) for \( t \in \mathbb{R} \) with \( \gamma, \theta > 0 \) and \( \gamma \theta < 1 \).

2. Let
\[ f(t) = \begin{cases} t^\gamma & \text{if } 0 \leq t \leq 1, \\ t^\theta & \text{if } t > 1, \end{cases} \]
and
\[ g(t) = \begin{cases} t^\theta & \text{if } 0 \leq t \leq 1, \\ t^{\gamma(n-1)} & \text{if } t > 1, \end{cases} \]
with \( \gamma, \theta > 0, \gamma \theta < m - 1 \), and \( f(t) = g(t) = 0 \) for \( t \leq 0 \).

**Proof of Theorem 2.** From Lemma 7, it is clear that \( [0, g(0)] \times [0, h(0)] \subset \mathcal{G} \) so that \( \mathcal{G} \) is non-empty. We shall show that \( \mathcal{G} \) is a bounded, closed set.

As a preliminary, note that if \((a, b) \in \mathcal{G}\) then any pair \((a_0, b_0)\) for which \( 0 \leq a_0 \leq a \) and \( 0 \leq b_0 \leq b \) must be in \( \mathcal{G} \) since the process used in Lemma 7 can be repeated with
\[ u_k(r) = a_0 + \int_0^r \left( t^{1-N}\int_0^t s^{N-1} p(s) g(v_{k-1}(s)) \, ds \right)^{1/(m-1)} \, dt, \]
\[ v_k(r) = b_0 + \int_0^r \left( t^{1-N}\int_0^t s^{N-1} q(s) f(u_{k-1}(s)) \, ds \right)^{1/(n-1)} \, dt. \]
and \( v_0 = b, u_0 = a \). Then, as in Lemma 7, the sequences \( \{u_k\} \) and \( \{v_k\} \) are monotonically increasing. Then, letting \((U, V)\) be the solution of (31) and (32) with central values \((a, b)\), we can easily prove, since \( b_0 \leq b \), that \( v_0 \leq V \). Thus, \( u_1 \leq U \) (since, also, \( a_0 \leq a \)), and consequently \( v_1 \leq V \), and so on. Hence we get \( u_k \leq U \) and \( v_k \leq V \), and therefore \( u \leq U \) and \( v \leq V \) where \((u, v) = \lim_{k \to \infty} (u_k, v_k)\) is a solution of (3) (with central values \((a_0, b_0)\)).

Set \( 0 < \lambda < \min\{\sigma, 1\} \) and let \( \delta = \delta(\lambda) \) be large enough so that
\[
f(t) \geq \lambda g(t), \quad \forall t \geq \delta.
\]

Lemma 3 ensures the existence of a positive explosive solution \( h_1, h_2 \) of the problem
\[
\begin{align*}
\text{div}(\{\nabla h_1\}^m) &= \lambda \eta(|x|) g(h_1) \quad \text{in } B(0, R), \\
h_1 &\to \infty, \quad |x| \to R, \\
\text{div}(\{\nabla h_2\}^m) &= \lambda \eta(|x|) g(h_2) \quad \text{in } B(0, R), \\
h_2 &\to \infty, \quad |x| \to R.
\end{align*}
\]

To prove that \( G \) is bounded, assume that it is not. Then, there exists \((a, b) \in G\) such that \( a + b > \max\{2\delta, h_1(0) + h_2(0)\} \). Let \((u, v)\) be the entire radial solution of (3) such that \((u(0), v(0)) = (a, b)\). Since \( u(x) + v(x) \geq a + b > 2\delta \) for all \( x \in \mathbb{R}^N \), by (40) we get
\[
\begin{align*}
\text{div}(\{\nabla u\}^m) &= p(|x|) g(v) \geq \lambda \eta(|x|) g(v), \\
\text{div}(\{\nabla v\}^m) &= q(|x|) f(u) \geq \lambda \eta(|x|) g(u).
\end{align*}
\]

On the other hand, \( h_1(x) \to \infty, h_2(x) \to \infty \) as \( |x| \to R \). Thus, using Lemma 1 we conclude that \( u + v \leq h_1 + h_2 \) in \( B(0, R) \). But this is impossible since \( u(0) + v(0) = a + b > h_1(0) + h_2(0) \).

To prove that \( G \) is closed, we let \((a_0, b_0) \in \partial G\) and show that \((a_0, b_0) \in G\). Let \((u, v)\) be the solution of (31) and (32) which corresponds to \( a = a_0 \) and \( b = b_0 \). Without loss of generality, we may assume that \( \max\{a_0, b_0\} > C = l(0) \) where the function \( l \) is given in Lemma 7. If \( \max\{a_0, b_0\} = a_0 \), then \( C \leq a_0 - 1/k \) for large \( k \) so that \( u_k(r) \geq C \) for all \( r \geq 0 \) and for all \( k \) sufficiently large where
\[
\begin{align*}
\lim_{k \to \infty} u_k &= a_0 - \frac{1}{k} + \int_0^r \left( \frac{1}{t(N - 1)} \int_0^t s^{N-1} p(s) g(v_{k-1}(s)) \, ds \right)^{1/(m-1)} \, dt, \\
\lim_{k \to \infty} v_k &= b_0 + \int_0^r \left( \frac{1}{t(N - 1)} \int_0^t s^{N-1} q(s) f(u_{k-1}(s)) \, ds \right)^{1/(m-1)} \, dt.
\end{align*}
\]

From (40), we have
\[
\begin{align*}
\text{div}(\{\nabla u_k\}^m) &\geq \lambda \eta(r) g(v_k), \\
\text{div}(\{\nabla v_k\}^m) &\geq \lambda \eta(r) g(u_k).
\end{align*}
\]

Let \( h_1, h_2 \) be positive solutions of
\[
\begin{align*}
\text{div}(\{\nabla h_1\}^m) &= \lambda \eta(r) g(h_1), \quad 0 \leq r < R_0, \\
h_1(r) &\to \infty, \quad r \to R_0^-.
\end{align*}
\]
and
\[
\text{div}\left(|\nabla h_2|^{n-2}\nabla h_2\right) = \lambda \eta(r) g(h_2), \quad 0 \leq r < R_0,
\]
\[h_2(r) \to \infty, \quad r \to R_0^-\]
where \(R_0\) is an arbitrary positive real number. It is now easy to show by Lemma 1 that \(u_k + v_k \leq h_1 + h_2\) in \([0, R_0]\). Hence \(u + v = \lim_{k \to \infty}(u_k + v_k) \leq h_1 + h_2\) on \([0, R_0]\).

Since \(R_0\) is arbitrary, the functions \(u, v\) exist on \(\mathbb{R}^N\) and hence are entire so that \((a_0, b_0) \in \mathcal{G}\). On the other hand, if \(\max\{a_0, b_0\} = b_0\), then \(C \leq b_0 - 1/k\) for large \(k\) so that \(v_k \geq C\) for all \(r \geq 0\) and for all sufficiently large \(k\). Then \(u_k(r) \geq C\alpha A(r)\) where \(A(r) = \int_0^r (t^{1-N} \int_0^t s^{N-1} p(s) \, ds)^{1/(m-1)} \, dt\) and the proof continues as before with \(C\) replaced by \(C\alpha A(r)\).

\[\Box\]

**Proof of Theorem 3.** The proof is similar to the Theorem 2 of [1,8], so we omit the detail. \[\Box\]

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**References**