Relative Cohen–Macaulayness of bigraded modules

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ABSTRACT

In this paper we study the local cohomology of finitely generated bigraded modules over a standard bigraded polynomial ring which have only one nonvanishing local cohomology with respect to one of the irrelevant bigraded ideals.

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Introduction

Let $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be the standard bigraded polynomial ring over a field $K$ with bigraded irrelevant ideals $P$ generated by all elements of degree $(1, 0)$, and $Q$ generated by all elements of degree $(0, 1)$. In other words, $P = (x_1, \ldots, x_m)$ and $Q = (y_1, \ldots, y_n)$. Let $q \in \mathbb{Z}$ and $M$ be a finitely generated bigraded $S$-module such that $H^i_Q(M) = 0$ for $i \neq q$. Thus grad$(Q, M) = cd(Q, M) = q$, where $cd(Q, M)$ denotes the cohomological dimension of $M$ with respect to $Q$. Our aim is to characterize all finitely generated $S$-modules which have this property. In this case, we call $M$ to be relative Cohen–Macaulay with respect to $Q$. We observe that ordinary Cohen–Macaulay modules are special cases of our definition. In fact, if we assume that $P = 0$, then $m = 0$, and $Q = m$ is the unique graded maximal ideal of $S$ where $\deg y_i = 1$ for $i = 1, \ldots, n$. Therefore depth $M = \text{grade}(m, M) = \text{cd}(m, M) = \dim M$. We set $K[y] = K[y_1, \ldots, y_n]$. We recall in the preliminaries section some basic definitions, known facts and examples. We show that $M$ is relative Cohen–Macaulay with respect to $Q$ with $\text{cd}(Q, M) = q$ if and only if $M_k = \bigoplus_{(k, j)} M_{k, j}$ is a finitely generated Cohen–Macaulay $K[y]$-module of dimension $q$ for all $k$ and this is also equivalent to say that $M$ is a not necessarily finitely generated Cohen–Macaulay as $K[y]$-module of dimension $q$. In Section 1, we set

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$H^3_Q(M)_j = \bigoplus_{k \in \mathbb{Z}} H^3_Q(M)_{(k, j)}$, and consider $H^3_Q(M)_j$ as a finitely generated graded $K[x]$-module where $K[x] = K[x_1, \ldots, x_m]$. As a main result of this section, we let $M$ be relative Cohen–Macaulay with respect to $Q$ with $\text{cd}(Q, M) = q$, then we give a free resolution of $H^3_Q(M)_j$ which has length at most $m$.

In particular, if $M$ is finitely generated bigraded $S$-module of finite length, then we give a free resolution for the graded components $M_j$. By using this result we show that the regularity of $H^3_Q(M)_j$ is bounded for all $j$, i.e., there exists an integer $c$ such that $-c \leq \text{reg} H^3_Q(M)_j \leq c$, for all $j$. In Section 2, we first assume that $M$ be a Cohen–Macaulay $S$-module. We show that $M$ is relative Cohen–Macaulay with respect to $Q$ if and only if $M$ is relative Cohen–Macaulay with respect to $P$, and this is equivalent to say that $M$ satisfies to the equation $\text{cd}(Q, M) + \text{cd}(P, M) = \dim M$. In the following we assume that $M$ is relative Cohen–Macaulay with respect to $Q$. We prove the bigraded version of prime avoidance theorem which gives us a bihomogeneous $M$-regular element $y \in Q$ for which $M/yM$ is relative Cohen–Macaulay with respect to $Q$ and cohomological dimension goes down by 1. As main result of this paper we prove that if $M$ is relative Cohen–Macaulay with respect to $Q$ then the following equality is always true: $\text{cd}(Q, M) + \text{cd}(P, M) = \dim M$. Some more applications of our main result are considered. Finally, we call $M$ to be maximal relative Cohen–Macaulay with respect to $Q$ if $M$ is relative Cohen–Macaulay with respect to $Q$ such that $\text{cd}(Q, M) = \dim K[y] = n$. We observe that maximal relative Cohen–Macaulay modules with respect to $Q$ are those finitely generated modules for which the sequence $y_1, \ldots, y_n$ is an $M$-sequence.

1. Preliminaries

Let $K$ be a field and $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be the standard bigraded polynomial ring. In other words, we set $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$ for all $i, j$. Let $M$ be a finitely generated bigraded $S$-module. We set $K[y] = K[y_1, \ldots, y_n]$ and

$$M_k = M_{(k, \ast)} = \bigoplus_{j \in \mathbb{Z}} M_{(k, j)}.$$  

Then we view $M = \bigoplus_{k \in \mathbb{Z}} M_k$ as a graded module where each graded component $M_k$ itself is a finitely generated graded $K[y]$-module with grading $(M_k)_j = M_{(k, j)}$ for all $j$. We shall use the following results [6, Lemma 1.2.2].

$$\dim_{K[y]} M = \sup\{\dim_{K[y]} M_k : k \in \mathbb{Z}\},$$  \hspace{1cm} (1)

and

$$\text{depth}_{K[y]} M = \inf\{\text{depth}_{K[y]} M_k : k \in \mathbb{Z} \text{ with } M_k \neq 0\}.  \hspace{1cm} (2)$$

We may view $M$ as graded $K[y]$-module which of course is not finitely generated in general. The module $M$ is called Cohen–Macaulay as $K[y]$-module if and only if $\text{depth}_{K[y]} M = \dim_{K[y]} M$.

Let $R$ be a graded Noetherian ring and $M$ be a finitely generated graded $R$-module. We set $R_+ = \bigoplus_{i > 0} R_i$ and denote by $\text{cd}(R_+, M)$ the cohomological dimension of $M$ with respect to $R_+$ which is the largest integer $i$ for which $H^i_{R_+}(M) \neq 0$. If $(R_0, m_0)$ is a local ring, then by [2, Lemma 3.4] we have $\text{cd}(R_+, M) = \dim_R M/m_0M$. This is also the case, if $R_0$ is a graded ring with unique graded maximal ideal $m_0$. In fact, we set $R'_0 = R_{0m_0}$. Then $R'_0$ is a Noetherian local ring with maximal ideal $m'_0 = m_0R'_0$. Set $R' = R'_0 \otimes_{R_0} R$ and $M' = R'_0 \otimes_{R_0} M$. Then the natural isomorphism of $R'$-modules $M'/m'_0 M' \cong R'_0/m'_0 \otimes_{R_0} M/m_0M$ implies that $\dim(M'/m'_0 M') = \dim(M/m_0 M)$. Moreover, by the graded flat base change property we have the natural isomorphisms of $R'$-modules $H^i_{R'_+}(M') \cong R'_0 \otimes_{R_0} H^i_{R_+}(M)$ for all $i$. This follows that $\text{cd}(R'_+, M') = \text{cd}(R_+, M)$. Therefore if $R_0$ is a graded ring with unique graded maximal ideal $m_0$, then by [2, Lemma 3.4] we have $\text{cd}(R_+, M) = \text{cd}(R'_+, M') = \dim(M'/m'_0 M') = \dim(M/m_0M)$. Thus, in our situation where $P = (x_1, \ldots, x_m)$ and $Q = (y_1, \ldots, y_n)$ are the irrelevant bigraded ideals of $S$, we always have the following
Proposition 1.2. \[
\text{cd}(P, M) = \dim S M/QM \quad \text{and} \quad \text{cd}(Q, M) = \dim S M/PM.
\] (3)

We shall use the following fact as remark:

**Remark 1.1.** Let \( R \) be a Noetherian graded ring with unique graded maximal ideal \( m \). Let \( I \) be a homogeneous ideal of \( R \) and \( M \) a finitely generated graded \( R \)-module. Then \( \text{cd}(I, M) = 0 \) if and only if \( H^0_I(M) = M \). This results from the following isomorphisms

\[
H^i_I(M) = H^i_I(M/H^0_I(M)) \quad \text{for all } i > 0
\]

(see [1, Corollary 2.1.7]) and the fact that \( M/H^0_I(M) \) is \( I \)-torsion free, i.e., \( H^0_I(M/H^0_I(M)) = 0 \). See [1, Lemma 2.1.2].

Let \( R \) be a Noetherian graded ring with unique graded maximal ideal \( m \) and \( I \) a homogeneous ideal of \( R \). We denote by \( \text{grade}(I, M) \) the grade of \( M \) with respect to \( I \) which is the smallest integer \( i \) for which \( H^i_I(M) \neq 0 \). Note that \( \text{grade}(I, M) \leq \text{cd}(I, M) \leq \dim M \). In the case of \( m \) the unique graded maximal ideal we have \( \text{grade}(m, M) = \text{depth} M \) and \( \text{cd}(m, M) = \dim M \). In fact, \( (R_m, mR_m) \) is a local ring and so we have \( \text{grade}(m, M) = \text{grade}(mR_m, M_m) = \text{depth} M_m = \text{depth} M \) and \( \text{cd}(m, M) = \text{cd}(mR_m, M_m) = \dim M_m = \dim M \) where the last equality follows from [3, Lemma 1.5.6] and [3, Theorem 1.5.8]. We also have the inequality \( \text{grade}(I, M) \leq \dim M - \dim M/IM \) and equality holds if \( M \) is Cohen–Macaulay. This is the graded version of [3, Theorem 2.1.2]. To prove this, it suffices to reduce the problem to the local case and use the facts that \( \text{grade}(I, M) = \text{grade}(I_m, M_m) \) (see [3, Proposition 1.5.5(e)]), \( \dim_M M = \dim M_m \) and \( \dim_R M/IM = \dim M/IM \). Thus in our situation and in view of (3) we always have

\[
\text{grade}(P, M) + \text{cd}(Q, M) \leq \dim M \quad \text{and} \quad \text{grade}(Q, M) + \text{cd}(P, M) \leq \dim M.
\] (4)

and if \( M \) is Cohen–Macaulay, we have

\[
\text{grade}(P, M) + \text{cd}(Q, M) = \dim M.
\] (5)

In the following we give a necessary and sufficient condition for a finitely generated bigraded \( S \)-module \( M \) which has only one nonvanishing local cohomology. This also generalizes [5, Corollary 4.6].

**Proposition 1.2.** Let \( M \) be a non-zero finitely generated bigraded \( S \)-module, \( q \in \mathbb{Z} \) and \( Q = (y_1, \ldots, y_n) \). Then the following statements are equivalent:

(a) \( H^i_Q(M) = 0 \) for all \( i \neq q \);
(b) \( M_k = \bigoplus_j M_{(k,j)} \) is finitely generated Cohen–Macaulay \( K[y] \)-module of dimension \( q \) for all \( k \);
(c) \( M \) is Cohen–Macaulay as \( K[y] \)-module of dimension \( q \);
(d) \( \text{grade}(Q, M) = \text{cd}(Q, M) = q \).

**Proof.** (a) \( \Rightarrow \) (b), (c): We first observe that

\[
H^i_Q(M) = \bigoplus_k H^i_Q(M)_{(k,s)} = \bigoplus_k H^i_Q(M_{(k,s)}).
\] (6)

The second equality follows from the fact that \( H^i_Q(M)_{(k,s)} = H^i_Q(M_{(k,s)}) \), as can be seen from the definition of local cohomology using the Čech complex. Now let \( H^i_Q(M) = 0 \) for all \( i \neq q \), it follows that \( H^i_Q(M_{(k,s)}) = 0 \) for all \( k \) and \( i \neq q \). Hence \( M_k = M_{(k,s)} \) is a finitely generated Cohen–Macaulay
Lemma 1.4. Macaulay. First we recall the following known lemma. See for example [7, Lemma 2.1].

In order to prove (a), in view of Lemma 1.4, we have the following isomorphisms of $A$-modules. Then for all $i$ we have the isomorphism of $A$-modules. We set $M$ be Cohen–Macaulay as $K[y]$-module of dimension $q$. Then by using (1) and (2), we have

$$
\dim_{K[y]} M_k \geq \text{depth}_{K[y]} M_k \geq \text{depth}_{K[y]} M = q = \dim_{K[y]} M \geq \dim_{K[y]} M_k.
$$

Thus $\text{depth}_{K[y]} M_k = \dim_{K[y]} M_k = q$, as required.

(a) $\Leftrightarrow$ (d): is obvious. $\Box$

Now we can make the following definition:

Definition 1.3. Let $M$ be a finitely generated bigraded $S$-module and $q \in \mathbb{Z}$. We call $M$ to be relative Cohen–Macaulay with respect to $Q$ if and only if $M$ satisfies one of the equivalent conditions of Proposition 1.2. Note that $0 \leq q \leq n$, because $H^i_Q(M) = 0$ for $i > n$.

In Section 3, we will show that if $M$ is a Cohen–Macaulay $S$-module which is relative Cohen–Macaulay with respect to $P$, then $M$ is relative Cohen–Macaulay with respect to $Q$ and vice versa. In the following we give some examples to show that this would not be the case if $M$ is not Cohen–Macaulay. First we recall the following known lemma. See for example [7, Lemma 2.1].

Lemma 1.6. Let $A$ and $B$ be two $K$-algebras. Assume that $M$ and $M'$ are $A$-modules and $N$ and $N'$ are $B$-modules. Then for all $i$ we have the isomorphism of $A \otimes_K B$-modules

$$
\text{Ext}_{A \otimes_K B}^i(M \otimes_K N, M' \otimes_K N') \cong \bigoplus_{s+t=i} \text{Ext}_{A}^s(M, M') \otimes_K \text{Ext}_{B}^t(N, N').
$$

Proposition 1.5. Let $M_1$ be a finitely generated graded $K[x]$-module and $M_2$ a finitely generated graded $K[y]$-module. We set $M = M_1 \otimes_K M_2$. Then the following statements hold:

(a) $M$ is relative Cohen–Macaulay with respect to $Q$ with $\text{cd}(Q, M) = q$ if and only if $M_2$ is Cohen–Macaulay of dimension $q$;

(b) $M$ is relative Cohen–Macaulay with respect to $P$ with $\text{cd}(P, M) = p$ if and only if $M_1$ is Cohen–Macaulay of dimension $p$.

Proof. In order to prove (a), in view of Lemma 1.4, we have the following isomorphisms of $S$-modules

$$
H^i_Q(M) \cong \lim_{\longrightarrow \ k \geq 0} \text{Ext}_S^i(S/Q^k, M) \cong \lim_{\longrightarrow \ n \geq 0} \text{Ext}_S^i(K[x] \otimes_K K[y]/Q^k, M_1 \otimes_K M_2)
$$

$$
\cong \bigoplus_{s+t=i} \text{Ext}_K^s(K[x], M_1) \otimes_K \lim_{\longrightarrow \ k \geq 0} \text{Ext}_K^t(K[y]/Q^k, M_2)
$$

$$
\cong M_1 \otimes_K H^i_Q(M_2).
$$

Now the assertion follows from this observation. Part (b) is proved the same way. $\Box$

Corollary 1.6. Let $I$ and $J$ be homogeneous ideals of $K[x]$ and $K[y]$, respectively. We set $R_0 = K[x]/I$, $R_1 = K[y]/J$ and $R = R_0 \otimes_K R_1$. Then the following statements hold:
(a) \( R \) is relative Cohen–Macaulay with respect to \( Q \) with \( cd(Q, M) = q \) if and only if \( R_1 \) is Cohen–Macaulay of dimension \( q \);
(b) \( R \) is relative Cohen–Macaulay with respect to \( P \) with \( cd(Q, M) = p \) if and only if \( R_0 \) is Cohen–Macaulay of dimension \( p \).

**Example 1.7.** We consider the following standard bigraded ring

\[
R = \frac{K[x_1, \ldots, x_m, y_1]}{(x_1 y_1, \ldots, x_m y_1, y_1^2)}.
\]

We observe that \( \text{depth} R = 0 \) and \( \dim R = m \), and so \( R \) is not Cohen–Macaulay. We see that \( \text{grade}(Q, R) \leq \text{grade}(m, R) = \text{depth} R = 0 \), and so \( \text{grade}(Q, R) = 0 \). One has \( \text{cd}(Q, R) = \dim R / \text{PR} R = 0 \). Thus \( R \) is relative Cohen–Macaulay with respect to \( Q \). On the other hand, \( \text{grade}(P, R) \leq \text{depth} R = 0 \), and so \( \text{grade}(P, R) = 0 \). One has \( \text{cd}(P, R) = \dim R / \text{QR} R = m \). Thus \( R \) is not relative Cohen–Macaulay with respect to \( P \).

**2. On the graded components of \( H^q_Q(M) \)**

Let \( M \) be a finitely generated bigraded \( S \)-module. Then the local cohomology modules \( H^i_Q(M) \) are naturally bigraded \( S \)-modules for all \( i \) and each graded component \( H^i_Q(M)_j \) is a finitely generated graded \( K[x] \)-module (see [1, Proposition 15.1.5]). The grading is given by

\[
(H^i_Q(M)_j)_k = H^i_Q(M)_{(k,j)}.
\]

Let \( F \) be a finitely generated bigraded free \( S \)-module. Hence \( F = \bigoplus_{i=1}^t S(-a_i, -b_i) \) and by using formula 1 in [4] we obtain

\[
H^n_Q(F)_j = \bigoplus_{i=1}^t \bigoplus_{|a|=-n-j+b_i} K[x](-a_i)z^a.
\]

Thus, we may consider \( H^n_Q(F)_j \) as finitely generated graded free \( K[x] \)-module. With this observation we prove the following

**Theorem 2.1.** Let \( M \) be a finitely generated bigraded \( S \)-module which has a minimal free resolution of the form

\[
\mathbb{F} : 0 \to F_l \xrightarrow{\psi_l} F_{l-1} \to \cdots \to F_1 \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} 0,
\]

where \( F_i = \bigoplus_{k=1}^{t_i} S(-a_{ik}, -b_{ik}) \) and where \( l \leq m + n \). Let \( M \) be relative Cohen–Macaulay with respect to \( Q \) with \( \text{cd}(Q, M) = q \). Then for all \( j \), the \( K[x] \)-module \( H^n_Q(M)_j \) has a free \( K[x] \)-resolution of length at most \( m \) and of the form

\[
0 \to H^n_Q(F_{m+n-q})_j \xrightarrow{\psi_{m+n-q}} \cdots \to H^n_Q(F_{n-q+1})_j \xrightarrow{\psi_{n-q+1}} \text{Ker} \psi_{n-q} \to H^n_Q(M)_j \to 0,
\]

where the maps \( \psi_i : H^n_Q(F_i)_j \to H^n_Q(F_i-1)_j \) are induced by \( \psi_i \) for all \( i \).

**Proof.** As \( M \) is relative Cohen–Macaulay with respect to \( Q \), Proposition 1.2 implies that \( M \) is Cohen–Macaulay as \( K[y] \)-module of dimension \( q \). Note that \( \text{depth}_S M \geq \text{depth}_{K[y]} M = q \). Hence \( \text{proj dim}_S M \leq m + n - q \) and so \( F_i = 0 \) for \( i > m + n - q \). Applying the functor \( H^n_Q(\cdot)_j \) to the resolution \( \mathbb{F} \) yields a graded complex of free \( K[x] \)-modules.
We first assume that with the same argument as above and see from the last split exact sequence finite length.

Now suppose that 0

To complete our proof we only need to show that Ker

This means that we have only one homology throughout of the complex \( H^n_Q(F) \). Thus the complex \( H^n_Q(F) \) breaks to the following exact sequences of \( K[x] \)-modules

where \( H^n_Q(M) = \text{Ker} \psi_{n-q}/\text{Im} \psi_{n-q+1} \). Combining these two exact sequences, we will obtain the following free resolution for \( H^n_Q(M) \) which has length at most \( m \)

To complete our proof we only need to show that Ker \( \psi_{n-q} \) is free. Here we distinguish two cases:

We first assume that \( q = n \). Thus Ker \( \psi_0 = H^n_Q(F_0) \) and so \( H^n_Q(M) \) has a free resolution of the form

Now suppose that \( 0 \leq q < n \). Thus we have the short exact sequence

in which \( H^n_Q(F_0) \) and \( H^n_Q(F_1) \) are free \( K[x] \)-modules. Hence the sequence is split exact and we have that Ker \( \psi_1 \) is a free \( K[x] \)-module. We also have the following split exact sequence

in which \( H^n_Q(F_2) \) and Im \( \psi_2 \) = Ker \( \psi_1 \) are free \( K[x] \)-modules. Then Ker \( \psi_2 \) = Im \( \psi_3 \) is free. We proceed with the same argument as above and see from the last split exact sequence

that Ker \( \psi_{n-q} \) is free. □

As a consequence we give a free resolution for each graded component of any finitely generated bigraded \( S \)-module \( M \) for which \( \text{cd}(Q,M) = 0 \). This also includes all finitely generated modules of finite length.
Corollary 2.2. Let \( M \) be a finitely generated bigraded \( S \)-module with \( \text{cd}(Q, M) = 0 \). Then for all \( j \), the finitely generated graded \( K[x] \)-module \( M_j \) has a free resolution of the form
\[
0 \to H^n_Q(F_{m+n})_j \to \cdots \to H^n_Q(F_{n+1})_j \to \text{Ker} \psi_n \to M_j \to 0.
\]

Proof. The assertion follows from Theorem 2.1 and Remark 1.1. \( \square \)

Let \( M \) be a finitely generated graded \( K[x] \)-module with graded minimal free resolution
\[
F : 0 \to F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 \to 0.
\]

The Castelnuovo–Mumford regularity of \( M \) is the invariant
\[
\text{reg}(M) = \max \{ b_i(F) - i : i \geq 0 \}
\]
where \( b_i(F) \) denotes the maximal degree of the generators of \( F_i \). In the following we show that if \( M \) is a relative Cohen–Macaulay \( S \)-module with respect to \( Q \) of relative dimension \( q \), then the regularity of \( H^q_Q(M)_j \) is bounded for all \( j \).

Proposition 2.3. Let \( M \) be a finitely generated \( S \)-module which is relative Cohen–Macaulay with respect to \( Q \) with \( \text{cd}(Q, M) = q \). Then the function \( f_M(j) = \text{reg} H^q_Q(M)_j \) is bounded.

Proof. In view of Theorem 2.1 and (8), we have \( H^n_Q(F_{n-q})_j = \text{Ker} \psi_{n-q} / \text{Im} \psi_{n-q+1} \) where \( \text{Ker} \psi_{n-q} \) is a free submodule \( H^n_Q(F_{n-q})_j \). By (7) we have
\[
H^n_Q(F_{n-q})_j = \bigoplus_{k=1}^{t_{(n-q)}} \bigoplus_{|a|=-n-j+b_{(n-q)k}} K[x](-a_{(n-q)k})z^a.
\]

By [4, Proposition 2.6] the smallest degree of generators of \( H^n_Q(F_{n-q})_j \) which is independent of \( j \) is a lower bound for the function \( f_M \). Thus it suffices to show that \( f_M \) is bounded above. We first assume that \( q = n \). By (9) the \( K[x] \)-module \( H^n_Q(M)_j \) has a free resolution of the form
\[
0 \to H^n_Q(F_m)_j \to \cdots \to H^n_Q(F_1)_j \to H^n_Q(F_0)_j \to H^n_Q(M)_j \to 0,
\]
where \( F_i = \bigoplus_{k=1}^{t_i} S(-a_{ik}, -b_{ik}) \). By (7) we have
\[
H^n_Q(F_i)_j = \bigoplus_{k=1}^{t_i} \bigoplus_{|a|=-n-j+b_{ik}} K[x](-a_{ik})z^a.
\]

Hence
\[
\text{reg} H^n_Q(M)_j \leq \max \{ b_i(H^n_Q(F)_j) - i : i \geq 0 \},
\]
where \( b_i(H^n_Q(F)_j) \) is the maximal degree of the generators of \( H^n_Q(F)_j \). Thus we conclude that
\[
\text{reg} H^n_Q(M)_j \leq \max_{i,k} \{ a_{ik} - i \} = c.
\]
Now let $0 \leq q < n$, and consider the following exact sequences of $K[x]$-modules which we observed in the proof of Theorem 2.1.

$$0 \to H^n_q(F_{m+n-q})_j \to \cdots \to H^n_q(F_{n-q+1})_j \to \text{Im } \psi_{n-q+1} \to 0,$$

and

$$0 \to \text{Im } \psi_{n-q+1} \to \text{Ker } \psi_{n-q} \to H^0_q(M)_j \to 0,$$

where $H^0_q(M)_j = \text{Ker } \psi_{n-q}/\text{Im } \psi_{n-q+1}$. Thus (10) yields

$$\text{reg } \text{Im } \psi_{n-q+1} \leq \max \{ b_i(H^n_q(F)_j) + n - q + 1 - i : i \geq n - q + 1 \}$$

$$= \max_{i \geq n - q + 1, k} (a_{ik} + n - q + 1 - i) = c,$$

for some number $c$, and by (11) we have

$$\text{reg } H^0_q(M)_j \leq \max(\text{reg } \text{Im } \psi_{n-q+1} - 1, \text{reg } \text{Ker } \psi_{n-q}).$$

Since $\text{reg } \text{Im } \psi_{n-q+1}$ is bounded above, to complete our proof, it suffices to show that $\text{reg } \text{Ker } \psi_{n-q}$ is bounded above. To do so, from the short exact sequence

$$0 \to \text{Ker } \psi_1 \to H^n_q(F_1)_j \to H^n_q(F_0)_j \to 0,$$

it follows that

$$\text{reg } \text{Ker } \psi_1 \leq \max \{ \text{reg } H^n_q(F_1)_j, \text{reg } H^n_q(F_0)_j + 1 \}.$$

Observing that $\text{reg } H^n_q(F_0)_j = \max_k a_{0k} = a$ and $\text{reg } H^n_q(F_1)_j = \max_k a_{1k} = b$ for some numbers $a$ and $b$. Thus the regularity of $\text{Ker } \psi_1$ is bounded above. As $\text{Ker } \psi_1 = \text{Im } \psi_2$, the exact sequence

$$0 \to \text{Ker } \psi_2 \to H^n_q(F_2)_j \to \text{Im } \psi_2 \to 0$$

yields that $\text{reg } \text{Ker } \psi_2$ is bounded above. We proceed with the same argument as above and observe from the last exact sequence

$$0 \to \text{Ker } \psi_{n-q} \to H^n_q(F_{n-q})_j \to \text{Im } \psi_{n-q} \to 0$$

that $\text{reg } \text{Ker } \psi_{n-q}$ is bounded above and therefore the regularity of $H^0_q(M)_j$ is bounded above for all $j$, as required. \qed
3. Relative Cohen–Macaulayness

In the following we give a very explicit criteria for all finitely generated bigraded Cohen–Macaulay modules which are relative Cohen–Macaulay with respect to one of the irrelevant bigraded ideals \( P \) and \( Q \).

**Proposition 3.1.** Let \( M \) be a finitely generated bigraded Cohen–Macaulay \( S \)-module. Then the following statements are equivalent:

(a) \( M \) is relative Cohen–Macaulay with respect to \( P \);
(b) \( M \) is relative Cohen–Macaulay with respect to \( Q \);
(c) \( M \) is relative Cohen–Macaulay with respect to \( P \) and \( Q \) with
\[
\text{cd}(P, M) + \text{cd}(Q, M) = \dim M;
\]
(d) \( \dim(M/QM) + \dim(M/PM) = \dim M \).

**Proof.** (a) \( \Rightarrow \) (b), (c): By (5) we have \( \text{grade}(Q, M) = \dim M - \dim M/QM \), since \( M \) is Cohen–Macaulay. In view of (3) we have \( \text{grade}(P, M) = \text{cd}(P, M) = \dim M/QM \) and \( \text{cd}(Q, M) = \dim M/PM \). By using these facts and again (5) we have
\[
\text{grade}(Q, M) = \dim M - \text{grade}(P, M) = \dim M - (\dim M - \dim M/PM) = \dim M/PM = \text{cd}(Q, M).
\]
Thus we conclude that \( M \) is relative Cohen–Macaulay with respect to \( Q \) and that \( \text{cd}(P, M) + \text{cd}(Q, M) = \dim M \).

(b) \( \Rightarrow \) (a), (c): is proved the same way, and (c) \( \Rightarrow \) (a), (b) is clear. Finally, (a) \( \iff \) (d) follows from (5). \( \Box \)

**Remark 3.2.** Proposition 3.1 can fail if \( M \) is not Cohen–Macaulay. See, Corollary 1.6 and Example 1.7. On the other hand, not all Cohen–Macaulay \( S \)-modules are relative Cohen–Macaulay. Obvious examples are hypersurface rings which are Cohen–Macaulay but have two nonvanishing local cohomology. In fact, let \( f \in S \) be a bihomogeneous form of degree \((a, b)\) with \( a > 0 \) and \( b > 0 \). Write
\[
f = \sum_{|\alpha| = a, |\beta| = b} c_{\alpha\beta} x^{\alpha} y^{\beta} \quad \text{where } c_{\alpha\beta} \in K.
\]
We may also write \( f = \sum_{|\beta| = b} f_{\beta} y^{\beta} \) where \( f_{\beta} \in K[x] \) with \( \deg f_{\beta} = a \) and set \( R = S/fS \). The ring \( R \) is Cohen–Macaulay of dimension \( m + n - 1 \). By (3) \( \text{cd}(Q, R) = \dim R/PR = \dim S/(P + (f)) = n \), and by (5) we have \( \text{grade}(Q, R) = \dim R - \text{cd}(P, R) = \dim R - \dim R/Q R = n - 1 \). But, since \( H^i_Q(S) = 0 \) for all \( i \neq n \), from the exact sequence \( 0 \rightarrow S(-a, -b) \xrightarrow{f} S \rightarrow S/fS \rightarrow 0 \), we get \( H^i_Q(R) = 0 \) for \( i \neq n, n - 1 \).

Let \( M \) be a relative Cohen–Macaulay module with respect to \( Q \). By (4) we have
\[
\text{cd}(Q, M) + \text{cd}(P, M) \leq \dim M.
\]
Our main result in this section is to prove that equality holds in general. We shall need to use the following bigraded version of prime avoidance theorem.
Lemma 3.3. Let $p_1, \ldots, p_r$ be prime ideals of $S$ such that $Q \not\subseteq p_i$ for $i = 1, \ldots, r$ and $|K| = \infty$. Then there exists a bihomogeneous element $y \in Q$ of degree $(0, 1)$ such that $y \notin p_1 \cup \cdots \cup p_r$.

Proof. Let $V$ be the $K$-vector space spanned by $y_1, \ldots, y_n$. Since $Q \not\subseteq p_i$ for $i = 1, \ldots, r$, it follows that $V_i = V \cap p_i$ is a proper linear subspace of $V$. Since $|K| = \infty$, the vector space $V$ cannot be the finite union of proper linear subspaces. Therefore, there exists $y \in V \setminus \bigcup_{i=1}^r V_i$. This is the desired element $y$ of degree $(0, 1)$. □

Lemma 3.4. Let $M$ be a finitely generated bigraded $S$-module with $\text{cd}(Q, M) > 0$ and $|K| = \infty$. Then there exists a bihomogeneous element $y \in Q$ of degree $(0, 1)$ such that

$$\text{cd}(Q, M/yM) = \text{cd}(Q, M) - 1.$$ 

Moreover, if $\text{grade}(Q, M) > 0$, then the element $y$ may be chosen to be also $M$-regular.

Proof. By our assumptions we have

$$\text{cd}(Q, M) = \dim(M/PM) > 0.$$ 

Let $\{p_1, \ldots, p_r\}$ be the minimal prime ideals of $\text{Supp}(M/PM)$. We claim that $Q \not\subseteq p_i$ for $i = 1, \ldots, r$. Assume that $Q \subseteq p_i$ for some $i$. Since $P \subseteq p_i$ for all $i$, it follows that $p_i = P + Q = m$, and so $\dim(M/PM) = 0$, a contradiction. By Lemma 3.3 we may choose a bihomogeneous element $y \in Q$ which does not belong to any minimal prime ideal of $\text{Supp}(M/PM)$. It follows that

$$\text{cd}(Q, M/yM) = \dim(M/yM)/P(M/yM) = \dim(M/PM)/y(M/PM) < \dim M/PM = \text{cd}(Q, M).$$

By the graded version of [3, Proposition A.4], we have

$$\text{cd}(Q, M/yM) = \dim(M/PM)/y(M/PM) \geq \dim M/PM - 1 = \text{cd}(Q, M) - 1.$$ 

Therefore $\text{cd}(Q, M/yM) = \text{cd}(Q, M) - 1$. Now let $\text{grade}(Q, M) > 0$. Thus by [3, Proposition 9.1.4(a)] it follows that $Q \not\subseteq \bigcup_{q \in \text{Ass}_M} q = Z(M)$ where $Z(M)$ denotes the set zero divisors of $M$. By Lemma 3.3 we may choose a bihomogeneous element $y \in Q$ which does not belong to any associated prime ideal of $M$ and not to any minimal prime ideal of $\text{Supp}(M/PM)$. In particular, this element is $M$-regular and we have $\text{cd}(Q, M/yM) = \text{cd}(Q, M) - 1$. □

Corollary 3.5. Let $M$ be a finitely generated bigraded $S$-module which is relative Cohen–Macaulay with respect to $Q$ with $\text{cd}(Q, M) > 0$ and $|K| = \infty$. Then there exists a bihomogeneous $M$-regular element $y \in Q$ such that $M/yM$ is relative Cohen–Macaulay with respect to $Q$ and we have

$$\text{cd}(Q, M/yM) = \text{cd}(Q, M) - 1.$$ 

As main result of this section we prove the following

Theorem 3.6. Let $M$ be a finitely generated bigraded $S$-module which is relative Cohen–Macaulay with respect to $Q$ and $|K| = \infty$. Then we have

$$\text{cd}(Q, M) + \text{cd}(P, M) = \dim M.$$
By the graded independence theorem [1, Theorem 13.1.6] we have $H_i^j(M) = H_{P+Ann(M)}^j(M)$. Therefore for all $i$ we have $H_i^j(M) \cong H_{P+Ann(M)}^j(M)$. Since $H_{P+Ann(M)}^j(M) \neq 0$, it follows that $H_i^j(M) \neq 0$. Hence $cd(P, M) \geq dim M$. We also have that $cd(P, M) \leq dim M$. Thus $cd(P, M) = dim M$. Now suppose that $cd(Q, M) > 0$, and our desired equality has been proved for all finitely generated bigraded $S$-modules $N$ such that $cd(Q, N) < cd(Q, M)$. We want to prove it for $M$. Since $cd(Q, M) > 0$, by Corollary 3.5 there exists a bihomogeneous $M$-regular element $y \in Q$ such that $M/yM$ is relative Cohen–Macaulay with respect to $Q$ with $cd(Q, M/yM) = cd(Q, M) - 1$. Thus the induction hypothesis implies that

$$cd(Q, M/yM) + cd(P, M/yM) = dim M/yM.$$ 

Since $y \in Q$, it follows that

$$cd(P, M/yM) = dim(M/yM)/Q(M/yM)$$

$$= dim M/(Q + (y))M$$

$$= dim M/QM = cd(P, M).$$

We also have $dim M/yM = dim M - 1$. Therefore the desired equality follows. 

**Remark 3.7.** Let $M$ be a finitely generated bigraded $S$-module which is relative Cohen–Macaulay with respect to $P$ and $Q$ with $cd(P, M) = p$ and $cd(Q, M) = q$, respectively and $|K| = \infty$. Then the modules $H_p^j(H_Q^q(M))$ and $H_m^j(H_p^q(M))$ are Artinian modules for all $i$ and we have the following isomorphism of non-zero bigraded modules

$$H_p^j(H_Q^q(M)) \cong H_q^i(H_p^j(M)).$$

In fact, we consider the following spectral sequence

$$H_p^j(H_Q^i(M)) \Rightarrow H_m^{i,j}(M),$$

where $m = P + Q$. As $M$ is relative Cohen–Macaulay with respect to $Q$ of relative dimension $q$, we have the following isomorphisms of bigraded $S$-modules

$$H_p^j(H_Q^q(M)) \cong H_m^{i+q}(M).$$

The modules $H_m^{i+q}(M)$ are Artinian for all $i$, and so the modules $H_p^j(H_Q^q(M))$ are Artinian for all $i$. Applying the isomorphism with $i = p$ yields

$$H_p^j(H_Q^q(M)) \cong H_m^{p+q}(M) = H_m^{dim M}(M) \neq 0.$$  (12)

The last equality follows from Theorem 3.6. By the similar argument as above we see that the modules $H_Q^i(H_p^q(M))$ are Artinian for all $i$ and

$$H_Q^q(H_p^i(M)) \cong H_m^{p+q}(M) = H_m^{dim M}(M) \neq 0.$$  (13)

The desired isomorphism follows from (12) and (13).
Lemma 3.8. Let $R$ be a positively graded ring for which $R_0$ is a local ring with maximal ideal $m_0$. Let $I$ and $J$ be homogeneous ideals of $R$ such that $I + J = m$ where $m$ is the unique graded maximal ideal $R$ and $M$ be a finitely generated graded $R$-module with $0 < \text{cd}(I, M) < \dim M$, $0 < \text{cd}(J, M) < \dim M$. Then

$$\text{cd}(I \cap J, M) = \dim M - 1.$$  

Proof. The graded Mayer–Vietoris sequence provides the long exact sequence of $R$-modules

$$\cdots \to H^{i-1}_{I \cap J}(M) \to H^i_m(M) \to H^i_I(M) \oplus H^i_J(M) \to H^i_{I \cup J}(M) \to \cdots.$$  

(See, [1, Exercise 13.1.4].) Applying the long exact sequence with $i = \dim M$, we have $H^\dim M_I(M) \oplus H^\dim M_J(M) = 0$. Thus we get the following exact sequence of $R$-modules

$$\cdots \to H^\dim M-I(M) \oplus H^\dim M-J(M) \to H^\dim M-I \cup J(M) \to H^\dim M_m(M) \to 0.$$  

Since $H^\dim M_m(M) \neq 0$, it follows that $H^\dim M-I \cup J(M) \neq 0$. Hence $\text{cd}(I \cap J, M) \geq \dim M - 1$. Thus we conclude that $\dim M - 1 \leq \text{cd}(I \cap J, M) \leq \dim M$. The equality $\text{cd}(I \cap J, M) = \dim M$ cannot be the case, because by putting $i = \dim M$ in the above exact sequence yields $H^\dim M_I(M) = 0$. Therefore $\text{cd}(I \cap J, M) = \dim M - 1$, as desired. \hfill $\Box$

Corollary 3.9. Let $M$ be a finitely generated bigraded $S$-module which is relative Cohen–Macaulay with respect to $Q$ with $\text{cd}(Q, M) > 0$. Assume that $\text{cd}(P, M) > 0$. Then we have

$$\text{cd}(P \cap Q, M) = \dim M - 1.$$  

Proof. The assertion follows from (4) and Lemma 3.8. \hfill $\Box$

Corollary 3.10. Let the assumption be as in Corollary 3.9, and assume in addition that $M$ is relative Cohen–Macaulay with respect to $P \cap Q$. Then $\text{cd}(Q, M) = \text{cd}(P, M) = 1$, and furthermore we have $\dim M = 2$. The converse holds when $M$ is relative Cohen–Macaulay with respect to $P$, as well.

Proof. Let $M$ be relative Cohen–Macaulay with respect to $P \cap Q$. Corollary 3.9 yields $\text{cd}(P \cap Q, M) = \dim M - 1 = \text{grade}(P \cap Q, M)$. Note that $H^i_{I \cap J}(-) = H^i_I(-)$ for all $i$ and for all graded ideals $I$ and $J$ of any graded Noetherian ring $R$. Hence

$$\text{grade}(P \cap Q, M) = \text{grade}(PQ, M) = \min\{\text{grade}(P, M), \text{grade}(Q, M)\}. \quad (14)$$

Here the second equality follows from [3, Proposition 9.1.3(b)]. Thus we conclude that $\dim M - 1 = \min\{\text{grade}(P, M), \text{grade}(Q, M)\}$. Here we consider two cases: Let grade$(P, M) \leq$ grade$(Q, M)$. Then, in view of (4) we have

$$\dim M - 1 = \text{grade}(P, M) \leq \text{grade}(Q, M) \leq \dim M - \text{cd}(P, M).$$

Thus $\text{cd}(P, M) = 1$. We also see that $\text{cd}(Q, M) = 1$, because

$$0 < \text{cd}(Q, M) = \dim M - 1 = \text{grade}(P, M) \leq \text{cd}(P, M) = 1.$$  

Similarly one obtains $\text{cd}(Q, M) = \text{cd}(P, M) = 1$, if grade$(Q, M) \leq$ grade$(P, M)$. Now let $M$ is relative Cohen–Macaulay with respect to both $P$ and $Q$ and $\text{cd}(Q, M) = \text{cd}(P, M) = 1$. Corollary 3.9 together with (14) yield that grade$(P \cap Q, M) = \text{cd}(P \cap Q, M) = 1$, as desired. \hfill $\Box$
We end this section by introducing a special class of relative Cohen–Macaulay modules.

**Definition 3.11.** Let $M$ be a finitely generated bigraded $S$-module. We call $M$ to be maximal relative Cohen–Macaulay with respect to $Q$ if $M$ is relative Cohen–Macaulay with respect to $Q$ such that $\text{cd}(Q, M) = \dim K[y] = n$. We also call $M$ to be maximal relative Cohen–Macaulay with respect to $P$ if $M$ is relative Cohen–Macaulay with respect to $P$ such that $\text{cd}(P, M) = \dim K[x] = m$.

**Remark 3.12.** In Corollary 1.6 we observe that if $J = 0$, then the ring $R$ is maximal relative Cohen–Macaulay with respect to $Q$ with $\text{cd}(Q, R) = \dim K[y] = n$ and if $I = 0$, then $R$ is maximal relative Cohen–Macaulay with respect to $P$ with $\text{cd}(P, R) = \dim K[x] = m$. In Example 1.7 the ring $R$ is relative Cohen–Macaulay with respect to $Q$ but not maximal relative Cohen–Macaulay with respect to $Q$. We note that if $M$ be a finitely generated bigraded $S$-module. Then [3, Corollary 1.6.19] and the fact that $\text{cd}(Q, M) \leq n$ show that $M$ is maximal relative Cohen–Macaulay with respect to $Q$ if and only if $y_1, \ldots, y_n$ is an $M$-sequence.

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**References**