The best $G^1$ cubic and $G^2$ quartic Bézier approximations of circular arcs

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**Abstract**

We obtain cubic and quartic Bézier approximations of circular arcs that respectively satisfy $G^1$ and $G^2$ end-point interpolation conditions. We identify the necessary and sufficient conditions for such approximations to be the best, in the sense that they have the minimum Hausdorff distance to the circular arc. We then establish the existence and uniqueness of these best approximations and present practical methods to calculate them, which are verified by examples.

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1. Introduction

The approximation of circular arcs by Bézier curves is a fundamental topic in CAGD, CAD and CAM areas, and many significant advances have been achieved [1–6]. The diversity of research on this subject reflects the different degrees of Bézier curves that can be used and the discontinuities required at end-points. We will refer to approximations which satisfy $G^k$ end-point interpolation conditions in a Hermite sense as $G^k$ approximations. Mörken [3] considered quadratic Bézier approximations of order four, while de Boor et al. [1] and Dokken et al. [2] proposed $G^2$ cubic Bézier approximations of order six. Goldapp [4] studied $G^0$, $G^1$, and $G^2$ cubic Bézier approximations, also of order six. Ahn and Kim [5] made a detailed study of Bézier approximations of quartic and higher degree. There has also been work on the optimal approximation order for Bézier approximation curves [7,8].

However, the present authors are unaware of any satisfactory research on the best cubic and higher degree Bézier approximations of circular arcs, and they were mainly interested in the approximation orders. In this paper, we identify the necessary and sufficient conditions for $G^1$ cubic and $G^2$ quartic approximations of circular arcs to be the best, and we also verify the existence and the uniqueness of such approximations. We assess the merit of an approximation using the Hausdorff distance function to measure its error, so that the best approximation has the minimum Hausdorff distance to the circular arc $c(\theta)$, which it is trying to approximate.

The remainder of this paper is organized as follows. In Section 2, we review the definition and properties of Hausdorff distance functions of two curves. In Section 3, we identify the necessary and sufficient conditions for the best $G^1$ cubic Bézier curve, and we also verify the existence and uniqueness of such an approximation. We obtain a similar result for the best $G^2$ quartic Bézier approximation in Section 4. We draw some conclusions and suggest directions in which to extend this work in Section 5.

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2. Preliminaries

We will use the following parametric representation of a unit circular arc $c : [0, \alpha] \rightarrow \mathbb{R}^2$:
\[
  \mathbf{c}(\theta) = (\cos \theta, \sin \theta), \quad 0 \leq \theta \leq \alpha < \pi,
\]
which we will write as $\mathbf{c}(\theta)$, and Bézier approximations to this circular arc will be written as $\mathbf{b}(t)$.

2.1. $C^k$ approximations

We say that a parametric curve $\mathbf{b}(t), 0 \leq t \leq 1$ is a $C^k$, $k = 0, 1, 2$ approximation to a circular arc $\mathbf{c}(\theta)$, if it satisfies $C^k$ end-point interpolation conditions in the Hermite sense described below. The $C^k$, $k = 0, 1, 2$, end-point interpolation conditions can be written $\mathbf{g}^k$, with the following interpretation:
- $\mathbf{g}^0$ - The two curves $\mathbf{b}$ and $\mathbf{c}$ have the same end-points.
- $\mathbf{g}^1$ - The two curves $\mathbf{b}$ and $\mathbf{c}$ satisfy $\mathbf{g}^0$, and have the same tangent lines at the end-points.
- $\mathbf{g}^2$ - The two curves $\mathbf{b}$ and $\mathbf{c}$ satisfy $\mathbf{g}^0, \mathbf{g}^1$, and have the same signed curvatures at the end-points.

We note that $C^k$ end-point interpolation conditions are equivalent to having contacts of order $k$ at the end-points of the two curves [9,10,5].

2.2. Hausdorff distance function

We will use the Hausdorff distance function to measure the error of approximation curve $\mathbf{b}(t)$ to a given circular arc $\mathbf{c}(\theta)$. The Hausdorff distance $d_H$ between two parametric curves $\mathbf{b}(t)$ and $\mathbf{c}(\theta)$ is defined as follows:
\[
d_H(\mathbf{b}, \mathbf{c}) = \max \{ \sup_{t} \inf_{\theta} d(\mathbf{b}(t), \mathbf{c}(\theta)), \sup_{\theta} \inf_{t} d(\mathbf{b}(t), \mathbf{c}(\theta)) \},
\]
where $d$ is the Euclidean distance between two points $\mathbf{p} = (x_1, y_1)$ and $\mathbf{q} = (x_2, y_2)$ on the plane:
\[
d(\mathbf{p}, \mathbf{q}) = \| \mathbf{p} - \mathbf{q} \| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]
Recent research in [11,12] has provided a better understanding of the general properties of the Hausdorff distance function.

A pair of points on two curves are said to be collinear normal points if their associated normal vectors lie on the same line. The Euclidean distance between a pair of collinear normal points is said to be a collinear normal distance corresponding to the pair of collinear normal points. If two curves $\mathbf{b}$ and $\mathbf{c}$ have the same end-points and are sufficiently similar, the Hausdorff distance between them is the maximum collinear normal distance [13,14,12,15]. To find all pairs of collinear normal points on $\mathbf{b}$ and $\mathbf{c}$, we will use the following collinear normal conditions [13,14]
\[
(\mathbf{b}(t) - \mathbf{c}(\theta)) \cdot \hat{\mathbf{c}}(\theta) = 0, \quad \hat{\mathbf{b}}(t) \wedge \hat{\mathbf{c}}(\theta) = 0,
\]
where $\mathbf{b}(t)$ and $\hat{\mathbf{c}}(\theta)$ are tangent vectors, and $(x_1, y_1) \wedge (x_2, y_2) = x_1y_2 - x_2y_1$. If $\mathbf{c}(\theta)$ is a circular arc, these conditions can be reduced to a simple equation in a single variable.

**Proposition 1.** If $\mathbf{c}(\theta)$ is a circular arc, the collinear normal conditions (3) can be written as follows:
\[
\mathbf{b}(t) \cdot \hat{\mathbf{c}}(\theta) = 0.
\]

**Proof.** Since $\mathbf{c}(\theta)$ is a circular arc, we have $\mathbf{c}(\theta) \cdot \hat{\mathbf{c}}(\theta) = 0$, so that the first of the collinear normal conditions can be written as follows:
\[
\mathbf{b}(t) \cdot \hat{\mathbf{c}}(\theta) = 0.
\]

This implies that $\mathbf{b}(t)$ and $\hat{\mathbf{c}}(\theta)$ are orthogonal, and from the second of the collinear normal conditions (3), we see that the two tangent vectors $\hat{\mathbf{b}}(t)$ and $\hat{\mathbf{c}}(\theta)$ are parallel. Consequently, $\mathbf{b}(t)$ and its tangent vector $\hat{\mathbf{b}}(t)$ are orthogonal which is expressed by Eq. (4) above. \[\square\]

A consequence of this proposition is that the Hausdorff distance between $\mathbf{b}(t)$ and $\mathbf{c}(\theta)$ will be the maximum absolute value of the signed distance function $d(t) = \| \mathbf{b}(t) \| - 1$. However, we will analyze the following signed distance function
\[
\psi(t) = \mathbf{b}(t) \cdot \mathbf{b}(t) - 1 = \| \mathbf{b}(t) \|^2 - 1, \quad 0 \leq t \leq 1
\]
instead of $d(t)$ to simplify subsequent calculations. This function is equivalent to the original signed distance function $d(t)$ in the following senses:
- $d(t) = 0$ and $\psi(t) = 0$ have the same solution sets.
- $d(t) = 0$ and $\psi(t) = 0$ have the same solution sets and $\psi'(t)d'(t) > 0$ for all $t$.
- $\psi(t)$ is a maximum or minimum of $\psi(t)$ if and only if $d(t)$ is a maximum or minimum of $d(t)$.

The above properties can be immediately verified from the equality $\psi(t) = d(t)(d(t) + 2)$. As a consequence of the last property, the Hausdorff distance occurs at $\mathbf{b}(\tau)$ on $\mathbf{b}(t)$ when the maximum absolute value of $\psi(t)$ occurs at the parametric value $t = \tau$. 


3. The best $G^1$ cubic Bézier approximations of a circular arc

We want to find the $G^1$ cubic Bézier approximation $\mathbf{b}(t)$, $0 \leq t \leq 1$ which has the minimum Hausdorff distance $d_H(\mathbf{b}, \mathbf{c})$.

3.1. One-parameter family of $G^1$ cubic Bézier approximations

A planar cubic Bézier curve $\mathbf{b}(t)$, $0 \leq t \leq 1$ is a cubic polynomial parametric curve determined by the four control points $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$, and $\mathbf{b}_3$ as follows:

$$\mathbf{b}(t) = (1-t)^3\mathbf{b}_0 + 3t(1-t)^2\mathbf{b}_1 + 3t^2(1-t)\mathbf{b}_2 + t^3\mathbf{b}_3, \quad 0 \leq t \leq 1.$$

We see that $\mathbf{b}(t)$ interpolates $\mathbf{b}_0$ and $\mathbf{b}_3$ at $t = 0$ and $t = 1$ respectively. So the condition $\mathbf{g}^0$ determines the two control points $\mathbf{b}_0$ and $\mathbf{b}_3$, which are the two end-points of $\mathbf{c}(\theta)$. That is,

$$\mathbf{b}_0 = \mathbf{c}(0) = (1, 0), \quad \text{and} \quad \mathbf{b}_3 = \mathbf{c}(\alpha) = (\cos \alpha, \sin \alpha).$$

The condition $\mathbf{g}^1$ restricts the locations of the two remaining control points $\mathbf{b}_1$ and $\mathbf{b}_2$ to be on each of the tangent lines at the two end-points of $\mathbf{c}(\theta)$. A further restriction can be obtained by considering the symmetry of the circular arc with respect to the line $\theta = \alpha/2$, and then making the cubic Bézier curve symmetric to this line. So the two remaining control points can be written with a single parameter:

$$\mathbf{b}_1 = (1, h), \quad \text{and} \quad \mathbf{b}_2 = (\cos \alpha, -\sin \alpha) + h(\sin \alpha, -\cos \alpha).$$

Thus we arrive at a one-parameter family of cubic Bézier approximations of $\mathbf{c}(\theta)$:

$$\mathbf{b}(h, t) = (x(h, t), y(h, t)), \quad 0 < h, \quad 0 \leq t \leq 1,$$

where

$$x(h, t) = (2 - 9h \sin \alpha - 2 \cos \alpha)t^3 + (3 \cos \alpha + 3h \sin \alpha - 3)t^2 + 1,$$

$$y(h, t) = (3h - 2 \sin \alpha + 3 \cos \alpha)t^3 + (3 \sin \alpha - 6h - 3 \cos \alpha)t^2 + 3ht.$$

Our aim now is to find the best approximation to the circular arc $\mathbf{c}(\theta)$ in this family of curves, based on the Hausdorff distance function. We will call $\mathbf{b}(h_0, t)$ the best $G^1$ cubic Bézier approximation if $d_H(\mathbf{b}(h_0, t), \mathbf{c}(\theta))$ is the minimum of $d_H(\mathbf{b}(h, t), \mathbf{c}(\theta))$ for all $h > 0$, and then $h_0$ will be called the best $G^1$ parameter.

3.2. Analysis of the signed distance function $\psi(h, t)$

If we consider the signed distance function

$$\psi(h, t) = x^2(h, t) + y^2(h, t) - 1$$

of the family of curves (5), we see that

$$\psi(h, t) = t^2(1-t)^2[f(h)t(t-1)+g(h)],$$

$$\frac{\partial}{\partial t} \psi(h, t) = t(t-1)(2t-1)(3f(h)t(t-1)+2g(h)).$$

with

$$f(h) = 2(9(1+\cos \alpha)h^2 - 12(\sin \alpha)h + 4(1-\cos \alpha)), $$

$$g(h) = 3(3h^2+2(\sin \alpha)h - 2(1-\cos \alpha)).$$

From the above equalities, we can deduce that:

(i) $f(h) > 0$ for $0 < h < 1$, $0 < \alpha < \pi$, because it is the sum of the squares of the two leading coefficients of $x(t)$ and $y(t)$.

(ii) A curve $\mathbf{b}(h, t)$ can intersect the circular arc $\mathbf{c}(\theta)$ at most two times in its interior counting multiplicity for any fixed $h > 0$, and the intersection points are $\mathbf{b}(h, t_1)$ and $\mathbf{b}(h, t_2)$, where $t_1$ and $t_2$ are solutions of the equation $f(h)t(t-1)+g(h) = 0$ with $t_1 \leq t_2 = 1 - t_1$.

(iii) A curve $\mathbf{b}(h, t)$ can have at most two collinear normal points, except for $\mathbf{b}(h, \frac{1}{2})$, in its interior for any fixed $h > 0$, and the collinear normal points are $\mathbf{b}(h, \tau_1)$ and $\mathbf{b}(h, \tau_2)$, where $\tau_1$ and $\tau_2$ are solutions of the equation $3f(h)t(t-1)+2g(h) = 0$ with $\tau_1 \leq \tau_2 = 1 - \tau_1$, and these points have the same collinear normal distances.

(iv) The points $\mathbf{b}(h, 0)$, $\mathbf{b}(h, 1)$, and $\mathbf{b}(h, \frac{1}{2})$ are always collinear normal points on $\mathbf{b}(h, t)$ for $h > 0$, and $\mathbf{b}(h, 0)$ and $\mathbf{b}(h, 1)$ always have zero collinear normal distances.

We note that $\tau_0 = \frac{1}{2}$, and remark that $\tau_1$ and $\tau_2$ are functions of $h$.

Now we will analyze the signed distance functions $\psi(h, \tau_k(h))$, $k = 0, 1, 2$ of the three collinear normal points $\mathbf{n}_k = \mathbf{b}(h, \tau_k)$, $k = 0, 1, 2$ on $\mathbf{b}(h, t)$. 

Lemma 2. We have
\[ \frac{d}{dh} \psi(h, \tau_k(h)) = \frac{\partial}{\partial h} \psi(h, \tau_k(h)), \quad k = 0, 1, 2. \]

Proof. This assertion follows immediately from the equality
\[ \frac{d}{dh} \psi(h, \tau_k(h)) = \frac{\partial}{\partial h} \psi(h, \tau_k(h)) + \frac{\partial}{\partial t} \psi(h, \tau_k(h)) \frac{d\tau_k}{dh}, \quad k = 0, 1, 2, \]
and the fact that \( \frac{\partial}{\partial t} \psi(h, \tau_k(h)) = 0. \)

Proposition 3. We have
\[ \frac{\partial}{\partial h} \psi(h, t) > 0, \quad \text{for all } 0 < t < 1, 0 < \alpha < \pi, \ h > 0. \] (6)

Proof. The value of this partial derivative is given by
\[ \frac{\partial}{\partial h} \psi(h, t) = 6\xi^2(1 - \xi)^2\psi_1(h, t), \] (7)
where
\[ \psi_1(h, t) = \sin \alpha (1 + 4\xi(1 - \xi)) + 3h(1 - 2\xi(1 - \xi)(1 + \cos \alpha)). \]
Since \( 6\xi^2(1 - \xi)^2 > 0 \) for \( 0 < \xi < 1, \) it suffices to show that \( \psi_1(h, t) > 0 \) for \( 0 < t < 1, 0 < \alpha < \pi, \ h > 0. \) First, we see that
\[ \sin \alpha (1 + 4\xi(1 - \xi)) > 0 \quad \text{for } 0 < t < 1, 0 < \alpha < \pi, \]
and second, we see that
\[ 3h(1 - 2\xi(1 - \xi)(1 + \cos \alpha)) > 0 \quad \text{for } h > 0, \]
because
\[ 0 < t(1 - t) \leq \frac{1}{4}, \quad \text{for } 0 < t < 1 \quad \text{and} \quad 0 < 1 + \cos \alpha < 2, \quad \text{for } 0 < \alpha < \pi. \]

Corollary 4. We have
\[ \frac{d}{dh} \psi(h, \tau_k(h)) > 0, \quad k = 0, 1, 2, \quad \text{for all } 0 < \alpha < \pi, \ h > 0. \]

3.3. Necessary and sufficient conditions for the best \( G^1 \) cubic Bézier approximation \( b(h_0, t) \) to \( c(\theta) \)

Consider the curve \( b(h_0, t) \), which satisfies the \( G^2 \) end-point interpolation conditions, because
\[ h_\ell = \frac{-\sin \alpha + \sqrt{\sin^2 \alpha - 6 \cos \alpha + 6}}{3} \]
is the positive solution of the equation \( g(h) = 0. \) Consider also \( b(h_\ell, t) \), which interpolates the mid-point \( (\cos \frac{\alpha}{2}, \sin \frac{\alpha}{2}) \) of \( c(\theta) \) at \( t = \frac{1}{2} \), so that
\[ h_\ell = \frac{4(2 \sin \frac{\alpha}{2} - \sin \alpha)}{3(1 - \cos \alpha)}. \]
We can verify the following facts using the properties of \( \psi(h, t) \) (see Fig. 1).

a. The curve \( b(h_0, t), 0 < t < 1 \) is located inside the unit circle if and only if \( 0 < h < h_\ell \), and \( b(h_\ell, t) \) is the best approximation to \( c(\theta) \) for \( 0 < h \leq h_\ell \).

b. The curve \( b(h_\ell, t), 0 < t < 1 \) is located outside the unit circle if and only if \( h \geq h_\ell \), and \( b(h_\ell, t) \) is the best approximation of \( c(\theta) \) for \( h \geq h_\ell \).

c. If \( h_\ell < h < h_\ell \), then \( b(h, t) \) has two transversal intersections with \( c(\theta) \) at the parametric values \( t_1 \) and \( t_2 \) with \( 0 < t_1 < t_2 = 1 - t_1 \), and \( b(h, t) \) has three collinear normal points at the parametric values \( t_0 = \frac{1}{2}, t_1 \) and \( t_2 \) with \( 0 < t_2 < t_1 = 1 - t_1 \).

d. If \( h_\ell < h < h_\ell \), then \( \psi(h, t_0) < 0 \) and \( \psi(h, t_1) = \psi(h, t_2) > 0. \)
Corollary 4

The curve $\gamma(8)$ is the unique intersection of two graphs at $d(\ell, u)$.

Theorem 5.

We can now state the necessary and sufficient conditions for the best parameter $h_b$ for the original signed distance function:

$$\psi(h, \tau_1) = d(h, \tau_1)$$

We see that neither $b(h, t)$ nor $b(h, t)$ can be the best $G^1$ approximation for all $h > 0$, and the best $G^1$ parameter $h_b$ is located in the interval $(h_l, h_u)$. Since the Hausdorff distance is the maximum collinear normal distance, we see that $h_b$ is the best $G^1$ parameter if and only if

$$\max_{k=0,1} |d(h, \tau_k)| = \min_{h_l < h < h_u} \max_{k=0,1} |d(h, \tau_k)|,$$

and we can now state the necessary and sufficient conditions for the best parameter $h_b$, and the existence and uniqueness of $h_b$.

Theorem 5. The curve $b(h, t)$ is the best $G^1$ cubic Bézier approximation if and only if $|d(h, \tau_0)| = |d(h, \tau_1)|$, or equivalently, $d(h, \tau_0) + d(h, \tau_1) = 0$, and such $h_b$ uniquely exists in the interval $(h_l, h_u)$ (see Fig. 1).

Proof. We see that $\inf_{h_l < h < h_u} |d(h, \tau_0)| = \inf_{h_l < h < h_u} |d(h, \tau_1)| = 0$. If we also consider the inequalities (8), there exists a unique parameter value $h_b$ which satisfies $|d(h, \tau_0)| = |d(h, \tau_1)|$. And at this parametric value $\max_{h=0,1} |d(h, \tau_k)|$ attains its minimum value for all $h \in (h_l, h_u)$ (see Fig. 2).

3.4. Finding the best $G^1$ parameter

A method of finding the parameter $h$ for which $b(h, t)$ satisfies the equioscillating condition has been described previously [2]. Here we present a simpler method.

Consider the function

$$\tilde{\psi}(h, t) = \frac{\psi(h, t)}{f(h)} = t^2(t - 1)^2 \left(t^2(t - 1) + \frac{g(h)}{f(h)}\right) = t^2(t - 1)^2(t^2(t - 1) + k(h))$$

where $k(h) = \frac{g(h)}{f(h)} - \frac{t^4(t - 1)^3}{f(h)}$.

The condition $|d(h, \tau_0)| = |d(h, \tau_1)|$ is called the equioscillating condition on $b(h, t)$. This was initially proposed in [2] without justification, and was discussed subsequently [4] using a different approach to ours.
There are two collinear normal points at the parameters
\[ t = \frac{1}{2} \pm \frac{\sqrt{9 - 24k}}{6}, \]
which can be obtained from the equation \( \frac{n}{m} \bar{\psi} = 0 \). The equioscillating condition implies that \( 256k^3 + 108k - 27 = 0 \), and this equation has a pair of complex conjugate solutions and one real positive solution, with the numerical value \( k = 0.2235268642 \). Now we have a quadratic equation in \( h \) of the form \( g(h) = \bar{k}(h) = 0 \) for \( h \), which is
\[ 9(1 - 2k \cos \alpha - 2k)h^2 + 6 \sin \alpha (1 + 4k)h + 2(\cos \alpha - 1)(3 + 4k) = 0. \]
We see that this equation has one negative and one positive solution, and the positive solution will be the best \( G^1 \) parameter. For instance, we have \( h_0 = 0.76808599 \) for \( \alpha = \frac{2\pi}{3} \) (see Fig. 1).

4. The best \( G^2 \) quartic Bézier approximations of a circular arc

We will now extend the result obtained in the previous section to \( G^2 \) quartic Bézier approximations, and identify the best \( G^2 \) quartic Bézier approximation \( \mathbf{b}(t) \) of a given circular arc \( \mathbf{c}(\theta) \), again based on the Hausdorff distance function. We note that a similar result has been achieved in [5] in which they identified two best \( G^2 \) quartic Bézier approximations of a circular arc which are the best in the proper subsets \( H_3 \) and \( H_4 \cup H_5 \) of parameter set \( \mathcal{H} \), respectively. They also improved their results in [6]. Now we are going to identify the best \( G^2 \) quartic Bézier approximation which is the best in the total parameter set \( \mathcal{H} \), and verify our result by comparing the best approximation with the previous results in [5,6] at the end of this section.

A quartic Bézier curve is determined by its five control points \( \mathbf{b}_k \), \( 0 \leq k \leq 4 \). The \( G^0 \) condition immediately fixes the two control points at the ends of the curve:
\[ \mathbf{b}_0 = \mathbf{c}(0) = (1, 0), \quad \text{and} \quad \mathbf{b}_4 = \mathbf{c}(\alpha) = (\cos \alpha, \sin \alpha). \]
The \( G^1 \) condition restricts the two control points \( \mathbf{b}_1 \) and \( \mathbf{b}_3 \) to lie respectively on the tangent lines at the two end-points \( \mathbf{b}_0 \) and \( \mathbf{b}_4 \), so that we can write:
\[ \mathbf{b}_1 = (1, h) \quad \text{and} \quad \mathbf{b}_3 = (\cos \alpha, \sin \alpha) + h(\sin \alpha, -\cos \alpha), \quad h > 0. \]
The symmetry of the circular arc restricts the middle control point \( \mathbf{b}_2 \) to lie on the line of symmetry, and this condition can be expressed as follows:
\[ \mathbf{b}_2 = r \frac{\mathbf{c}(0) + \mathbf{c}(\alpha)}{2} = \left( \frac{r}{2} (1 + \cos \alpha), \frac{r}{2} \sin \alpha \right), \quad r > 1. \]
The \( G^2 \) condition allows us to express \( r \) as a function of \( h \):
\[ r = \frac{6 - 8h^2}{3(1 + \cos \alpha)}. \]
Now we have a one-parameter family of \( G^2 \) quartic Bézier approximations of \( \mathbf{c}(\theta) \):
\[ \mathbf{b}(h, t) = (x(h, t), y(h, t)) \quad 0 < h, \quad 0 \leq t \leq 1, \tag{10} \]
where
\[ x(h, t) = (-8h^2 - 4h \sin \alpha - 3 \cos \alpha + 3)t^4 + 4(4h^2 + h \sin \alpha + \cos \alpha - 1)t^3 - 8h^2t^2 + 1, \]
\[ y(h, t) = \frac{1}{1 + \cos \alpha} \left( (-8h^2 \sin \alpha + 4h \cos^2 \alpha - 4h - 3 \sin \alpha \cos \alpha + 3 \sin \alpha)t^4 \\
+ 4(4h^2 \sin \alpha - h \cos^2 \alpha + 2h \cos \alpha + 3h + \sin \alpha \cos \alpha - 2 \sin \alpha)t^3 \\
+ (-8h^2 \sin \alpha - 12h \cos \alpha - 12h + 6 \sin \alpha)t^2 + 4(h \cos \alpha + 1)t \right). \]
This leads to the following expression for the signed distance function:
\[ \psi(h, t) = x^2(h, t) + y^2(h, t) - 1 = 2t^3(t - 1)^3 \frac{(a(h)t(t - 1) + b(h))}{1 + \cos \alpha}, \]
where
\[ a(h) = 64h^4 + 64(\sin \alpha)h^2 - 16(\cos^2 \alpha - 3 \cos \alpha + 2)h^2 + 24 \sin \alpha(\cos \alpha - 1)h + 9(\cos \alpha - 1)^2, \]
\[ b(h) = 4(8(\sin \alpha)h^3 + 8(\cos \alpha + 1)h^2 - \sin \alpha(\cos \alpha + 7)h + (1 - \cos^2 \alpha)). \]
We can easily verify the following properties of \( \mathbf{e}(h, t) = a(h)t(t - 1) + b(h): \)
We have established previously that for $k = 1, 2, 3$, $\psi(u_k, t) > 0$, for all $0 < t < 1$, that is, in the interior of the curve $b(u_k, t)$ is located outside the unit circle.

Now we analyze the discriminant $a(h) - 4a(h)b(h) = a(h)(a(h) - 4b(h))$ of $e(h, t)$. We can factorize the second term $a(h) - 4b(h) = p(h)q(h)$,

with

\[
p(h) = -8h^2 + 4h \sin \alpha + 5 \cos \alpha + 11 - 16 \cos(\alpha/2),
\]

\[
q(h) = -8h^2 + 4h \sin \alpha + 5 \cos \alpha + 11 + 16 \cos(\alpha/2).
\]

The quadratic equation $p(h) = 0$ has two real solutions $\mu_1$ and $\mu_2$ for $h$, and the quadratic equation $q(h) = 0$ also has two real solutions $\mu_3$ and $\mu_4$ for $h$, which are ordered as follows:

\[
\mu_3 < \mu_1 < -\frac{1}{4} \sin \alpha < \mu_2 < \mu_4, \quad \text{for} \quad \alpha > 0.
\]

We see that $a - 4b = 0$ if and only if the quadratic equation $at(t - 1) + b = 0$ has a double root, which must be $t = \frac{1}{2}$ by the symmetry from $b(h, t)$. We note that $\mu_1$ and $\mu_2$ are the same as (3.14) in [5], and $b(\mu_1, t)$ and $b(\mu_2, t)$ have tangential intersections with the circular arc $c(\theta)$ at $\theta = \frac{\pi}{2}$, whereas $b(\mu_3, t)$ and $b(\mu_4, t)$ have tangential intersections with the circular arc at $\theta = \pi + \frac{\pi}{2}$. Ahn and Kim [5] also verified the following:

- For $k = 1, 2, \psi(\mu_k, t) \leq 0$, that is, in the interior of the curve $b(\mu_k, t)$ is located inside the unit circle except for the point $b(\mu_k, \frac{1}{2})$.
- $b(\mu_2, t)$ is the best $G^2$ Bézier approximation that satisfies $\|b(\mu_k, t)\| \leq 1$ for all $0 \leq t \leq 1$.

We have seen that the two approximations $b(u_3, t)$ and $b(\mu_2, t)$ are the best within proper subsets of the set of all $G^2$ quartic Bézier approximations. Now, we will find the best $G^2$ quartic Bézier approximation $b(h, t)$ in the set of all $G^2$ quartic Bézier approximations.

**Proposition 6.** We have

\[
0 < u_1 < \mu_1 < u_2 < \frac{1}{4} \sin \alpha < u_3 < \mu_2 \quad \text{for} \quad 0 < \alpha < \pi.
\]

**Proof.** We know that $u_i$, $i = 1, 2, 3$ are solutions of $b(h) = 0$, and $\mu_j$, $j = 1, 2, 3, 4$ are solutions of $(a - 4b)(h) = 0$. We can see that $u_1 < \mu_1 < u_2 < u_3 < \mu_2$ from the inequalities

\[
(a - 4b)(u_i) = a(u_i) > 0, \quad \text{for} \quad i = 1, 2, 3,
\]

\[
4b(\mu_j) = a(\mu_j) > 0, \quad \text{for} \quad j = 1, 2, 3, 4.
\]

It is sufficient to show that $b\left(\frac{1}{4} \sin \alpha\right) < 0$ for the assertion $u_2 < \frac{1}{4} \sin \alpha < u_3$. We have

\[
b\left(\frac{1}{4} \sin \alpha\right) = -\frac{1}{2} \sin^2 \alpha(\cos \alpha - 1)^2 < 0 \quad \text{for} \quad 0 < \alpha < \pi. \quad \Box
\]

Now we consider a partition of the set $\mathcal{H} = \{h : h > 0\}$ of parameter space

\[
\mathcal{H} = \bigcup_{k=1}^{5} \mathcal{H}_k,
\]

where $\mathcal{H}_k = \mathcal{H} \cap \tilde{\mathcal{H}}_k$ are disjoint subsets of $\mathcal{H}$ with

\[
\tilde{\mathcal{H}}_1 = \{h : \mu_1 \leq h\}, \quad \tilde{\mathcal{H}}_2 = \{\mu_1 < h < u_2\}, \quad \tilde{\mathcal{H}}_3 = \{u_2 \leq h \leq u_3\}, \quad \tilde{\mathcal{H}}_4 = \{u_3 < h < \mu_2\}, \quad \tilde{\mathcal{H}}_5 = \{\mu_2 \leq h\}.
\]

We note that $\tilde{\mathcal{H}}_k = \mathcal{H}_k$ for $k = 3, 4, 5$, and $\mathcal{H}_1 = \emptyset$, $\mathcal{H}_2 \subseteq \tilde{\mathcal{H}}_2$ if $\mu_1 < 0$, which is equivalent to $\cos \frac{\alpha}{2} < \frac{1}{2}$. The following facts have been established previously [5]:

- $h \in \mathcal{H}_3$ if and only if $\psi(h, t) \geq 0$ for all $0 \leq t \leq 1$, and $h = u_3$ is the best $G^2$ parameter in $\mathcal{H}_3$.
- $h \in \mathcal{H}_1 \cup \mathcal{H}_5$ if and only if $\psi(h, t) \leq 0$ for all $0 \leq t \leq 1$, and $h = \mu_2$ is the best $G^2$ parameter in $\mathcal{H}_1 \cup \mathcal{H}_5$. 
Corollary 7. If \( h \in \widetilde{H}_2 \cup H_4 \), then \( b(h, t) \) has two transversal intersections with \( c(\theta) \), \( 0 < \theta < \alpha \) in the interior of the curve \( b(h, t) \), \( 0 < t < 1 \).

Proof. If \( \mu_1 < h < \mu_2 \), we know that \( b(h, t) \) has two transversal intersections with the circular arc \( c(\theta) \). These two transversal intersections occur in the interior of the curve \( b(h, t) \), \( 0 < t < 1 \) if \( e(h, 0) = e(h, 1) = b(h) > 0 \). The condition \( b(h) > 0 \) is equivalent to \( u_1 < h < u_2 \) or \( h > u_3 \), and if we also apply Proposition 6, the corollary is established. \( \square \)

If \( h \in \widetilde{H}_2 \cup H_4 \), we know that \( e(h, t) = 0 \) has two solutions \( 0 < t_1 < \frac{1}{2} \) and \( \frac{1}{2} < t_2 = 1 - t_1 < 1 \), which correspond to the transversal intersections of the two curves \( b(h, t) \) and \( c(\theta) \) at parametric values \( t_1 \) and \( t_2 \) of \( b(h, t) \). Consequently, there will be three collinear normal points on \( b(h, t) \) at the parameter values \( t = t_0, t_1 \) and \( t_2 \), where \( t_0 = \frac{1}{2} \) and \( 0 < t_1 < t_1, t_1 + t_2 = 1 \), and these points satisfy
\[
\psi(h, t_0) > 0, \quad \text{and} \quad \psi(h, t_1) = \psi(h, t_2) < 0.
\]
The following lemma was verified in the previous section.

Lemma 8. We have
\[
\frac{d}{dh} \psi(h, \tau_k(h)) = \frac{\partial}{\partial h} \psi(h, \tau_k(h)), \quad k = 0, 1, 2.
\]
And it follows that \( \psi(h, t) \) is monotone with respect to \( h \) in \( \widetilde{H}_2 \) and \( H_4 \).

Proposition 9. We have
\[
\frac{\partial}{\partial h} \psi(h, t) > 0 \quad \text{for} \quad h \in \widetilde{H}_2, \quad 0 < t < 1, \quad 0 < \alpha < \pi,
\]
and
\[
\frac{\partial}{\partial h} \psi(h, t) < 0 \quad \text{for} \quad h \in H_4, \quad 0 < t < 1, \quad 0 < \alpha < \pi.
\]

Proof. The first assertion
\[
\frac{\partial \psi}{\partial h} = \frac{2t^3(t - 1)^3}{1 + \cos \alpha} (a'(h)t(t - 1) + b'(h)) > 0 \quad \text{for all} \quad h \in \widetilde{H}_2, \quad 0 < t < 1
\]
is equivalent to
\[
a'(h)t(t - 1) + b'(h) < 0 \quad \text{for} \quad h \in \widetilde{H}_2, \quad 0 < t < 1.
\]
If \( a'(h) \leq 0 \), then \( a'(h)t(t - 1) + b'(h) \) has its maximum \( -\frac{1}{4}a'(h) + b'(h) \) at \( t = \frac{1}{2} \), and it is sufficient to show that
\[
a'(h) - 4b'(h) = (a - 4b)'(h) > 0, \quad \text{for} \quad h \in \widetilde{H}_2.
\]
If we consider the inequality in Proposition 6, we can easily verify this inequality. If \( a'(h) > 0 \), then \( a'(h)t(t - 1) + b'(h) \) has its maximum \( b'(h) \) at \( t = 0, 1 \), and it is sufficient to show that \( b'(h) < 0 \) for \( h \in H_2 \). If we consider
\[
b'(h) = 4(24\sin \alpha)h^2 + 16(1 + \cos \alpha)h - \sin \alpha(\cos \alpha + 7),
\]
we see that the two solutions of \( b'(h) = 0 \) are composed of one negative and one positive real numbers and \( u_2 \) is located between them. So it is sufficient to show that \( b'(\mu_1) < 0 \), which can be verified from Proposition 6.

Similarly, the second assertion is equivalent to
\[
a'(h)t(t - 1) + b'(h) > 0 \quad \text{for} \quad h \in H_4, \quad 0 < t < 1.
\]
If \( a'(h) \geq 0 \), then \( a'(h)t(t - 1) + b'(h) \) has its minimum \( -\frac{1}{4}a'(h) + b'(h) \) at \( t = \frac{1}{2} \), and it suffices to show that
\[
a'(h) - 4b'(h) < 0, \quad \text{for all} \quad h \in H_4.
\]
We see that this is true because \( \frac{1}{2} \sin \alpha < u_3 < \mu_2 \). If \( a'(h) < 0 \), then \( a'(h)t(t - 1) + b'(h) \) has its minimum \( b'(h) \) at \( t = 0, 1 \), and it is sufficient to show that \( b'(h) > 0 \) for all \( h \in H_4 \). Since \( u_3 \) is the largest solution of \( b(h) = 0 \), we see that \( b'(h) > 0 \) if \( h > u_3 \). \( \square \)

Lemma 8 and Proposition 9 allow us to state the following corollary.
Corollary 10. We have
\[
\frac{d}{dh} \psi(h, \tau_h) > 0, \quad k = 0, 1, 2 \quad \text{for all } h \in \tilde{H}_2, \quad 0 < \alpha < \pi,
\]
and
\[
\frac{d}{dh} \psi(h, \tau_h) < 0, \quad k = 0, 1, 2 \quad \text{for all } h \in H_4, \quad 0 < \alpha < \pi.
\]

Now we see that the situation is almost the same as that in the previous section. Thus we can assert the existence and uniqueness of the best parameters \( h_{b2} \) and \( h_{b4} \) in \( H_2 \) and \( H_4 = \tilde{H}_4 \) respectively, without proof.

Theorem 11. For \( k = 2, 4 \), \( h_{b_k} \) is the best \( G^2 \) parameter in \( \tilde{H}_k \) if and only if \( |d(h_{b_k}, \tau_0)| = |d(h_{b_k}, \tau_1)| \), or equivalently, \( d(h_{b_k}, \tau_0) + d(h_{b_k}, \tau_1) = 0 \), and such \( h_{b_k} \) is unique in \( H_k \).

We know that \( b(h_{b4}, t) \) is a better \( G^2 \) approximation than \( b(u_3, t) \) and \( b(\mu_2, t) \), which are the best that can be found in the parameter-sets \( H_3 \) and \( H_1 \cup H_5 \) respectively. Consequently, we can say that the best \( G^2 \) Bézier approximation of the circular arc \( c(\theta) \) is whichever of \( b(h_{b2}, t) \) and \( b(h_{b4}, t) \) is the better.

4.1. Finding the best \( G^2 \) parameter

We now present a method of determining the best parameter which is similar to that given in the previous section. Consider the function
\[
\tilde{\psi}(h, t) = \frac{\psi(h, t)}{a(h)} = t^3(t - 1)^3 (t^2(t - 1) + b(h)\frac{a(h)}{a(h)}) = t^3(t - 1)^3(t^2(t - 1) + k(h)).
\]

There are two collinear normal points at the parameters
\[
t = \frac{1}{2} \pm \sqrt{1 - 3k}.
\]
which can be obtained from the equation \( \frac{\partial}{\partial h} \tilde{\psi} = 0 \). The equioscillating condition implies the equation \( 27k^4 + 4k - 1 = 0 \), which has a pair of complex conjugate solutions, one negative solution, and one positive solution with the numerical value \( k = 0.2308349475 \). Since \( a/b = k \) is positive in \( H_2 \cup H_4 \), we use the positive solution to calculate the best \( G^2 \) parameter. Now we have a quartic equations in \( h \) of the form \( b(h) - ka(h) = 0 \), which is given as follows
\[
-64kh^4 + 32\sin\alpha(1 - 2k)h^3 + 16(k\cos\alpha^2 - 3k\cos\alpha + 2k + 2\cos\alpha + 2)h^2
\]
\[
+ 4\sin\alpha(-6k\cos\alpha + 6k - \cos\alpha - 7)h - 9k(1 - \cos\alpha)^2 + 4(1 - \cos^2\alpha) = 0.
\]

There exist exactly one solution \( h_{b2} \in \tilde{H}_2 \) and \( h_{b4} \in H_4 \) in each of the parameter sets \( H_2 \) and \( H_4 \). For instance, we have the solutions
\[
h_{b2} = 0.1062156511 \quad \text{and} \quad h_{b4} = 0.4025869310.
\]
for \( \alpha = \frac{\pi}{5} \). The corresponding Hausdorff distances are
\[
d_H(h_{b2}) = 1.850734382 \times 10^{-03} \quad \text{for} \ h_{b2} \ (\text{see Fig. 3}), \quad \text{and}
\]
\[
d_H(h_{b4}) = 2.592341606 \times 10^{-06} \quad \text{for} \ h_{b4} \ (\text{see Fig. 4}).
\]

Hence the best \( G^2 \) parameter is \( h_{b4} \), which has the Hausdorff distance \( d_H(h_{b4}) \) (see Fig. 5). If we compare this distance to the result given in Table 1 of [6], we see that \( d_H(h_{b4}) \) is much smaller than their result \( d_H(\mu_2) = 3.55 \times 10^{-06} \) (see Fig. 6). And the Hausdorff distance \( d_H = 2.03 \times 10^{-06} \) for the \( G^1 \) quartic approximation presented in [6] is similar to \( d_H(h_{b4}) \).

5. Conclusions

We have identified the necessary and sufficient conditions needed to obtain the best \( G^1 \) cubic and \( G^2 \) quartic Bézier approximations of circular arcs based on the Hausdorff distance function, and we have presented practical methods of calculating such approximations. We want to extend this work to Bézier approximations of higher degree with different end-point continuity conditions. For instance, we plan to seek the best \( G^1 \) quartic, and the best \( G^2 \) and \( G^3 \) quintic Bézier approximations of a circular arc. Furthermore, we think that the method described in this paper can be applied for any one parameter family of approximation curves. So we are going to continue our research for identifying the best Bézier approximation of conic sections and for the best rational Bézier approximation of a given curve.
Fig. 3. Graphs of $\psi(h, t)$ with $\alpha = \pi/2$, dotted line (red): $h = u_2$, solid line (green): $h = h_{b2}$, dashed line (blue): $h = \mu_1$.

Fig. 4. Graphs of $\psi(h, t)$ with $\alpha = \pi/2$, dotted line (red): $h = u_3$, solid line (green): $h = h_{b4}$, dashed line (blue): $h = \mu_2$.

Fig. 5. Signed curvature plot of the best quartic approximation for $\alpha = \pi/2$.

Fig. 6. Table 1 in [6]. $b_{u_3}$, $b_{\mu_2}$ are results of [5], and $b$ is the result of [6].

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