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Well-posedness for equilibrium problems and for optimization problems with equilibrium constraints[☆]

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Abstract

In this paper we generalize the concepts of well-posedness to equilibrium problems and to optimization problems with equilibrium constraints. We establish some metric characterizations of well-posedness for equilibrium problems and for optimization problems with equilibrium constraints. We prove that under suitable conditions, the well-posedness is equivalent to the existence and uniqueness of solutions. The corresponding concepts of well-posedness in the generalized sense are also introduced and investigated for equilibrium problems and for optimization problems with equilibrium constraints.

Keywords: Equilibrium problems; Optimization problems with equilibrium constraints; Well-posedness; Metric characterizations; Monotonicity

1. Introduction

The importance of well-posedness is widely recognized in the theory of variational problems. The first concept of well-posedness was introduced by Tykhonov [1] for a global minimization problem having a unique solution, which has been known as Tykhonov well-posedness. The concept of Tykhonov well-posedness in the generalized sense was also introduced for a global minimization problem having more than one solution. Roughly speaking, the Tykhonov well-posedness of a global minimization problem means the existence and uniqueness of minimizers, and the convergence of every minimizing sequence toward the unique minimizer, and the Tykhonov well-posedness in the generalized sense means the existence of minimizers and the convergence of some subsequence of every minimizing sequence toward a minimizer. There are in the literature a very large number of papers dealing with the well-posedness of global minimization problems. For details, we refer readers to [1–6] and the references therein. The concept of well-posedness has also been generalized to several related variational problems: variational inequality problems [7–10], Nash equilibrium problems [11–16], and fixed point problems [17,18]. A more general formulation for the above variational problems is the equilibrium problem, which has recently had a deep development and impact

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in variational analysis. For details, we refer readers to [19–21]. But, to the best of our knowledge, the well-posedness issue of equilibrium problems has not been extensively studied yet. So it is interesting and important to study the well-posedness of equilibrium problems.

On the other hand, optimization problems with constraints defined by variational problems have been attracting increasing attentions of mathematician in past years since they provide unified mathematical models for some important problems arising in economics and engineering science. Such problems include bilevel programming problems [22,23], optimization problems with variational inequality constraints [8,24,25], optimization problems with Nash equilibrium constraints [26,27], optimization problems with equilibrium constraints [28,29], etc. Morgan [22] investigated the well-posedness of bilevel programming problems. Lignola and Morgan [8] studied the well-posedness of optimization problems with variational inequality constraints. Recently, Lignola and Morgan [27] further investigated the well-posedness of optimization problems with Nash equilibrium constraints.

Motivated and inspired by the above works, in this paper we shall investigate the well-posedness of equilibrium problems and of optimization problems with equilibrium constraints. We establish some metric characterizations of well-posedness. We also show that under suitable conditions, the well-posedness is equivalent to the existence and uniqueness of solutions, and that the well-posedness in the generalized sense is equivalent to the existence of solutions.

2. Preliminaries and notations

Let E be a real Banach space, K be a nonempty subset of E, and $\varphi : K \times K \to R$ be a bifunction. Blum and Oettli [19] understood the equilibrium problem (denoted by EP) by finding

$$u \in K \text{ such that } \varphi(u, v) \geq 0, \quad \forall v \in K.$$

The equilibrium problem provides a very general formulation of variational problems such as:

- Minimization problem: find $u \in K$ such that $f(u) \leq f(v)$ for all $v \in K$, where $f : K \to R$ is a functional. In this case, we define $\varphi(u, v) = f(v) - f(u)$ for all $u, v \in K$.
- Variational inequality: find $u \in K$ such that

  $$\langle Au, v - u\rangle \geq 0, \quad \forall v \in K,$$

  where $A : K \to X^*$ is a map and X^* denotes the topological dual of X. In this case, we define $\varphi(u, v) = \langle Au, v-u\rangle$ for all $u, v \in K$.

Now we recall some concepts and results.

Definition 2.1 (*[19]*). A bifunction $\varphi : K \times K \to R$ is said to be

(i) monotone iff for all $u, v \in K$,

$$\varphi(u, v) + \varphi(v, u) \leq 0.$$

(ii) strictly monotone iff it is monotone and the equality holds if and only if $u = v$.
(iii) hemicontinuous iff for each $x, y \in K$,

$$\limsup_{t\to 0+} \varphi(x + t(y - x), y) \leq \varphi(x, y).$$

Lemma 2.1 ([19]). *Let K be convex, $\varphi : K \times K \to R$ be a monotone and hemicontinuous bifunction. Assume that*

(i) $\varphi(u, u) \geq 0$ *for all* $u \in K$.
(ii) *for every* $u \in K$, $\varphi(u, \cdot)$ *is convex.*

Then for given $u^* \in K$,

$$\varphi(u^*, v) \geq 0, \quad \forall v \in K$$

if and only if

$$\varphi(v, u^*) \leq 0, \quad \forall v \in K.$$

Lemma 2.2 (*[19]*). *Let K be a compact and convex subset of E and $\varphi : K \times K \to R$ be a monotone and hemicontinuous bifunction. Assume that*

(i) $\varphi(u, u) = 0$ *for all* $u \in K$.
(ii) *for every* $u \in K$, $\varphi(u, \cdot)$ *is lower semicontinuous and convex.*

Then there exists $u^* \in K$ *such that*

$$\varphi(u^*, v) \geq 0, \quad \forall v \in K.$$

In addition, if φ is strictly monotone, the solution is unique.

Next we recall the formulation of optimization problems with equilibrium constraints. Let X be a nonempty closed subset of a parametric normed space, $f : X \times K \to R$, $h : X \times K \times K \to R$. The optimization problem with an equilibrium constraint (denoted by OPEC) is formulated as:

$$\min f(x, u) \quad s.t.\ (x, u) \in X \times K \text{ and } u \in T(x),$$

where $T(x)$ is the solution set of the parametric equilibrium problem EP(x) defined by, $u \in T(x)$ iff

$$h(x, u, v) \geq 0, \quad \forall v \in K.$$

When EP(x) reduces to a parametric minimization problem, OPEC coincides with the bilevel programming problems [22,23]. When EP(x) reduces to a parametric variational inequality problem, OPEC coincides with the optimization problems with variational inequality constraints [8,24,25].

We also need the concepts of noncompactness measure and Hausdorff metric.

Definition 2.2 (*See [30]*). Let A be a nonempty subset of E. The measure of noncompactness μ of the set A is defined by

$$\mu(A) = \inf\{\epsilon > 0 : A \subset \cup_{i=1}^{n} A_i,\ \text{diam } A_i < \epsilon, i = 1, 2, \ldots, n\},$$

where diam means the diameter of a set.

Definition 2.3. Let A, B be nonempty subsets of E. The Hausdorff metric $\mathcal{H}(\cdot, \cdot)$ between A and B is defined by

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} \|a - b\|$. Let $\{A_n\}$ be a sequence of nonempty subsets of E. We say that A_n converges to A in the sense of Hausdorff metric if $\mathcal{H}(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all selection $a_n \in A_n$. For more details on this topic, we refer readers to [31,30].

3. Well-posedness for equilibrium problems

In this section we shall investigate the well-posedness of equilibrium problems. With notations of Section 2 we consider the following parametric equilibrium problem:

$$\text{EP}(x)\text{: find } u \in K \text{ such that } h(x, u, v) \geq 0, \quad \forall v \in K.$$

Denote by $\mathcal{EP}$ the family $\{\text{EP}(x): x \in X\}$.

Definition 3.1. Let $x \in X$ and $\{x_n\} \subset X$ with $x_n \to x$. A sequence $\{u_n\} \subset K$ is said to be approximating for EP(x) corresponding to $\{x_n\}$ iff, there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$h(x_n, u_n, v) + \epsilon_n \geq 0, \quad \forall v \in K,\ \forall n \in N.$$

Definition 3.2. We say that $\mathcal{EP}$ is well-posed iff for every $x \in X$, EP(x) has a unique solution u_x, and for every sequence $\{x_n\}$ with $x_n \to x$, every approximating sequence for EP(x) corresponding to $\{x_n\}$ converges strongly to u_x, and that $\mathcal{EP}$ is well-posed in the generalized sense iff for every $x \in X$, EP(x) has a nonempty solution set S_x, and for every sequence $\{x_n\}$ with $x_n \to x$, every approximating sequence for EP(x) corresponding to $\{x_n\}$ has some subsequence which converges strongly to some point of S_x.

Remark 3.1. (i) When EP(x) reduces to the parametric minimization problem, Definitions 3.1 and 3.2 reduce to the corresponding definitions of [5,6].

(ii) When EP(x) reduces to the parametric variational inequality problem, Definition 3.1 coincides with Definition 2.1 of [8], and Definition 3.2 coincides with Definition 2.2 of [8].

The approximating solution set of EP(x) is defined by

$$\Omega_x(\delta,\epsilon) = \bigcup_{x'\in B(x,\delta)} \{u \in K: h(x', u, v) + \epsilon \geq 0, \forall v \in K\}, \quad \forall \delta, \epsilon \geq 0,$$

where $B(x, \delta)$ denotes the closed ball centered at x with radius δ.

The following theorem shows that the well-posedness of $\mathcal{EP}$ can be characterized by considering the behavior of the diameter of the approximating solution set.

Theorem 3.1. *Let K be a nonempty, closed and convex subset of E and $h : X \times K \times K \to R$. Assume that*

(i) $h(x, u, u) \geq 0$ *for all* $x \in X, u \in K$.

(ii) *for every* $x \in X$, $h(x, \cdot, \cdot)$ *is monotone and hemicontinuous.*

(iii) *for every* $(x, u) \in X \times K$, $h(x, u, \cdot)$ *is convex.*

(iv) *for every* $u \in K$, $h(\cdot, u, \cdot)$ *is lower semicontinuous.*

Then $\mathcal{EP}$ is well-posed if and only if for every $x \in X$,

$$\Omega_x(\delta,\epsilon) \neq \emptyset, \quad \forall \delta, \epsilon > 0, \quad \textit{and} \quad \operatorname{diam} \Omega_x(\delta,\epsilon) \to 0 \quad \textit{as } (\delta,\epsilon) \to (0,0). \tag{1}$$

Proof. Suppose that $\mathcal{EP}$ is well-posed. Then EP(x) has a unique solution u_x for all $x \in X$. Clearly $\Omega_x(\delta,\epsilon) \neq \emptyset$ since $u_x \in \Omega_x(\delta,\epsilon)$ for all $\delta, \epsilon > 0$. If diam $\Omega_x(\delta,\epsilon) \not\to 0$ as $(\delta,\epsilon) \to (0,0)$, then there exist $l > 0$ and $\delta_n > 0, \epsilon_n > 0$ with $\delta_n \to 0, \epsilon_n \to 0$, and $u_n, v_n \in \Omega_x(\delta_n, \epsilon_n)$ such that

$$\|u_n - v_n\| > l, \quad \forall n \in N. \tag{2}$$

Since $u_n, v_n \in \Omega_x(\delta_n, \epsilon_n)$, for each $n \in N$, there exist $x_n, y_n \in B(x, \delta_n)$ such that

$$h(x_n, u_n, v) + \epsilon_n \geq 0, \quad \forall v \in K$$

and

$$h(y_n, v_n, v) + \epsilon_n \geq 0, \quad \forall v \in K.$$

So $\{u_n\}$ and $\{v_n\}$ are approximating sequences for EP(x) corresponding to $\{x_n\}$ and $\{y_n\}$ respectively. By the well-posedness of $\mathcal{EP}$, they have to converge strongly to the unique solution u_x of EP(x), a contradiction to (2).

Conversely, suppose that condition (1) holds. Let $x_n \to x \in X$ and $\{u_n\} \subset K$ be an approximating sequence for EP(x) corresponding to $\{x_n\}$. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$h(x_n, u_n, v) + \epsilon_n \geq 0, \quad \forall v \in K, \ \forall n \in N. \tag{3}$$

This yields that $u_n \in \Omega_x(\delta_n, \epsilon_n)$ with $\delta_n = \|x_n - x\| \to 0$. It follows from (1) that $\{u_n\}$ is a Cauchy sequence and so it converges strongly to a point $\bar{u} \in K$. It follows from (3) and conditions (ii) and (iv) that

$$h(x, v, \bar{u}) \leq \liminf_{n\to\infty} h(x_n, v, u_n) \leq \liminf_{n\to\infty} \{-h(x_n, u_n, v)\} \leq \liminf_{n\to\infty} \epsilon_n = 0, \quad \forall v \in K.$$

This together with Lemma 2.1, $\bar{u}$ solves EP(x).

To complete the proof, it is sufficient to prove that EP(x) has a unique solution. If EP(x) has two distinct solutions u_1 and u_2, it is easily seen that $u_1, u_2 \in \Omega_x(\delta,\epsilon)$ for all $\delta, \epsilon > 0$. It follows that

$$0 < \|u_1 - u_2\| \leq \operatorname{diam} \Omega_x(\delta,\epsilon) \to 0,$$

a contradiction to (1). □

Remark 3.2. Theorem 3.1 generalizes Proposition 2.3 of [8].

For the well-posedness in the generalized sense, we give the following characterization by considering the noncompactness of approximate solution set.

Theorem 3.2. *Let X be finite dimensional and K be closed. Let $h : X \times K \times K \to R$ be such that $h(\cdot, \cdot, u)$ is upper semicontinuous for every $u \in K$. Then $\mathcal{EP}$ is well-posed in the generalized sense if and only if for every $x \in X$,*

$$\Omega_x(\delta, \epsilon) \neq \emptyset, \quad \forall \delta, \epsilon > 0, \quad \text{and} \quad \mu(\Omega_x(\delta, \epsilon)) \to 0 \quad \text{as } (\delta, \epsilon) \to (0, 0). \tag{4}$$

Proof. Suppose that $\mathcal{EP}$ is well-posed in the generalized sense. Let S_x be the solution set of EP(x) for all $x \in X$. Then S_x is nonempty compact. Indeed, let $\{u_n\}$ be any sequence in S_x. Then $\{u_n\}$ is an approximating sequence for EP(x). By the well-posedness in the generalized sense of $\mathcal{EP}$, $\{u_n\}$ has a subsequence which converges strongly to some point of S_x. Thus S_x is compact. Clearly $\Omega_x(\delta, \epsilon) \supset S_x$ for all $\delta, \epsilon > 0$. Now we show that

$$\mu(\Omega_x(\delta, \epsilon)) \to 0 \quad \text{as } (\delta, \epsilon) \to (0, 0).$$

Observe that for every $\delta, \epsilon > 0$,

$$\mathcal{H}(\Omega_x(\delta, \epsilon), S_x) = \max\{e(\Omega_x(\delta, \epsilon), S_x), e(S_x, \Omega_x(\delta, \epsilon))\} = e(\Omega_x(\delta, \epsilon), S_x).$$

Taking into account the compactness of S_x, we get

$$\mu(\Omega_x(\delta, \epsilon)) \leq 2\mathcal{H}(\Omega_x(\delta, \epsilon), S_x) + \mu(S_x) = 2e(\Omega_x(\delta, \epsilon), S_x).$$

To prove (4), it is sufficient to show

$$e(\Omega_x(\delta, \epsilon), S_x) \to 0 \quad \text{as } (\delta, \epsilon) \to (0, 0).$$

If $e(\Omega_x(\delta, \epsilon), S_x) \not\to 0$ as $(\delta, \epsilon) \to (0, 0)$. Then there exist $l > 0$ and $\delta_n > 0, \epsilon_n > 0$ with $\delta_n \to 0, \epsilon_n \to 0$, and $u_n \in \Omega_x(\delta_n, \epsilon_n)$ such that

$$u_n \notin S_x + B(0, l), \quad \forall n \in N. \tag{5}$$

As $u_n \in \Omega_x(\delta_n, \epsilon_n)$, $\{u_n\}$ is an approximating sequence for EP(x). By the well-posedness in the generalized sense of $\mathcal{EP}$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging strongly to some point of S_x. This contradicts (5) and so

$$e(\Omega_x(\delta, \epsilon), S_x) \to 0 \quad \text{as } (\delta, \epsilon) \to (0, 0).$$

Conversely, assume that (4) holds. We first show that $\Omega_x(\delta, \epsilon)$ is closed for all $\delta, \epsilon > 0$. Let $u_n \in \Omega_x(\delta, \epsilon)$ with $u_n \to u$. Then, for each $n \in N$, there exists $x_n \in B(x, \delta)$ such that

$$h(x_n, u_n, v) + \epsilon \geq 0, \quad \forall v \in K.$$

Without loss of generality, we can suppose that $x_n \to \bar{x} \in B(x, \delta)$ since X is finite dimensional. By the upper semicontinuity of $h(\cdot, \cdot, v)$, we get

$$h(\bar{x}, u, v) + \epsilon \geq 0, \quad \forall v \in K,$$

which together with $\bar{x} \in B(x, \delta)$ yields $u \in \Omega_x(\delta, \epsilon)$ and so $\Omega_x(\delta, \epsilon)$ is nonempty closed for all $\delta, \epsilon > 0$. Observe that

$$S_x = \bigcap_{\delta > 0, \epsilon > 0} \Omega_x(\delta, \epsilon).$$

Since

$$\mu(\Omega_x(\delta, \epsilon)) \to 0,$$

the theorem on p. 412 in [30] can be applied and one concludes that S_x is nonempty, compact, and

$$e(\Omega_x(\delta, \epsilon), S_x) = \mathcal{H}(\Omega_x(\delta, \epsilon), S_x) \to 0 \quad \text{as } (\delta, \epsilon) \to (0, 0).$$

Let $x_n \to x \in X$ and $\{u_n\} \subset K$ be an approximating sequence for EP(x) corresponding to $\{x_n\}$. Then there exist $\delta_n > 0, \epsilon_n > 0$ with $\delta_n \to 0, \epsilon_n \to 0$ such that

$$h(x_n, u_n, v) + \epsilon_n \geq 0, \quad \forall v \in K, \ \forall n \in N.$$

This means that $u_n \in \Omega_x(\delta_n, \epsilon_n)$ with $\delta_n = \|x_n - x\|$. It follows from (4) that

$$d(u_n, S_x) \le e(\Omega_x(\delta_n, \epsilon_n), S_x) \to 0.$$

Since S_x is compact, there exists $\bar{u}_n \in S_x$ such that

$$\|u_n - \bar{u}_n\| = d(u_n, S_x) \to 0.$$

Again from the compactness of S_x, $\{\bar{u}_n\}$ has a subsequence $\{\bar{u}_{n_k}\}$ converging strongly to $\bar{u} \in S$. Hence the corresponding subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges strongly to $\bar{u}$. Thus $\mathcal{EP}$ is well-posed in the generalized sense. □

Remark 3.3. Theorem 3.3 generalizes Theorem 4.2 of [10].

The following theorem shows that under suitable conditions, the well-posedness of $\mathcal{EP}$ is equivalent to the existence and uniqueness of the solution.

Theorem 3.3. *Let E be finite dimensional, K be a nonempty, closed and convex subset of E and $h : X \times K \times K \to R$. Assume that*

(i) $h(x, u, u) \ge 0$ *for all* $x \in X, u \in K$.
(ii) *for every* $x \in X$, $h(x, \cdot, \cdot)$ *is monotone and hemicontinuous.*
(iii) *for every* $(x, u) \in X \times K$, $h(x, u, \cdot)$ *is convex.*
(iv) *for every* $u \in K$, $h(\cdot, u, \cdot)$ *is continuous.*

Then $\mathcal{EP}$ is well-posed if and only if for every $x \in X$, EP(x) has a unique solution.

Proof. The necessity is obvious. For the sufficiency, suppose that EP(x) has a unique solution u_x for all $x \in X$. Let $x \in X$, $x_n \to x$ and $\{u_n\} \subset K$ be an approximating sequence for EP(x) corresponding to $\{x_n\}$. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$h(x_n, u_n, v) + \epsilon_n \ge 0, \quad \forall v \in K, \ \forall n \in N. \tag{6}$$

We assert that $\{u_n\}$ is bounded. Indeed, if $\{u_n\}$ is unbounded, without loss of generality, we can suppose that $\|u_n\| \to +\infty$. Let $t_n = \frac{1}{\|u_n - u_x\|}$ and $w_n = u_x + t_n(u_n - u_x)$. Without loss of generality, we can suppose that $t_n \in (0, 1)$ and $w_n \to w (\ne u_x)$ since E is finite dimensional. Since $h(x_n, \cdot, \cdot)$ is monotone, from (6) we get

$$h(x_n, v, u_n) \le -h(x_n, u_n, v) \le \epsilon_n, \quad \forall v \in K.$$

It follows from conditions (iii) and (iv) that

$$\begin{aligned} h(x, v, w) &= \lim_{n\to\infty} h(x_n, v, w_n) \\ &\le \lim_{n\to\infty} \{t_n h(x_n, v, u_n) + (1 - t_n) h(x_n, v, u_x)\} \\ &\le \lim_{n\to\infty} \{t_n \epsilon_n + (1 - t_n) h(x_n, v, u_x)\} \\ &= h(x, v, u_x), \quad \forall v \in K. \end{aligned} \tag{7}$$

Since u_x solves EP(x), from Lemma 2.1 we obtain

$$h(x, v, u_x) \le 0, \quad \forall v \in K,$$

which together with (7) implies that

$$h(x, v, w) \le 0, \quad \forall v \in K.$$

Again from Lemma 2.1, w is a solution of EP(x). This is a contradiction to the uniqueness of solution. Thus $\{u_n\}$ is bounded.

Let $\{u_{n_k}\}$ be any subsequence of $\{u_n\}$ such that $u_{n_k} \to \bar{u}$ as $k \to \infty$. It follows that

$$h(x, v, \bar{u}) = \lim_{k\to\infty} h(x_{n_k}, v, u_{n_k}) \le -\lim_{k\to\infty} h(x_{n_k}, u_{n_k}, v) \le \lim_{k\to\infty} \epsilon_{n_k} = 0, \quad \forall v \in K.$$

This together with Lemma 2.1 yields that $\bar{u}$ solves EP(x). By the uniqueness of the solution to EP(x), we have $\bar{u} = u_x$. Thus u_n converges to u_x. Therefore, $\mathcal{EP}$ is well-posed. □

Remark 3.4. Theorem 3.3 generalizes Proposition 2.8 of [8] and Theorem 5.2 of [10].

For the well-posedness in the generalized sense, we have the following result.

Theorem 3.4. *Let E be finite dimensional, K be a nonempty, closed and convex subset of E and $h : X \times K \times K \to R$. Assume that*

(i) $h(x, u, u) \geq 0$ *for all* $x \in X, u \in K$.
(ii) *for every* $x \in X$, $h(x, \cdot, \cdot)$ *is monotone and hemicontinuous.*
(iii) *for every* $(x, u) \in X \times K$, $h(x, u, \cdot)$ *is convex.*
(iv) *for every* $u \in K$, $h(\cdot, u, \cdot)$ *is continuous.*

If for each $x \in X$ there exists some $\epsilon > 0$ such that $\Omega_x(\epsilon, \epsilon)$ is nonempty and bounded, then $\mathcal{EP}$ is well-posed in the generalized sense.

Proof. Let $x \in X$, $x_n \to x$ and $\{u_n\}$ be an approximating sequence for EP(x) corresponding to $\{x_n\}$. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$h(x_n, u_n, v) + \epsilon_n \geq 0, \quad \forall v \in K, \ \forall n \in N. \tag{8}$$

Let $\epsilon > 0$ be such that $\Omega_x(\epsilon, \epsilon)$ is nonempty bounded. Then there exists n_0 such that $u_n \in \Omega_\alpha(\epsilon, \epsilon)$ for all $n > n_0$, and so $\{u_n\}$ is bounded. Then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to \bar{u}$ as $k \to \infty$. By the same arguments as Theorem 3.3, $\bar{u}$ solves EP(x). Thus $\mathcal{EP}$ is well-posed in the generalized sense. □

Remark 3.5. Theorem 3.4 says nothing but that, under suitable conditions, the well-posedness in the generalized sense of $\mathcal{EP}$ is equivalent to the existence of solutions.

As applications of Theorems 3.3 and 3.4, we give some classes of functionals that ensure this type of well-posedness.

Theorem 3.5. *Let E be finite dimensional, K be a nonempty, compact and convex subset of E and $h : X \times K \times K \to R$. Assume that*

(i) $h(x, u, u) = 0$ *for all* $x \in X, u \in K$.
(ii) *for every* $x \in X$, $h(x, \cdot, \cdot)$ *is monotone and hemicontinuous.*
(iii) *for every* $(x, u) \in X \times K$, $h(x, u, \cdot)$ *is convex.*
(iv) *for every* $u \in K$, $h(\cdot, u, \cdot)$ *is continuous.*

Then $\mathcal{EP}$ is well-posed in the generalized sense. In addition, if $h(x, \cdot, \cdot)$ is strictly monotone for all $x \in X$, then $\mathcal{EP}$ is well-posed.

Proof. The conclusion follows directly from Lemma 2.2 and Theorems 3.3 and 3.4. □

4. Well-posedness for optimization problems with equilibrium constraints

With notations of Section 2 we consider the well-posedness of the optimization problem with an equilibrium constraint (denoted by OPEC):

$$\min f(x, u) \quad s.t.\ (x, u) \in X \times K \text{ and } u \in T(x),$$

where $T(x)$ is the solution set of the parametric equilibrium problem EP(x) defined by, $u \in T(x)$ iff

$$h(x, u, v) \geq 0, \quad \forall v \in K.$$

We first give some concepts.

Definition 4.1. A sequence $\{(x_n, u_n)\} \subset X \times K$ is called an approximating sequence for OPEC iff:

(i) $\limsup_{n\to\infty} f(x_n, u_n) \leq \inf_{y\in X, v\in T(y)} f(y, v)$.

(ii) there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$h(x_n, u_n, v) + \epsilon_n \geq 0, \quad \forall v \in K, \ \forall n \in N.$$

Definition 4.2. We say that OPEC is well-posed iff OPEC has a unique solution, and every approximating sequence for OPEC converges strongly to the unique solution, and that OPEC is well-posed in the generalized sense iff OPEC has a nonempty solution set S, and every approximating sequence for OPEC has some subsequence which converges strongly to some point of S.

Remark 4.1. When the constraint EP(x) reduces to the parametric variational inequality problem, Definitions 4.1 and 4.2 collapse to the corresponding definitions of optimization problems with variational inequality constraints [8].

To obtain metric characterizations of well-posedness of OPEC, consider the following approximating solution set of OPEC:

$$M(\delta, \epsilon) = \{(x, u) \in X \times K : f(x, u) \leq \inf_{y \in X, v \in T(y)} f(y, v) + \delta, \text{ and } h(x, u, v) + \epsilon \geq 0, \forall v \in K\}.$$

Theorem 4.1. *Let K be a nonempty, closed and convex subset of E, $f : X \times K \to R$ and $h : X \times K \times K \to R$. Assume that*

(i) $h(x, u, u) \geq 0$ *for all* $x \in X, u \in K$.
(ii) *for every* $x \in X$, $h(x, \cdot, \cdot)$ *is monotone and hemicontinuous.*
(iii) *for every* $(x, u) \in X \times K$, $h(x, u, \cdot)$ *is convex.*
(iv) *for every* $u \in K$, $h(\cdot, u, \cdot)$ *is lower semicontinuous.*
(v) f *is lower semicontinuous.*

Then OPEC is well-posed if and only if

$$M(\delta, \epsilon) \neq \emptyset, \quad \forall \delta, \epsilon > 0, \quad \textit{and} \quad \operatorname{diam} M(\delta, \epsilon) \to 0 \quad \textit{as } (\delta, \epsilon) \to (0, 0). \tag{9}$$

Proof. Let OPEC be well-posed and (x^*, u^*) be the unique solution of OPEC. Clearly $(x^*, u^*) \in M(\delta, \epsilon) \neq \emptyset$ for all $\delta, \epsilon > 0$. If $\operatorname{diam} M(\delta, \epsilon) \not\to 0$ as $(\delta, \epsilon) \to (0, 0)$, then there exist $l > 0$ and $\delta_n > 0, \epsilon_n > 0$ with $\delta_n \to 0, \epsilon_n \to 0$, and $(x_n, u_n), (y_n, v_n) \in M(\delta_n, \epsilon_n)$ such that

$$\|(x_n, u_n) - (y_n, v_n)\| > l, \quad \forall n \in N. \tag{10}$$

Taking into account $(x_n, u_n), (y_n, v_n) \in M(\delta_n, \epsilon_n)$, both $\{(x_n, u_n)\}$ and $\{(y_n, v_n)\}$ are approximating sequences for OPEC. Since OPEC is well-posed, they have to converge strongly to the unique solution (x^*, u^*) of OPEC, a contradiction to (10).

For the converse, let $\{(x_n, u_n)\} \subset X \times K$ be an approximating sequence for OPEC. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\begin{cases} \limsup_{n \to \infty} f(x_n, u_n) \leq \inf_{y \in X, v \in T(y)} f(y, v), \\ h(x_n, u_n, v) + \epsilon_n \geq 0, \quad \forall v \in K, \ \forall n \in N. \end{cases}$$

This means that $(x_n, u_n) \in M(\delta_n, \epsilon_n)$ for some $\delta_n \to 0$. By (9), $\{(x_n, u_n)\}$ is a Cauchy sequence in $X \times K$ and so it converges strongly to a point $(\bar{x}, \bar{u}) \in X \times K$. By the same arguments as in Theorem 3.1, we have

$$h(\bar{x}, \bar{u}, v) \geq 0, \quad \forall v \in K. \tag{11}$$

Since f is lower semicontinuous,

$$f(\bar{x}, \bar{u}) \leq \liminf_{n \to \infty} f(x_n, u_n) \leq \limsup_{n \to \infty} f(x_n, u_n) \leq \inf_{y \in X, v \in T(y)} f(y, v).$$

This together with (11) implies that $(\bar{x}, \bar{u})$ solves OPEC.

To complete the proof, it is sufficient to prove that OPEC has a unique solution. Let (y^*, v^*) be another solution of OPEC. It is easily see that $\{(\bar{x}, \bar{u}), (y^*, v^*)\} \subset M(\delta, \epsilon)$ for all $\delta, \epsilon > 0$. It follows from (9) that

$$0 < \|(\bar{x}, \bar{u}) - (y^*, v^*)\| \leq \operatorname{diam} M(\delta, \epsilon) \to 0.$$

This yields $(\bar{x}, \bar{u}) = (y^*, v^*)$ and so OPEC has a unique solution. $\square$

For the well-posedness in the generalized sense, we have the following characterization by considering the noncompactness of the approximate solution set of OPEC.

Theorem 4.2. *Let X be finite dimensional and K be closed. Let $f : X \times K \to R$ be a lower semicontinuous functional and $h : X \times K \times K \to R$ be such that $h(\cdot, \cdot, u)$ is upper semicontinuous for every $u \in K$. Then OPEC is well-posed in the generalized sense if and only if*

$$M(\delta, \epsilon) \neq \emptyset, \quad \forall \delta, \epsilon > 0, \quad \text{and} \quad \mu(M(\delta, \epsilon)) \to 0 \quad \text{as } (\delta, \epsilon) \to (0, 0).$$

Proof. The conclusion follows from similar arguments as in Theorem 3.2. □

The following results show that the well-posedness of $\mathcal{EP}$ is closely related to the well-posedness of OPEC.

Theorem 4.3. *Assume that X is compact, K is closed, f is lower semicontinuous, and OPEC has at least one solution. If $\mathcal{EP}$ is well-posed in the generalized sense, then OPEC is well-posed in the generalized sense.*

Proof. Let $\{(x_n, u_n)\} \subset X \times K$ be an approximating sequence for OPEC. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\begin{cases} \limsup\limits_{n\to\infty} f(x_n, u_n) \leq \inf\limits_{y\in X, v\in T(y)} f(y, v), \\ h(x_n, u_n, v) + \epsilon_n \geq 0, \quad \forall v \in K, \ \forall n \in N. \end{cases} \tag{12}$$

Since X is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $k \to \infty$. By (12), $\{u_{n_k}\}$ is an approximating sequence for EP($\bar{x}$) corresponding to $\{x_{n_k}\}$. Since $\mathcal{EP}$ is well-posed in the generalized sense, $\{u_{n_k}\}$ has a subsequence (with loss of generality, denoted still by $\{u_{n_k}\}$), such that $u_{n_k} \to \bar{u}$ as $k \to \infty$, where $\bar{u} \in T(\bar{x})$. Since f is lower semicontinuous, it follows from (12) that

$$f(\bar{x}, \bar{u}) \leq \liminf_{k\to\infty} f(x_{n_k}, u_{n_k}) \leq \limsup_{k\to\infty} f(x_{n_k}, u_{n_k}) \leq \inf_{y\in X, v\in T(y)} f(y, v).$$

This together with $\bar{u} \in T(\bar{x})$ implies that $(\bar{x}, \bar{u})$ solves OPEC. Thus $\{(x_n, u_n)\}$ has a subsequence which converges strongly to some solution of OPEC and so OPEC is well-posed in the generalized sense. □

Theorem 4.4. *Assume that X is compact, K is closed, f is lower semicontinuous, and OPEC has a unique solution. If $\mathcal{EP}$ is well-posed in the generalized sense, then OPEC is well-posed.*

Proof. Let $(\bar{x}, \bar{u})$ be the unique solution of OPEC and let $\{(x_n, u_n)\} \subset X \times K$ be an approximating sequence for OPEC. Taking into account the uniqueness of the solution to OPEC, by the same arguments as in Theorem 4.3, we can prove that $\{(x_n, u_n)\}$ has a subsequence which converges strongly to $(\bar{x}, \bar{u})$. Since any convergent subsequence of $\{(x_n, u_n)\}$ is convergent to $(\bar{x}, \bar{u})$, the whole sequence $\{(x_n, u_n)\}$ is convergent to $(\bar{x}, \bar{u})$. So OPEC is well-posed. □

Remark 4.2. Theorem 4.3 generalizes Theorem 3.4 of [8] and Theorem 4.4 generalizes Theorem 3.5 of [8].

The following theorem shows that under suitable conditions, the well-posedness of OPEC is equivalent to the existence and uniqueness of solutions.

Theorem 4.5. *Let X be closed, convex and finite dimensional, E be finite dimensional, K be a nonempty, closed and convex subset of E, $f : X \times K \to R$ and $h : X \times K \times K \to R$. Assume that*

(i) *$h(x, u, u) \geq 0$ for all $x \in X, u \in K$.*
(ii) *for every $x \in X$, $h(x, \cdot, \cdot)$ is monotone and hemicontinuous.*
(iii) *for every $u \in K$, $h(\cdot, u, \cdot)$ is convex and lower semicontinuous.*
(iv) *f is convex and lower semicontinuous.*

Then OPEC is well-posed if and only if it has a unique solution.

Proof. The necessity is obvious. For the sufficiency, suppose that OPEC has a unique solution (x^*, u^*). It follows that

$$\begin{cases} f(x^*, u^*) = \inf_{y\in X, v\in T(y)} f(y, v), \\ h(x^*, u^*, v) \geq 0, \quad \forall v \in K,\ \forall n \in N. \end{cases} \tag{13}$$

Let $\{(x_n, u_n)\} \subset X \times K$ be an approximating sequence for OPEC. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\begin{cases} \limsup_{n\to\infty} f(x_n, u_n) \leq \inf_{y\in X, v\in T(y)} f(y, v), \\ h(x_n, u_n, v) + \epsilon_n \geq 0, \quad \forall v \in K,\ \forall n \in N. \end{cases} \tag{14}$$

If $\{(x_n, u_n)\}$ is unbounded, without loss of generality, we can suppose that $\|(x_n, u_n)\| \to +\infty$. Let $t_n = \frac{1}{\|(x_n,u_n)-(x^*,u^*)\|}$ and

$$(z_n, w_n) = (x^*, u^*) + t_n[(x_n, u_n) - (x^*, u^*)] = (x^* + t_n(x_n - x^*), u^* + t_n(u_n - u^*)).$$

Without loss of generality, we can suppose that $t_n \in (0, 1)$ and $(z_n, w_n) \to (z, w)(\neq (x^*, u^*))$ since both X and E are finite dimensional. Taking into account the closedness and convexity of X and K, one has $(z_n, w_n) \in X \times K$ and $(z, w) \in X \times K$. It follows from (iv) and (14) that

$$\begin{aligned} f(z, w) &\leq \liminf_{n\to\infty} f(z_n, w_n) \leq \limsup_{n\to\infty} f(z_n, w_n) \\ &\leq \limsup_{n\to\infty}\{t_n f(x_n, u_n) + (1 - t_n) f(x^*, u^*)\} \\ &\leq \limsup_{n\to\infty} t_n f(x_n, u_n) + \limsup_{n\to\infty}(1 - t_n) f(x^*, u^*) \\ &= f(x^*, u^*). \end{aligned} \tag{15}$$

Further, it follows from conditions (ii)–(iii) and (13) and (14) that

$$\begin{aligned} h(z, v, w) &\leq \liminf_{n\to\infty} h(z_n, v, w_n) \\ &\leq \liminf_{n\to\infty}\{t_n h(x_n, v, u_n) + (1 - t_n)h(x^*, v, u^*)\} \\ &\leq \liminf_{n\to\infty}\{-t_n h(x_n, u_n, v) - (1 - t_n)h(x^*, u^*, v)\} \\ &\leq \liminf_{n\to\infty} t_n\epsilon_n = 0, \quad \forall v \in K. \end{aligned}$$

This together with Lemma 2.1 implies

$$h(z, w, v) \geq 0, \quad \forall v \in K. \tag{16}$$

By (15) and (16), (z, w) solves (OPEC), a contradiction.

So $\{(x_n, u_n)\}$ is bounded. Let $\{(x_{n_k}, u_{n_k})\}$ be any subsequence of $\{(x_n, u_n)\}$ such that $(x_{n_k}, u_{n_k}) \to (\bar{x}, \bar{u})$ as $k \to \infty$. It follows that

$$h(\bar{x}, v, \bar{u}) \leq \liminf_{k\to\infty} h(x_{n_k}, v, u_{n_k}) \leq \liminf_{k\to\infty}\{-h(x_{n_k}, u_{n_k}, v)\} \leq \liminf_{k\to\infty} \epsilon_{n_k} = 0, \quad \forall v \in K.$$

This together with Lemma 2.1 yields

$$h(\bar{x}, \bar{u}, v) \geq 0, \quad \forall v \in K. \tag{17}$$

Since f is lower semicontinuous, it follows from (14) that

$$f(\bar{x}, \bar{u}) \leq \liminf_{k\to\infty} f(x_{n_k}, u_{n_k}) \leq \limsup_{k\to\infty} f(x_{n_k}, u_{n_k}) \leq \inf_{y\in X, v\in T(y)} f(y, v),$$

which together with (17) means that $(\bar{x}, \bar{u})$ solves OPEC. Taking into account the uniqueness of the solution, we have $(\bar{x}, \bar{u}) = (x^*, u^*)$. Thus (x_n, u_n) converges to (x^*, u^*). Therefore, OPEC is well-posed. □

Theorem 4.6. *Let X be closed, convex and finite dimensional, E be finite dimensional, K be a nonempty, closed and convex subset of E, $f : X \times K \to R$ and $h : X \times K \times K \to R$. Assume that*

(i) *$h(x, u, u) \geq 0$ for all $x \in X, u \in K$.*
(ii) *for every $x \in X$, $h(x, \cdot, \cdot)$ is monotone and hemicontinuous.*
(iii) *for every $u \in K$, $h(\cdot, u, \cdot)$ is convex and lower semicontinuous.*
(iv) *f is convex and lower semicontinuous.*

If there exists some $\epsilon > 0$ such that $M(\epsilon, \epsilon)$ is nonempty and bounded, then OPEC is well-posed in the generalized sense.

Proof. Let $\{(x_n, u_n)\} \subset X \times K$ be an approximating sequence for OPEC. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\begin{cases} \limsup_{n\to\infty} f(x_n, u_n) \leq \inf_{y\in X, v\in T(y)} f(y, v), \\ h(x_n, u_n, v) + \epsilon_n \geq 0, \quad \forall v \in K, \ \forall n \in N. \end{cases}$$

Let $\epsilon > 0$ be such that $M(\epsilon, \epsilon)$ is nonempty and bounded. Then there exists n_0 such that $(x_n, u_n) \in M(\epsilon, \epsilon)$ for all $n \geq n_0$. Taking into account the boundedness of $M(\epsilon, \epsilon)$, there exists some subsequence $\{(x_{n_k}, u_{n_k})\}$ of $\{(x_n, u_n)\}$ such that $(x_{n_k}, u_{n_k}) \to (\bar{x}, \bar{u})$ as $k \to \infty$. As proved in Theorem 4.5, $(\bar{x}, \bar{u})$ solves OPEC. Therefore, OPEC is well-posed in the generalized sense. □

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