Effect of rotation on generalized thermo-viscoelastic Rayleigh–Lamb waves

J.N. Sharma a, Mohamed I.A. Othman b,*

a Department of Applied Sciences, National Institute of Technology, Hamirpur 177 005, India
b Faculty of Education, Department of Mathematics, Salalah-211, P.O. Box 2801, Sultanate of Oman, Oman

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Abstract

The present paper is aimed at studying the effects of rotation on the thermoelastic interaction in an infinite Kelvin–Voigt-type viscoelastic, thermally conducting plate rotating about the normal to its faces with uniform angular velocity. This facilitates the decoupling of anti-plane/in-plane motion which is not possible, in general. The upper and lower surfaces of the plate are subjected to stress-free, thermally insulated or isothermal conditions. The formulation is applied according to three theories of the generalized thermoelasticity: Lord-Shulman with one relaxation time, Green–Lindsay with two relaxation times, as well as the coupled theory. Secular equations are derived for the plate in closed form isolated mathematical conditions for symmetric and skew-symmetric wave mode propagation in completely separate terms. In the absence of mechanical relaxations (Rotation and viscous effects), the results for generalized and couple theories of thermoelasticity were obtained as particular cases from the derived secular equations. In the absence of thermomechanical coupling, the analysis for a viscoelastic plate can be deduced from the present one. Finally the numerical solution is carried out for copper material. The function iteration numerical scheme is used to solve the complex secular equations, in order to obtain phase velocity and attenuation coefficients of propagating wave mode. The dispersion curves and attenuation coefficients profiles so obtained for symmetric and skew-symmetric wave modes are presented graphically to illustrate and compare the theoretical results in the presence and absence of rotation. The study may be useful in the construction and design of gyroscopes and rotation sensors as well as in the application in diverse fields.
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1. Introduction

Several mathematical models have been used by Ewing et al. (1957), Hunter et al. (1960) and Flugge (1967) to accommodate the energy dissipation in vibrating solids where it is observed that internal friction produces attenuation and dispersion; hence, the effect of the viscoelastic nature of material medium in the process of
wave propagation cannot be neglected. The viscoelastic nature of a medium has special significance in wave propagation in a solid medium. Acharya and Mondal (2002) investigated the propagation of Rayleigh surface waves in a Voigt-type (1887) viscoelastic solid under the linear theory of non local elasticity. As pointed out by Freudenthal (1954), most of the solids, when subjected to dynamic loading, exhibit a viscous effect. The Kelvin–Voigt model is one of the macroscopic mechanical models often used to describe the viscoelastic behavior of material. The model represents the delayed elastic response subjected to stress when the deformation is time dependent but recoverable. The dynamic interaction of thermal and mechanical fields in solids has great practical applications in modern aeronautics, astronautics, nuclear reactors, and high-energy particle accelerators, for example. The generalized theory of thermoelasticity has drawn widespread attention because it removes the physically unacceptable situation of the classical theory of thermoelasticity, which is that the thermal disturbance is propagated with infinite velocity. Lord and Shulman (LS) (1967) and Green and Lindsay (GL) (1972) are responsible for two important generalized theories of thermoelasticity that have become the center of interest of recent research in this area. In both of these theories, the introduction of thermal relaxation parameters modifies the basic equations of thermoelasticity. These theories are more realistic because they consider the second sound effects, that is, the actual occurrence of wave link heat propagation with finite speed, and they have practical importance, especially in problems involving high heat fluxes and/or small time intervals. Chandrashekhariah (1986) published a comprehensive review of literature on the subject. Sharma et al. (2000) and Sharma (2001) studied the thermoelastic waves in homogeneous isotropic plates subjected to stress-free and rigidly fixed insulated or isothermal boundary conditions in the context of conventional coupled thermoelasticity (CT), (LS) and (GL) theories of thermoelasticity. Song et al. (2006) studied the problems of a plane harmonic wave at the interface between two viscoelastic media under generalized thermo-viscoelasticity when the media permeate a uniform magnetic field. Othman et al. (2002) studied the generalized thermo-viscoelastic plane wave with two relaxation times, Othman (2004a) established the electromagneto-thermo-viscoelasticity coupled equation in a finite conducting medium with one relaxation time; Othman (2004b) studied the uniqueness and reciprocity theorems for generalized thermo-viscoelasticity with thermal relaxation times. Sharma et al. (2004) and Sharma (2005) employed Kelvin–Voigt model of viscoelasticity to study Rayleigh–Lamb waves in thermoelastic plates in the context of generalized (GL and LS) and coupled theories of thermoelasticity. The significance of the viscoelastic nature of a material medium and nonexistence of systematic investigation of waves in thermo-viscoelastic plates and gyroscopic structures has motivated the authors to study the propagation of Rayleigh–Lamb waves in Kelvin–Voigt-type thermo-viscoelastic rotating solid plates in the context of recently developed nonclassical theories of thermoelasticity.

Othman (2005a, 2004c) has established the model of two-dimensional equations of generalized thermoelasticity with one and two relaxation times under the effect of rotation. Othman (2005b) applied the normal mode analysis to a two-dimensional generalized thermo-viscoelastic plane wave with one relaxation time under the effect of rotation. Roy-Choudhuri and Mukhopdhay (2000) studied the effect of rotation and relaxation time on plane waves in an infinite generalized thermo-viscoelastic solid of Kelvin–Voigt-type with the entire medium rotating with a uniform angular velocity. The effect of rotation on elastic waves, both partial and surface, has been studied by many authors such as Schoenberg and Censor (1973), Clark and Burdness (1994), Destrade (2004) and Ting (2004). Fang et al. (2000, 2002) investigated the effect of rotation on surface acoustic waves in a piezoelectric halfspace. Zhou and Jiang (2001) studied the effects of Coriolis and centrifugal forces on the acoustic waves in a piezoelectric halfspace. Recently, Sharma and Thakur (2006) studied the effect of rotation on Rayleigh–Lamb waves in magneto-thermoelastic plate rotating about the normal to its faces with uniform angular velocity under the action of a uniform magnetic field. Sharma and Walia (2006) investigated the effect of rotation on Rayleigh waves in piezothermoelastic half-space rotating about an axis normal to its surface with uniform angular velocity.

The aim of the present paper is to study the effect of rotation on the thermo-viscoelastic interactions in an infinite Kelvin–Voigt-type viscoelastic thermally conducting plate rotating about the normal to its faces with uniform angular velocity. The upper and lower surfaces of the plate were subjected to stress-free, thermally insulated or isothermal conditions. The Voigt model of linear viscothermoelasticity, earlier used by Kaliski (1963), was employed to consider the viscoelastic behavior of the solid although such a study can also possibly be carried out in the context of the Maxwell model of visco-elasticity along similar lines. Secular equations for symmetric and skew-symmetric modes of wave propagation in completely separate
terms were derived. The amplitudes of temperature and displacements were also obtained. Finally, the numerical solution was carried out for copper material and the dispersion curves and amplitudes of temperature change and displacements for symmetric and skew-symmetric wave modes are presented to illustrate and compare the theoretical results in the presence and absence of rotation.

2. Formulation of the problem

We consider an infinite homogeneous isotropic, thermally conducting viscoelastic plate of thickness \( 2L \) initially at uniform temperature \( T_0 \). We take the origin of the coordinate system \((x, y, z)\) on the middle surface of the plate. The \( x-y \) plane is chosen to coincide with the middle surface and the \( z \)-axis normal to it along the thickness. The surfaces \( z = \pm L \) are assumed to be stress-free insulated or isothermal. The viscoelastic plate is rotating uniformly with an angular velocity \( \Omega = (\Omega \mathbf{n}) \), where \( \mathbf{n} \) is a unit vector representing the direction of the axis of rotation. The displacement equation of motion in the rotating frame has two additional terms [Roy-Choudhuri and Mukhopdhyay, 2000]:

Centripetal acceleration is \( \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}) \) due to time varying motion only and \( 2\mathbf{\Omega} \times \dot{\mathbf{u}} \), where \( \mathbf{u} = (u, v, w) \) is the dynamic displacement vector. These terms do not appear in non-rotating media (Fig. 1).

The basic governing equations of linear generalized visco-thermoelasticity with rotation in the absence of body forces and heat sources are

\[
\rho [\ddot{\mathbf{u}} + (\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u})) + (2\mathbf{\Omega} \times \dot{\mathbf{u}})] = \left( \lambda^* + \mu^* \right) \nabla (\nabla \cdot \mathbf{u}) + \mu^* \nabla^2 \mathbf{u} - \beta^* \nabla (T + \nu_o \dot{T})
\]

\[
K \nabla^2 T - \rho C_v \left( n_1 + \tau_o \frac{\partial}{\partial t} \right) \dot{T} = \beta^* T_0 \left( n_1 + n_o \tau_o \frac{\partial}{\partial t} \right) \dot{e}
\]

The constitutive relations are given by

\[
\sigma_{ij} = \lambda^* e_{kk} \delta_{ij} + 2\mu^* e_{ij} - \beta^* \left( 1 + \nu_o \frac{\partial}{\partial t} \right) T \delta_{ij}
\]

where \( T(x, y, z) \) is the temperature deviation from the reference temperature \( T_0 \); \( K, \rho \) and \( C_v \), which are, respectively, the thermal conductivity, density and specific heat at constant strain; \( \nu_o, \tau_o \) are the thermal relaxation times, \( e \) is the dilatation; and \( \delta_{ij} \) is Kronecker’s delta. The dot notation is used to denote time differentiation. The parameters \( \lambda^*, \mu^* \) and \( \beta^* \) are defined as

\[
\lambda^* = \lambda_c \left( 1 + \alpha_o \frac{\partial}{\partial t} \right), \quad \mu^* = \mu_c \left( 1 + \alpha_1 \frac{\partial}{\partial t} \right), \quad \beta^* = \beta_c \left( 1 + \beta_o \frac{\partial}{\partial t} \right)
\]

where \( \beta_c = (3\lambda_c + 2\mu_c) \alpha_T, \beta_o = (3\lambda_c \alpha_o + 2\mu_c \alpha_1) \alpha_T/\beta_c; \lambda_c, \mu_c \) are Lame’ parameters; \( \alpha_o, \alpha_1 \) are viscoelastic relaxation times; and \( \alpha_T \) is the coefficient of linear thermal expansion. The strain tensor is

\[
e_{ij} = \frac{1}{2} (u_{ij} + u_{ji}), \quad i, j = 1, 2, 3
\]

Eqs. (1) and (2) are the field equations of the generalized visco-thermoelasticity for a rotating media, applicable to the coupled theory and nonclassical theories of thermoelasticity, as follows: (i) The equation of the coupled viscothermoelasticity (CT theory) for a rotating media can be obtained when

\[
n_1 = 1, \quad \nu_o = n_o = 0
\]

Eqs. (1) and (2) have the following forms

![Fig. 1. Geometry of the problem.](image-url)
\[ \rho [\ddot{\mathbf{u}} + (\mathbf{Q} \times (\mathbf{Q} \times \mathbf{u}))] + (2\mathbf{Q} \times \dot{\mathbf{u}}) = (\lambda^{\prime} + \mu^{\prime}) \nabla (\nabla \cdot \mathbf{u}) + \mu^{\prime} \nabla^2 \mathbf{u} - \beta^{\prime} \nabla T \]  
(7)

\[ K \nabla^2 T - \rho C_e \dot{\mathbf{e}} = \beta T_0 \dot{\mathbf{e}} \]  
(8)

(ii) Lord–Shulman (L–S theory) can also be obtained when

\[ m_1 = n_0 = 1, \quad v_o = 0, \quad \tau_o > 0 \]  
(9)

where \( \tau_o \) is the relaxation time Eq. (1) is the same as Eq. (7), and Eq. (2) has the following form

\[ K \nabla^2 T - \rho C_e \left( 1 + \tau_o \frac{\partial}{\partial t} \right) \dot{\mathbf{e}} = \beta T_0 \left( 1 + \tau_o \frac{\partial}{\partial t} \right) \dot{\mathbf{e}} \]  
(10)

(iii) Green–Lindsay (G–L theory) is obtained when

\[ m_1 = 1, \quad n_0 = 0, \quad v_o \geq \tau_o > 0 \]  
(11)

where \( v_o, \tau_o \) are the two relaxation times. Eq. (1) remains without change and Eq. (2) has the form

\[ K \nabla^2 T - \rho C_e \left( 1 + \tau_o \frac{\partial}{\partial t} \right) \dot{\mathbf{e}} = \beta T_0 \dot{\mathbf{e}} \]  
(12)

(iv) The corresponding equations for the generalized thermoelasticity without rotation results from the above mentioned cases can be obtained by taking \( \Omega = 0 \).

For simplifications we shall use the following non-dimensional variables

\[ x_i^\prime = \frac{m_i}{c_1} x_i, \quad u_i^\prime = \frac{\rho c_1 m_i}{\beta_o T_0} u_i, \quad \ell = \sigma t, \quad v_o^\prime = \sigma v_o, \quad \tau_o^\prime = \sigma \tau_o, \quad T^\prime = \frac{T}{T_0}, \quad \sigma^\prime = \frac{\sigma}{c_1}, \quad \Omega^\prime = \frac{\Omega}{c_1}, \quad \lambda_i^\prime = \sigma \lambda_i, \quad \lambda_o^\prime = \sigma \lambda_o, \quad h^\prime = \frac{h c_1}{\sigma}, \quad \zeta^\prime = \frac{c_1}{\sigma}, \]  
(13)

\[ c^\prime = \frac{c}{c_1}, \quad L^\prime = \frac{\sigma}{c_1} L, \quad \delta^2 = \frac{c_2}{c_1^2}, \quad \sigma = \frac{C_e (\lambda_e + 2 \mu_e)}{\rho}, \quad c_1^2 = \frac{\lambda_e + 2 \mu_e}{\rho}, \quad c_2^2 = \frac{\mu_e}{\rho} \]

where \( \sigma \) is the characteristic frequency of the material and \( c_1, c_2 \) are the longitudinal and shear wave velocities in the medium, respectively.

We take the \( x-z \) plane as the plane of incidence and we assume that the solutions are explicitly independent of \( y \). However, implicit dependence is involved, so that the transverse component of displacement, \( v \), is non-vanishing in Eqs. (1) and (2). We also assume that the plate is rotating about an axis normal to the plate so that \( \Omega = (0, \Omega, 0) \). In view of this and quantities (3), Eqs. (1) and (2) can be rewritten in the non-dimensional form as follows

\[ \frac{\partial^2 u}{\partial t^2} - \Omega^2 u + 2 \Omega \frac{\partial w}{\partial t} = \left( 1 + \delta_o \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial x^2} + \delta^2 \left( 1 + \lambda_1 \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial x^2} - \left( 1 + \beta_o \frac{\partial}{\partial t} \right) \frac{\partial T}{\partial x} \]  
(14)

\[ \frac{\partial^2 w}{\partial t^2} - \Omega^2 w - 2 \Omega \frac{\partial u}{\partial t} = \left( 1 + \delta_o \frac{\partial}{\partial t} \right) \frac{\partial^2 w}{\partial x^2} + \delta^2 \left( 1 + \lambda_1 \frac{\partial}{\partial t} \right) \frac{\partial^2 w}{\partial x^2} - \left( 1 + \beta_o \frac{\partial}{\partial t} \right) \frac{\partial T}{\partial z} \]  
(15)

\[ \frac{\partial^2 v}{\partial t^2} = \delta^2 \left( 1 + \lambda_1 \frac{\partial}{\partial t} \right) \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} \]  
(16)

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} - \left( n_1 + \tau_o \frac{\partial}{\partial t} \right) \frac{\partial T}{\partial t} = \varepsilon \left( 1 + \beta_o \frac{\partial}{\partial t} \right) \left( n_1 + n_o \tau_o \frac{\partial}{\partial t} \right) \frac{\partial e}{\partial t} \]  
(17)

where \( \delta_o = \lambda_o + 2 \beta (x_1 - x_o) \) and \( \varepsilon = \frac{\tau_o \beta^2}{\rho C_e (\lambda_e + 2 \mu_e)} \) is the thermoelastic coupling constant (primes have been suppressed for convenience). The non-dimensional constitutive relations are given by
\[
\sigma_{ij} = (1 - 2\delta^2) \left( 1 + \alpha_o \frac{\partial}{\partial t} \right) \delta_{ik} \delta_{ij} + 2\delta^2 \left( 1 + \alpha_1 \frac{\partial}{\partial t} \right) e_{ij} - \left( 1 + \beta_o \frac{\partial}{\partial t} \right) \left( 1 + \nu_o \frac{\partial}{\partial t} \right) T \delta_{ij}, \quad i, j = 1, 2, 3
\]  

(18)

Eq. (16) corresponds to decoupled purely transverse (shear) wave motion and has already been discussed by many authors and hence will not be considered here unless stated otherwise. The rotation of plate about an axis normal to its plane has the advantage of decoupling of purely transverse (shear horizontal) waves motion which is not in general possible when the rotation takes place about an arbitrary direction.

3. Boundary conditions

The boundaries of the plate are assumed to be stress-free thermally insulated/isothermal. Therefore, we consider following types of boundary conditions:

(i) Mechanical conditions The non-dimensional mechanical boundary conditions at \( z = \pm L \) for stress-free and rigidly fixed boundaries respectively, are given by

\[
\sigma_{zz} = 0, \quad \sigma_{xz} = 0,
\]

(19)

(ii) Thermal conditions The non-dimensional thermal boundary conditions at \( z = \pm L \) are given by

\[
\frac{\partial T}{\partial z} + h T = 0
\]

(20)

where \( h \) is the surface heat transfer coefficient; \( h \to 0 \) corresponds to thermally insulated boundaries and \( h \to \infty \) refers to isothermal boundaries.

4. Solution of the problem

To solve the problem, we introduce the potential \( \Phi(x, z, t) \) and \( \Psi(x, z, t) \) through the relations

\[
u = \Phi_z + \Psi_z, \quad w = \Phi_x - \Psi_x.
\]

(21)

Substituting Eqs. (21) and (22) into Eqs. (14), (15) and (17) we obtain

\[
\begin{align*}
[\Phi - \Omega^2 \Phi - 2\Omega \Psi] &= \left( 1 + \delta_o \frac{\partial}{\partial t} \right) \nabla^2 \Phi - \left( 1 + \beta_o \frac{\partial}{\partial t} \right) \left( 1 + \nu_o \frac{\partial}{\partial t} \right) T, \\
[\Psi - \Omega^2 \Psi - 2\Omega \Phi] &= \delta^2 \left( 1 + \alpha_1 \frac{\partial}{\partial t} \right) \nabla^2 \Psi, \\
\nabla^2 T - \left( n_1 + \tau_o \frac{\partial}{\partial t} \right) \hat{T} &= \varepsilon \left( 1 + \beta_o \frac{\partial}{\partial t} \right) \left( n_1 + n_o \tau_o \frac{\partial}{\partial t} \right) \nabla^2 \Phi
\end{align*}
\]

(22)

(23)

(24)

where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \) is the Laplace operator.

5. Normal mode analysis

The solution of the considered physical variables can be decomposed in terms of normal modes as in the following form

\[
[\Phi, \Psi, T] = [f(z), g(z), \Theta(z)] \exp[ik(x - c t)]
\]

(25)

where \( c = \omega/k \) is the non-dimensional phase velocity and \( \omega, \alpha \) are, respectively, the non-dimensional circular frequency and wave number.

Upon using solutions (25) in Eqs. (22) and (23), the resulting system of equations leads to the expression for \( \Phi, \Psi, \) and \( T \) as
\[ \Phi = \sum_{i=1}^{3} (A_i \sin m_i z + B_i \cos m_i z) e^{i(k(x-cr))} \] (26)

\[ \Psi = \sum_{i=1}^{3} V_i (A_i \sin m_i z + B_i \cos m_i z) e^{i(k(x-cr))} \] (27)

\[ T = \sum_{i=1}^{3} S_i (A_i \sin m_i z + B_i \cos m_i z) e^{i(k(x-cr))} \] (28)

where

\[ m_i^2 = k^2(\omega^{-1} a_i^2 c^2 - 1), \quad i = 1, 2, 3 \] (29)

\[ \alpha^2 = k^2 \left( \frac{\omega^{-1}(1 + I^2)}{\delta_o^2} c^2 - 1 \right) \] (30)

\[ \beta^2 = k^2 \left( \frac{\omega^{-1}(1 + I^2)}{\delta_i^2} c^2 - 1 \right) \] (31)

\[ V_i = \frac{2\omega I}{\alpha_i^2 (m_i^2 - \beta^2)} \] (32)

\[ S_i = \left( m_i^2 - \alpha^2 - \frac{4\omega^2 I^2}{\alpha_i^2 \delta_i^2 (m_i^2 - \beta^2)} \right) / \alpha \beta_o v_o, \quad i = 1, 2, 3. \] (33)

\[ \sum a_i^2 = 1 + I^2 \frac{1 + I^2}{\delta_o^2} + \frac{\tau_o - \nu_o \omega^2 \epsilon \beta_o \tau_o}{\tau_1} \] (34)

\[ \sum a_i^2 a_j^2 = \frac{(1 + I^2)^2}{\alpha_i^2 \alpha_j^2 \delta_o^2} + \frac{\tau_o (1 + I^2)}{\alpha_i^2 \alpha_j^2 \delta_o^2} (\alpha_i^2 \delta_i^2 + 1) + \frac{4I^2}{\alpha_i^2 \delta_i^2} - \frac{\tau_o - \nu_o \omega^2 \epsilon \beta_o \tau_o (1 + I^2)}{\tau_1 \alpha_i^2 \delta_i^2} \] (35)

\[ \sum a_i^2 a_j^2 a_k^2 = \frac{\tau_o}{\tau_1 \delta_o \alpha_i^2} [(1 + I^2) - 4\delta_o I^2] \] (36)

\[ \tau_o = t_o + i\omega^{-1} n_o, \quad \tau_o' = n_o \tau_o + i\omega^{-1} n_1, \quad \tau_1 = t_1 + i\omega^{-1} n_1 \] (37)

\[ \alpha_i^2 = z_1 + i\omega^{-1}, \quad \delta_o^2 = \delta_o + i\omega^{-1}, \quad \nu_o = \nu_o + i\omega^{-1}, \quad \beta_o^2 = \beta_o + i\omega^{-1}, \quad \Gamma = \frac{\Omega}{\omega} \] (38)

In the absence of rotation (\( \Omega = 0 \)), we have

\[ V_i = \begin{cases} 0, & i = 1, 3 \\ 1, & i = 2 \end{cases} \] (39)

\[ S_i = \begin{cases} (m_i^2 - \alpha^2) / i\omega \beta_o v_o^*, & i = 1, 3 \\ 0, & i = 2 \end{cases} \] (40)

The displacements and stresses are obtained as

\[ u = \sum_{i=1}^{3} [(ik \sin m_i z + m_i V_i \cos m_i z) A_i + (ik \cos m_i z - m_i V_i \sin m_i z) B_i] e^{i(k(x-cr))} \] (41)

\[ w = \sum_{i=1}^{3} [(m_i \cos m_i z - ik V_i \sin m_i z) A_i - (m_i \sin m_i z + ik V_i \cos m_i z) B_i] e^{i(k(x-cr))} \] (42)

\[ \sigma_{zz} = \sum_{i=1}^{3} [(p \sin m_i z + q m_i V_i \cos m_i z) A_i + (p \cos m_i z - q m_i V_i \sin m_i z) B_i] F e^{i(k(x-cr))} \] (43)

\[ \sigma_{xz} = \sum_{i=1}^{3} [(p V_i \sin m_i z + q m_i \cos m_i z) A_i + (p V_i \cos m_i z - q m_i \sin m_i z) B_i] F e^{i(k(x-cr))} \] (44)
where \( p = \beta^2 - k^2 \), \( q = 2i k \), \( F = i\omega \xi_0 \sigma^2 \).

Invoking the boundary conditions (19) and (20) at the surfaces \( z = \pm L \) of the plate, we obtain a system of six simultaneously equations. This system will have a non-trivial solution if the determinant of the coefficients of amplitudes \( A_i, B_j \), \( i = 1, 2, 3 \) vanishes. After applying some simplifications the determinantal equation leads to the secular equation for stress free, thermally insulated and isothermal boundaries of the plate as

\[
\begin{vmatrix}
qm_1 V_1 & qm_2 V_2 & qm_3 V_3 & p & p & p \\
-pT_1 & -pT_2 & -pT_3 & qm_1 V_1 T_1 & qm_2 V_2 T_2 & qm_3 V_3 T_3 \\
qm_1 & qm_2 & qm_3 & pV_1 & pV_2 & pV_3 \\
-pV_1 T_1 & -pV_2 T_2 & -pV_3 T_3 & qm_1 T_1 & qm_2 T_2 & qm_3 T_3 \\
m_1 S_1 & m_2 S_2 & m_3 S_3 & hS_1 & hS_2 & hS_3 \\
-hS_1 T_1 & -hS_2 T_2 & -hS_3 T_3 & m_1 S_1 T_1 & m_2 S_2 T_2 & m_3 S_3 T_3
\end{vmatrix} = 0 \quad (45)
\]

Here \( h \to 0 \) corresponds to thermally insulate and \( h \to \infty \) refers to isothermal boundaries of the plate. After lengthy algebraic manipulations and reductions, the determinantal Equation (45) reduces to

\[
D_{11}(D_{22}D_{33} - D_{23}D_{32}) - D_{12}(D_{21}D_{33} - D_{31}D_{23}) + D_{13}(D_{21}D_{32} - D_{31}D_{22}) = 0 \quad (46)
\]

in case of thermally insulated (\( h \to 0 \)) surfaces of plate and

\[
D'_{11}(D'_{22}D'_{33} - D'_{23}D'_{32}) - D'_{12}(D'_{21}D'_{33} - D'_{31}D'_{23}) + D'_{13}(D'_{21}D'_{32} - D'_{31}D'_{22}) = 0 \quad (47)
\]

for isothermal (\( h \to \infty \)) boundaries of the plate.

Here \((ij = 1, 2, 3)\) and \((ij = 1, 2, 3)\) are defined as below.

\[
D_{11} = V_3 \left\{ \frac{V_3 S_1 - V_1 S_3}{S_3 - S_1} - \frac{V_2 S_1 - V_1 S_2}{S_2 - S_1} \right\}
\]

\[
D_{12} = (V_2 - V_1) - F_1 m_1 S_1 V_1 \left\{ \frac{S_2 - S_1 + V_3 (S_1 V_2 - V_1 S_2)}{S_2 - S_1} \right\}
\]

\[
D_{13} = (V_3 - V_2) - F_2 m_3 S_2 V_2 \left\{ \frac{S_2 - S_1 + V_3 (S_1 V_2 - V_1 S_2)}{S_2 - S_1} \right\}
\]

\[
D_{21} = V_3 \left\{ \frac{S_3 F_4}{m_1 (S_3 - S_1)} - \frac{S_2 F_3}{(S_2 - S_1)} \right\}
\]

\[
D_{22} = F_5 \left( \frac{m_1 V_1 V_2 q^2}{p^2} \right) - F_3 F_1 \left( \frac{m_1 S_1 S_2 V_1 V_3}{(S_2 - S_1) m_3 S_3} \right)
\]

\[
D_{23} = F_6 \left( \frac{m_2 V_2 V_3 q^2}{p^2} \right) - F_3 F_2 \left( \frac{m_2 S_2 V_2}{(S_2 - S_1) m_3 S_3} \right)
\]

\[
D_{31} = \frac{V_1}{m_1} \left\{ (1 + V_3 S_1) F_8 - \left( \frac{S_2 - S_1 + V_3 S_1}{S_2 - S_1} \right) F_7 \right\}
\]

\[
D_{32} = F_9 \left( \frac{m_1 V_2 q^2}{p^2} \right) - F_1 F_7 \left( \frac{V_1 V_2 S_2 + (V_3 - 1) S_1}{m_1 (S_2 - S_1)} \right) \frac{m_1 S_1}{m_3 S_3}
\]

\[
D_{33} = F_{10} \left( \frac{m_2 V_3 q^2}{p^2} \right) - F_2 F_7 \left( \frac{F_2 m_2 S_2 V_2}{m_3 S_3} \right) \frac{V_1 V_2 S_2 + (V_3 - 1) S_1}{m_1 (S_2 - S_1)}
\]

and

\[
D'_{11} = \left( \frac{T_1}{T_3} \right)^{\pm} + F'_1 \left( \frac{T_2}{T_3} \right)^{\pm} - F'_2, \quad D'_{21} = \left( \frac{T_1}{T_3} \right)^{\pm} + F'_3 \left( \frac{T_2}{T_3} \right)^{\pm} - F'_4
\]
6. Special cases

In the absence of rotation ($I = 0$), the secular Equations (46) and (47) respectively, become

$$
\left[ \frac{\tan m_3 d}{\tan \beta a} \right]^{\pm 1} - \frac{m_1 (x^2 - m_1^2)}{m_3 (x^2 - m_3^2)} \left[ \frac{\tan m_3 d}{\tan \beta a} \right]^{\pm 1} = \frac{4k^2 \beta m_1(m_1^2 - m_3^2)}{(k^2 - \beta^2)(x^2 - m_3^2)}
$$

(50)

for stress free, thermally insulated ($h \to 0$) boundaries and (51)

$$
\left[ \frac{\tan m_3 d}{\tan \beta a} \right]^{\pm 1} - \frac{m_3 (x^2 - m_3^2)}{m_1 (x^2 - m_1^2)} \left[ \frac{\tan m_3 d}{\tan \beta a} \right]^{\pm 1} = \frac{(k^2 - \beta^2)(m_1^2 - m_3^2)}{4k^2 \beta m_1(x^2 - m_3^2)}
$$

(51)

for stress free isothermal ($h \to \infty$) boundaries of the plate. The secular Equations (50) and (51) have already been discussed in detail by Sharma et al. (2004) and Sharma (2005), respectively, in the context of generalized (LS and GL) and coupled (CT) theories of thermoelasticity.

7. Numerical results and discussion

In order to illustrate the theoretical results obtained in the preceding sections, we now present some numerical results. The material chosen for this purpose is copper, the physical data for which are given as (Mukhopadhyay, 2000)

$$
\lambda_e = 8.2 \times 10^{10} \text{ N/m}^2, \quad \mu_e = 4.2 \times 10^{10} \text{ N/m}^2, \quad K = 1.13 \times 10^3 \text{ W/m/s/K}, \quad T_o = 300 \text{ K}
\rho = 8.950 \times 10^3 \text{ kg/m}^3, \quad \alpha_T = 1.0 \times 10^{-8}/\text{K}, \quad \omega = 1.11 \times 10^{11} \text{ s}^{-1}, \quad \tau_o = v_o = 6.131 \times 10^{-13} \text{ s},
\alpha_o = \alpha_1 = 6.8831 \times 10^{-13} \text{ s}, \quad C_e = 2.2 \times 10^{-6} \text{ J/kg}^{-1}\text{K}^{-1}, \quad \varepsilon = 0.05.
$$

Here the thermal relaxation time $\tau_o$ has been estimated based on Eq. (2.5) of ChandraShekar (1986) and the values of other relaxation times including mechanical ones have been selected proportionally.

In general, wave number and hence the phase velocity of the waves is a complex quantity, therefore the waves are attenuated in space. If we write

$$
c^{-1} = V^{-1} + i \omega^{-1} Q
$$

(52)

so that $k = R + iQ$, where $R = \omega/V$ and $Q$ are real numbers. Then the exponent in plane wave solution (26) becomes $-Qx_1 + iR(x_1 - Vt)$. This shows that $V$ is the propagation speed and $Q$ the attenuation coefficient of the waves. Since $c' = c/c_1$ is the non-dimensional complex phase velocity, so $V' = V/c_1$ and $Q' = c_1 Q$ are the non-dimensional phase speed and attenuation coefficient, respectively. Here dashes have been omitted for convenience. The complex quantities $a_{ij}, k = 1, 2, 3$ have been successfully computed by solving a cubic equation whose coefficients are given by Eqs. (34)–(36) with the help of Cardano’s method along with DeMoivre’s
theorem in order to obtain the characteristic roots $m_i^2, k = 1, 2, 3$ from Eq. (29). Then the complex secular Equations (46) and (47) have been solved by using the functional iteration numerical technique after obtaining characteristic roots given by Eqs. (29)–(31) via representation (52) to obtain the values of propagation speed $V$ and attenuation coefficient $Q$ for different modes of wave propagation. The non-dimensional phase velocity $(V/c_1)$ and attenuation coefficient of symmetric and skew symmetric modes of waves propagation were computed for various values of non-dimensional wave number ($R$) from the dispersion relations (46) and (47) for stress-free, thermally insulated and isothermal boundary conditions. The corresponding dispersion curves and attenuation coefficient profiles for Rayleigh–Lamb type modes are presented in Figs. 2–5 in the context of LS theory of thermoelasticity in the case of rotating and non-rotating plates. The solid curves correspond to rotating viscous material plate and the broken line curve to that of a non-rotating viscous plate. The results in the case of conventional coupled thermoelasticity ($\tau_o = 0 = v_o$) have also been computed and found to be quite
close to those of LS-theory of thermoeelasticity but are not presented here keeping in view the resolution and clarity of plots. This shows that the thermal relaxation time has negligible small effects on various considered functions although these provide a significant advantage in numerical computations.

It is observed from Figs. 2 and 4 that the phase velocity of lowest (acoustic) skew symmetric \((A_0)\) mode in the case of a non-rotating plate is observed to increase from zero value at vanishing wave number and increases to become closer to Rayleigh wave velocity at higher values of the wave number. In the case of the lowest (acoustic) symmetric \((S_0)\) mode the phase velocity decreases from a value greater than unity towards the Rayleigh wave velocity with the increasing wave number both for the stress-free thermally insulated non-rotating plate. However, both symmetric and skew symmetric acoustic modes become dispersionless due to rotation in this case. The phase velocity of higher modes (optical modes) of propagation, symmetric and skew symmetric, attain quite large values at the vanishing wave number. The magnitudes of velocity of higher symmetric and skew symmetric modes are observed to develop at a rate, which is approximately, \(n\)-times the magnitude of the
velocity of the first mode \((n = 1)\) in the case of rotation as well as non-rotation. The optical modes also exhibit significant effect and sensitivity due to rotation of the plate. These numerically computed results are found to be in considerable agreement with the corresponding analytical results, and their trends are similar to those reported by Graff (1991) in the case of homogeneous isotropic elastic plates and also to those of Sharma et al. (2004), and Sharma (2005) for non-rotating viscothermoelastic plates except in the case of modification due to rotation effects.

Figs. 3 and 5 represent profiles of attenuation coefficients of acoustic \((n = 0)\) and the first two optical \((n = 1,2)\) modes in the case of stress-free insulated and rigidly fixed insulated plates, respectively. It is noticed that the mechanical relaxation time viz. the time dependence of elastic parameters has a significant effect on the attenuation coefficient of different modes of wave propagation. The attenuation attains significantly large values in the case of a rotating viscothermoelastic plate as compared to a non-rotating one. These values increase with increase in non-dimensional wave number in the former case while they die out in the latter one. This reveals that the thicker rotating viscothermoelastic plates are subjected to more attenuation, whereas non-rotating plate structures, in comparison, have quite small attenuation. The effects of stress-free thermally insulated and isothermal boundaries of the plate are also quite pertinent and can easily be noticed from the dispersion and attenuation curves in Figs. 2–5.

8. Conclusions

In this work the effect of rotation on the thermoelastic interactions in an infinite Kelvin–Voigt-type viscoelastic, thermally conducting plate rotating about normal to its faces has been investigated in the context of nonclassical theories of thermoelasticity. The secular equations for plate vibrations in different situations and conditions are derived and discussed.

In the absence of rotation, the secular equations reduce to those of Sharma et al. (2004) and Sharma (2005) in the case of generalized (LS and GL) and coupled (CT) theories of thermoelasticity respectively. The real phase speeds and attenuation coefficients of different symmetric and skew symmetric modes have been computed numerically by using function iteration numerical technique to solve complex secular equations after obtaining the complex characteristics roots with the help of reducible Cardano’s method. The numerically simulated results have been presented graphically in the form of dispersion curves and attenuation coefficient profiles in the case of a rotating and non-rotating, plates with stress-free, thermally insulated and isothermal boundaries. These results are found to be in agreement with those reported by Graff (1991) and Sharma et al. (2004) and Sharma (2005) except modifications due to rotation effect. It is revealed that the thicker rotating viscothermoelastic plates are subjected to more attenuation than non-rotating one having quite small attenuation. The phase velocity profiles of lowest modes are noticed to become dispersionless in due to rotation in contrast to their counterparts in the absence of rotation. The study may be useful in construction and design of gyroscope and rotation sensors.

Appendix A

\[
F_1 = \left( \frac{T_1}{T_3} - \frac{m_2 S_2}{m_1 S_1} \frac{T_2}{T_3} \right), \quad F_2 = \left( \frac{T_1}{T_3} - \frac{m_3 V_2 S_3}{m_2 S_2} \right),
\]
\[
F_3 = \left( \frac{T_1}{T_3} - \frac{S_1 T_2}{m_2 S_2} + \left( \frac{q}{p} \right)^2 \frac{m_2 (S_2 - S_1)}{S_2} \right),
\]
\[
F_4 = \left( \frac{T_1}{T_3} - \frac{m_1 S_1}{m_3 S_3} + \frac{m_1 m_3 (S_2 - S_1)}{S_3} \right)^2, \quad F_5 = \left( \frac{T_1}{T_3} - \frac{m_2}{m_1} \frac{T_2}{T_3} \right), \quad F_6 = \left( \frac{T_2}{T_3} - \frac{m_3}{m_2} \right),
\]
\[
F_7 = \left[ \frac{T_1}{T_3} - \frac{m_1 V_2 S_1}{m_2 V_1 (S_2 + (V_3 - 1) S_3)} \frac{T_2}{T_3} + \frac{q^2}{p^2} \frac{m_1 m_3 (S_2 - S_1)}{V_1 (S_2 + (V_3 - 1) S_3)} \right], \quad F_8 = \left( \frac{T_1}{T_3} - \frac{m_3 V_1}{m_1 V_3} \right).
\]
\[ F_8 = \left( \frac{T_1}{T_3} - \frac{m_1(V_3^2 - 1)}{m_3 V_1 (1 + V_3^2 S_1)} \right), \quad F_{10} = \left( \frac{T_2}{T_3} - \frac{m_3 V_2}{m_3 V_3} \right) \]

\[ F_1' = \frac{m_1 S_2(V_3 - V_1)}{m_2 S_2(V_2 - V_3)}, \quad F_2' = \frac{m_1 S_3(V_2 - V_1)}{m_2 S_3(V_2 - V_3)}, \quad F_3' = \frac{m_1(V_3 - V_1)}{m_3(V_2 - V_3)}, \quad F_4' = \frac{m_1(V_2 - V_1)}{m_3(V_2 - V_3)} \]

\[ F_5' = \frac{m_1 V_2(V_3 - V_1)}{m_2 V_2(V_2 - V_3)}, \quad F_6' = \frac{m_1 V_3(V_2 - V_1)}{m_2 V_3(V_2 - V_3)}, \quad F_7' = \frac{m_1 S_2[S_3 - S_1 + V_1(V_3 S_1 - V_1 S_3)]}{m_2 S_1[S_3 - S_1 + V_2(V_3 S_1 - V_1 S_3)]} \]

\[ F_8' = V_2(V_3 - S_1 V_3) + S_1 - S_3 \]

\[ F_9' = \frac{m_1[S_3 - S_1 + V_1(V_3 S_1 - V_1 S_3)]}{m_2\left[V_1(V_2 - V_1)S_3 m_1\left(\frac{q}{p} \right)^2 - V_2(V_3 S_1 - V_1 S_3) + (S_1 - S_3) \right]} \]

\[ F_{10}' = \frac{V_1 S_3 m_1 m_3\left(\frac{q}{p} \right)^2(V_2 - V_1)}{V_1(V_2 - V_1)S_3 m_1\left(\frac{q}{p} \right)^2 - V_2(V_3 S_1 - V_1 S_3) + (S_1 - S_3)} \]

\[ F_{11}' = V_1(V_2 - V_1)S_3 m_1\left(\frac{q}{p} \right)^2 + (S_1 - S_3) - V_2(V_3 S_1 - V_1 S_3) \]

\[ F_{12}' = \frac{m_1 V_2}{m_2 V_1} \left[\frac{m_2}{S_3 m_3(V_2 - V_1)\left(\frac{q}{p} \right)^2 - V_1 V_2(V_3 S_1 - V_1 S_3) + (S_1 - S_3)} \right] \]

\[ F_{13}' = \frac{m_1 S_2}{V_1} \left[\frac{S_3 m_1(V_2 - V_1)\left(\frac{q}{p} \right)^2 - V_1 V_2(V_3 S_1 - V_1 S_3) + (S_1 - S_3)} \right] \]

\[ F_{14}' = \frac{m_1 S_2}{m_2 S_1} \left[\frac{S_3 m_1(V_2 - V_1)\left(\frac{q}{p} \right)^2 - V_1 V_2(V_3 S_1 - V_1 S_3) + (S_1 - S_3)} \right] \]

\[ F_{15}' = \frac{m_1 S_2}{m_2 S_1} \left[\frac{S_3 V_2 m_1 m_2(V_2 - V_1)\left(\frac{q}{p} \right)^2 + m_1 [(S_3 - S_1) + V_1(V_3 S_1 - V_1 S_3)] m_2}{(V_3 S_1 - V_2 S_1)(V_2 - V_1) + (S_1 - S_3) - V_1(V_3 S_1 - V_1 S_3)} \right] \]

\[ F_{16}' = \frac{m_2}{(V_2 S_3 - V_3 S_2)(V_2 - V_1) + (S_1 - S_3) - V_1(V_3 S_1 - V_1 S_3)} \]

\[ F_{17}' = \frac{m_1 S_2}{m_2 S_1} \left[\frac{S_3 V_2 m_1 m_2(V_2 - V_1)\left(\frac{q}{p} \right)^2 + m_1 [(S_3 - S_1) + V_1(V_3 S_1 - V_1 S_3)] m_2}{(V_2 S_3 - V_3 S_2)(V_2 - V_1) + (S_1 - S_3) - V_1(V_3 S_1 - V_1 S_3)} \right] \]

\[ F_{18}' = \frac{S_2 m_1 m_3(V_2 - V_1)\left(\frac{q}{p} \right)^2}{V_1(V_2 S_3 - V_3 S_2)(V_2 - V_1) + (S_1 - S_3) - V_1(V_3 S_1 - V_1 S_3)} \]

\[ F_{19}' = \frac{S_2 m_1 m_3(V_2 - V_1)\left(\frac{q}{p} \right)^2}{V_1(V_2 S_3 - V_3 S_2)(V_2 - V_1) + (S_1 - S_3) - V_1(V_3 S_1 - V_1 S_3)} \]

References


