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Āryabhaṭa's Rule and Table for Sine-Differences

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in the second chapter of his *Āryabhaṭīya* (A.D. 499/510), one of the oldest astronomical texts in India, and a hypothesis about the origin of his table of sine-differences given in the first chapter of the same work. © 1997 Academic Press

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1. INTRODUCTION

Āryabhaṭa (born A.D. 476) gave a rule for the second-order sine-differences in the second chapter, called "Mathematics" (*gaṇita-pāda*), of his Sanskrit astronomical work, *Āryabhaṭīya* (A.D. 499/510, abbr. AB). As will be summarized in Section 3, various interpretations of the rule have been proposed by historians of astronomy and mathematics, but none of these interpretations seems to be correct. His rule expressed in an Āryā verse is this:

> *prathamāc cāpajyārdhād yair ūnaṃ khaṇḍitaṃ dvitīyārdham,*
> *tatprathamajyārdhāṃśais tais tair ūnāni śeṣāṇi.*
> When the second half-⟨chord⟩[1] partitioned is less than the first half-chord, which is ⟨approximately equated to⟩ the ⟨corresponding⟩ arc, by a certain amount, the remaining ⟨sine-differences⟩ are less ⟨than the previous ones⟩ each by that amount of that (i.e., the corresponding half-chord)[2] divided by the first half-chord. (AB 2.12 [1, 83])

Āryabhaṭa also gave a table of sine-differences in the first chapter, called "Ten

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[1] A pair of angular brackets in translations indicates a word/words supplied by me.
[2] A pair of parentheses in translations indicates an explanation of the preceding word(s).

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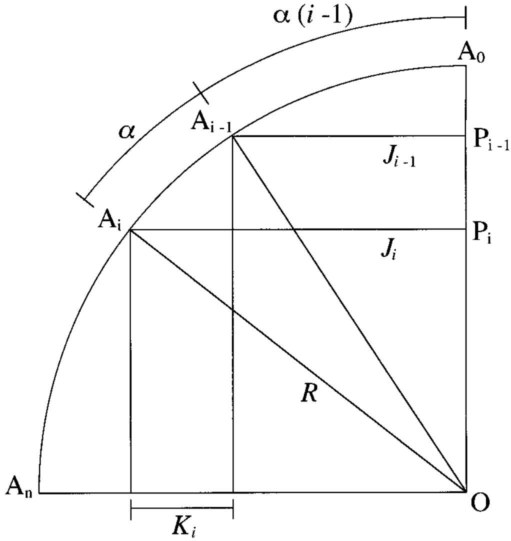

FIG. 1. Notation.

Gīti verses" (*daśagītikā-pāda*), of the same work. Various conjectures have been made concerning the origin of the table, but, as I will show, these conjectures are incorrect. The table expressed in a Gīti verse by means of Āryabhaṭa's own alphabetical notation of numbers (defined in AB 1.2 [1, 7]) is this:

> *makhi-bhakhi-phakhi-dhakhi-ṇakhi-ñakhi-ṅakhi-hasjha-skaki-kiṣga-śghaki-kighva, ghlaki-kigra-hakya-dhaki-kica-sga-jhaśa-ṅva-kla-pta-pha-cha-kalārdhajyāḥ.*
> The half-chords (i.e., sines) consist of 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, and 7 *kalās* (minutes of arc). (AB 1.12 [1, 41]; AB 1.10 in the earlier editions)

The present paper aims at giving both a new translation of AB 2.12 based on the interpretation of Nīlakaṇṭha (born 1444) and a hypothesis about the origin of the table in AB 1.12. Nīlakaṇṭha's interpretation of AB 2.12 itself seems not to have been understood correctly thus far. Nīlakaṇṭha's interpretation is not only precise from the points of view of Sanskrit grammar and mathematics, but also offers us a key to solving the problem of origin of Āryabhaṭa's table of sine-differences.

2. NOTATION

Notation is given in Fig. 1. Let OA_0A_n be a quadrant (*pāda*) of a circle (*vṛtta*) with the radius or semi-diameter (*vyāsārdha*) R and with the circumference (*paridhi*) C. Let the arc (*cāpa*) A_0A_n be divided into n equal arcs: $A_0A_1, A_1A_2, \ldots, A_{n-1}A_n$. Each small arc, measured by α, is called "an arc-part" (*cāpakhaṇḍa*), i.e., a unit of arc. We have $C = 2\pi R = 4n\alpha$. Let J_i be the Rsine (i.e., R times the sine) of the arc αi ($i = 1, 2, \ldots, n$), that is, $J_i = R\sin(\alpha i)$, and K_i the corresponding sinedifference, $K_i = J_i - J_{i-1}$ ($J_0 = 0$). The quantity J_i is usually called "a chord" (*jīvā/ jyā/maurvikā*/etc.) or, more precisely, "a half-chord" (*ardha-*), and K_i "a partial (half)-chord" (*khaṇḍa-*). Thus we have,

$$J_i = K_1 + K_2 + \cdots + K_i. \tag{1}$$

The chord J_i is, therefore, sometimes called "an accumulated (half)-chord" (*piṇḍa-*).

3. PREVIOUS INTERPRETATIONS OF *ĀRYABHAṬĪYA* 2.12

Before giving my own translation and interpretation of AB 2.12, I summarize here various interpretations (expressed in my notation) given thus far by different scholars. All these interpretations, except the last one, are equivalent to, or based on, the approximate relationship

$$K_i - K_{i+1} = \frac{J_i}{J_1}, \tag{2}$$

with or without the assumption $J_1 = 225$. This relationship has a mathematical defect, that is, inconsistency of dimension, since the unit for K_i and J_i is taken to be a linear measure.

Shukla and Sarma's interpretation (the last in the following list) based on Nīlakaṇṭha's is correct insofar as the second half of the verse is concerned, but Nīlakaṇṭha does not read any rule for K_2 in AB 2.12. In fact, such a rule is not necessary at all because K_2 can be calculated by (6). See Section 5 below.

$$K_{i+1} = K_1 - \left\{(K_1 - K_i) + \frac{\sum_{j=1}^{i} K_j}{K_1}\right\}$$ [5, 29; 12, 264–265; 21, 311–312].[3]

$$K_{i+1} = K_1 - (K_1 - K_i) - \frac{\sum_{j=1}^{i} K_j}{K_1}$$ [2, 121; 26, 50].

$$K_i - K_{i+1} = \frac{\sum_{j=1}^{i} K_j}{K_1}$$ [23, 18–19].

$$K_{i+1} = K_i - \frac{\sum_{j=1}^{i} K_j}{J_1}$$ [4, 194–195; 25, 88].

$$K_2 = K_1 - \frac{K_1}{K_1}, \quad K_{i+1} = K_i - \sum_{j=1}^{i} \frac{K_j}{K_1}$$ [7, 110–115].

$$K_2 = J_1 - \frac{J_1}{J_1}, \quad K_{i+1} = K_i - \frac{\sum_{j=1}^{i} K_j}{J_1}$$ [6, 79].[4]

[3] Clark [5, 29] and Sen [21, 311–312] have based this interpretation on the Sanskrit commentary of Parameśvara (fl. 1380/1460).

[4] Datta and Singh (revised by K. S. Shukla) [6, 79], too, have based this interpretation on the Sanskrit commentary of Parameśvara.

$$K_{i+1} = K_i - \frac{J_i}{J_1} \qquad [9, 82; 13, 123; 20, 399, 413].$$

$$K_{i+1} = K_i - \frac{J_i}{225} \qquad [30, 8; 31, 100–101].$$

$$J_{i+1} - J_i = J_i - J_{i-1} - \frac{J_i}{225} \qquad [22, 199; 29, 89].$$

$$J_2 = J_1 + \left(J_1 - \frac{J_1}{J_1}\right), \quad J_{i+1} = J_i + J_1 - \sum_{j=1}^{i} \frac{J_j}{J_1} \qquad [14, 113].$$

$$J_2 = J_1 + \left(J_1 - \frac{J_1}{J_1}\right), \quad J_{i+1} = J_i + \left(J_1 - \frac{\sum_{j=1}^{i} J_j}{J_1}\right) \qquad [6, 80].$$

$$K_2 = J_1 - \frac{J_1}{J_1}, \quad K_{i+1} = J_1 - \sum_{j=1}^{i} \frac{J_j}{J_1} \qquad [24, 51].^5$$

$$K_2 = J_1 - \frac{J_1}{J_1}, \quad K_{i+1} = K_i - \frac{J_i}{J_1}(K_1 - K_2) \qquad [24, 52–53].^6$$

4. NEW TRANSLATION OF *ĀRYABHAṬĪYA* 2.12

The following is my translation[7] of AB 2.12 based on Nīlakaṇṭha's interpretation.

> When the second half-⟨chord⟩ partitioned (K_2) is less than the first half-chord ($J_1 = K_1$), which is ⟨approximately equated to⟩ the ⟨corresponding⟩ arc (α), by a certain amount, the remaining ⟨sine-differences⟩ are less ⟨than the previous ones⟩ each by that amount of that (i.e., the corresponding half-chord, J_i) divided by the first half-chord. (AB 2.12)

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[5] Shukla and Sarma [24, 52] say that "the above translation (hence this interpretation too — Hayashi) is based on Prabhākara's interpretation of the text," but, more precisely, it is based on what has been handed down to us by his not always faithful pupil, Bhāskara I (fl. 629), as no work of Prabhākara (fl. ca. 600) is extant. Moreover, two errors seem to have been committed here, one by Shukla and Sarma and the other by Bhāskara I. First, the rule Bhāskara I [1, 84] actually ascribed to Prabhākara is not that which has been given by Shukla and Sarma but this:

$$K_{i+1} = J_1 - \left\{(J_1 - K_2) + \sum_{j=2}^{i} \frac{J_j}{J_1}\right\}.$$

Bhāskara I admits no rule for K_2 in Prabhākara's interpretation. In fact, according to Bhāskara I, "this computation ⟨of sine-differences prescribed in AB 2.12⟩ is made with the J_1 and J_2 (hence, also with the K_1 and K_2) obtained by means of the graphic procedure prescribed in the previous Āryā verse (i.e., AB 2.11)" (*pūrvāryābhihitachedyakavidhinā nirjñātābhyāṃ prathamadvitīyacāpajyārdhābhyām idaṃ karma kriyate*). He, therefore, criticizes AB 2.12 as not being independent of AB 2.11. See footnote 10 for AB 2.11. Second, the first term within the braces of the above formula, $(J_1 - K_2)$, is nonsense, although it is equivalent to J_1/J_1 (=1) if we assume here the first two sine-differences, K_1 (=J_1) = 225 and K_2 = 224, of Āryabhaṭa's table. It is very likely that Bhāskara I misunderstood his teacher's instruction about AB 2.12.

[6] Shukla and Sarma [24, 52–53] have based this on Nīlakaṇṭha's commentary.

[7] A Japanese version of this translation has been given in [10, 173–175].

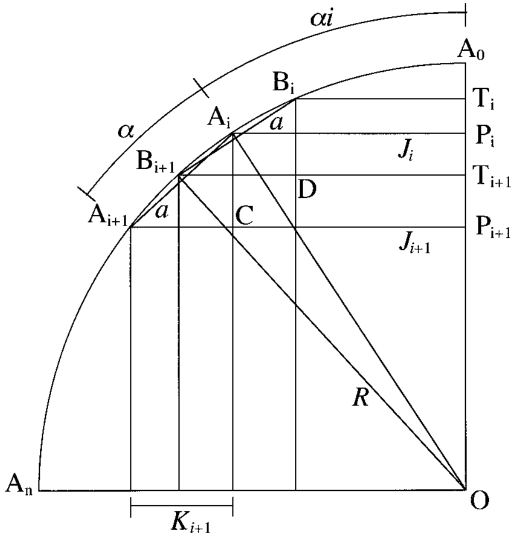

FIG. 2. Proof of Eq. (4).

That is to say,

$$K_i - K_{i+1} = (K_1 - K_2)\frac{J_i}{J_1}. \tag{3}$$

As was pointed out by Nīlakaṇṭha [15, 45–53, 83–84] in the first half of the 16th century A.D., this relationship can be derived immediately from the equation

$$\frac{K_i - K_{i+1}}{J_i} = \left(\frac{a}{R}\right)^2, \tag{4}$$

where a is the "whole chord" (*samasta-jyā*) subtending one unit-arc (α). In particular, we have,

$$\frac{K_1 - K_2}{J_1} = \left(\frac{a}{R}\right)^2.$$

Equation (4) has been elegantly verified by Nīlakaṇṭha [15, 45–53]. It has already been explained and commented on at least twice, that is, by Gupta [9] and by Shukla and Sarma [24, 55–56], but is restated here for the sake of convenience.

Let B_i be the middle point of the arc $A_{i-1}A_i$ (Fig. 2). Then, $A_iA_{i+1} = B_iB_{i+1} = a$. Since OB_{i+1} is perpendicular to A_iA_{i+1}, the right triangles $A_iA_{i+1}C$ and $B_{i+1}OT_{i+1}$ are similar to each other, and hence $A_{i+1}C : A_iA_{i+1} = OT_{i+1} : B_{i+1}O$. Hence, $K_{i+1} = A_{i+1}C = (a/R)\cdot OT_{i+1}$. Likewise, since the right triangles $B_iB_{i+1}D$ and A_iOP_i are similar to each other, $B_iD : B_iB_{i+1} = A_iP_i : A_iO$. Hence, $T_iT_{i+1} = B_iD = (a/R) \cdot J_i$. Therefore,

$$K_i - K_{i+1} = \frac{a}{R} \cdot (OT_i - OT_{i+1}) = \frac{a}{R} \cdot T_iT_{i+1} = \left(\frac{a}{R}\right)^2 \cdot J_i.$$

Equation (4) now follows.

5. ORIGIN OF ĀRYABHAṬA'S TABLE OF *R*SINE-DIFFERENCES

Equation (4) employed for AB 2.12 is also a key to solving the problem of origin of Āryabhaṭa's table of *R*sine-differences given in AB 1.12.

We have the relationships, $J_1 = K_1 < a < \alpha$. Suppose that the first sine (J_1) is approximately equal to the unit-arc (α) as has been alluded to in AB 2.12. Then we have $a \approx \alpha$ and, from (4), $(K_i - K_{i+1})/J_i \approx (\alpha/R)^2$, or

$$K_{i+1} \approx K_i - \left(\frac{\alpha}{R}\right)^2 \times J_i, \tag{5}$$

for $i = 1, 2, \ldots, n - 1$. As a particular case, we have[8]

$$K_2 \approx K_1 - \left(\frac{\alpha}{R}\right)^2 \times J_1. \tag{6}$$

If, therefore, α is given,[9] all the K_i's (for $i = 2, 3, \ldots, n$) can be calculated successively by means of relations (5) and (1), or using the formula

$$K_{i+1} = K_i - \frac{(K_1 - K_2)J_i}{J_1}, \tag{7}$$

together with (1) and (6). (Note that (7) is equivalent to (3).)

Let us suppose with Āryabhaṭa that $C = 21600$, $R = 3438$, and $K_1 = J_1 \approx a \approx \alpha = 225$. Table 1 shows the results of my computation carried out on these conditions with formulas (5)–(7). The first column shows the serial numbers. In the second and third columns, the K_i's obtained respectively by means of relations (5) and (7) are listed. They have fractional parts, which are expressed here in the decimal place-value system, although they must have been expressed either in the Indian way (i.e., with a numerator placed above a denominator) or in the sexagesimal notation as in the case of Govindasvāmin [8]. The fourth column shows the nearest integers to them obtained by rounding, for which the abbreviation $\overline{K}_i$ will be used. That the practice of "rounding" existed in the early seventh century A.D. is known

[8] Nīlakaṇṭha [15, 75–77] seems to be explaining a method for getting K_1 and K_2, but his intention is not clear to me. He says, "Calculation of the first and the second ⟨sine-differences⟩, too, is accomplished by means of this same principle (stated in AB 2.12). ⟨That is, ⟩ it is accomplished by means of that when assisted by the instruction on the half-chords (given in AB 2.11)." See below for AB 2.11.

[9] My previous statement (in Japanese), "When, therefore, K_1 (=J_1) and K_2 are given, ..." [10, 175], is incorrect.

TABLE 1
COMPUTATION OF ĀRYABHAṬA'S *R*SINE-DIFFERENCES

i	K_i by (5)	K_i by (7)	$\overline{K}_i$	$\overline{J}_i$	K_i in AB	J_i	J_i^*	K_i by (2)
1	225	225	225	225	225	225	224.86	225
2	224.036	224.036	224	449	224	449	448.75	224
3	222.113	222.112	222	671	222	671	670.72	222.004
4	219.238	219.236	219	890	219	890	889.82	219.022
5	215.424	215.421	215	1105	215	1105	1105.11	215.066
6	210.688	210.683	211	1316	210	1315	1315.66	210.154
7	205.049	205.043	205	1521	205	1520	1520.59	204.308
8	198.532	198.524	199	1720	199	1719	1719	197.554
9	191.165	191.155	191	1911	191	1910	1910.05	189.922
10	182.979	182.967	183	2094	183	2093	2092.92	181.446
11	174.010	173.995	174	2268	174	2267	2266.84	172.164
12	164.295	164.277	164	2432	164	2431	2431.03	162.117
13	153.877	153.855	154	2586	154	2585	2584.82	151.349
14	142.800	142.774	143	2729	143	2728	2727.55	139.909
15	131.111	131.082	131	2860	131	2859	2858.59	127.847
16	118.861	118.828	119	2979	119	2978	2977.40	115.217
17	106.102	106.065	106	3085	106	3084	3083.45	102.074
18	92.889	92.847	93	3178	93	3177	3176.30	88.478
19	79.278	79.232	79	3257	79	3256	3255.55	74.488
20	65.327	65.277	65	3322	65	3321	3320.85	60.168
21	51.098	51.043	51	3373	51	3372	3371.94	45.580
22	36.649	36.589	37	3410	37	3409	3408.59	30.789
23	22.044	21.979	22	3432	22	3431	3430.64	15.862
24	7.344	7.275	7	3439	7	3438	3438	0.864

from the following statement of Bhāskara I: "When this (449) is divided by the first half-chord of arc (225), the quotient is two units 'because of its being greater than half' (*ardhādhikena*)" [1, 84]. We can therefore assume the same practice in Āryabhaṭa's computation. The fifth column gives *R*sines ($\overline{J}_i$) obtained from the $\overline{K}_i$ by means of relation (1).

Now, the main criteria known to Āryabhaṭa must have been $J_{24} = R = 3438$ and $J_8 = R/2 = 1719$. But the fifth column of Table 1 shows that the values obtained for both (underlined) each exceed the expected values by one. Āryabhaṭa, therefore, seems to have assumed that the $\overline{J}_i$ between the two also exceed the correct values by one each. The excess was caused primarily by "rounding up." In order to reconcile them he must therefore have subtracted one from $\overline{K}_6$, the first integer that had been obtained by rounding up. Āryabhaṭa must have obtained his K_i in this way and versified them in AB 1.12. They are listed in the sixth column of Table 1. The next (seventh) column shows the 24 J_i's obtained from these K_i's by means of (1), but it should be noted that those J_i's have not been stated anywhere by Āryabhaṭa himself.

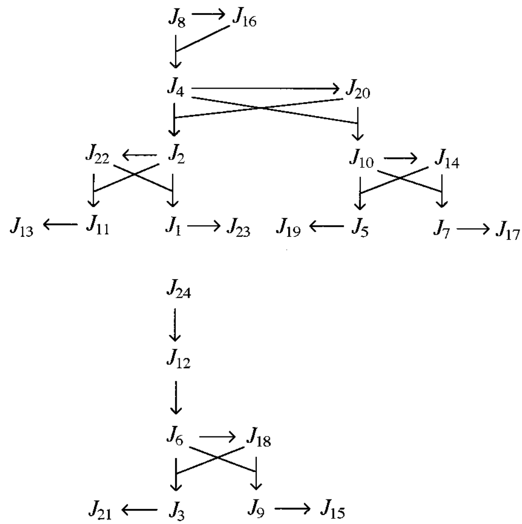

FIG. 3. Derivation of *R*sines (J_i^* in Table 1) by the "graphic procedure."

It suffices for the purpose of the reconciliation mentioned above to decrease not $\overline{K}_6$ but $\overline{K}_8$ by one. In fact, that would have brought him better results, since the original values of $\overline{J}_6$ and $\overline{J}_7$, 1316 and 1521, are closer to the correct ones than their values, 1315 and 1520. It is therefore most probable that he did not compare the values of J_i thus obtained with the J_i^* (listed in the eighth column of Table 1), which, starting from $J_8^* = 1719$ and $J_{24}^* = 3438$, we can calculate by the successive application of the so-called "graphic procedure" (*chedyaka-vidhi*), which consists of the two formulas,

$$J_{24-i} = \sqrt{R^2 - J_i^2} \qquad \text{and} \qquad J_{i/2} = \frac{\sqrt{J_i^2 + (R - J_{24-i})^2}}{2}.$$

The process of the derivation of the 24 *R*sines (J_i^*) according to this method is shown in Fig. 3, where the use of these two formulas is indicated respectively by the horizontal and the vertical arrows. This method, together with a sine table for $R = 120$, has been recorded by Varāhamihira in his *Pañcasiddhāntikā* (ca. 550) [28, 52–56]. The commentator Bhāskara I [1, 77–85] assumes the same method to be implied in the ambiguous verse of AB 2.11[10] and asserts that Āryabhaṭa actually obtained the values of J_i in this way. However, if Bhāskara I is correct, Āryabhaṭa

[10] AB 2.11 reads as follows: "One should divide a quarter of the circumference of an even circle (*samavṛtta,* perhaps as against an *āyatavṛtta* or 'a long circle', that is, an ellipse-like figure) by means of triangles and quadrilaterals ⟨so that⟩ as many half-chords of equal arcs as desired ⟨may be obtained⟩ for the ⟨given⟩ semi-diameter."

would have obtained more precise values not only for J_6 and J_7, but also for J_{16}, J_{17}, and J_{18}, on the basis of the corresponding J_i^* (underlined).

The last column of Table 1 lists the values of K_i calculated by means of (2) so that a comparison can be made.

6. CONCLUDING REMARKS

The wrong notion that AB 2.12 must prescribe a rule based on relationship (2) has a long list of adherents, to which belong most of the traditional commentators beginning with Bhāskara I (A.D. 629). The correct meaning of AB 2.12 remained forgotten from, perhaps, soon after Āryabhaṭa's time until Nīlakaṇṭha rediscovered it with his great insight nearly one millennium later. Even those modern scholars who admitted the correctness of Nīlakaṇṭha's interpretation did not regard it as Āryabhaṭa's own intention because it was different from the traditional interpretation, which has been supported also by the *Sūryasiddhānta* (ca. A.D. 800), one of the most popular astronomical works in India. This work prescribes the formula

$$J_2 = J_1 + \left(J_1 - \frac{J_1}{J_1}\right), \qquad J_{i+1} = J_i + \left(K_i - \frac{J_i}{J_1}\right),$$

with $J_1 = 225$ (*Sūryasiddhānta* 2.15–2.16) [27, 27].

The *Paitāmahasiddhānta* (as handed down to us in the *Viṣṇudharmottarapurāṇa* 2.166–2.174) of uncertain date[11] also explains a step-by-step procedure in order to apply the formula

$$K_{i+1} = K_i - \frac{J_i}{J_1}, \qquad J_{i+1} = J_i + K_{i+1},$$

with $J_1 = K_1 = 225$ (in a section of 2.168 [16, 215b–216a]; III.12 in [17]).

These approximate formulas, as well as (2), can be easily derived from Āryabhaṭa's exact formula, (3) or (7), by assuming the first two values of his table of sine-differences, $K_1 = 225$ and $K_2 = 224$, as has been rightly pointed out by Nīlakaṇṭha [15, 47, 75–76]. The inverse derivation, from (2) to (3), would have been quite difficult, if not impossible, given the fact that it took nearly one millennium for Indian mathematicians to rediscover (3). This strongly suggests the possibility that the present *Paitāmahasiddhānta* was compiled after the *Āryabhaṭīya,* although we cannot completely deny the possibility that Āryabhaṭa, like Nīlakaṇṭha, unaffected by the earlier, wrong formula of the *Paitāmahasiddhānta,* discovered the correct one.[12] From the purely logical point of view, there is also the possibility that the exact formula (3) was contained in an earlier work, now lost, and was simply copied by Āryabhaṭa, on the one hand, and adapted in the *Paitāmahasiddhānta* on the other.

[11] Hazra [11, 205–212] dates the *Viṣṇudharmottarapurāṇa* to the fifth century A.D. Billard [3, 114] dates the *Paitāmahasiddhānta* to some time between the *Brāhmasphuṭasiddhānta* (A.D. 628) and the *Rājamṛgāṅka* (A.D. 1042), while Pingree [18, 17; 19, 71] dates it to the early fifth century (ca. 425).

[12] For a discussion of the relationship of the *Paitāmahasiddhānta* and the *Āryabhaṭīya,* see [19].

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