Bifurcation of periodic solutions of delay differential equation with two delays

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Abstract
In this paper, we develop Kaplan–Yorke’s method and consider the existence of periodic solutions for delay differential equations with two delays. Especially, we study Hopf and saddle-node bifurcations of periodic solutions for the equation with parameters, and give conditions under which the bifurcations occur.
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1. Introduction
In recent years, the existence of nontrivial periodic solutions of differential difference equations has attracted attention of many mathematicians. Fixed point theorems are the principle tools to study the existence of such solutions. In 1974, Kaplan and Yorke introduced a new technique to relate a special class of scalar delay differential equations to the corresponding ordinary differential systems. The equation they studied is

\[ \dot{x}(t) = -\left[ f(x(t-1)) + f(x(t-2)) \right], \quad (1.1) \]

where \( f \) is odd with \( f(x) = 0 \) for \( x = 0 \). They gave a sufficient condition under which (1.1) has a periodic solution with period 6. Later this method was developed widely, and many results have been established on the existence of periodic solutions for some delay differ-
ential equations such as [2–11]. Xu [5] gave conditions for the existence and uniqueness of \((6/(1 + 6N))\)-periodic solution for \((1.1)\), where \(N\) is a nonnegative integer. Ge [11] found conditions for the existence of \(\left(6r_i/(6m_i + i)\right)\)-periodic solution for equation
\[
\dot{x}(t) = -\left[f\left(x(t - r_1)\right) + f\left(x(t - r_2)\right)\right],
\]
where \(6r_i/(6m_1 + 1) = 6r_2/(6m_2 + 2)\). Cima et al. [3] found conditions under which the equations
\[
\dot{x}(t) = -f\left(x(t - r_1)\right)g\left(x(t - r_2)\right) - f\left(x(t - r_2)\right)g\left(x(t - r_1)\right) \tag{1.2}
\]
and
\[
\dot{x}(t) = f\left(x(t - r_1)\right)g\left(x(t - r_2)\right) + f\left(x(t - r_2)\right)g\left(x(t - r_1)\right) \tag{1.3}
\]
have \(6r_i/(6m_1 + i)\) and \(6r_i/(6m_2 - i)\) periodic solutions, respectively, where \(r_1 = (6m_1 + i)\mu_1\) for \((1.2)\) and \(r_1 = (6m_2 - i)\mu_2\) for \((1.3)\), \(i = 1, 2, m_1\) and \(m_2\) are nonnegative integers.

In this paper, we study two more general delay differential equations and obtain conditions for the existence of periodic solutions. The conditions in our paper are new and different from those appeared in [3,11]. We also study the Hopf and saddle node bifurcations of these kinds of periodic solutions for the delay differential equation with parameters.

2. Existence of periodic solutions

Consider the scalar delay differential equation
\[
\dot{x}(t) = -F\left(x(t), x(t - r_1), x(t - r_2)\right) \tag{2.1}
\]
and
\[
\dot{x}(t) = F\left(x(t), x(t - r'_1), x(t - r'_2)\right), \tag{2.2}
\]
where \(F : (x, y, z) \rightarrow F(x, y, z)\) is of class \(C^l\), \(l \geq 3\), and is odd in \((x, y, z)\), \(r_1, r'_1 > 0\).

Suppose that there exist nonnegative integers \(k_i, k'_i\) (\(i = 1, 2\)) such that \(r_1/(6k_1 + 1) = r_2/(6k_2 + 2)\) and \(r'_1/(6k'_1 - 1) = r'_2/(6k'_2 - 2)\). To study the existence of periodic solutions of \((2.1)\) and \((2.2)\), we introduce the following three dimensional ordinary differential systems:
\[
\dot{x}(t) = -F(x, y, z), \quad \dot{y}(t) = -F(y, z, -x), \quad \dot{z}(t) = -F(z, -x, -y), \tag{2.3}
\]
and
\[
\dot{x}(t) = F(x, y, z), \quad \dot{y}(t) = F(y, z, -x), \quad \dot{z}(t) = F(z, -x, -y). \tag{2.4}
\]

By assumption on \(F\) we have
\[
F(-x, -y, -z) = -F(x, y, z). \tag{2.5}
\]

We also suppose that there exists \(\gamma > 0\) such that
\[
F(0, \gamma, \gamma) > 0 \quad \text{and} \quad F(x, x + z, z) - F(x + z, z, -x) - F(-z, x, x + z) \equiv 0. \tag{2.6}
\]
To establish the relationship between periodic solutions of (2.1) (respectively, (2.2)) and (2.3) (respectively, (2.4)), we first consider the linear map $T_3 : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$T_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$ 

It is easy to see that $T_3^6 = I$, where $I$ is the identity matrix. The following result is easy to prove by (2.5).

**Lemma 2.1.** If $\phi(t) = (x(t), y(t), z(t))$ is a periodic solution of (2.3) (respectively, (2.4)) with period $T$, so is $T_3^l \phi(t)$ ($l = 1, 2, \ldots, 6$) with the same period.

**Remark 2.1.** The similar result of above lemma was mentioned in [3–5] for special cases of (2.3) and (2.4).

From (2.6), we know that $G$: $x − y + z = 0$ is an invariant plane. Then we can prove

**Lemma 2.2.** Suppose (2.5) and (2.6) hold. If (2.3) (respectively, (2.4)) has a $T$-periodic orbit $L$: $(x, y, z) = (x(t), y(t), z(t)), 0 \leq t \leq T$, which starts at $(0, \gamma, \gamma) \in G$ and surrounds the origin on $G$. Then the following formula (2.7) (respectively, (2.8)) holds:

$$y(t) = x\left(t - \frac{T}{6}\right), \quad z(t) = y\left(t - \frac{T}{6}\right), \quad x(t) = -z\left(t - \frac{T}{6}\right). \quad (2.7)$$

$$x(t) = y\left(t - \frac{T}{6}\right), \quad y(t) = z\left(t - \frac{T}{6}\right), \quad z(t) = -x\left(t - \frac{T}{6}\right). \quad (2.8)$$

**Proof.** From Lemma 1 we know that $\phi(t)$ and $\phi_3(t) = T_3(\phi(t))$ are both solutions of (2.3), where $\phi(t) = (x(t), y(t), z(t))$. Let $L$ and $L_1$ denote the corresponding orbits. Note that $\phi(t) \in G$ implies $T_3\phi(t) \in G$. Also note that both $L$ and $L_1$ surround the origin on $G$ and that $|\phi(t)| = |T_3\phi(t)|$ for all $t$. It follows that $L$ and $L_1$ have points in common. Hence $L = L_1$. The uniqueness of solutions implies that there exists $t_1 \in R$ such that

$$T_3\phi(t) = \phi(t + t_1) \quad \text{for all } t.$$ 

(2.9)

Since $\phi$ is $T$-periodic, without loss of generality, we may assume $t_1 \in [0, T)$. By (2.9) we know that

$$-\phi(t) = T_3^3\phi(t) = \phi(t + 3t_1). \quad (2.10)$$

Therefore,

$$\phi(t) = T_3^6\phi(t) = T_3^3\phi(t + 3t_1) = \phi(t + 6t_1).$$

It follows that $6t_1 = kT$ for an integer $k$ satisfying $0 \leq k < 6$. From (2.10), it is easy to know $k \neq 0, 2, 4$. If $k = 3$, we have from (2.9) and (2.10) that $-\phi(t) = \phi(t + 3T/2) = \phi(t + T/2) = T_3\phi(t)$ or $y(t) = -x(t) = -z(t)$. This gives that $\phi(t) \in \{(x, -x, x): x \in \mathbb{R}\}$, a contradiction. Hence $k = 1$ or $k = 5$. If $k = 1$, i.e., $t_1 = T/6$, we know from (2.9),

$$x(t) = y\left(t - \frac{T}{6}\right), \quad y(t) = z\left(t - \frac{T}{6}\right), \quad z(t) = -x\left(t - \frac{T}{6}\right).$$
Thus
\[ x(t) = y\left(t - \frac{T}{6}\right) = z\left(t - \frac{T}{3}\right) = -x\left(t - \frac{T}{2}\right). \quad (2.11) \]

Note that \( \varphi(0) = (x(0), y(0), z(0)) = (0, \gamma, \gamma) \). From (2.11), it is easy to see
\[ x(0) = x\left(\frac{T}{2}\right) = 0, \quad -x\left(\frac{T}{6}\right) = x\left(\frac{2T}{3}\right) = -x\left(\frac{T}{3}\right) = x\left(\frac{5T}{6}\right) = -\gamma. \]

Similarly
\[ -y(0) = y\left(\frac{T}{2}\right) = -y\left(\frac{T}{6}\right) = y\left(\frac{2T}{3}\right) = -\gamma, \quad y\left(\frac{T}{3}\right) = y\left(\frac{5T}{6}\right) = 0. \]

Thus, we have
\[ \varphi(0) = (0, \gamma, \gamma), \quad \varphi\left(\frac{T}{6}\right) = (\gamma, \gamma, 0), \quad \varphi\left(\frac{T}{3}\right) = (\gamma, 0, -\gamma). \]

By (2.6) we have \( x'(0) = -F(0, \gamma, \gamma) < 0 \). Hence, the orientation of \( L \) in the first octant is from \( A(\gamma, \gamma, 0) \) to \( B(0, \gamma, \gamma) \), as shown in Fig. 1.

Note that \( B = \varphi(0), A = \varphi(T/6) \). Thus \( \varphi(T/3) \) must belong to the part \( AB \) of \( L \), contradicting to \( \varphi(T/3) = (\gamma, 0, -\gamma) \). Hence \( t_1 = 5T/6 \). From (2.9), it is easy to obtain (2.7).

Similarly, we can prove that (2.8) holds for Eq. (2.4).

Using these lemmas, we can obtain the following theorems.

**Theorem 2.1.** Let (2.5) and (2.6) hold. Equation (2.3) (respectively, (2.4)) has a periodic orbit \( L \) with period \( T = 6r_i/(6k_i + i) \) (respectively, \( 6r'_i/(6k'_i - i) \)), which passes through the point \( (0, \gamma, \gamma) \) and surrounds the origin on \( G \) for an integer \( k_i \geq 0 \) (respectively, \( k'_i \geq 0 \)) if and only if (2.1) (respectively, (2.2)) has a periodic solution \( x(t) \) with period \( T = 6r_i/(6k_i + i) \) (respectively, \( 6r'_i/(6k'_i - i) \)), which satisfies \( x(t - T/2) = -x(t) \) and \( x(t_0), x(t_0 - T/6), x(t_0 - T/3) = (0, \gamma, \gamma) \) for \( t_0 \).

**Proof.** We only consider the first case. In a similar way, we can prove the other case.

**Necessity.** Assume that \( \varphi(t) = (x(t), y(t), z(t)) \) is a solution of (2.3) with period \( T = 6r_i/(6k_i + i) \), \( i = 1, 2 \). By Lemma 2.2, we have
\[ y(t) = x\left(t - \frac{T}{6}\right), \quad z(t) = y\left(t - \frac{T}{6}\right) = x\left(t - \frac{T}{3}\right). \]
Then
\[
y(t) = x\left(t - \frac{T}{6}\right) = x\left(t - k_1 T - \frac{T}{6}\right) = x(t - r_1),
\]
\[
z(t) = y\left(t - \frac{T}{6}\right) = x\left(t - k_2 T - \frac{T}{3}\right) = x(t - r_2),
\]
\[
z(t) = y\left(t - \frac{T}{6}\right) = x\left(t - 2k_1 T - \frac{2T}{6}\right) = x(t - 2r_1),
\]
\[-x(t) = z\left(t - \frac{T}{6}\right) = x\left(t - \frac{T}{2}\right) = x(t - r_1 - r_2),
\]
\[-y(t) = -z\left(t + \frac{T}{6}\right) = x\left(t + \frac{T}{6} + \frac{T}{6}\right) = x\left(t - \frac{2T}{3}\right) = x(t - 2r_2).
\]
From above, it is easy to prove that \(x(t - T/2) = -x(t)\). Hence, \(x(t)\) is a \((6r_1/(6k_1 + i))\)-periodic solution of (2.1) with \(x(t - T/2) = -x(t), i = 1, 2\).

**Sufficiency.** Suppose (2.1) has a nontrivial \((6r_1/(6k_1 + i))\)-periodic solution \(x(t)\) with \(x(t - T/2) = -x(t)\). Let \(y(t) = x(t - r_1), z(t) = x(t - r_2)\). Then we can obtain
\[
z(t - r_2) = x(t - 2r_2) = x\left(t - \frac{(6k_2 + 2)T}{3}\right) = x\left(t - \frac{T}{2} - \frac{T}{6}\right)
\]
\[-x(t - r_1) = -y(t),
\]
\[
y(t - r_2) = x(t - r_1 - r_2) = x\left(t - \frac{(6k_1 + 1)T}{6} - \frac{(6k_2 + 2)T}{6}\right)
\]
\[
= x\left(t - \frac{T}{2}\right) = -x(t),
\]
\[
y(t - r_1) = x(t - 2r_1) = x\left(t - \frac{T}{3}\right) = x\left(t - k_2 T - \frac{2T}{6}\right) = x(t - r_2) = z(t).
\]
Hence \((x(t), y(t), z(t))\) is a \(T\)-periodic solution of (2.3). Note that \(x(t_0) - x(t_0 - T/6) + x(t_0 - T/3) = 0\). This yields \((x(t_0), y(t_0), z(t_0)) \in G\). Thus \((x(t), y(t), z(t))\) is a periodic solution of (2.3) on the invariant plane \(x - y + z = 0\). This completes the proof. □

Using this theorem, it is easy to know the following result.

**Theorem 2.2.** Let \(F\) satisfy (2.5) and (2.6). Suppose (2.3) (respectively, (2.4)) has a family of periodic orbits \(L_h\) for \(h \in J \subset R\) with \(F(0, y(h), y(h)) > 0\) for \((0, y(h), y(h)) \in L_h, h \in J\). Set \(\alpha = \inf J, \beta = \sup J\), where \(T_h\) denotes the period of \(L_h\). If there exists integers \(k_i \geq 0\) (respectively, \(k_i' \geq 0\)) such that \(6r_i/(6k_i + i)\) (respectively, \(6r_i'//(6k_i' - i)\)) \(\in (\alpha, \beta)\), \(i = 1, 2\), then (2.1) (respectively, (2.2)) has a periodic solution with period \(6r_i/(6k_i + 1)\) (respectively, \(6r_i'//(6k_i' - i)\), \(i = 1, 2\).

**Remark 2.2.** Under (2.5) and (2.6), on the plane \(G\) we can reduce (2.3) (respectively, (2.4)) to a two-dimensional system by which we may estimate \(T_h\).
Example 2.1. Consider the delay differential equation
\[ \dot{x}(t) = \sin(2x(t - r_1)) + \sin(2x(t - r_2)), \] (2.12)
where \( r_2 = ((6k_2 + 2)/(6k_1 + 1))r_1 \) with \( k_1, k_2 \geq 0 \) integers. The corresponding planar system is
\[
\begin{align*}
\dot{x}(t) &= 2y + 2z, \\
\dot{y}(t) &= 2z - 2x, \\
\dot{z}(t) &= -2x - 2y.
\end{align*}
\] (2.13)

Obviously, (2.13) has two integrals \( (x - y + z)' = 0 \) and \( \sin^2x + \sin^2y + \sin^2z = 0 \). That is, (2.13) has a family of periodic orbits
\[
L_h: \quad \frac{V(x, y)}{2} = \sin^2x + \sin^2y + \sin^2(x - y) = h, \quad 0 < h < h_0, \quad z = y - x, \quad (2.14)
\]
where \( h_0 \) is a to-be-determined constant. Noting that \( G: x - y + z = 0 \) is an invariant plane and
\[
\max\{\sin^2x + \sin^2y + \sin^2(x - y)\} = 2\sin^2\frac{\pi}{3} + \sin^2\frac{2\pi}{3} = \frac{3 + \sqrt{3}}{2}.
\]
Then for \( h_0 = (3 + \sqrt{3})/2 \), the closed curve \( L_{h_0} \) exists finitely and contains a singular point. Hence \( \lim_{h \to h_0} T(h) = \infty \). Thus the period function \( T(h) \) satisfies
\[
\alpha = \inf_{0 < h < h_0} T(h) \leq \lim_{h \to 0^+} T(h) = \frac{\pi}{\sqrt{3}}, \quad \beta = \sup_{0 < h < h_0} T(h) \geq \lim_{h \to h_0^-} T(h) = \infty.
\] (2.15)

From Theorem 2.2, we know that for any \( k_i \) such that \( 6r_1/(6k_1 + 1) = 6r_2/(6k_2 + 2) \geq \pi/\sqrt{3} \), (2.12) has a periodic solution with period \( 6r_i/(6k_i + i) \), \( i = 1, 2 \).

Now we consider the following equation with a vector parameter:
\[ \dot{x}(t) = -F(x(t), x(t - r_1), x(t - r_2), a). \] (2.16)
where \( a \in I \subset R^n, n \geq 1, F \in C^1, l \geq 3 \). For the other equation
\[ \dot{x}(t) = F(x(t), x(t - r_1), x(t - r_2), a) \] (2.17)
there are similar results as above, which will be omitted here. Then the corresponding planar systems of (2.16) is
\[
\begin{align*}
\dot{x} &= -F(x, y, z, a), \\
\dot{y} &= -F(y, z, -x, a), \\
\dot{z} &= -F(z, -x, -y, a).
\end{align*}
\] (2.18)
As before, suppose
\[ F(x, y, z, a) = -F(-x, -y, -z, a). \] (2.19)
We also suppose there exists \( \gamma = \gamma(a) > 0 \) such that
\[ F(0, \gamma(a), \gamma(a), a) > 0 \quad \text{and} \quad F(x, x + z, z, a) - F(x + z, z, -x, a) - F(-z, x, x + z, a) = 0 \]  
(2.20)

for all \( a \in I \). From above we can obtain the following two theorems in the same way as before.

**Theorem 2.3.** Let (2.19) and (2.20) hold. For \( i = 1, 2 \), we suppose there exists an open subset \( I^* \subset I \) such that for each \( a \in I^* \), (2.18) has a periodic orbit \( L(a) \) passing through the point \( (0, \gamma(a), \gamma(a)) \) and surrounding the origin on \( G \) for \( a \in I^* \). Let \( T(a) \) denote the period of \( L(a) \). Then (2.16) has a periodic solution \( \psi(t, a) \) for \( a \in I^* \) such that \( T(a) = 6r_i/(6k_i + i) \), \( k_i \geq 0 \).

**Theorem 2.4.** Let (2.19) and (2.20) hold. Suppose that there exist a subset \( I^* \) of \( I \), an interval \( J(a) \) with \( a \in I^* \) and \( \gamma(a, h) > 0 \) with \( h \in J(a) \) such that

(i) For each \( a \in I^* \), (2.18) has a family of periodic orbits \( L(a, h) \) with \( F(0, \gamma(a, h), \gamma(a, h)) > 0 \) for \( (0, \gamma(a, h), \gamma(a, h)) \in L(a, h), h \in J(a) \);

(ii) For \( a \in I^* \), we let \( \alpha(a) = \inf_{h \in J(a)} T(a, h), \beta(a) = \sup_{h \in J(a)} T(a, h) \), where \( T(a, h) \) denotes the period of \( L(a, h) \). If there exist integers \( k_1 \geq 0, k_2 \geq 0 \) such that \( 6r_1/(6k_1 + 1) = 6r_2/(6k_2 + 2) \in (\alpha(a), \beta(a)) \), then for all \( a \in I^* \), (2.16) has a periodic solution \( x_k(t, a) \) with period \( T_k = 6r_i/(6k_i + i) \).

**Example 2.2.** Consider the delay equation

\[ i(t) = -x(t - r_1)(a^2 - x^2(t - r_1)) - x(t - r_2)(a^2 - x^2(t - r_2)), \]  
(2.21)

where \( a > 0, r_2 = ((6k_2 + 2)/(6k_1 + 1))r_1 \) with \( k_i \geq 0 \) integers. The corresponding ordinary differential system is

\[
\begin{align*}
\dot{i} &= -y(a^2 - y^2) - z(a^2 - z^2), \\
\dot{y} &= -z(a^2 - z^2) + x(a^2 - x^2), \\
\dot{z} &= x(a^2 - x^2) + y(a^2 - y^2).
\end{align*}
\]  
(2.22)

From (2.22), it is easy to see \( x - y + z = 0 \). That is \( G(x, y, z) = x - y + z \) is the first integral. Especially, the plane \( x - y + z = 0 \) is invariant. Then (2.22) is reduced to

\[
\begin{align*}
\dot{i} &= a^2 i - 2a^2 y + y^3 + (y - x)^3, \\
\dot{y} &= 2a^2 x - a^2 y - x^3 + (y - x)^3,
\end{align*}
\]  
(2.23)

which can be written as

\[
\begin{align*}
\dot{i} &= -H_i(a^2 - H), \\
\dot{y} &= H_x(a^2 - H),
\end{align*}
\]  
(2.23)

where \( H = x^2 - xy + y^2 \). Let \( A = a^2 \). It is easy to see that (2.22) has a family of closed orbits

\[
L_h: \begin{cases} 
H(x, y) = h, & h \neq A, \\
z = y - x.
\end{cases}
\]  
(2.24)

It is obvious that
\[ \alpha_1 = T_0 = \inf_{0 < h < A} T(h, a) = \lim_{h \to 0} T(h, a) = \frac{1}{A} \int_0^{2\pi} \frac{d\theta}{2 - \sin 2\theta} = \frac{2\pi}{\sqrt{3}A}, \]
\[ \beta_1 = \sup_{0 < h < A} T(h, a) \leq \lim_{h \to A} T(h, a) = \infty, \]
\[ \alpha_2 = \inf_{h > A} T(h, a) = \lim_{h \to \infty} T(h, a) = 0, \]
\[ \beta_2 = \sup_{h > A} T(h, a) = \lim_{h \to A} T(h, a) = \infty. \]

Let there exist integers \( k_1 \geq 0 \) and \( k_2 \geq 0 \) such that \( r_1 = ((6k_1 + 1)/(6k_2 + 2))r_2 \). Then from Theorem 2.4, we know that if \( r_1 \geq ((6k_1 + 1)/(3\sqrt{3}A))\pi \), (2.22) has two periodic solutions with period \( T_{k_1} = 6r_1/(6k_1 + 1) \); if \( 0 < r_1 < ((6k_1 + 1)/(3\sqrt{3}A))\pi \), (2.22) has a period solution with period \( T_{k_1} = 6r_1/(6k_1 + 1) \). For the graph of \( T(a, h) \) for fixed \( a > 0 \), see Fig. 2.

3. Hopf and saddle-node bifurcations

In this section, we still consider (2.16). As an application of the above theorems, first we consider \( F(x, y, z) \) with two parameters \( F(x, y, z, a, b) \), suppose
\[ F(x, y, z, a, b) = f(y, z, a, b) + f(z, x, a, b). \]
(3.1)
If \( f(x, y, a, b) \) is odd in \( x \) and even in \( y \), then the identity in (2.19) is satisfied. Let \( f \) be a polynomial of degree 5 and it has the form
\[ f(x, y, a, b) = bx + ax^2 + A_{30}x^3 + A_{32}x^3y^2 + A_{14}xy^4 + A_{50}x^5, \]
where \( A_{ij} \) are constant. Let \( k \) be any nonnegative integer and \( k_2 \) an integer satisfying \( 6r_2/(6k_2 + 2) = 6r_1/(6k_1 + 1) = 2\pi/\sqrt{3}b_k \), where \( b_k = (6k + 1)\pi/(3\sqrt{3}r_1) > 0 \). Take \( a \) as a small parameter and let \( b \) vary near \( b_k \). Then
\[ F(x, y, z, a, b) = by + bz + a(yz^2 + zx^2) + A_{30}(y^3 + z^3) \]
\[ + A_{32}(y^3z^2 + z^3x^2) + A_{14}(yz^4 + zx^4) + A_{50}(y^5 + z^5). \]
It is easy to see that $F(0, γ, γ, a, b) = 2by + αγ^5 + (2A_{30} + A_{32} + A_{14} + 2A_{50})γ^5 > 0$ for $|b − b_k| + |a|$ and $γ > 0$ small. Then the corresponding ordinary system is

$$
\begin{aligned}
\dot{x} &= b(y + z) + a(yz^2 + z^2) + A_{30}(y^3 + z^3) \\
&+ A_{32}(y^3 z^2 + z^2 x^2) + A_{14}(yz^4 + xz^4) + A_{50}(y^5 + z^5), \\
\dot{y} &= b(z - x) + a(zx^2 - xy^2) + A_{30}(z^3 - x^3) \\
&+ A_{32}(z^3 x^2 - x^3 y^2) + A_{14}(zx^4 - xy^4) + A_{50}(z^5 - x^5), \\
\dot{z} &= -b(x + y) + a(−x^2 − yz^2) + A_{30}(−x^3 − y^3) \\
&+ A_{32}(−x^3 y^2 − y^3 z^2) + A_{14}(−x^4 − yz^4) + A_{50}(−x^5 − y^5).
\end{aligned}
$$

(3.2)

It is easy to see that $(x − y + z) = 0$. Inserting $z = y − x$ into the first two equations of (3.2), we obtain the following plane system:

$$
\begin{aligned}
\dot{x} &= −bx + 2by − (a + A_{30})x^3 + (2a + 3A_{30})x^2y \\
&− (2a + 3A_{30})xy^2 + (a + 2A_{30})y^3 − (A_{14} + A_{32} + A_{50})x^5 \\
&+ (2A_{14} + 3A_{32} + 5A_{50})x^4y − (4A_{14} + 3A_{32} + 10A_{50})x^3y^2 \\
&+ (6A_{14} + 2A_{32} + 10A_{50})x^2y^3 − (4A_{14} + 2A_{32} + 5A_{50})xy^4 \\
&+ (A_{14} + A_{32} + 2A_{50})y^5, \\
\dot{y} &= −2bx + by − (a + 2A_{30})x^3 + (a + 3A_{30})x^2y \\
&− (a + 3A_{30})xy^2 + A_{30}y^3 − (A_{14} + A_{32} + 2A_{50})x^5 \\
&+ (A_{14} + 3A_{32} + 5A_{50})x^4y − (4A_{14} + 3A_{32} + 10A_{50})x^3y^2 \\
&+ (A_{14} + 2A_{32} + 10A_{50})x^2y^3 − (A_{14} + 5A_{30})xy^4 + A_{50}y^5. \\
\end{aligned}
$$

(3.3)

Noting that (3.3) has always a weak focus at the origin, we have

**Theorem 3.1.** Let $2A_{14} + A_{32} ≠ 0$. For any nonnegative integer $k ≥ 0$ and $k_2$ satisfying $6r_2/(6k_2 + 2) = 6r_1/(6k + 1) = 2π/(√3b_k)$, there exists a $C^1$ function $ϕ(a) = b_k + \frac{3A_{30}a}{(2A_{14} + A_{32})}$ + $O(a^2)$ such that for $|b − b_k| + |a|$ small (2.16) has a periodic solution of period $T_k = 6r_1/(6k + 1)$ if $b = ϕ(a)$ and $a(2A_{14} + A_{32}) < 0$.

**Proof.** Set $x_1 = y$, $x_2 = (2x − y)/√3$; then (3.3) becomes

$$
\begin{aligned}
x_1 &= \sqrt{3}bx_2 − \frac{3√3}{2}a^2x_1^3 − \frac{3√3}{2}a^2x_1x_2^2 − \frac{3√3}{8}(a + 2A_{30})x_1^4x_2 \\
&− \frac{3√3}{8}(a + 2A_{30})x_1^5x_2^2 − \frac{3√3}{16}(15A_{14} + 3A_{32})x_1^3x_2 \\
&+ \frac{3√3}{32}(13A_{14} + 13A_{32} + 10A_{50})x_1^4x_2^2 + \frac{3√3}{16}(A_{14} − 7A_{32})x_1^3x_2^2 \\
&− \frac{3√3}{16}(A_{14} + A_{32} + 10A_{50})x_1^2x_2^3 + \frac{3√3}{64}(−3A_{14} + A_{32})x_1^2x_2 \\
&− \frac{9√3}{32}(A_{14} + A_{32} + 2A_{50})x_1x_2^4 \\
\dot{x}_2 &= \sqrt{3}bx_1 + \frac{3√3}{2}(a + 2A_{30})x_1^2 + \frac{3√3}{8}(a + 2A_{30})x_1x_2^2 − \frac{3√3}{8}ax_1^3x_2 \\
&− \frac{3√3}{16}(7A_{14} + 7A_{32} + 22A_{50})x_1^4x_2 \\
&+ \frac{3√3}{16}(A_{14} − 7A_{32})x_1^3x_2^2 + \frac{3√3}{16}(13A_{14} + 13A_{32} + 10A_{50})x_1^2x_2^3 \\
&− \frac{9√3}{16}(3A_{14} − A_{32})x_1x_2^4 + \frac{3√3}{32}(A_{14} + A_{32} + 10A_{50})x_1x_2^5 \\
&− \frac{9√3}{32}(A_{14} + A_{32})x_2^6.
\end{aligned}
$$

(3.4)
Let \( z = x_1 + ix_2, \bar{z} = x_1 - ix_2; \) then
\[
\dot{z} = \frac{1}{32} \left( 32i\sqrt{3}bz + \bar{z}(12i(i + \sqrt{3})\bar{z}^2 + 24i\sqrt{3}A_{30}\bar{z}^2 \\
+ \bar{z}(2i\sqrt{3}A_{30}(10z^3 + \bar{z}^3) + A_{14}((-12 + 8i\sqrt{3})z^3 + (-3 - i\sqrt{3})\bar{z}^3) \\
+ A_{32}((-6 + 8i\sqrt{3})z^3 + (3 - i\sqrt{3})\bar{z}^3)) \right).
\]

Hence from the formula of [1, p. 252], [6], using software 4.1, the first and second focus values of the origin for (3.2) are given by
\[
\Delta_3(a, b) = -\frac{\sqrt{3}a\pi}{4b}, \quad \Delta_5(a, b) = \frac{\sqrt{3}(3a^2 + 6aA_{30} - 4b(2A_{14} + A_{32}))\pi}{32b^2}.
\]

We have
\[
\Delta_3(0, b_k) = 0, \quad \Delta_5(0, b_k) = -\frac{3\pi}{8b_k}(2A_{14} + A_{32}) \neq 0.
\]

The Hopf bifurcation theorem for ODE implies that (3.3) has one limit cycle \( L(a) \) when 
\( a(2A_{14} + A_{32}) < 0 \) and \( |a| + |b - b_k| \) small. Let \( x_1 = r \cos \theta, x_2 = r \sin \theta. \) Then (3.4) becomes
\[
\begin{align*}
\frac{dr}{d\theta} &= -\frac{1}{2}a r^3 - \frac{1}{3}B_1(\theta)r^5, \\
\frac{d\theta}{d\theta} &= \sqrt{3}b + \frac{1}{4}\sqrt{3}A_{30}r^2 + B_2(\theta)r^4,
\end{align*}
\]

where
\[
\begin{align*}
B_1(\theta) &= 12A_{14} + 6A_{32} + 3(A_{14} - A_{32}) \cos 6\theta + \sqrt{3}(A_{14} + A_{32} - 2A_{30}) \sin 6\theta, \\
B_2(\theta) &= \frac{\sqrt{3}}{32}(4A_{32} + 10A_{30} - (A_{14} + A_{32} + 2A_{30}) \cos 6\theta \\
& \quad + (\sqrt{3}A_{14} - \sqrt{3}A_{32}) \sin 6\theta).
\end{align*}
\]

Hence
\[
\frac{dr}{d\theta} = -\frac{(3/8)ar^3 - (1/32)B_1(\theta)r^5}{\sqrt{3}b + (3/4)\sqrt{3}A_{30}r^2 + B_2(\theta)r^4}
\]
\[
= -\frac{12ar^3 + B_1(\theta)r^5}{32\sqrt{3}b} \left( 1 - \frac{3A_{30}r^2}{4b} \right) + O(r^7)
\]
\[
= -\frac{\sqrt{3}ar^3}{8b} + \left( \frac{9aA_{30}}{32\sqrt{3}b^2} - \frac{B_1(\theta)}{32\sqrt{3}b} \right) r^5 + O(r^7)
\]
\[
= A(a, b) r^3 + B(a, b, \theta) r^5 + O(r^7),
\]

where
\[
A(a, b) = -\frac{\sqrt{3}a}{8b}, \quad B(a, b, \theta) = \frac{9aA_{30}}{32\sqrt{3}b^2} - \frac{B_1(\theta)}{32\sqrt{3}b}.
\]
Let
\[ r(\theta, \theta_0) = r_1(\theta) r_0 + r_2(\theta) r_0^2 + r_3(\theta) r_0^3 + r_4(\theta) r_0^4 + r_5(\theta) r_0^5 + O(r_0^6) \]
be a solution of (3.8) satisfying \( r(0, r_0) = r_0 \). Then \( r_1(0) = 1, r_i(0) = 0 \) for \( i \geq 2 \). Inserting the above into (3.8) we obtain
\[
\begin{aligned}
r_1'(\theta) r_0 + r_2'(\theta) r_0^2 + r_3'(\theta) r_0^3 + r_4'(\theta) r_0^4 + r_5'(\theta) r_0^5 + O(r_0^6) \\
= A(a, b)(r_1^3(\theta) r_0^3 + 3 r_2^2(\theta) r_0^2 + 3 r_3(\theta) r_0^3) + 3 r_1(\theta) r_2^2(\theta) r_0^5 \\
+ B(a, b, \theta) r_1^3(\theta) r_0^5 + O(r_0^6) \\
= A(a, b) r_1^3(\theta) r_0^3 + 3 A(a, b) r_1^2(\theta) r_2(\theta) r_0^4 \\
+ (3 A(a, b) r_1(\theta) r_2^2(\theta) + 3 A(a, b) r_2^3(\theta) r_3(\theta) + B(a, b, \theta) r_1^3(\theta) r_0^5 + O(r_0^6).
\end{aligned}
\]
Thus \( r_1'(\theta) = 0, r_2'(\theta) = 0, r_3'(\theta) = A(a, b) r_1^3(\theta), r_4'(\theta) = 3 A(a, b) r_1^2(\theta) r_2(\theta), \) and \( r_5'(\theta) = 3 A(a, b) r_1(\theta) r_2^2(\theta) + B(a, b, \theta) r_1^3(\theta) + 3 A(a, b) r_2^3(\theta) r_3(\theta) \). Hence
\[
\begin{aligned}
r_1(\theta) = 1, \\
r_2(\theta) = 0, \\
r_3(\theta) = A(a, b) \theta, \\
r_4(\theta) = 0, \\
r_5(\theta) = \int_0^\theta B(a, b, \theta) d\theta + \frac{3}{2} A^2(a, b) \theta^2 = \tilde{B}(a, b, \theta) + \frac{9 a^2 \theta^2}{128 b^2},
\end{aligned}
\]
where
\[
\begin{aligned}
\tilde{B}(a, b, \theta) = & \frac{1}{32 \sqrt{3} b^2} \left( 9 a^2 A_{30} \theta - b (6 (2 A_{14} + A_{32}) \theta) + \frac{1}{2} (A_{14} - A_{32}) \sin 6 \theta \\
& - \frac{\sqrt{3}}{6} (A_{14} + A_{32} - 2 A_{30}) (\cos 6 \theta - 1) \right).
\end{aligned}
\]
Hence, the Poincare map \( P(r_0) = r(2\pi, r_0) \) has the form
\[
P(r_0) = r_0 - \frac{\sqrt{3} a \pi}{4 b} r_0^3 + \frac{6 \pi}{32 \sqrt{3} b^2} (3 a^2 A_{30} - 2 b (2 A_{14} + A_{32})) r_0^5 + O(r_0^6).
\]
Near \( r_0 = 0 \) the map has a unique fixed point
\[
r_0^* = \sqrt{\frac{-2a}{2 A_{14} + A_{32}}} (1 + O(|a|)).
\]
It follows that the corresponding periodic solution of (3.7) is
\[
r_0^*(\theta) = \sqrt{\frac{-2a}{2 A_{14} + A_{32}}} (1 + O(|a|)).
\]
Hence
\[
\frac{d \theta}{d t} = \sqrt{b} - \frac{3 \sqrt{3} A_{30} a}{2 (2 A_{14} + A_{32})} + O(a^2).
\]
Then
Thus consider the equation $T(a, b)$. Suppose that $T(a, b) = 6r_1/(6k + 1) = 2\pi/(\sqrt{3}b_k)$. Then we have the following result.

Set

$$\varphi(a) = b_k + \frac{3A_{30a}}{2(2A_{14} + A_{32})} + O(a^2).$$

Then the conclusion is clear by Theorem 2.1. This finishes the proof.

In the following, we consider $F(x, y, z)$ with a parameter $a$. As before, we still assume $F(x, y, z, a) = f(y, z, a) + f(z, x, a)$, where $f(x, y, z, a)$ is odd in $x$ and even in $y$. Let $f$ be a polynomial of degree 5. Then it has the form

$$f(x, y, a) = A_{10}x + A_{12}xy^2 + A_{30}x^3 + A_{32}x^3y^2 + A_{14}xy^4 + A_{50}x^5,$$

where $A_{ij} = A_{ij}(a)$ are continuous functions of $a \in \mathbb{R}^n$, $n \geq 1$. Then

$$F(x, y, z, a) = A_{10}y + A_{10}z + A_{12}(yz^2 + zx^2) + A_{30}(y^3 + z^3) + A_{32}(y^3z^2 + z^3x^2) + A_{14}(yz^4 + zx^4) + A_{50}(y^5 + z^5).$$

Hence the corresponding ordinary equations are

\[
\begin{align*}
\dot{x} &= A_{10}(y + z) + A_{12}(yz^2 + zx^2) + A_{30}(y^3 + z^3) + A_{32}(y^3z^2 + z^3x^2) + A_{14}(yz^4 + zx^4) + A_{50}(y^5 + z^5), \\
\dot{y} &= A_{10}(z - x) + A_{12}(zx^2 - xy^2) + A_{30}(z^3 - x^3) + A_{32}(z^3x^2 - x^3y^2) + A_{14}(zx^4 - xy^4) + A_{50}(z^5 - x^5), \\
\dot{z} &= -A_{10}(x + y) + A_{12}(-xy^2 - yz^2) + A_{30}(-x^3 - y^3) + A_{32}(-x^3y^2 - y^3z^2) + A_{14}(-xy^4 - yz^4) + A_{50}(-x^5 - y^5). 
\end{align*}
\]

Then we have the following result.

**Theorem 3.2.** Suppose that

(i) There exists an open subset $I_0 \subset I$ such that $2A_{14}(a) + A_{32}(a) = A_{12}(a) \equiv 0$ for $\forall a \in I_0$;
There exists \( a_k \in I_0 \) such that \( A_{10}(a_k) > 0 \), \( 2\pi / (\sqrt{3} A_{10}(a_k)) = T_k \), where \( k \geq 0 \) is an integer and \( T_k = 6r_i / (6k_i + i) \) (\( i = 1, 2 \)).

Then we have

1. If

\[
A_{30}(a_k) \neq 0 \quad \text{and} \quad \left( A_{10}(a_k) - |A_{10}(a)| \right) A_{30}(a) > 0, \quad \forall a \in I_0,
\]

then (3.2) has a Hopf bifurcation at \( a = a_k \) and has a periodic solution with period \( T_k \) for \( a \in I_0 \) near \( a_k \).

2. If \( A_{30}(a_k) = 0 \) and \( 18A_{30}^2 + A_{10}(3A_{30}^2 + 8A_{14} - 20A_{50}) \equiv B_k \neq 0 \) for \( a_k \in I_0 \), then there exists

\[
\Delta^*(k, a) = B_k \left( \frac{2\pi}{\sqrt{3}} \left( \frac{A_{10}(a_k) - A_{10}(a)}{A_{10}(a)A_{10}(a_k)} \right) - O(A_{30}^2) \right)
\]

such that for \( \Delta^*(k, a) < 0 \) (\( = 0, > 0 \)) and a near \( a_k \), (3.2) has two periodic solutions (a periodic solution of saddle-node type, no periodic solution) of period \( T_k \).

**Proof.** Note that \( A_{10}(a_k) > 0 \), it is easy to know there exist \( \gamma > 0 \) small such that \( F(0, \gamma, a) = 2A_{10} + O(\gamma^3) > 0 \). Insert \( y = z - x \) to (3.10). We still introduce the transformation \( x_1 = y, x_2 = (2x - y) / \sqrt{3} \), noting (i), we have

\[
\begin{aligned}
\dot{x}_1 &= -\sqrt{3} A_{10} x_2 - \frac{3\sqrt{3}}{4} A_{30} x_1^2 x_2 + \frac{3\sqrt{3}}{4} A_{30} x_1^3 - \frac{9\sqrt{3}}{32} A_{14} x_1^5 \\
&\quad + \frac{\sqrt{3}}{32} (13A_{14} - 10A_{50}) x_1^4 x_2 + \frac{45}{16} A_{14} x_1^3 x_2^2 + \frac{\sqrt{3}}{16} (A_{14} - 10A_{50}) x_1^7 x_2^3 \\
&\quad - \frac{45}{32} A_{14} x_1^5 x_2^4 + \frac{9\sqrt{3}}{32} (A_{14} - 2A_{50}) x_2^5 \\
\dot{x}_2 &= \sqrt{3} A_{10} x_1 + \frac{3\sqrt{3}}{4} A_{30} x_1 x_1^2 + \frac{3\sqrt{3}}{4} A_{30} x_1 x_2^2 + \frac{\sqrt{3}}{32} (-7A_{14} + 22A_{50}) x_1^5 \\
&\quad + \frac{45}{32} A_{14} x_1^4 x_2 + \frac{\sqrt{3}}{32} (-13A_{14} + 10A_{50}) x_1^3 x_2^2 - \frac{45}{16} A_{14} x_1^3 x_2^3 \\
&\quad - \frac{\sqrt{3}}{32} (A_{14} - 10A_{50}) x_1 x_2^5 + \frac{9\sqrt{3}}{32} A_{14} x_1^5.
\end{aligned}
\]

It is easy to see the above system has a family of periodic orbits \( L(a, h) \) given by

\[
H(x_1, x_2, a) = h \quad \text{for} \quad a \in I_0, \quad 0 < h \ll 1, \quad \text{where} \quad H \quad \text{has the form}
\]

\[
\begin{aligned}
H(x_1, x_2, a) &= \frac{\sqrt{3}}{2} A_{10} (x_1^3 + x_2^3) + \frac{\sqrt{3}}{16} A_{30} (x_1^4 + 2x_1^2 x_2^2 + x_2^4) \\
&\quad + \frac{\sqrt{3}}{64} (22A_{50} - 7A_{14}) x_1^6 + \frac{9\sqrt{3}}{32} A_{14} x_1^5 x_2 \\
&\quad + \frac{\sqrt{3}}{64} (-13A_{14} + 10A_{50}) x_1^4 x_2^2 - \frac{15}{16} A_{14} x_1^3 x_2^3 \\
&\quad + \frac{3\sqrt{3}}{64} (A_{14} - 10A_{50}) x_1^2 x_2^4 - \frac{9\sqrt{3}}{32} A_{14} x_1 x_2^5 + \frac{3\sqrt{3}}{64} (A_{14} - 2A_{50}) x_2^6.
\end{aligned}
\]
Let \( (x_1(t, a, h), x_2(t, a, h)) \) be a representation of \( L(a, h) \) with period \( T(a, h) \) on the \((x_1, x_2)\) plane. It also can be represented in polar coordinate, \( x_1 = r \cos \theta, \ x_2 = r \sin \theta \).

Substitute them into \( H(r \cos \theta, r \sin \theta, a) = h \); then we can obtain

\[
r^2 = \frac{2h}{\sqrt{3}A_{10}} - \frac{A_{30}h^2}{6A_{10}^2} + O(h^3).
\]

Then we can compute \( T(a, h) \) as follows:

\[
T(a, h) = \left. \frac{\frac{d\theta}{T(a, h)}}{2\pi} \right|_{0}^{2\pi} \int_{0}^{2\pi} \frac{d\theta}{\Delta(\theta) \sqrt{3}A_{10} + \Delta_1(\theta) r^2 + \Delta_2(\theta) r^4 + o(h)}
\]

\[
= \frac{1}{\sqrt{3}A_{10}} \left( 2\pi - \int_{0}^{2\pi} \left( \frac{\Delta(\theta)}{A_{10}} - \frac{\Delta_1(\theta)^2}{A_{10}^2} + o(r^6) \right) d\theta \right)
\]

\[
= \frac{2\pi}{\sqrt{3}A_{10}} \left( 1 - \frac{\sqrt{3}A_{30}}{2A_{10}^2} h + \frac{18A_{30}^2 + A_{10}(3A_{30}^2 + 8A_{14} - 20A_{50})}{24A_{10}^3} h^2 \right),
\]

where \( \Delta(\theta) = \Delta_1(\cos \theta, \sin \theta) + \Delta_2(\cos \theta, \sin \theta), \ \Delta_1(\cos \theta, \sin \theta) = (3/4)\sqrt{3}A_{30}r^2 \), and

\[
\Delta_2(\cos \theta, \sin \theta) = \frac{\sqrt{2}}{32} \left( 4(A_{32} + 5A_{50}) + 3(A_{14} - A_{32}) \sin 6\theta \right).
\]

Consider the equation \( T(a, h) = T_k \). If \( A_{30}(a_k) \neq 0 \), by (3.11) and (3.12), then

\[
\frac{\sqrt{3}A_{30}}{2A_{10}^2} h - o(h) = \frac{A_{10}(a_k) - A_{10}(a)}{A_{10}(a_k)}.
\]

The implicit function theorem implies that the above equation has a solution

\[
h = h_k(a) = \frac{2A_{10}^2(a)(A_{10}(a_k) - A_{10}(a))}{\sqrt{3}A_{10}(a_k)A_{30}(a)} + o(|a - a_k|).
\]

Note that (3.11), then there exist \( h_k(a) > 0 \) for \( a \in I_0 \) near \( a_k \). That is \( x_{1k}(t, a) = x_1(t, a, h_k(a)), \ x_{2k}(t, a) = x_2(t, a, h_k(a)) \) is the periodic solution of (3.2) with period \( T_k(a, h_k(a)) \) for \( a \in I \) near \( a_k \). Hence \( T(a, h) = T_k \) for \( |a - a_k| \ll 0 \) and \( a_k \) is a Hopf bifurcation value.

If \( A_{30}(a_k) = 0 \) note that (2.12) and \( B_k \neq 0 \), it is easy to see that

\[
T_k'(a, h) = \frac{2\pi}{\sqrt{3}A_{10}} \times \left( \frac{-\sqrt{3}A_{30}}{2A_{10}} + \frac{18A_{30}^2 + A_{10}(3A_{30}^2 + 8A_{14} - 20A_{50})}{12A_{10}^3} h + O(h^2) \right).
\]
Let $G(a, h) = T'_h(a, h)$. Then $G(a_k, 0) = 0$ and $G'_h(a_k, 0) \neq 0$. The implicit function theorem implies that

$$h_k = h_k(a) = \frac{6\sqrt{3}A_{10}^2 A_{30}}{18A_{30}^2 + A_{10}(3A_{30}^2 + 8A_{14} - 20A_{50})}.$$

Let $C_k(a)$ be the maximal value of $T(a, h)$. Then

$$C_k(a) = \frac{2\pi}{\sqrt{3}A_{10}}(1 - O(A_{30}^2)).$$

Set

$$\Delta(a, k) = C_k(a) - T_k = 2\pi \left( \frac{A_{10}(a_k) - A_{10}(a)}{A_{10}(a)A_{10}(a_k)} \right) - O(A_{30}^2).$$

Let $\Delta^*(k, a) = B_k A_k(10)(a - a_k)$. Then the conclusion of (2) is obvious by Theorem 2.1. If we take cases one, two, and three as case of $\Delta^*(k, a) < 0$, $\Delta^*(k, a) = 0$, and $\Delta^*(k, a) > 0$, respectively. Then the graph of the function $T = T(a, h)$ of the three cases on the plane $(h, T)$ are as shown in Fig. 3.

**Corollary.** Assume all conditions of Theorem 3.2 hold. We also suppose $A_{30}(a_k) = 0$, $18A_{30}^2 + A_{10}(3A_{30}^2 + 8A_{14} - 20A_{50}) \neq 0$ for $a_k \in I_0$, and $A_{10}(a)$ are continuous and differentiable in $a$. If $A'_{10}(a) \neq 0$, then there exist $\delta^*(k, a) = B_k A'_{10}(a)(a - a_k)$ such that for $\delta^*(k, a) < 0$ ($= 0$, $> 0$) and a near $a_k$, (3.2) has two periodic solutions (a periodic solution of saddle-node type, no periodic solution) of period $T_k$.

**Proof.** From the proof of (2) in Theorem 3.2, it is easy to know that

$$\Delta^*(k, a) = \frac{2\pi B_k}{\sqrt{3}} \left( \frac{1}{A_{10}(a)} - \frac{1}{A_{10}(a_k)} \right) = \frac{2\pi B_k}{\sqrt{3}} \left( \frac{A_{10}(a_k)(a_k - a)}{(A_{10}(a_k))^2} + |a - a_k|^2 \right).$$

Let $\delta^*(k, a) = B_k A'_{10}(a_k)(a_k - a)$. Then the conclusion is obvious.
References

[10] J. Kaplan, J. Yorke, On the nonlinear differential delay equation $\dot{x}(t) = -f(x(t), x(t - 1))$, J. Differential Equations 23 (1977) 293–314.