Every divisor class of Krull monoid domains contains a prime ideal

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Let $D$ be an integral domain, $\Gamma$ be a torsion-free grading monoid, and $D[\Gamma]$ be the monoid domain of $\Gamma$ over $D$. Suppose that $D[\Gamma]$ is a Krull domain, and let $\text{Cl}(D[\Gamma])$ be the divisor class group of $D[\Gamma]$. We show that every divisor class of $D[\Gamma]$ contains a prime ideal. As a corollary, we have that $D[\Gamma]$ is a half-factorial domain if and only if $|\text{Cl}(D[\Gamma])| \leq 2$; hence in this case, either $D$ or $\Gamma$ is factorial. We also show that if $T$ is the set of non-homogeneous prime elements of $D[\Gamma]$, then $D[\Gamma]_T$ is a $\pi$-domain with $\text{Cl}(D[\Gamma]) = \text{Cl}(D[\Gamma]_T)$.

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0. Introduction

Let $D$ be an integral domain with quotient field $K$, $\Gamma$ be a nontrivial torsion-free grading monoid with quotient group $G$ and $D[\Gamma]$ be the monoid domain of $\Gamma$ over $D$ (hence, $D[\Gamma]$ is an integral domain).

Let $A$ be an index set of given cardinality, and let $G_0 = \sum_{i \in A} \mathbb{Z}_i$, where each $\mathbb{Z}_i$ is the additive group of integers. Let $F$ be a field and $X_i, Y_i, T_i, U_i$ be indeterminates over $F$ with $X_i U_i = Y_i T_i$, and let $A = F[[X_i, Y_i, T_i, U_i]_{i \in A}]$. Claborn showed that $A$ is a Krull domain with $\text{Cl}(A) = G_0$. Also, he showed that each divisor class of $A[X]$, the polynomial ring over $A$, contains a prime ideal. Hence for
a subgroup $H$ of $G_0$, if we set $R = \bigcap \{A[X]_0 \mid Q \text{ is a height-one prime ideal of } A[X] \text{ and } cl(Q) \notin H\}$, then $R$ is a Krull domain with $cl(R) = G_0/H$. Note that each abelian group is of the form $G_0/H$ for some free abelian group $G_0$ and its subgroup $H$. Thus, every abelian group is the divisor class group of a Krull domain [3, Propositions 4, 5 and 6].

The ring $D$ is called a half-factorial domain (HFD) if for every nonzero $d \in D$, any two factorizations of $d$ into irreducible factors have the same number of terms. Zaks showed that if $D$ is a Krull domain, then $D[X]$ is an HFD if and only if $|cl(D[X])| \leq 2$ [15, Theorem 2.4]. The proof depends on the fact that each divisor class of $D[X]$ contains a prime ideal. Anderson–Anderson proved that if each divisor class of a Krull domain $D$ contains a prime ideal, then $D$ is an HFD if and only if $|cl(D)| \leq 2$ [1, Corollary 2.3(c)]. Kim showed that if $D[\Gamma]$ is a Krull domain, where either $D$ is a factorial domain but not a field or $\Gamma$ is a group, then each nonzero divisor class of $D[\Gamma]$ contains a prime ideal [12, Theorems 7 and 11].

The purpose of this paper is to show that every divisor class of Krull monoid domains contains a prime ideal. More precisely, in Section 1, we show that $D[\Gamma]$ is integrally closed and $G$ is of type $(0, 0, 0, \ldots)$ if and only if $fK[G] \cap D[\Gamma] = fA[1/\gamma_1]^{1/\gamma_1}$ for all nonzero $f \in K[G]$ and $G$ is of type $(0, 0, 0, \ldots)$, if and only if each $t$-ideal $A$ of $D[\Gamma]$ is of the form $A = \frac{1}{\gamma_i}JD$ for some nonzero $\gamma_i, g \in K[G]$ and $t$-ideals $I$ and $J$ of $D$ and $\Gamma$, respectively. Let $D[\Gamma]$ be a Krull domain. We show, in Section 2, that each divisor class of $D[\Gamma]$ contains a prime ideal. As a corollary, we have that $D[\Gamma]$ is an HFD if and only if $|cl(D[\Gamma])| \leq 2$; hence in this case, either $D$ or $\Gamma$ is factorial. Let $T$ be the set of non-homogeneous prime elements of $D[\Gamma]$. Finally, in Section 3, we show that $D[\Gamma]_1$ is a $\pi$-domain and $cl(D[\Gamma]_1) = cl(D[\Gamma])$.

0.1. Definitions related to the $t$-operation

Let $F(D)$ be the set of nonzero fractional ideals of $D$. For each $l \in F(D)$, let $l^{-1} = \{x \in K \mid xl \subseteq D\}$, $I_l = (l^{-1})^{-1}$, and $I_l = \bigcup \{J \mid J \subseteq I$ and $J$ is a nonzero finitely generated ideal$\}$. An $l \in F(D)$ is called a $v$-ideal (resp., $t$-ideal) if $I_v = I$ (resp., $I_t = I$), while a $t$-ideal $P$ of $D$ is a maximal $t$-ideal if $P$ is maximal among proper integral $t$-ideals of $D$. It is well known that a prime ideal minimal over a $t$-ideal is a $t$-ideal; a maximal $t$-ideal is a prime ideal; and each proper integral $t$-ideal is contained in a maximal $t$-ideal. An $l \in F(D)$ is said to be $t$-invertible if $(l^{-1})_l = D$; equivalently, $l^{-1} \nsubseteq P$ for all maximal $t$-ideals $P$ of $D$. The $(t)$-class group of $D$ is an abelian group $cl(D) = T(D)/\text{Prin}(D)$, where $T(D)$ is the group of $t$-invertible fractional $t$-ideals of $D$ under the $t$-multiplication $I*J = (IJ)_l$ and $\text{Prin}(D)$ is the subgroup of $T(D)$ of nonzero principal fractional ideals. If $D$ is a Krull domain, then $cl(D)$ is just the usual divisor class group of $D$; and if $D$ is a Prüfer domain, then $cl(D)$ is the ideal class group of $D$. We denote by $cl(A)$ the divisor class of $D$ containing a $t$-invertible $t$-ideal $A$. So if $A, B$ are $t$-invertible $t$-ideals, then $cl(A) = cl(B)$ if and only if $A = uB$ for some nonzero $u \in K$.

Let $\Gamma$ be a torsion-free grading monoid with quotient group $G$, and let $D[\Gamma]$ be the semigroup ring of $\Gamma$ over $D$. It is well known that $D[\Gamma]$ is an integral domain [8, Theorem 8.1] and $\Gamma$ admits a total order $<$ compatible with its monoid operation [8, Corollary 3.4]. Hence each $f \in D[\Gamma]$ is uniquely written in the form

$$f = a_0X^{\alpha_0} + a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n},$$

where $a_i \in D$ and $\alpha_j \in \Gamma$ with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. For any $f \in K[G]$, we denote by $A_f$ (resp., $E_f$) the fractional ideal of $D$ (resp., $\Gamma$) generated by the coefficients (resp., exponents) of $f$; hence $A_f = (a_0, a_1, \ldots, a_n)$ and $E_f = (\alpha_0 + \Gamma) \cup (\alpha_1 + \Gamma) \cup \cdots \cup (\alpha_n + \Gamma)$. The torsion-free abelian group $G$ is said to be of type $(0, 0, 0, \ldots)$ if $G$ satisfies the ascending chain condition on cyclic subgroups. As in the domain case, one can define the $v$- and $t$-operation; maximal $t$-ideals; $t$-invertibility; and the $(t)$-class group for $\Gamma$. The reader can refer to [7, §32 and §34] for the $v$- and $t$-operation on integral domains; to [8, §16] or [10] for the $v$- and $t$-operation on monoids; and to [8,10] for monoids and monoid domains.
1. The class semigroup of $D[\Gamma]$

Let $D$ be an integral domain with quotient field $K$ and $\Gamma$ be a torsion-free grading monoid with quotient group $G$. Let $Div(D)$ be the semigroup of $t$-ideals of $D$ under $A \ast B = (AB)_t$. Then $Prin(D)$ is a subgroup of $Div(D)$, and hence $Cl_t(D) = Div(D)/Prin(D)$, called the class semigroup of $D$, is a semigroup. Clearly, $Cl(D)$ is a subgroup of $Cl_t(D)$, and $Cl(D) = Cl_t(D)$ if and only if $D$ is a Krull domain. Similarly, we can define the class semigroup $Cl_t(\Gamma)$ of $\Gamma$.

In this section, we prove that $Cl_t(D[\Gamma]) = Cl_t(D) \oplus Cl_t(\Gamma)$ naturally as monoids if and only if $f K[\Gamma] \cap D[\Gamma] = f A^{-1}_f [E_f^{-1}]$ for all nonzero $f \in D[\Gamma]$ and $G$ is of type $(0, 0, 0, \ldots)$, if and only if $D[\Gamma]$ is integrally closed and $G$ is of type $(0, 0, 0, \ldots)$. This is the combination of Kang's and Kim-Park's results which state that $Cl_t(D) = Cl_t(D[G])$ (resp., $Cl_t(\Gamma) = Cl_t(K[\Gamma])$) naturally as monoids if and only if $f K[\Gamma] \cap D[G] = f A^{-1}_f [G]$ (resp., $f K[\Gamma] \cap K[\Gamma] = f K[E_f^{-1}]$) for all nonzero $f \in K[G]$ and $G$ is of type $(0, 0, 0, \ldots)$ [11, Theorem] (resp., [13, Theorem 5]).

Lemma 1. (See [14, Lemma 1.4].) Let $S$ be a multiplicative subset of $D$ and let $I$ be a nonzero fractional ideal of $D$.

(1) If $I$ is finitely generated, then $(I S^{-1})^{-1} = I^{-1} D_S$ and $(I S)_V = (I V D_S)_V$.

(2) $(I S)_t = (I_t D_S)_t$.

Lemma 2. If $A$ is a nonzero fractional ideal of $D[\Gamma]$, then $A_t = (AD[G])_t \cap (AK[\Gamma])_t$.

Proof. Let $T = D \setminus \{0\}$ and $N = \{X^\alpha \mid \alpha \in \Gamma\}$. Then $K[\Gamma] = D[\Gamma]_T, D[G] = D[\Gamma]_N$, and $D[\Gamma] = D[G] \cap K[\Gamma]$. Hence by Lemma 1, $A_t \subseteq (AD[G])_t \cap (AK[\Gamma])_t$. For the reverse containment, let $f \in (AD[G])_t \cap (AK[\Gamma])_t$. Then there exists a nonzero finitely generated ideal $I$ of $D[\Gamma]$ such that $I \subseteq A$ and $f \in (ID[G])_t \cap (IK[\Gamma])_t$; so by Lemma 1, $f I^{-1} \subseteq f^{-1} D[G] \cap f I^{-1} K[\Gamma] = f (ID[G]^{-1} \cap f (IK[\Gamma]^{-1}) \subseteq D[G] \cap K[\Gamma] = D[\Gamma]$. Hence $f \in I_V \setminus A_t$. Thus, $(AD[G])_t \cap (AK[\Gamma])_t \subseteq A_t$. □

It is known that $D[\Gamma]$ is integrally closed if and only if $D$ and $\Gamma$ are integrally closed [8, Corollary 12.11]. Let $I$ be a nonzero fractional ideal of $D$ and let $J$ be a fractional ideal of $\Gamma$. Then $(I J)^{-1} = I^{-1} J^{-1}$ and $(I J)_t = I_t J_t$ [5, Lemma 2.3]. Hence $I J$ is a $t$-ideal if and only if $I$ and $J$ are $t$-ideals; and $I J$ is $t$-invariant if and only if $I$ and $J$ are $t$-invariant [5, Corollary 2.4].

Let $\varphi : Cl_t(D) \oplus Cl_t(\Gamma) \to Cl_t(D[\Gamma])$ be the map defined by $\varphi (cl(I), cl(J)) = cl(I J)$. Then $\varphi$ is a semigroup homomorphism, and we mean by $Cl_t(D[\Gamma]) = Cl_t(D) \oplus Cl_t(\Gamma)$ that $\varphi$ is an isomorphism. Thus, $Cl_t(D[\Gamma]) = Cl_t(D) \oplus Cl_t(\Gamma)$ if and only if each $t$-ideal of $D[\Gamma]$ is of the form $A = \frac{1}{n} I J$ for some nonzero $h, g \in K[G]$ and $t$-ideals $I$ and $J$ of $D$ and $\Gamma$, respectively. In particular, the equality $Cl_t(D[\Gamma]) = Cl_t(D) \oplus Cl_t(\Gamma)$ implies that $D[\Gamma]$ is integrally closed [5, Proposition 2.2].

Theorem 3. The following statements are equivalent.

(1) $Cl_t(D[\Gamma]) = Cl_t(D) \oplus Cl_t(\Gamma)$.

(2) $f K[G] \cap D[\Gamma] = f A^{-1}_f [E_f^{-1}]$ for all nonzero $f \in K[G]$ and $G$ is of type $(0, 0, 0, \ldots)$.

(3) $f K[G] \cap D[G] = f A^{-1}_f [G]$ and $f K[G] \cap K[\Gamma] = f K[E_f^{-1}]$ for all nonzero $f \in K[G]$ and $G$ is of type $(0, 0, 0, \ldots)$.

(4) $D[\Gamma]$ is integrally closed and $G$ is of type $(0, 0, 0, \ldots)$.

Proof. Let $T = D \setminus \{0\}$ and $N = \{X^\alpha \mid \alpha \in \Gamma\}$; so $D[\Gamma]_T = K[\Gamma]$ and $D[\Gamma]_N = D[G]$.

(1) $\Rightarrow$ (2) Let $f \in K[G]$ be a nonzero element. Then $Q_f = f K[G] \cap D[\Gamma]$ is a $t$-ideal of $D[\Gamma]$. Note that $Q_f \subseteq K[G]$; hence by (1), $Q_f = h I J$ for some nonzero $h \in K[G]$ and $t$-ideals $I$ and $J$ of $D$ and $\Gamma$, respectively. Clearly, $f K[G] = h K[G]$, and hence $h = u X^\alpha f$ for some $u \in K$ and $\alpha \in \Gamma$; so $Q_f = u I J (w + J)$. Obviously, $u I J = A_f^{-1}$ and $\alpha + J = E_f^{-1}$.

Next, assume that $G$ is of type $(0, 0, 0, \ldots)$, and let $(\alpha_1) \subseteq (\alpha_2) \subseteq (\alpha_3) \subseteq \cdots$ be an infinite sequence of cyclic subgroups of $G$. Put $A = \bigcup_{n=1}^\infty (1 - X^{\alpha_n})$, and note that, for each $\alpha_n$, there exists a.
positive integer \( k = k(\alpha_n) \) such that \( \alpha_1 = k\alpha_n \). If \( \alpha_1 = \alpha - \beta \) for \( \alpha, \beta \in \Gamma \), then \( k(\beta + \alpha_n) = k\beta + \alpha_1 = \alpha + (k - 1)\beta \in \Gamma \), and since \( \Gamma \) is integrally closed, \( \beta + \alpha_n \in \Gamma \). Hence \( X^\beta A \subseteq D[\Gamma] \), and thus \( A \) is a fractional ideal of \( D[\Gamma] \). Note that \( (1 - X^{\alpha_1}) \subseteq (1 - X^{\alpha_2}) \subseteq (1 - X^{\alpha_3}) \subseteq \cdots \) and each \( (1 - X^{\alpha_n}) \) is a \( t \)-ideal; so \( A \) is a \( t \)-ideal. However, \( A \) is not of the form \( \frac{h}{g} I[J] \) for all nonzero \( g, h \in K[G] \).

\[
(2) \Rightarrow (3) \quad fK[G] \cap D[G] = (fK[G] \cap D[\Gamma])_N = (fA_{f^{-1}}E_f^{-1})_N = fA_{f^{-1}}[G] \quad \text{and} \quad fK[G] \cap K[\Gamma] = (fK[G] \cap D[\Gamma])_T = (fA_{f^{-1}}E_f^{-1})_T = fK[E_f^{-1}] \quad \text{by (2)}.
\]

\[
(3) \Rightarrow (4) \quad \text{This appears in [11, Theorem] and [13, Theorem 5].}
\]

\[
(4) \Rightarrow (1) \quad \text{Let } A \text{ be a } t \text{-ideal of } D[\Gamma]. \text{ Since } A \text{ is a fractional ideal, there exists a nonzero } k \in D[\Gamma] \text{ such that } kA \subseteq D[\Gamma]. \text{ Hence we may assume that } A \subseteq D[\Gamma] \subseteq K[G]. \text{ Note that } D \text{ and } \Gamma \text{ are integrally closed; so } (AD[G])_T = fI[G] \text{ [11, Lemma 3] and } (AK[\Gamma])_T = gK[J] \text{ [13, Lemma 4] for some } f, g \in K[G] \text{ and t-ideals } I \text{ and } J \text{ of } D \text{ and } \Gamma, \text{ respectively. Hence } fK[G] = ((AD[G])_T)_T = (((AD[G])_N)_T)_T = (((AD[G])_N)_T)_T = ((AK[\Gamma])_T)_N = (AK[\Gamma])_T = (AK[\Gamma])_T = gK[J] \text{ by Lemma 1}; \text{ so } f = uX^\beta g \text{ for some } u \in K \text{ and } \alpha \in G. \text{ Let } h = ug. \text{ Then } (AK[\Gamma])_T = hK[J] = hI[J] = hI[\Gamma] = hI[G] \text{ and } (AD[G])_T = hI[G] = hI[J]D[G]. \text{ Thus by Lemma 2, } A = (AD[G])_T \cap (AK[\Gamma])_T = (hI[J])D[G] \cap (hI[J])K[\Gamma] = hI[J], \text{ because } hI[J] \text{ is a } t \text{-ideal of } D[\Gamma]. \quad \blacksquare
\]

**Corollary 4.** (See [8, Theorem 15.6].) \( D[\Gamma] \) is a Krull domain if and only if \( D \) and \( \Gamma \) are Krull and \( G \) is of type (0, 0, 0, \ldots).

**Proof.** Suppose that \( D[\Gamma] \) is a Krull domain, and let \( I \) (resp., \( J \)) be a \( t \)-ideal of \( D \) (resp., \( \Gamma \)). Then \( I[J] \), and hence \( I \) and \( J \) are \( t \)-invertible. Thus, \( D \) and \( \Gamma \) are Krull. Note that a Krull domain satisfies the ascending chain condition on principal ideals, thus \( G \) is of type (0, 0, 0, \ldots) (cf. the proof of (1) \( \Rightarrow \) (2) in Theorem 3). For the converse, let \( A \) be a \( t \)-ideal of \( D[\Gamma] \). Note that \( D[\Gamma] \) is integrally closed. Hence by Theorem 3, \( A = \frac{h}{g} I[J] \) for some nonzero \( g, h \in K[G] \) and \( t \)-ideals \( I \) and \( J \) of \( D \) and \( \Gamma \), respectively. Since \( I \) and \( J \) are \( t \)-invertible, \( A \) is \( t \)-invertible. Thus, \( D[\Gamma] \) is a Krull domain. \( \quad \blacksquare \)

2. The class group of Krull monoid domains

Throughout \( D \) is an integral domain with quotient field \( K \) and \( \Gamma \) is a torsion-free grading monoid with quotient group \( G \).

In this section, we prove that if \( D[\Gamma] \) is a Krull domain, then each divisor class of \( D[\Gamma] \) contains a prime ideal. Our first result is a generalization of the Eisenstein Criteria whose proof is the same as the usual proof.

**Lemma 5.** Let \( D \) be a factorial domain and \( G \) be a torsion-free abelian group. Let \( f = a_0 + a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n} \in D[G] \) such that \( n \geq 1 \) and \( 0 < \alpha_1 < \cdots < \alpha_n \). Let \( p \in D \) be a prime, and assume \( p \nmid a_0 \), \( p \nmid a_i \) for \( i \leq n - 1 \), \( p^2 \nmid a_0 \), and \( (A_f)_v = D \). Then \( f \) is a prime in \( D[G] \).

**Proof.** Suppose that \( f \) is reducible, and let \( f = gh \) for some \( g, h \in D[\Gamma] \). Since \( (A_f)_v = D \), we can write \( g \) and \( h \) as follows

\[
g = b_0 + b_1X^{\beta_1} + \cdots + b_dX^{\beta_d}
\]

and

\[
h = c_0 + c_1X^{\gamma_1} + \cdots + c_mX^{\gamma_m},
\]

where \( 0 < \beta_1 < \cdots < \beta_d < 0 < \gamma_1 < \cdots < \gamma_m \), \( d, m \geq 1 \) and \( b_d \neq 0 \).

Note that \( pD[G] \) is a prime ideal, and so \( D[G]/pD[G] \cong (D/pD)[G] \) is an integral domain. Note also that in \( (D/pD)[G] \), we have \( \bar{f} = \bar{g} \bar{h} \). Hence \( \bar{a}_1X^{\alpha_n} = (\bar{b}_0 + \bar{b}_1X^{\beta_1} + \cdots + \bar{b}_dX^{\beta_d})(\bar{c}_0 + \bar{c}_1X^{\gamma_1} + \cdots + \bar{c}_mX^{\gamma_m}) \), and thus \( g \) and \( h \) are both homogeneous in \( (D/pD)[G] \). However, since \( p^2 \nmid a_0 = \bar{b}_0 = 0 \), we
have \( p \mid b_0 \) or \( p \mid c_0 \), and hence \( \bar{g} \) or \( \bar{h} \) is not homogeneous, a contradiction. Thus, \( f \) is irreducible in \( D[G] \), and since \( D[G] \) is a GCD-domain \([8, \text{Theorem 14.5}]\), \( f \) is a prime in \( D[G] \). □

Let \( X^1(D) \) denote the set of height-one prime ideals of \( D \). It is well known that if \( D \) is a Krull domain, then \( X^1(D) \) is the set of maximal \( t \)-ideals of \( D \) and also \( D_P \) is a rank-one DVR for all \( P \in X^1(D) \). We denote by \( v_P \) the valuation on \( K \) associated with \( D_P \). We also use the same notations for Krull monoids.

**Lemma 6.** Let \( D \) be a Krull domain that is not a factorial domain, and let \( a, b \in K \) be nonzero elements. Then there exist a nonzero \( c \in D \) and a prime \( P \in X^1(D) \) such that \( (a, b)_P = (a, bc)_P \) and \( bc \notin D_P = PD_P \).

**Proof.** It is clear that if \( |X^1(D)| < \infty \), then \( D \) is a semi-local PID. Hence \( |X^1(D)| = \infty \), because \( D \) is not factorial, and so there exists a prime \( P \in X^1(D) \) such that \( aD_P = bD_P = D_P \). By the approximation theorem for Krull domains \([7, \text{Theorem 44.1}]\), there exists a nonzero \( c \in K \) such that, for \( Q \in X^1(D) \),

\[
v_Q(c) = \begin{cases} 
0 & \text{if } v_Q(a) \neq 0 \text{ or } v_Q(b) \neq 0, \\
1 & \text{if } Q = P 
\end{cases}
\]

and \( v_Q(c) \geq 0 \) if otherwise.

Clearly, \( c \in D \). Next, if \( v_Q(a) = v_Q(b) = 0 \), then \( (a, b)D_Q = D_Q = (a, bc)D_Q \). Also, if \( v_Q(a) \neq 0 \) or \( v_Q(b) \neq 0 \), then \( cD_Q = D_Q \), and hence \( (a, bc)D_Q = (a, b)D_Q \). Thus, \( (a, b)_P = \bigcap_{Q \in X^1(D)} (a, b)D_Q = (a, bc)_P \) \([7, \text{Theorem 44.2}]\). Moreover, \( v_P \left( \frac{bc}{a} \right) = v_P(b) + v_P(c) - v_P(a) = 1 \) or \( \frac{bc}{a} \notin D_P = PD_P \). □

**Lemma 7.** Let \( \Gamma \) be a Krull monoid that is not a factorial monoid, and let \( \alpha_1, \alpha_2 \in G \). Then there exist an \( \alpha \in \Gamma \) and a prime \( P \in X^1(\Gamma) \) such that \((\langle \alpha_1 + \Gamma \rangle \cup \langle \alpha_2 + \Gamma \rangle)_P = ((\alpha_1 + \Gamma) \cup ((\alpha_2 + \alpha) + \Gamma))_P \) and \( v_P(\alpha_2 + \alpha - \alpha_1) = 1 \).

**Proof.** This can be proved in the same way as the proof of Lemma 6 using the approximation theorem for Krull monoids \([10, \text{Theorem 26.4}]\). □

We next give the main result of this paper.

**Theorem 8.** Each divisor class of a Krull domain \( D[\Gamma] \) contains a prime ideal.

**Proof.** Let \( A \) be a \( v \)-ideal of \( D[\Gamma] \). Note that \( G \), the quotient group of \( \Gamma \), is of type \((0, 0, 0, \ldots)\) by Corollary 4. Hence by Theorem 3, \( A = h_1h_2[I/J] \) for some \( h_1, h_2 \in K[G] \) and \( v \)-ideals \( I \) and \( J \) of \( D \) and \( \Gamma \), respectively. Note also that \( I^{-1} \) and \( J^{-1} \) are \( v \)-ideals. Hence \( I^{-1} = (a, b)_V \) and \( J^{-1} = ((\alpha + \Gamma) \cup (\beta + \Gamma))_V \) for some nonzero \( a, b \in K \) and some \( \alpha, \beta \in G \).

**Case 1.** \( \text{cl}(A) = \text{cl}(D[\Gamma]) \). Since \( \Gamma \) is a Krull monoid and \( G \) is of type \((0, 0, 0, \ldots)\), we can choose a nonzero \( \gamma \in \Gamma \) such that \( \gamma = nh \) for any \( h \in \Gamma \) and integer \( n \geq 0 \), then \( \gamma = h \). So if we set \( f = 1 - X^\gamma \), then \( f \) is a prime element of \( K[G] \) \([9, \text{Corollary 7.7}]\). Hence \( Q_f = fK[G] \cap D[\Gamma] = fA_f^{-1}[E_f^{-1}] = fD[\Gamma] \) by Theorem 3, and thus \( Q_f \) is a prime ideal and \( \text{cl}(A) = \text{cl}(Q_f) \).

**Case 2.** \( \text{cl}(A) = \text{cl}(D[J]) \). By Lemma 7, we may assume that \( v_Q(\alpha - \beta) = 1 \) for some prime ideal \( Q \) of \( \Gamma \) with \( \Gamma_Q \) a discrete valuation monoid of rank-one. Let \( g = \alpha - \beta \), and note that if \( g = nh \) for some \( h \in G \) and integer \( n \geq 0 \), then \( 1 = v_Q(g) = v_Q(nh) = nv_Q(h) \). Hence \( n = 1 \). Thus, \( f := 1 - X^g \) is a prime element of \( K[G] \) \([9, \text{Corollary 7.7}]\). Also, note that \( E_f = \Gamma \cup (g + \Gamma) = -\beta + ((\alpha + \Gamma) \cup (\beta + \Gamma)) \). Hence by Theorem 3, \( Q_f = fK[G] \cap D[\Gamma] = fA_f^{-1}[E_f^{-1}] = fD[\beta + ((\alpha + \Gamma) \cup (\beta + \Gamma))^{-1}] = fX^\beta D[J] \). Thus, \( Q_f \) is a prime ideal and \( \text{cl}(A) = \text{cl}(Q_f) \).
Case 3. \( cl(A) = cl(I[\Gamma]) \). By Lemma 6, we may assume that \( \frac{p}{q}D_P = PD_P \) for some prime \( P \in X^1(D) \). Let \( p = \frac{a}{b} \). Note that \( D_P \) is a rank-one DVR, and so \( D_P \) is a factorial domain. Hence if \( \alpha \in \Gamma \) with \( \alpha > 0 \), then \( f := p + \alpha \) is a prime in \( D_P[G] \) by Lemma 5. Note that \( A_f^{-1} = (\frac{p}{q}, 1)^{-1} = b(a, b)^{-1} = bI \) and \( E_f^{-1} = (\Gamma \cup (\alpha + \Gamma))^{-1} = \Gamma \). Hence \( Q_f = fD_P[G] \cap D[\Gamma] = (fK[G] \cap D_P[\Gamma]) \cap D[\Gamma] = fK[G] \cap D[\Gamma] = fA_f^{-1}[E_f^{-1}] = bf(a, b)^{-1}[\Gamma] = bfI[\Gamma] \). Thus, \( Q_f \) is a prime ideal of \( D[\Gamma] \) and \( cl(A) = cl(Q_f) \).

Case 4. \( cl(A) = cl(I[\Gamma]) \). Let \( p \) and \( g \) be as in Cases 2 and 3, and let \( f = p + \alpha \). Then \( f \) is a prime in \( D_P[G] \) by Lemma 5, and so \( f \) is a prime in \( K[G] \). Hence \( Q_f = fK[G] \cap D[\Gamma] = fA_f^{-1}[E_f^{-1}] = bfX^\alpha(a, b)^{-1}[\alpha + \Gamma] = bfX^\alpha[I[\Gamma]] \). Thus, \( Q_f \) is a prime ideal of \( D[\Gamma] \) and \( cl(A) = cl(Q_f) \).

Anderson–Anderson showed that if each divisor class of a Krull domain \( D \) contains a prime ideal, then \( D \) is an HFD if and only if \( |Cl(D)| \leq 2 \) [1, Corollary 2.3(c)]. Also, if \( D[\Gamma] \) is a Krull domain, then \( Cl(D[\Gamma]) = Cl(D) \odot Cl(\Gamma) \); hence by Theorem 8, we have

**Corollary 9.** If \( D[\Gamma] \) is a Krull domain, then \( D[\Gamma] \) is an HFD if and only if \( |Cl(D[\Gamma])| \leq 2 \); hence, in this case, either \( D \) or \( \Gamma \) is factorial.

Let \( U(D) \) be the group of units of \( D \), \( \Delta^* = D - \{0\} \), \( K^* = K - \{0\} \), \( \Delta = \Delta^*/U(D) \) and \( G = K^*/U(D) \). For each \( aU(D), bU(D) \in \Gamma \), define \( aU(D) + bU(D) = abU(D) \). Clearly \( \Gamma \) is a commutative cancellative monoid with quotient group \( G \). Moreover, if \( D \) is integrally closed, then \( \Gamma \) is torsion-free [2, Lemma 1]. Hence \( \Gamma \) and \( G \) are totally ordered. Note that \( G(D) \), the group of divisibility of \( D \), is partially ordered under \( "aU(D) \leq bU(D) \Leftrightarrow \frac{a}{b} \in D^*" \); hence \( G(D) \) is totally ordered if and only if \( D \) is a valuation domain [7, Theorem 16.3]. Thus, if \( D \) is not a valuation domain, then the order of \( G \) is different from that of \( G(D) \).

**Corollary 10.** The following statements are equivalent for a Krull domain \( D \).

1. \( |Cl(D)| \leq 2 \).
2. \( K[\Delta^*/U(D)] \) is an HFD.
3. \( D[K^*/U(D)] \) is an HFD.

**Proof.** Let \( \Gamma = \Delta^*/U(D) \) and \( G = K^*/U(D) \). Then \( \Gamma \) is a Krull monoid with quotient group \( G \) [10, Theorem 23.4]. Note that \( \Gamma \) has a unique unit \( U(D) \), and so \( G \) is of type \( (0, 0, 0, \ldots) \). Hence \( D[G] \) and \( K[\Gamma] \) are Krull domains (Corollary 4) and \( Cl(D[G]) = Cl(K[\Gamma]) = Cl(D) \) [2, Theorem 4]. Thus, the result is an immediate consequence of Corollary 9.

**3. \( \pi \)-Domain overrings of Krull monoid domains**

An integral domain \( D \) is called a \( \pi \)-domain if each nonzero ideal of \( D \) is a finite product of prime ideals. It is known that \( D \) is a \( \pi \)-domain if and only if \( D \) is a Krull domain and each minimal prime ideal of \( D \) is invertible [7, Theorem 46.7].

Let \( D[\Gamma] \) be a Krull domain, and let \( T \) be the set of non-homogeneous prime elements of \( D[\Gamma] \). In this section, we show that \( D[\Gamma]_T \) is a \( \pi \)-domain and that if \( \Gamma \) is a factorial monoid with \( \Gamma \cap (\neg \Gamma) = \{0\} \), then \( D[\Gamma]_T \) is a Dedekind domain.

**Lemma 11.** Let \( D \) be a Krull domain that is not a factorial domain. If \( I \) is a nonzero ideal of \( D \) with \( I_D = D \), then \( I[\Gamma] \) contains a non-homogeneous prime element.

**Proof.** Since \( D \) is a Krull domain, there exist some nonzero \( a, b \in I \) such that \( (a, b)_D = D \). Also, by Lemma 6, we may assume that there exists a prime \( P \in X^1(D) \) such that \( \frac{b}{a}D_P = PD_P \). For any \( \alpha \in \Gamma \) with \( \alpha > 0 \), put \( f = b + aX^\alpha \). Clearly, \( f \in (a, b)D[\Gamma] \subseteq I[\Gamma] \). \( A_f^{-1} = D \) and \( E_f^{-1} = \Gamma \). Note that \( \frac{b}{a} \)
is a prime element of \( D_P \); so by Lemma 5, \( \frac{b}{a} + X^a \) is a prime in \( D_P[G] \) (hence, in \( K[G] \)). Hence 
\[
\left( \frac{b}{a} + X^a \right) K[G] \cap D[\Gamma] = f K[G] \cap D[\Gamma] = f A_f^{-1}[E_f^{-1}] = f D[\Gamma].
\]
Thus, \( f \) is a non-homogeneous prime element of \( D[\Gamma] \). □

**Lemma 12.** Let \( \Gamma \) be a Krull monoid that is not a factorial monoid. If \( f \) is an ideal of \( \Gamma \) with \( f \Gamma = \Gamma \), then \( D[f] \) contains a non-homogeneous prime element.

**Proof.** This can be proved in the same way as the proof of Lemma 11 using Lemma 7 and [9, Corollary 7.7]. (Cf. The proof of Case 2 in the proof of Theorem 8.) □

**Theorem 13.** Let \( D[\Gamma] \) be a Krull domain, and let \( T \) be the set of non-\text{homogeneous} prime elements of \( D[\Gamma] \). Then \( D[\Gamma]_T \) is a \( \pi \)-domain and \( \text{Cl}(D[\Gamma]) = \text{Cl}(D[\Gamma]_T) \).

**Proof.** By the Nagata’s theorem [6, Corollary 7.3], \( \text{Cl}(D[\Gamma]) = \text{Cl}(D[\Gamma]_T) \). So it suffices to show that each height-one prime ideal of \( D[\Gamma]_T \) is invertible [7, Theorem 46.7]. Note that each height-one prime ideal \( Q \) of \( D[\Gamma]_T \) is of the form \( P[\Gamma]_T \), \( D[S]_T \) or \((f A_f^{-1}[E_f^{-1}])_T\), where \( P \in X^1(D), S \in X^1(\Gamma), \) and \( f \in D[\Gamma] \).

Case 1. \( Q = P[\Gamma]_T \). Clearly \( P \subseteq P P^{-1} \), and hence \((PP^{-1})_T = D[\Gamma]_T \). Next, if \( P \) is factorial, then \( PP^{-1} = D \), and hence \((PP^{-1})[\Gamma]_T = D[\Gamma]_T \). By Lemma 11, \((PP^{-1})[\Gamma]_T \) contains a non-\text{homogeneous} prime element, and so \((PP^{-1})[\Gamma]_T = D[\Gamma]_T \). Hence \( D[\Gamma]_T = (PP^{-1})[\Gamma]_T = (P[\Gamma]_T)(PP^{-1}[\Gamma]_T) \subseteq \{ Q Q^{-1} \subseteq D[\Gamma]_T \}. \) Thus, \( Q Q^{-1} = D[\Gamma]_T \).

Case 2. \( Q = D[S]_T \). By Lemma 12 and using the same argument as in Case 1, we have \( Q Q^{-1} = D[\Gamma]_T \).

Case 3. \( Q = (f A_f^{-1}[E_f^{-1}])_T \). If \( A_f^{-1} \) is principal, then \( A_f^{-1}[E_f^{-1}]_T \) is invertible. Next if \( A_f^{-1} \) is not principal, then \( D \) is not factorial, and since \((A_f A_f^{-1})_T = D \), we have \((A_f A_f^{-1})[\Gamma]_T = (A_f A_f^{-1})_T = D[\Gamma]_T \). A similar argument using Lemma 12 shows \((D[E_f]_T)(D[E_f^{-1}]_T) = D[E_f + E_f^{-1}]_T = D[\Gamma]_T \). Note that \( Q = (f D[\Gamma]_T)(A_f^{-1} D[\Gamma]_T)(D[\Gamma]^{-1}_T) \). Thus \( Q \) is invertible. □

The next result was proved by Claborn when \( \{X_a\} \) is an infinite set [3, Proof of Theorem 7] and by Costa, Gallardo and Querre when \( \{X_a\} \) is a singleton set and \( D \) has infinitely many height-one prime ideals [4].

**Proposition 14.** Let \( \{X_a\} \) be a nonempty set of indeterminates over a Krull domain \( D \), and let \( T \) be the set of prime polynomials of degree \( \geq 1 \). Then \( D[\{X_a\}]_T \) is a Dedekind domain with \( \text{Cl}(D) = \text{Cl}(D[\{X_a\}]_T) \).

**Proof.** The equality \( \text{Cl}(D) = \text{Cl}(D[\{X_a\}]_T) \) is an immediate consequence of the Nagata’s theorem [6, Corollary 7.3]. Next, let \( Q \) be a prime ideal of \( D[\{X_a\}] \) with \( \text{ht} Q \geq 2 \). It suffices to show that \( Q \cap T \neq \emptyset \). Choose \( x \in \{X_a\} \), and let \( Y = \{X_a\} - \{x\} \); hence \( D[\{X_a\}] = D[Y][x] \) and \( Q \cap D[Y] \neq \emptyset \). Let \( p \in Q \cap D[Y] \) be a nonzero element, and choose \( h = a_0 + a_1 x + \cdots + a_n x^n \in Q \), where \( a_i \in D[Y], \) such that \( (p, h)_V = D[\{X_a\}] \). Note that \( x \in T \) and \( h \neq 0 \). So we assume that \( x \notin Q \) and \( a_0 \neq 0 \).

Case 1. \( D[Y] \) is a factorial domain. Then we can assume that \( p \) is a prime in \( D[Y] \). Let \( f \in Q \) be a prime factor of \( h + px^{n+1} \). Note that \( p \) does not divide at least one of the \( a_i \) in \( D[Y] \); so \( p \mid h + px^{n+1} \) in \( D[Y][x] \). Hence the degree of \( f \) in \( x \) is greater than or equal to 1, and thus \( f \in Q \cap T \).

Case 2. \( D[Y] \) is not a factorial domain. Then \( \{X^1(D[Y])\} = \infty \), and hence there exist a prime \( p \in X^1(D[Y]) \) and \( q \in D[Y] \) such that \( p, a_0 \notin P \), \( (p, q) D[Y]_V = D[Y], \) and \( q D[Y]_P = PD[Y]_P \). So if we set \( g = qh + px^{n+1} \), then \( g \) is a prime in \( D(Y)_P[x, x^{-1}] \) by Lemma 5. Note that \((q a_0, \ldots, q a_n, p) D(Y)_V = D[Y], \) because \((p, q) D[Y]_V = D[Y] \) and \( (p, h)_V = D[\{X_a\}] \). Hence if \( F \) is the quotient field of \( D[Y] \), then \( g F[x, x^{-1}] \cap D[Y][x] = g A_g^{-1}[E_g^{-1}] = g D[Y][x] \) by Theorem 3, and so \( g \) is a prime in \( D[Y][x] \). Thus \( g \in Q \cap T. \) □
It is known that if $\Gamma$ is a factorial monoid with $\Gamma \cap (-\Gamma) = \{0\}$, then $\Gamma$ is the sum of copies of $\mathbb{Z}^+$, the additive semigroup of nonnegative integers. Hence $D[\Gamma] \cong D[\{X_\alpha\}]$ for some indeterminates $\{X_\alpha\}$, and thus $D[\Gamma]_T$ is a Dedekind domain by Proposition 14. However, the proof of Theorem 13 does not show if the ring $D[\Gamma]_T$ is a Dedekind domain for a Krull monoid $\Gamma$.

**Question 15.** Let $D[\Gamma]$ be a Krull domain, and let $T$ be the set of non-homogeneous prime elements of $D[\Gamma]$. Is $D[\Gamma]_T$ a Dedekind domain?

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