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Operations preserving the global rigidity of graphs and frameworks in the plane

Tibor Jordán *,[1], Zoltán Szabadka [2]

Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary

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ABSTRACT

A straight-line realization of (or a bar-and-joint framework on) graph G in $\mathbb{R}^d$ is said to be globally rigid if it is congruent to every other realization of G with the same edge lengths. A graph G is called globally rigid in $\mathbb{R}^d$ if every generic realization of G is globally rigid. We give an algorithm for constructing a globally rigid realization of globally rigid graphs in $\mathbb{R}^2$. If G is triangle-reducible, which is a subfamily of globally rigid graphs that includes Cauchy graphs as well as Grünbaum graphs, the constructed realization will also be infinitesimally rigid.

Our algorithm relies on the inductive construction of globally rigid graphs which uses edge additions and one of the Henneberg operations. We also show that vertex splitting, which is another well-known operation in combinatorial rigidity, preserves global rigidity in $\mathbb{R}^2$.

1. Introduction

We shall consider finite graphs without loops, multiple edges or isolated vertices. A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Two frameworks (G, p) and (G, q) are *equivalent* if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ holds for all pairs u, v with $uv \in E$, where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ holds for all pairs u, v with $u, v \in V$. This is the same as saying that (G, q) can be obtained from (G, p) by an isometry of $\mathbb{R}^d$.

We say that (G, p) is *rigid* if there exists an $\epsilon > 0$ such that if (G, q) is equivalent to (G, p) and $\|p(v) - q(v)\| < \epsilon$ for all $v \in V$ then (G, q) is congruent to (G, p). Intuitively, this means that if we think of a d-dimensional framework (G, p) as a collection of bars and joints where points correspond to joints and each edge to a rigid bar joining its end-points, then the framework is rigid if it has no non-trivial continuous deformations (see also [8,19]). The framework (G, p) is called *globally rigid* if every framework (G, q) which is equivalent to (G, p) is congruent to (G, p).

It seems to be a hard problem to decide if a given framework is rigid or globally rigid. Indeed Saxe [15] has shown that it is NP-hard to decide if even a 1-dimensional framework is globally rigid. These problems become more tractable, however, if we assume that there are no algebraic dependencies between the coordinates of the points of the framework. A framework (G, p) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. It is known [19] that rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G, p)

* Corresponding author.
E-mail addresses: jordan@cs.elte.hu (T. Jordán), sinus@cs.elte.hu (Z. Szabadka).

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depends only on the graph G and not the particular realization p, if (G, p) is generic. We say that the graph G is *rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid. The characterization of rigid graphs in $\mathbb{R}^d$ is known only for $d \leqslant 2$, see [14]. Similarly, we say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every generic realization of G in $\mathbb{R}^d$ is globally rigid. The characterization of globally rigid graphs in $\mathbb{R}^d$ (and the fact that global rigidity is a generic property) is known only for $d \leqslant 2$. The 2-dimensional characterization was recently completed by Jackson and Jordán [12], relying on previous results of Hendrickson [10] and Connelly [5]. We say that G is *redundantly rigid* if $G - e$ is rigid for all edges e of G.

Theorem 1.1. *(See [5,12].) Let G be a graph. Then G is globally rigid in $\mathbb{R}^2$ if and only if either G is a complete graph on two or three vertices, or G is 3-connected and redundantly rigid.*

The 1-*extension* operation (which is one of the two well-known Henneberg operations [11]) *on edge uw and vertex t* deletes an edge uw from a graph G and adds a new vertex v and new edges vu, vw, vt for some vertex $t \in V(G) - \{u, w\}$. A key step in the proof of the above combinatorial characterization is the following inductive construction.

Theorem 1.2. *(See [12].) Let G be a 3-connected and redundantly rigid graph. Then G can be obtained from K_4 by a sequence of 1-extensions and edge additions.*

We shall consider two problems related to globally rigid graphs and frameworks in $\mathbb{R}^2$:

(1) Develop an efficient deterministic algorithm which takes a globally rigid graph G as its input and outputs a 'non-trivial' globally rigid realization (G, p) of G in $\mathbb{R}^2$.
(2) Prove that if graph G' is obtained from a globally rigid graph G by a vertex splitting operation then G' is also globally rigid.

We next describe the main results and the proof methods.

1.1. Constructing a globally rigid framework

One may define 'non-trivial' in the construction problem in a number of ways. For instance, if G is globally rigid (and hence connected) then the realization of G in which all vertices are mapped to the same point is globally rigid. By using the fact that G must also be 2-connected (and hence it has an 'ear-decomposition') it is not difficult to construct a globally rigid realization of G in which no two points coincide but all points lie on the same line. To avoid these 'trivial' solutions, we call a realization of G *non-trivial* if the points $p(v)$, $v \in V(G)$, are not collinear. Our solution to problem (1) will always output a non-trivial realization. Nevertheless, the constructed framework may be rather degenerate: the positions of several vertices may coincide and certain edges may have length zero.

For a special family of globally rigid graphs, however, our algorithm is able to build a truly non-trivial globally rigid realization. Given a graph $G = (V, E)$ we say that a 1-extension on the edge uw and vertex t is a *triangle-split* if $\{ut, wt\} \subseteq E$ (that is, if u, w, t induce a triangle of G). A graph will be called *triangle-reducible* if it can be obtained from K_4 by a sequence of triangle-splits. For example, Cauchy-graphs and Grünbaum-graphs (defined in Section 3) are triangle-reducible. Furthermore, we shall prove that testing triangle-reducibility and finding an inductive construction for triangle-reducible graphs can be done efficiently.

When G is triangle-reducible, the algorithm provides a realization (G, p) in general position which will also be infinitesimally rigid (defined in Section 3). Infinitesimal rigidity ensures the 'stability' of (G, p), since it is known (see e.g. [3]) that if (G, p) is an infinitesimally and globally rigid framework then there exists an $\epsilon > 0$ such that if (G, q) is a realization of G for which $\|p(v) - q(v)\| < \epsilon$ for all $v \in V$ then (G, q) is also globally rigid.

The only result about the construction problem of globally rigid frameworks that we are aware of is the observation that it is easy to construct a globally rigid realization when G is a trilateration graph, see [6]. Note that trilateration graphs satisfy $|E| \geqslant 3|V| - 6$, while triangle-reducible graphs are sparser: they have $2|V| - 2$ edges, which is the smallest possible number in a globally rigid graph in $\mathbb{R}^2$.

One difficulty in the construction problem is due to the fact that there is no 'simple' sufficient condition for the global rigidity of a non-generic framework. We shall use a sufficient condition for global rigidity which is based on stress matrices, described in Section 2. The algorithm will construct the framework inductively: by using the inductive construction of globally rigid graphs from Theorem 1.2 it builds a 'Gale framework' on G which satisfies the stress condition.

We note that the problem of constructing a rigid realization of a rigid graph is, in some sense, simpler. In [7] a polynomial algorithm was given which creates a rigid realization on a small grid for any given rigid graph in $\mathbb{R}^2$.

1.2. Vertex splitting

Another familiar operation in combinatorial rigidity is vertex splitting. Given a graph $G = (V, E)$, an edge $uv \in E$, and a bipartition F_1, F_2 of the edges incident to v (except uv), the (2-dimensional) *vertex splitting* operation *on edge uv at vertex v*

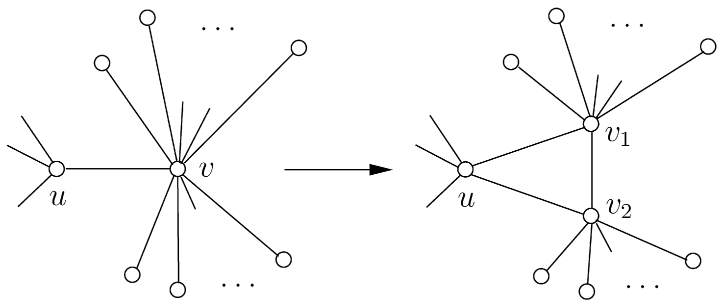

Fig. 1. The vertex splitting operation on edge uv and vertex v.

replaces vertex v by two new vertices v_1 and v_2, replaces the edge uv by three new edges uv_1, uv_2, v_1v_2, and replaces each edge $wv \in F_i$ by an edge wv_i, $i = 1, 2$, see Fig. 1. The vertex splitting operation is said to be *non-trivial* if F_1, F_2 are both non-empty, or equivalently, if each of the split vertices v_1, v_2 has degree at least three.

Cheung and Whiteley [3] conjecture that vertex splitting also preserves global rigidity in the following sense. For the definition of the d-dimensional vertex splitting operation see [18].

Conjecture 1.3. *(See [3].) If G is globally rigid in $\mathbb{R}^d$ and G' is obtained from G by a (d-dimensional) vertex splitting operation, so that each of the split vertices has degree at least $d+1$, then G' is globally rigid in $\mathbb{R}^d$.*

We shall verify the 2-dimensional version of Conjecture 1.3 by proving that non-trivial vertex splitting preserves global rigidity in $\mathbb{R}^2$. Our proof relies on Theorem 1.1: we prove that vertex splitting preserves both graph properties (3-connectivity and redundant rigidity) when applied to a 3-connected and redundantly rigid graph.

The structure of the paper is as follows. In Section 2 we state the sufficient condition for the global rigidity of a framework in terms of its stress matrix and introduce Gale frameworks. Section 3 contains the description of the operations on (Gale) frameworks that we shall use to create a globally rigid realization as well as our main algorithmic result. We devote Sections 3.1 and 3.2 to the special cases when G is triangle reducible or is a Cauchy or Grünbaum graph, respectively. In Section 4 we prove that a non-trivial vertex splitting operation preserves global rigidity.

2. Sufficient conditions for global rigidity of frameworks

The sufficient conditions known for the global rigidity of frameworks are in terms of stresses. Let $G = (V, E)$ be a graph, where V is the set of vertices labeled $1, 2, \ldots, n$. A *stress* is a map $\omega : E \to \mathbb{R}$. The stress is *non-zero* (*nowhere-zero*), if $w_{ij} \neq 0$ for at least one (resp., for all) $ij \in E$. The *stress matrix* Ω associated with a stress ω is an n-by-n symmetric matrix defined by

$$\Omega_{ij} = \begin{cases} \sum_{ki \in E} \omega_{ki} & \text{if } i = j \\ -\omega_{ij} & \text{if } i \neq j \text{ and } ij \in E \\ 0 & \text{if } i \neq j \text{ and } ij \notin E \end{cases}$$

Let (G, p) be a framework. We say that $\omega : E \to \mathbb{R}$ is a *self stress* for a framework (G, p) if for each $i \in V$,

$$\sum_{ij \in E} \omega_{ij}(p_i - p_j) = 0$$

It is easy to see that Ω is the stress matrix of a self stress of framework (G, p) if and only if Ω is symmetric, $\Omega_{ij} = 0$ whenever $ij \notin E$ $(i \neq j)$, and $P\Omega = 0$, where

$$P = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

is the *augmented configuration matrix* of p and $p_i = (p_{i1}, p_{i2})$ for all $i \in V$.

The proof of the following theorem can be extracted from papers by Connelly [4] and Whiteley [17]. See the preliminary version of our paper [13] for proof details. We say that a framework (G, p) is *bidirectional* if there exist vectors $v_1, v_2 \in \mathbb{R}^2$ such that for each $ij \in E$ either $p_i - p_j = \lambda v_1$ or $p_i - p_j = \lambda v_2$ holds for some $\lambda \in \mathbb{R}$. Otherwise (G, p) is said to be *multidirectional.*

Theorem 2.1. *Let (G, p) be a multidirectional framework on n vertices for which there is a self-stress ω, such that the associated stress matrix Ω is positive semi-definite and has rank $n-3$. Then (G, p) is globally rigid.*

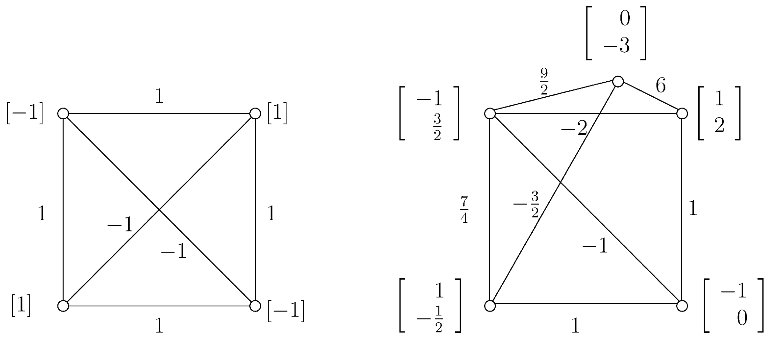

Fig. 2. The first framework is a Gale framework on K_4 with $p(v_1) = (0, 0)$, $p(v_2) = (1, 0)$, $p(v_3) = (1, 1)$, and $p(v_4) = (0, 1)$. The second framework is the Gale framework obtained from the first one by a 1-extension on edge v_1v_3 and vertex v_4 with parameters $\frac{2}{3}, -\frac{1}{6}, \frac{1}{2}, 3$. The position of the new vertex is $(\frac{2}{3}, \frac{7}{6})$. The values of the (nowhere-zero) stress are shown on the corresponding edges and the column vectors of the Gale transform are indicated at the vertices. Both frameworks are multidirectional and infinitesimally rigid.

We note that if a framework (G, p) satisfies the conditions of Theorem 2.1 then it is in fact *universally globally rigid*, which means that it is globally rigid in $\mathbb{R}^d$ for all $d \geqslant 2$. This is an unpublished result of Connelly. Since we use Theorem 2.1 to verify the global rigidity of the frameworks output by our algorithm, it follows that the constructed frameworks are also universally globally rigid.

2.1. Gale transforms

Let (G, p) be a framework and suppose that the points in p affinely span $\mathbb{R}^2$. Let A be an $(n-3) \times n$ matrix with linearly independent rows, satisfying $AP^\top = 0$. Then we say that the columns of A, treated as points $a_1, \ldots, a_n \in \mathbb{R}^{n-3}$, form the *Gale transform* of the original points $p_1, \ldots, p_n \in \mathbb{R}^2$ [17]. We say that the four-tuple (G, p, ω, A) is a *Gale framework* if (G, p) is a framework, ω is a stress for (G, p) and $A = (a_1, \ldots, a_n) \in \mathbb{R}^{(n-3)\times n}$ is a Gale transform of p satisfying $a_i^\top a_j = -\omega_{ij}$ for all $ij \in E$ and $a_i^\top a_j = 0$ for all $i, j \in V$, $i \neq j$, $ij \notin E$. As we shall see, $A^\top A$ is the positive semi-definite stress matrix of rank $n-3$ that we can use to certify the global rigidity of framework (G, p).

For example, the following is a multidirectional Gale framework on K_4, given by its augmented configuration matrix P, A, and a self-stress ω, see Fig. 2.

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A = [\,1 \quad -1 \quad 1 \quad -1\,]$$

$$\omega_{12} = \omega_{23} = \omega_{34} = \omega_{14} = 1$$

$$\omega_{13} = \omega_{24} = -1$$

Lemma 2.2. *Let (G, p, ω, A) be a Gale framework on n vertices. Then ω is a self-stress for (G, p) with a positive semi-definite stress matrix of rank $n-3$.*

Proof. Let $\Omega = A^\top A$. By the definition of Gale frameworks and the fact that $\Omega P^\top = A^\top AP^\top = A^\top 0 = 0$ we get that Ω is the stress matrix of ω and ω is a self-stress for (G, p). Ω has rank $n-3$ since A has $n-3$ independent rows and it is positive semi-definite since $q^\top \Omega q = q^\top A^\top Aq = (Aq)^\top(Aq) \geqslant 0$ for all $q \in \mathbb{R}^n$. $\square$

A Gale framework is *multidirectional* if (G, p) is multidirectional. By Theorem 2.1 and Lemma 2.2 we obtain:

Theorem 2.3. *Let (G, p, ω, A) be a multidirectional Gale framework. Then (G, p) is globally rigid.*

3. Globally rigid realizations

In this section we describe our algorithm which creates a globally rigid realization of a globally rigid graph G. The algorithm builds a multidirectional Gale framework on G inductively, following the local operations edge addition and 1-extension that we can use to construct G from K_4. In what follows we describe the 'parameterized' versions of these operations which work on (Gale) frameworks and prove that if the parameters are chosen appropriately, they take a multidirectional Gale framework to a multidirectional Gale framework. We shall also prove that when the 1-extension happens to be a triangle-split, the parameters can be chosen so that the operation preserves infinitesimal rigidity, too. This will yield the desired extra property for triangle-reducible graphs.

Let (G, p) be a framework, let $uw \in E(G)$, $t \in V(G) - \{u, w\}$, and let $\alpha_u, \alpha_w, \alpha_t$ be real numbers with $\alpha_u + \alpha_w + \alpha_t = 1$. The 1-*extension* operation *on edge* uw *and vertex* t *with parameters* $\alpha_u, \alpha_w, \alpha_t$ consists of performing a 1-extension on G which adds a new vertex v, as well as extending the realization p by letting $p(v) = \alpha_u p(u) + \alpha_w p(w) + \alpha_t p(t)$.

Lemma 3.1. *Let (G, p) be a multidirectional framework and (G^*, p^*) its 1-extension with parameters $\alpha_u, \alpha_w, \alpha_t$. If $\alpha_t = 0$ or $\alpha_u\alpha_w \neq 0$, then (G^*, p^*) is multidirectional.*

Proof. If p_u, p_w, p_t are collinear or $\alpha_t = 0$, then the set of edge directions of (G^*, p^*) are the same as that of (G, p). Otherwise, p_u, p_w, p_t are affinely independent and $\alpha_u, \alpha_w, \alpha_t \neq 0$. In this case the edges vu, vw, vt define three independent directions, so (G^*, p^*) is multidirectional. □

Let (G, p, ω, A) be a Gale framework and let $\beta \neq 0$ be a real number. The 1-*extension* operation on edge uw and vertex t *with parameters* $\alpha_u, \alpha_w, \alpha_t, \beta$ consists of performing a 1-extension of (G, p) with parameters $\alpha_u, \alpha_w, \alpha_t$, as defined above, as well as replacing ω and A by ω^* and A^* by letting

$$\omega^*_{ij} = \begin{cases} \omega_{ij} & \text{if } ij \in E - \{uw, ut, wt\} \\ \omega_{ij} - \beta^2\alpha_i\alpha_j & \text{if } ij \in E \cap \{ut, wt\} \\ \beta^2\alpha_j & \text{if } i = v \text{ and } j \in \{u, w, t\} \end{cases}$$

$$A^* = \begin{bmatrix} a_1 & \cdots & a_u & a_w & a_t & \cdots & a_n & 0 \\ 0 & \cdots & \beta\alpha_u & \beta\alpha_w & \beta\alpha_t & \cdots & 0 & -\beta \end{bmatrix}$$

Lemma 3.2. *Let (G, p, ω, A) be a Gale framework and let $(G^*, p^*, \omega^*, A^*)$ be its 1-extension with parameters $\alpha_u, \alpha_w, \alpha_t, \beta$. If $\alpha_u\alpha_w = \omega_{uw}/\beta^2$, and if $\alpha_t = 0$ whenever $\{ut, wt\} \nsubseteq E$, then $(G^*, p^*, \omega^*, A^*)$ is a Gale framework.*

Proof. Let a^*_i denote the columns of A^*, $1 \leqslant i \leqslant n+1$. It is easy to check that A^* is a Gale transform of p^* and $a^{*\top}_i a^*_j = -\omega^*_{ij}$ if $ij \in E^*$. Let us suppose now that $ij \notin E^*$ for some $i, j \in V^*$, $i \neq j$. Then either $i = v$ and $j \in V - \{u, w, t\}$, or $i \in \{u, w, t\}$ and $j \in V - \{u, w, t\}$ and $ij \notin E$, or $ij \in \{ut, wt\} - E$, or $ij = uw$. In the first case $a^{*\top}_i a^*_j = 0^\top a_j + \beta 0 = 0$. In the second case $a^{*\top}_i a^*_j = a^\top_i a_j + \beta\alpha_i 0 = a^\top_i a_j = 0$. In the third case $a^{*\top}_i a^*_j = a^\top_i a_t + \beta^2\alpha_i\alpha_t = 0$. In the last case $a^{*\top}_i a^*_j = a^\top_u a_w + \beta^2\alpha_u\alpha_w = -\omega_{uw} + \omega_{uw} = 0$. □

We next describe how the parameters are defined when the algorithm applies edge addition or 1-extension to a Gale framework (G, p, ω, A).

- **Edge addition** In this case G^* is obtained from G by adding an edge uw. We define $(G^*, p^*, \omega^*, A^*)$ by letting $p^* = p$, $A^* = A$, $\omega^*_{ij} = \omega_{ij}$ if $ij \in E(G)$ and $\omega^*_{uw} = 0$.
- **1-Extension** In this case G^* is obtained from G by a 1-extension on edge uw and vertex t. We define $(G^*, p^*, \omega^*, A^*)$ by defining the parameters $\alpha_u, \alpha_w, \alpha_t, \beta$ of the 1-extension operation on (G, p, ω, A). This will also determine $p(v)$. We consider three cases.

  Case 1 $\omega_{uw} = 0$.

  Let $\alpha_t = 0$ and let $\alpha_u = 0$ or $\alpha_w = 0$. Let $\beta = 1$.
  (Note that in this case the choice of the parameters implies that $p(v) = p(w)$ or $p(v) = p(u)$ must hold.)

  Case 2 $\omega_{uw} \neq 0$ and $\{ut, wt\} \nsubseteq E$.

  Let $\alpha_t = 0$ and let α_u, α_w be chosen so that $\alpha_u\alpha_w$ has the same sign as ω_{uw}. Let $\beta^2 = \frac{\omega_{uw}}{\alpha_u\alpha_w}$.
  (In this case $p(v) = \alpha_u p(u) + \alpha_w p(w)$ and hence $p(v) = p(u) = p(w)$ or $p(v)$ lies on the line L_{uw} through $p(u), p(w)$. If $\omega_{uw} > 0$ ($\omega_{uw} < 0$) then $p(v)$ lies on the segment $[p(u), p(w)]$ (resp. it lies on $L_{uw} - [p(u), p(w)]$).)

  Case 3 $\omega_{uw} \neq 0$ and $\{ut, wt\} \subseteq E$.

  Let $\alpha_u, \alpha_w, \alpha_t$ be chosen so that $\alpha_u\alpha_w$ has the same sign as ω_{uw}, and so that $\alpha_t \notin \{0, \frac{\omega_{ut}}{\omega_{uw}}\alpha_u, \frac{\omega_{wt}}{\omega_{uw}}\alpha_w\}$. Let $\beta^2 = \frac{\omega_{uw}}{\alpha_u\alpha_w}$. See Fig. 2.
  (Note that if $p(u), p(w), p(t)$ are not collinear then, depending on the sign of ω_{uw}, it is possible to choose the parameters so that $p(v)$ becomes equal to any given point in the interior of two of the four regions determined by the lines through $p(u), p(t)$ and $p(w), p(t)$ (with the exception of three lines corresponding to the non-admissible values of α_t), see Fig. 3.)

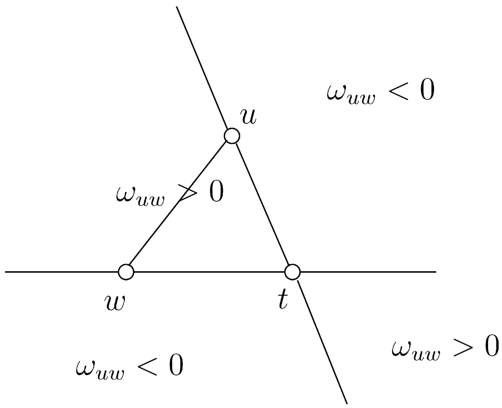

Fig. 3. The admissible regions of $p(v)$ in Case 3.

As mentioned in the Introduction, when G is triangle-reducible, our algorithm will construct a realization which is also infinitesimally rigid. To define this notion let us recall another basic concept. The *rigidity matrix* of the framework is the matrix $R(G,p)$ of size $|E| \times d|V|$, where, for each edge $v_iv_j \in E$, in the row corresponding to v_iv_j, the entries in the d columns corresponding to vertices i and j contain the d coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros. We say that a framework (G,p) on n vertices in $\mathbb{R}^d$ is *infinitesimally rigid* if $\operatorname{rank} R(G,p) = \max\{\operatorname{rank} R(K_n,q)\colon\ q \in \mathbb{R}^{dn}\}$, where K_n is the complete graph on n vertices. It is known that the infinitesimal rigidity of (G,p) implies rigidity, and that the reverse implication holds if the realization is generic. We refer the reader to [9,19] for more details.

Lemma 3.3. *Suppose that (G,p,ω,A) is a multidirectional Gale framework for which (G,p) is infinitesimally rigid, ω is nowhere-zero, and the points $p(v)$, $v \in V$, are in general position. Let (G^*,p^*,ω^*,A^*) be obtained from (G,p,ω,A) by a 1-extension as described in Case 3. Then (G^*,p^*,ω^*,A^*) is a multidirectional Gale framework, for which (G^*,p^*) is infinitesimally rigid and ω^* is nowhere-zero.*

Proof. By Lemmas 3.1 and 3.2 (G^*,p^*,ω^*,A^*) is a multidirectional Gale framework. Since ω is nowhere-zero, we have $\omega_{uw} \neq 0$. Thus we must have $\alpha_u \neq 0$ and $\alpha_w \neq 0$. Hence $\omega^*_{vi} = \beta^2\alpha_i \neq 0$ for $i \in \{u,w,t\}$. Furthermore, the choice of α_t implies that $\omega^*_{ut} = \omega_{ut} - \beta^2\alpha_u\alpha_t = \omega_{ut} - \omega_{uw}\alpha_t/\alpha_w \neq 0$. Similarly, $\omega^*_{wt} \neq 0$. Thus ω^* is a nowhere-zero stress.

Since the self stress ω is also a linear dependency of the rows of the rigidity matrix $R(G,p)$, and since ω is nowhere-zero, each row of $R(G,p)$ can be expressed as a linear combination of the remaining rows, thus deleting any row will not decrease its rank. This shows that $(G - uw, p)$ is infinitesimally rigid. To show that (G^*,p^*) is infinitesimally rigid we first observe that $p(v)$ is not on the line through $p(u), p(w)$, since $p(u)$, $p(w)$ and $p(t)$ are in general position and $\alpha_t \neq 0$. Thus the addition of the new point $p(v)$ and the new edges $p(v)p(u)$ and $p(v)p(w)$ increases the rank of the rigidity matrix by two and hence preserves infinitesimal rigidity (see [19, Lemma 2.1.3]). □

By applying the operations described in the previous lemmas we can now deduce the main result of this section.

Theorem 3.4. *Let G be a globally rigid graph on at least four vertices. Then one can construct, in polynomial time, a non-trivial globally rigid realization (G,p). Furthermore, if G is triangle-reducible, such a realization can be constructed so that it is infinitesimally rigid and its points are in general position.*

Proof. Let $K_4 = H_1, H_2, \dots, H_m = G$ be an inductive construction of G from K_4 using edge-additions and 1-extensions. Such a sequence exists by Theorem 1.2. Furthermore, if G is triangle-reducible, we may assume that H_{i+1} is obtained from H_i by a triangle-split, $1 \leqslant i \leqslant m-1$. Since 3-connectivity and redundant rigidity can be tested in polynomial time, see e.g. [2], the inductive construction of G can be obtained in polynomial time. See also the discussion about triangle-reducible graphs before Lemma 3.5 below.

Let (H_1,p_1,ω_1,A_1) be a multidirectional Gale framework on $H_1 = K_4$. If G is triangle-reducible, we choose one with a nowhere-zero stress and for which (H_1,p_1) is infinitesimally rigid and is in general position. The K_4 example in Fig. 2 satisfies all these conditions.

To compute a globally rigid framework on G we follow the inductive construction and perform edge additions and 1-extensions as described in Cases 1–3, to create multidirectional Gale frameworks (H_i,p_i,ω_i,A_i) for $1 \leqslant i \leqslant m$. By Lemmas 3.1, 3.2, and Theorem 2.3, the framework (H_m,p_m) will be a globally rigid realization of G.

If, in addition, G is triangle-reducible, we only perform 1-extensions as described in Case 3, by choosing the parameters so that the points in each framework (H_i,p_i), $1 \leqslant i \leqslant m$, are in general position. In this case Lemma 3.3 implies that (H_m,p_m) will also be infinitesimally rigid.

Observe that the algorithm does not need to compute the Gale transforms A_i but only updates the stress and the realization. It is not difficult to see that the numbers (the values of the self-stress and the coordinates of the vertices) occurring in the algorithm can always be chosen to be of polynomial size. $\square$

3.1. Testing triangle-reducibility

In this subsection we show that testing triangle-reducibility (and finding an inductive construction for triangle-reducible graphs) can be done efficiently in a greedy fashion. Let $G = (V, E)$ be a graph. If G is triangle-reducible and $|V| > 4$ then there must be a vertex v with neighbors x, y, z spanning exactly two edges in G. The following lemma says that we can easily eliminate such a vertex to obtain a smaller triangle-reducible graph. Thus triangle-reducibility can be tested with a simple greedy algorithm which also provides a sequence of triangle-splits which generates G.

Lemma 3.5. *Let $G = (V, E)$ be a triangle-reducible graph with $|V| \geqslant 5$ and let $v \in V$ be a vertex with three neighbors x, y, z. Suppose that $xz, yz \in E$ and $xy \notin E$. Then $G' = G - v + xy$ is triangle-reducible.*

Proof. Let $K_4 = H_1, H_2, \dots, H_m = G$ be a sequence of graphs, where H_{i+1} is obtained from H_i by a triangle-split, $1 \leqslant i \leqslant m-1$. Consider the first graph H_k in the sequence which contains v. It is easy to see that, by modifying H_1 and H_2, if necessary, we may assume that $k \geqslant 2$. Thus v is created by a triangle split operation on H_{k-1}.

Since a triangle split does not decrease the degree of any vertex, v must have degree three in H_l for all $k \leqslant l \leqslant m$. Furthermore, observing that a triangle split does not increase the number of edges induced by $N_l(v)$, it follows that $N_l(v)$ induces exactly two edges in H_l for all $k \leqslant l \leqslant m$, where $N_i(v)$ denotes the set of neighbors of v in some H_i.

Let $N_k(v) = \{u, w, t\}$ and suppose that $ut, wt \in E(H_k)$ and $uw \notin E(H_k)$. Next observe that as long as t remains a neighbor of v, the other two neighbors of v must be non-adjacent. In fact, t must remain a neighbor of v in the rest of the sequence.

Claim 3.6. *$vt \in E(H_l)$ for all $k \leqslant l \leqslant m$.*

Proof. Let $i \geqslant k$ be the largest index for which $vt \in E(H_i)$. For a contradiction suppose that $i \leqslant m-1$. Let $N_i(v) = \{u_i, w_i, t\}$. It follows from the previous observation that we must have $u_i w_i \notin E(H_i)$. Since $vt \notin E(H_{i+1})$, it follows that H_{i+1} is obtained from H_i by 'splitting' the edge vt by a new vertex t' of degree three. Hence $N_{i+1}(v) = \{u_i, w_i, t'\}$ induces at most one edge in H_{i+1}. This contradicts the fact that the neighbors of v induce exactly two edges in H_l for all $k \leqslant l \leqslant m$. $\square$

It follows from Claim 3.6 that $N_i(v) = \{u_i, w_i, t\}$ and $u_i w_i \notin E(H_i)$ for all $k \leqslant i \leqslant m$. Thus $z = t$ holds. Let $H_i' = H_i - v + u_i w_i$, $k \leqslant i \leqslant m$. Next we show, by induction on i, that H_i' is triangle-reducible. Since $H_k' = H_{k-1}$, it is true for $i = k$. Suppose that $N_{i+1}(v) = N_i(v)$, i.e. the triangle-split, applied to H_i, leaves the neighbor set of v unchanged. Then H_{i+1}' can be obtained from H_i' by the same triangle split, and hence, by induction, H_{i+1}' is also triangle-reducible. Otherwise H_{i+1} is obtained from H_i by 'splitting' the edge vu_i (or vw_i). Then, without loss of generality, we have $H_{i+1} = H_i + u_{i+1} - vu_i + \{u_{i+1}v, u_{i+1}u_i, u_{i+1}t\}$. Then $w_{i+1} = w_i$ and $H_{i+1}' = H_i' + u_{i+1} - u_i w_i + \{u_{i+1}w_i, u_{i+1}u_i, u_{i+1}t\}$. So H_{i+1}' can be obtained from H_i' by a triangle-split. By induction, this gives that H_{i+1}' is triangle-reducible. Thus $G' = H_m'$ is triangle-reducible, which completes the proof. $\square$

3.2. Cauchy and Grünbaum graphs

Another sufficient condition for global rigidity, due to Connelly, is based on stresses as well as convexity. Here we formulate a 2-dimensional version of his result for bar-and-joint frameworks, which can be deduced from Corollary 1 and Theorem 5 of [4].

Theorem 3.7. *(See [4].) Let (G, p) be a framework whose edges form a convex polygon P in $\mathbb{R}^2$ with some chords. Suppose that there is a non-zero self-stress ω for (G, p) for which $\omega_{ij} \geqslant 0$ if $ij \in E$ is an edge on the boundary of P and $\omega_{ij} \leqslant 0$ if $ij \in E$ is an edge which is a chord of P. Then (G, p) is globally rigid.*

The *Cauchy-graphs* C_n and *Grünbaum graphs* G_n are both defined on vertex set $\{1, \dots, n\}$ and both contain the edges $\{i, i+1\}$, $i = 1, 2, \dots, n$ (modulo n). In addition, the Cauchy graph contains the chords $\{i, i+2\}$, $i = 1, \dots, n-2$, and the Grünbaum graph has the edges $\{1, 3\}$ and $\{2, i\}$ for $i = 4, \dots, n$. Thus G_n is a wheel graph centered on vertex 2.

A *Cauchy-polygon* (*Grünbaum polygon*) is a framework (C_n, p) (resp. (G_n, p)) whose vertices $p_1, \dots, p_n$ are in general position and, in this order, they correspond to the vertices of a convex polygon in the plane. See Fig. 4.

It is easy to check that Cauchy-graphs as well as Grünbaum-graphs are triangle-reducible. One can also show, by induction on n, that any given Cauchy-polygon (C_n, p) (or Grünbaum-polygon (G_n, p)) can be obtained as the output of our algorithm. This gives a different proof of the first part of the next theorem.

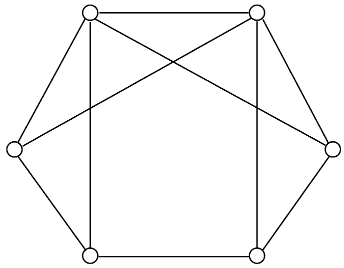

Fig. 4. A Cauchy-polygon on 6 vertices.

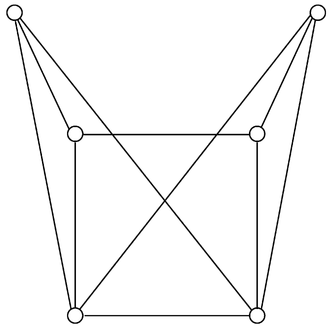

Fig. 5. A non-convex globally and infinitesimally rigid realization of the Cauchy-graph C_6.

Theorem 3.8. (i) [4, Lemma 4, Theorem 5] *Every Cauchy-polygon* (C_n, p) *is globally rigid.*
(ii) *Every Grünbaum-polygon* (G_n, p) *is globally rigid.*

Note that our algorithm may also generate non-convex globally rigid realizations of Cauchy-graphs, see Fig. 5. Thus, in this sense, it gives an extension of Theorem 3.8(i).

4. Vertex splitting

In what follows we shall prove that non-trivial vertex splitting preserves global rigidity in $\mathbb{R}^2$. First we recall some additional basic notions and results concerning rigid graphs in $\mathbb{R}^2$. Let $G = (V, E)$ be a graph. Then G is *minimally rigid* if G is rigid but $G - e$ is not rigid for all $e \in E$. An edge $e \in E$ is *redundant* in a rigid graph G if $G - e$ is rigid. An edge is redundant in G if and only if it belongs to a *circuit*, i.e. a minimal subgraph in G in which each edge is redundant. It can be deduced from Laman's characterization of minimally rigid graphs [14] that C is a circuit if and only if $|E(C)| = 2|V(C)| - 2$ and $i_C(X) \leqslant 2|X| - 3$ holds for all subsets $X \subset V(C)$ with $2 \leqslant |X| \leqslant |V(C)| - 1$, where $i_C(X)$ denotes the number of edges induced by X in C. Note that a graph G is redundantly rigid if and only if G is rigid and each edge of G belongs to a circuit. A graph G is called *M-connected* if for each pair of edges $e, f \in E$ there is a circuit C in G with $e, f \in E(C)$. We refer the reader to [9,12,19] for more details and related results in terms of the rigidity matroid.

It is known that vertex splitting preserves rigidity [18,19]. For completeness we sketch a different proof of this fact, which uses the result that a graph G is minimally rigid if and only if it can be built up from an edge by applying 0-extensions and 1-extensions [11,16]. The 0-*extension* operation on a graph H adds a new vertex v and new edges vu, vw for two distinct vertices $u, w \in V(H)$. The sequence of graphs in this inductive construction of G is called a *Henneberg sequence*. It is also known that any designated edge of G can be chosen as the first graph in this sequence.

Lemma 4.1. *Let G be a rigid graph and let G' be obtained from G by a vertex splitting operation. Then G' is rigid.*

Proof. Let G' be obtained from G by a vertex splitting on edge uv at vertex v, with bipartition F_1, F_2. Let H be a minimally rigid spanning subgraph of G which contains the edge uv and consider a Henneberg sequence $H_1, H_2, \dots, H_m$ of H with $H_1 = uv$ and $H_m = H$. Let us define a bipartition F_1^j, F_2^j of the edges incident to v (except uv) in each H_j, starting with H_m, as follows. Let $F_i^m = F_i$ for $i = 1, 2$. Now suppose that $j < m$ and let wv be an edge different from uv in H_j. If $wv \in E(H_j) \cap E(H_{j+1})$ then let wv belong to the same class of the bipartition as in H_{j+1}. Otherwise, if $wv \in E(H_j) - E(H_{j+1})$, then H_{j+1} is obtained from H_j by a 1-extension which replaces wv by two edges yv, yw, where y is a new vertex. In this case let the bipartition class of wv be defined to be the same as that of yv in H_{j+1}.

To see that G' is rigid apply the 'same' Henneberg sequence to build a graph but start with a triangle on vertices u, v_1, v_2 instead of the edge uv and so that whenever a new edge incident to v is to be added in H_j, connect the corresponding

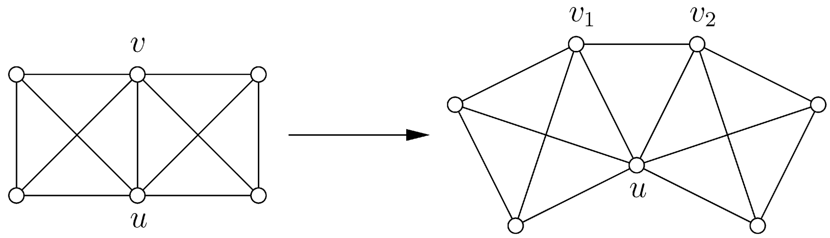

Fig. 6. Non-trivial vertex split may destroy redundant rigidity.

edge to either v_1 or v_2 according to the bipartition F_1^j, F_2^j. The graph H'_m obtained this way is a minimally rigid spanning subgraph of G'. Thus G' is rigid. $\square$

Next we show that a non-trivial vertex splitting operation takes a circuit to a circuit.

Lemma 4.2. *Let C be a circuit and let C' be obtained from G by a non-trivial vertex splitting. Then C' is a circuit.*

Proof. Suppose that the vertex splitting is made on edge uv at vertex v with bipartition F_1, F_2. Since the splitting is non-trivial, F_1 and F_2 are both non-empty. We shall use the above mentioned characterization of circuits to show that C' is indeed a circuit. Since C is a circuit, it is clear that $|E(C')| = 2|V(C')| - 2$. Consider a proper subset X' of $V(C')$ and let X denote the corresponding subset of $V(C)$ obtained by identifying vertices v_1 and v_2. Clearly, if $|X' \cap \{u, v_1, v_2\}| \in \{0, 1, 3\}$ or $X' \cap \{u, v_1, v_2\} = \{v_1, v_2\}$ then X is a proper subset of $V(C)$ and we have $2|X'| - i_{C'}(X') \geqslant 2|X| - i_C(X) \geqslant 3$. If $X' \cap \{u, v_1, v_2\} = \{u, v_i\}$ for some $i = 1, 2$ then either X is a proper subset of $V(C)$ and we have $2|X'| - i_{C'}(X') \geqslant 2|X| - i_C(X) \geqslant 3$, or $V(C') - X' = \{v_j\}$, $j \neq i$. In the latter case we can use the fact that $|E(C')| = 2|V(C')| - 2$ and $F_j \neq \emptyset$ to deduce that $2|X'| - i_{C'}(X') \geqslant 3$. This completes the proof. $\square$

Applying a vertex splitting operation to an arbitrary redundantly rigid graph G may destroy redundant rigidity, even if the operation is non-trivial, see Fig. 6. We shall prove that this cannot happen when G is 3-connected. First we need the following observation.

Lemma 4.3. *Let G be a 3-connected graph and let G' be obtained from G by a non-trivial vertex splitting operation. Then G' is 3-connected.*

Proof. Suppose that the vertex splitting is made on edge uv at vertex v with bipartition F_1, F_2. Since the splitting is non-trivial, F_1 and F_2 are both non-empty. For a contradiction suppose that G' is not 3-connected. Then there is a small separator, i.e. a set $S \subset V(G')$ with $|S| \leqslant 2$ for which $G' - S$ is disconnected. Since each vertex has degree at least three in G', it follows that each connected component of $G' - S$ contains at least two vertices. Furthermore, since u, v_1, v_2 induce a triangle in G', there is exactly one component of $G' - S$ which intersects $\{u, v_1, v_2\}$. This implies that G, which can be obtained from G' by contracting the edge v_1v_2 (i.e. by performing the inverse of the vertex splitting operation) also has a separator of size at most two, a contradiction. $\square$

We say that a graph G is *nearly* 3*-connected* if G can be made 3-connected by adding at most one new edge. We need the following result on M-connected graphs. The first part appears as [12, Lemma 3.1]. The second part was proved in [12, Theorem 3.2] for redundantly rigid graphs. The same proof goes through under the weaker hypothesis that each edge of G is in a circuit.

Theorem 4.4. *(See [12].)* (a) *If G is M-connected then G is redundantly rigid.*
(b) *If G is nearly* 3*-connected and each edge of G is in a circuit then G is M-connected.*

We are now ready to prove the main result of this section.

Theorem 4.5. *Let G be a* 3*-connected and redundantly rigid graph and let G' be obtained from G by a non-trivial vertex splitting operation. Then G' is also* 3*-connected and redundantly rigid.*

Proof. Suppose that the vertex splitting is made on edge uv at vertex v with bipartition F_1, F_2. Since the splitting is non-trivial, F_1 and F_2 are both non-empty. It follows from Lemmas 4.1 and 4.3 that G' is 3-connected and rigid. It remains to prove that $G' - xy$ is rigid for all edges $xy \in E(G')$. If $xy \notin \{uv_1, uv_2, v_1v_2\}$ then this follows from Lemma 4.1 and the

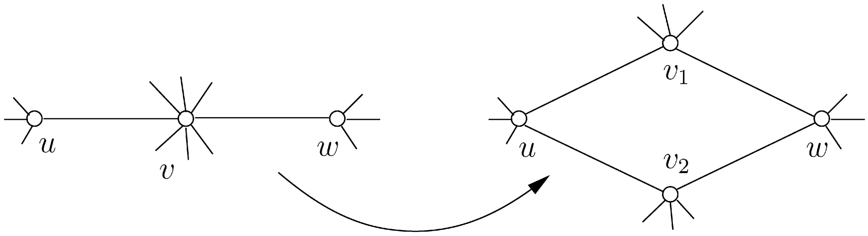

Fig. 7. The diamond split operation.

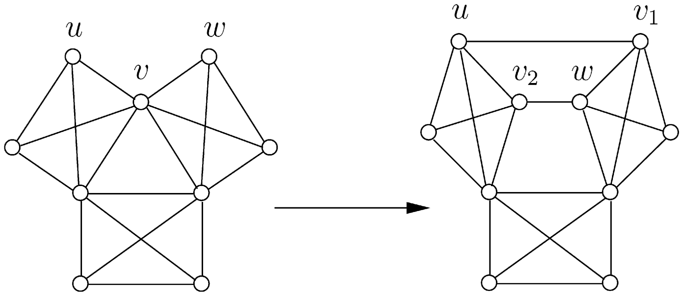

Fig. 8. Diamond split may destroy redundant rigidity.

hypothesis that G is redundantly rigid, since, for such an edge, $G' - xy$ can be obtained from $G - xy$ by a vertex splitting operation on edge uv at v.

To deal with the remaining edges let us choose and edge $va \in F_1$ and consider a circuit C in G with $\{uv, va\} \subset E(C)$. Such a circuit exists, since G is M-connected by Theorem 4.4(b). First suppose that $E(C) \cap F_2 \neq \emptyset$. In this case G' contains, as a subgraph, a graph C' obtained from C by a non-trivial vertex splitting on edge uv at v with bipartition F_1', F_2', where $F_1' = F_1 \cap E(C)$ and $F_2' = F_2 \cap E(C)$. Since $\{uv_1, uv_2, v_1v_2\} \subset E(C')$, it follows from Lemma 4.2 that uv_1, uv_2, and v_1v_2 are also redundant edges in G' and hence we are done.

Next suppose that $E(C) \cap F_2 = \emptyset$. In this case G' contains a subgraph which is isomorphic to C and contains the edge uv_1 (i.e. the subgraph obtained from C by replacing uv by uv_1, and replacing each edge $wv \in E(C) \cap F_1$ by wv_1). Thus uv_1 is a redundant edge in G'. By symmetry we obtain that uv_2 is redundant as well. Hence we are done if v_1v_2 is also redundant in G'.

Otherwise, when $G' - v_1v_2$ is not rigid, and hence v_1v_2 is in no circuit in G', the above arguments imply that each edge of $G' - v_1v_2$ belongs to a circuit. Since G' is 3-connected, $G' - v_1v_2$ is nearly 3-connected. Theorem 4.4 now implies that $G' - v_1v_2$ is M-connected, and hence rigid, a contradiction. This completes the proof of the theorem. □

Theorems 1.1 and 4.5 imply the two-dimensional version of Conjecture 1.3.

Theorem 4.6. *Let G be a globally rigid graph in $\mathbb{R}^2$ and let G' be obtained from G by a non-trivial vertex splitting. Then G' is also globally rigid in $\mathbb{R}^2$.*

4.1. Diamond split

There is a second form of vertex splitting in two dimensions. Let uv, vw be two adjacent edges and let F_1, F_2 be a bipartition of the edges incident to v (except uv, vw). The operation *diamond split* replaces vertex v by two new vertices v_1, v_2, replaces the edges uv, vw by a four-cycle uv_1, uv_2, wv_1, wv_2, and then replaces each edge $zv \in F_i$ by an edge zv_i, for $i = 1, 2$. See Fig. 7. It is known that diamond split preserves rigidity [1].

Whiteley [20] asked whether diamond-split preserves redundant rigidity or global rigidity, provided each of the two new vertices has degree at least three. It is not difficult to show, just like in Lemma 4.2, that such a diamond split operation takes a circuit to a circuit. In general, however, it may destroy redundant rigidity, see Fig. 8. Since the diamond split operation may also destroy 3-connectivity (when u, v, w form a separating set in G, u, w may become a separating pair after the split), it follows that it does not always preserve global rigidity either. Further useful operations as well as results about the effect of the above split operations to planar circuits and their duals can be found in [1].

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